Chapter 7

Modeling binary, nominal and count data

Modeling binary variables

Practical question: a bank should decide about granting loans to new clients, i.e. forecast of the solvency

$$Y_i = \begin{cases} 0, & \text{the client } i \text{ is solvent} \\ 1, & \text{the client } i \text{ is insolvent} \end{cases}$$

X_{1i}	–	debt-to-income ratio ($\times 100$);
X_{2i}	–	years with the current employer;
X_{3i}	–	other debts (in 1000 Euro);
X_{4i}	–	age (in years).

x_1 large $\Rightarrow$ higher chances of being insolvent

x_1 small $\Rightarrow$ lower chances of insolvency.

Question: can we use a linear regression model for binary variales? $\rightsquigarrow$ linear probability model

$$\hat{\beta} = (X'X)^{-1}X'\underline{y} \leftarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{not a problem can be estimated.}$$

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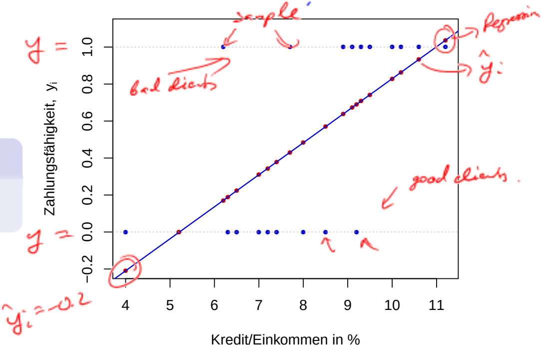
scatter plot: on two horizontal lines ⇒ unusual



Linear Prob.-model

$$Y_i = \beta_0 + \beta_1 \cdot X_i + u_i$$

{0,1}.   debt/income.

Note:

- $(+)$ the forecast $\hat{Y}_i$ can be seen as probability

  expectation for discreet RV

$$E(Y_i|X_i) = 1 \cdot P(Y_i = 1|X_i) + 0 \cdot P(Y_i = 0|X_i) = p_i$$

  ⇒ $\hat{Y}_i$ can be seen as the probability of being insolvent!

- $(-)$ $\hat{Y}_i$ may lie outside of [0,1] ⇒ not a probability.
- $(-)$ R^2 is useless as a goodness-of-fit measure ($R^2=1$ cannot be attained)
- $(-)$ the residuals are not normally distributed Y-Binary ⇒ u Binary ⇒ not normal.
- $(-)$ $Var(Y_i|X_i) = p_i(1-p_i) \neq const \rightsquigarrow$ heteroscedastic

⇒ $Var(u_i) \neq$

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Transition to Logit/Probit

$\{0,1\}$

Let Y_i be the observed binary variable and Y_i^* the corresponding unobserved metric variable. For Y_i^* it holds:

continuous ⇒ LR is OK

$$Y_i^* = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_k X_{ki} + u_i = \boldsymbol{X}_i'\boldsymbol{\beta} + u_i.$$

a continuous variable

Example: Y_i^* is an unobserved solvency of the client i with

$$\underline{Y_i = 1} \text{ if } Y_i^* > 0 \text{ and } \underline{Y_i = 0} \text{ if } Y_i^* \leq 0.$$

debt/income

$$\begin{aligned} P(Y_i = 1|\boldsymbol{X}_i) &= P(Y_i^* > 0|\boldsymbol{X}_i) = P(\cancel{\boldsymbol{X}}_i'\boldsymbol{\beta} + u_i > 0|\boldsymbol{X}_i) \\ &= \underbrace{P(-u_i < \boldsymbol{X}_i'\boldsymbol{\beta}|\boldsymbol{X}_i)}_{\text{definition of the cdf}} = F(\boldsymbol{X}_i'\boldsymbol{\beta}), \end{aligned}$$

⇒ $P(Y=1)$ = to the cdf of the residuals u evaluated at $X'\beta$.

where $F(\cdot)$ is the cdf of the residuals.

- $F(z) = \frac{1}{1+e^{-z}}$ - the cdf of the logistic distribution $\rightsquigarrow$ logit
- $F(z)$ - the cdf of the normal distribution $\rightsquigarrow$ probit

$$P(Y_i = 1|X_i) = F(X_i'\beta)$$

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Logistic regression

$\lim_{z\to\infty} \frac{1}{1+e^{-z}} = 1$

Idea: transformation with the logistic function

F (or cdf of the logistic distribution)

Logistic model

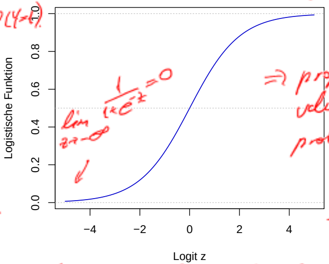
$$p_i = P(Y_i = 1|\boldsymbol{X}_i) = \frac{1}{1+e^{-z_i}}; \quad z_i \to X_i'\beta$$

for logits z_i it holds

$$z_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_k X_{ki} \cancel{+ u_i}$$

$P(Y=1)$

$\lim_{z\to-\infty} \frac{1}{1+e^{-z}} = 0$



$\Rightarrow$ proper values for probabilities.

z_i

important: no residuals here!!! $P(Y_i=1)$ is a probability, so no RV's allowed on the right-hand-side.

Note: Alternatively we may use the CDF $\Phi(z_i)$ of $N(0,1)$ $\rightsquigarrow$ probit-model

$$P(Y_i = 1|X_i) = \Phi(z_i)$$

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Estimation of the parameters

The parameters are estimated using ML:

$y_i = 1$ $\qquad y_i = 0$

$$L = \prod_{i=1}^{n} \underbrace{\left(\frac{1}{1+e^{-z_i}}\right)^{y_i}}_{P(Y_i=1)} \cdot \underbrace{\left(1 - \frac{1}{1+e^{-z_i}}\right)^{1-y_i}}_{P(Y_i=0)} \longrightarrow max, \text{ w.r.t. } \beta_0, \ldots, \beta_k.$$

Note:

- In contrary to the LR the estimation is always numeric.
- Likelihood-Ratio tests can be used to check the significance of the parameters.
- R: `glm(y ~ X,data=data, family=binomial(logit))`

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.434735   0.482326  -2.975  0.00293 **
debtinc      0.121391   0.019023   6.381 1.76e-10 ***
employer    -0.161795   0.023742  -6.815 9.44e-12 ***
debts        0.093460   0.045045   2.075  0.03801 *
age         -0.004397   0.014212  -0.309  0.75701
---
```

significant

insignificant

$\beta_0, \dots \beta_4$ $\Rightarrow$ age has no impact on the probability of being solvent / insolvent.

Interpretation: LR : x_i changes by 1 $\Rightarrow$ y changes by β_i

Here : difficult x_i changes by 1 $\Rightarrow$ $P(Y=1)$ – changes by ???

Example: a data set with 700 observations

odds = 3 ⇒ the probability of being insolvent is 3 times higher than probability of being solvent.

x_i changes by 1, then odds changes by e^{β_i}

	debtinc	employer	debts	age
$\hat{\beta}_i$	0.121*	-0.162*	0.093*	-0.004
$e^{\hat{\beta}_i}$	1.129	0.851	1.098	0.996

(*) - significant with $\alpha = 0.05$

$\frac{1}{1+e^{-z}}$

Odds of the logistic regression

$$\text{Odds} = \frac{P(Y=1|\boldsymbol{X})}{P(Y=0|\boldsymbol{X})} = e^z$$

$\Rightarrow 1 - \frac{1}{1+e^{-z}} = \frac{1+e^{-z}-1}{1+e^{-z}} = \frac{e^{-z}}{1+e^{-z}}$

	Logit (z)	Odds	$P(Y=1\|\boldsymbol{X})$
⇒ $\beta > 0$	rises by β	rises by $e^\beta > 1$	rises
⇒ $\beta < 0$	falls by β	falls by $e^\beta < 1$	falls

⇒ debtinc and debts ⇒ increase the prob. of insolvency.

⇒ the years with the current employer decrease the prob. of insolvency.

⇒ age is insignificant.

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Forecasts:

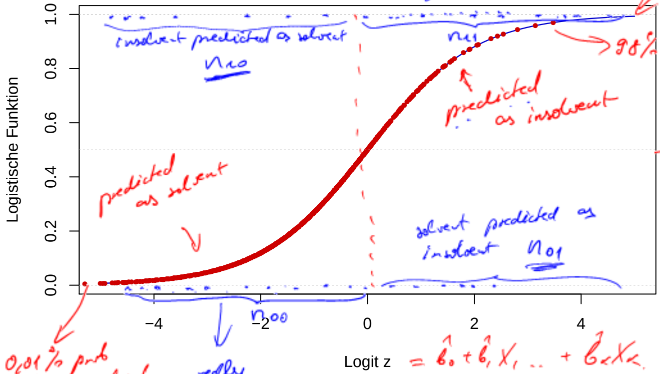
plug-in estimates β's

predicted probability of being insolvent

$$P(\widehat{Y_0 = 1|\boldsymbol{X}_0}) = \hat{\pi}_0 = \frac{1}{1 + e^{-\hat{b}_0 - \hat{b}_1 X_{10} - \ldots - \hat{b}_k X_{k0}}}.$$

really insolvent.

logistic line



98% prob of being insolvent.

0.5

0,01% prob of being insolvent.

really solvent.

$= \hat{b}_0 + \hat{b}_1 X_1 \ldots + \hat{b}_k X_k$

! in contrary to a LR: here we can visualize the forecasts because of z.

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Goodness of the model

$R^2 = 1$ cannot be attained

Problem: classical measures, such as R^2, cannot be used $\rightsquigarrow$ pseudo-R^2; classification tables; graphical measures (ROC-curve)

assume we do not have any X-Variables; 75% good; 25% bad clients $\Rightarrow$ for a new client there is a prob. of 75% of being good.

- pseudo-R^2:
  - Let LL_0 be the Log-Likelihood of the null model ($b_1 = \cdots = b_k = 0$)
  - Let LL_v be the Log-Likelihood of the full model (with all variables)
  - Let LL_s be the Log-Likelihood of the $\underbrace{\text{saturated model}}_{\text{oracle model}}$ (model with perfect fit, here $LL_s = 0$)
  - Deviance: $D = -2 \cdot LL_v$ (close 0)
  - Mc-Faddens-R^2: $1 - LL_v/LL_0$ (starting from 0.4)

slide 392

our / worst model

LL_0

```
Null deviance: 804.36  on 699  degrees of freedom
Residual deviance: 626.49  on 695  degrees of freedom
```

$LL_0 - LL_v$

$R^2 = \frac{626}{804} = 0.75 > 0.4.$

- Classification table

really insolvent | predicted to be insolvent | predicted to be solvent

		predicted		
		$\hat{Y}=1$	$\hat{Y}=0$	
truth	$y=1$	$n_{11}=TP$	$n_{01}=FN$	$n_{\cdot 1}=P$
	$y=0$	$n_{10}=FP$	$n_{00}=TN$	$n_{\cdot 0}=N$
		$n_{1\cdot}$	$n_{0\cdot}$	

really solvent

Let $\hat{y}_i = 1$ if $\widehat{P(Y_i = 1|\boldsymbol{X}_i)} \geq 0.5$ and 0 else.

$n_{10} = 38 \Rightarrow$ solvent predicted as insolvent
$n_{01} = 111 \Rightarrow$ insolvent predicted as solvent

	$\hat{Y}=1$	$\hat{Y}=0$
$Y=1$	72	111 $\Rightarrow$ wrong
$Y=0$	38	479

$\Rightarrow$ correct decisions

$n_{11} = 72$ – the number of insolvent, who are predicted to be insolvent.
$n_{00} = 479$ – solvent predicted as solvent.

$\rightsquigarrow$ $(479+72)/700 = 78{,}71\%$ are correctly predicted. $\Rightarrow$ in 79% of cases we made a correct decision with our logit.

But: there are 73,86% solvent clients in the sample.

$\frac{38+479}{700} \approx 75\% \Rightarrow$ toss 2 coins $\rightarrow$ N N $\Rightarrow$ insolvent (25%); N H, H H, H N $\Rightarrow$ solvent. (75%) $\Rightarrow$ 75% of correct decision with 2 coins.

Quaestion: is the threshold 0.5 a good choice? $\Rightarrow$ objective is to find another threshold 30%? 70%?

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Goodness of the model and the choice of the threshold

- ROC (*receiver operating characteristics*), Lift and Gain curves are used to visualize and to quantify the goodness of the classification algorithms.

number of insolvent classified as insolvent

$$
\begin{aligned}
\text{sensitivity} &= \frac{n_{11}}{n_{\cdot 1}} = \frac{n_{11}}{(n_{11} + n_{01})} \\
\text{specifity} &= \frac{n_{00}}{n_{\cdot 0}} = \frac{n_{00}}{(n_{10} + n_{00})}
\end{aligned}
$$

number of insolvent clients

solvent as solvent

number of solvent

Sensitivity: the fraction of correctly classified 1-values among all true 1-objects.

Specifity: the fraction of correctly classified 0-values among all true 0-object.

⇒ 61% of insolvent are classified as solvent ⇒ these get money from the bank, but cannot pay it back!!!

- Sensitivity= $\overset{n_{11}}{72}/(\overset{n_{11}}{72}+\overset{n_{01}}{111})=0.39$ - only 39% of insolvent clients are classified as insolvent
- Specifity= $\overset{n_{00}}{479}/(\overset{n_{00}}{479}+\overset{n_{10}}{38})=0.92$ - 92% of solvent clients are classified as solvent

⇒ only 8% of solvent are classified as insolvent.

⇒ that our model works well for solvent clients, but has a very low detection rate for insolvent!!!

Aim: increase sensitivity of the model. (this will obviously decrease the specifity).

number of clients predicted as being insolvent.

$$\text{PPV or PV+} = \frac{n_{11}}{n_{1\cdot}} = \frac{n_{11}}{(n_{11}+n_{10})}$$

$$\text{NPV or PV-} = \frac{n_{00}}{n_{0\cdot}} = \frac{n_{00}}{(n_{01}+n_{00})}$$

predicted solvent clients.

PPV: the fraction of correctly classified 1-values among all objects classified as 1.

NPV: the fraction of correctly classified 0-values among all objects classified as 0.

(PPV-positive predicted value, NPV-negative predicted value)

⇒ 35% are in fact solvent, but classified as insolvent
→ will get no credit, despite of being able to pay it back
(bank looses good clients).

- PPV= $\overset{n_{11}}{72}/(\overset{n_{11}}{72}+\overset{n_{10}}{38}) = 0.65$ - only 65% of all as insolvent classified clients are really insolvent
- NPV= $\overset{n_{00}}{479}/(\overset{n_{00}}{479}+\overset{n_{01}}{111}) = 0.81$ - 81% of all as solvent classified clients are really solvent

⇒ 19% of classified as solvent are insolvent
bank looses the whole credits

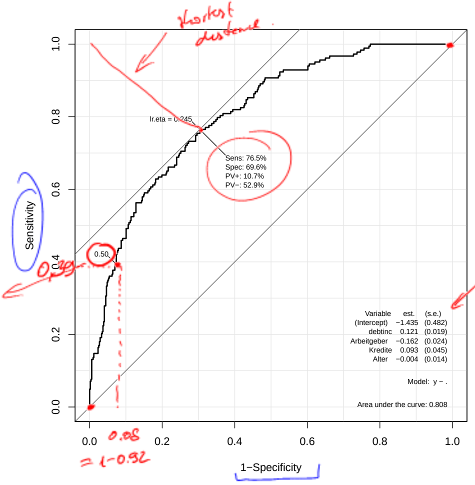
ROC-curve: senstitivity values as a function of specifity

- The steeper the function, the better the algorithm. ROC-value is the square under the curve.
- If the curve is close to the diagonal, then the algorithm is as good as random assignments.

R: `roc`-function from the `pROC`-package

optimal threshold = 0.245.
⇒ every client with prob. of insolvency ≥ 0.245 is classified as insolvent, and else as solvent.

natural threshold



the same parameters as before.

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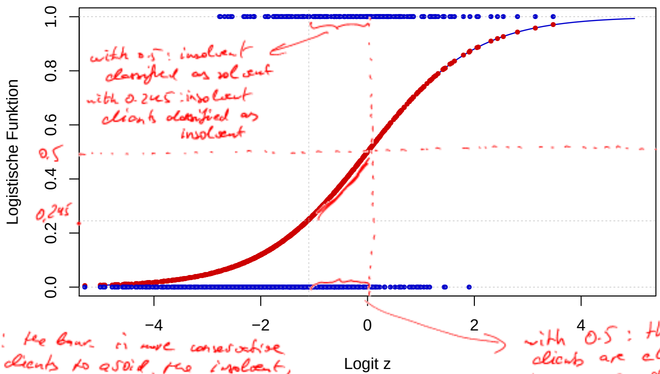
Now let $\hat{y}_i = 1$ if $\widehat{P(Y_i = 1|\boldsymbol{X}_i)} > 0.245$ and 0 else.

72 111
38 479.

479 ↘ 359
⇒ worse job in detecting solvent clients
spec : 92% ↘ 69,6%

	$\hat{Y}=1$	$\hat{Y}=0$
$Y=1$	140	43
$Y=0$	158	359

72 ↗ 140 ⇒ better job in detecting insolvent
sens : 39% ↗ 76.5%.
⇒ price
sens ↗ ⇒ spec ↘
and other way around.



new threshold: the bank is more conservative
⇒ reject more clients to avoid the insolvent, but also takes the risk of rejecting good clients

with 0.5 : these good clients are classified as good
with 0.245 : these good are classified as bad.

Gain-curve:

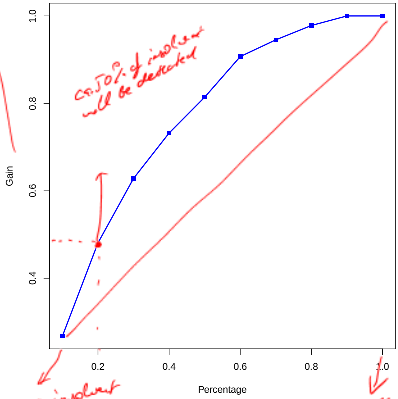
- If Gain equals 40% for 20%, this implies that if 20 % of the clients are classified as insolvent, then the algorithm will detect 40% of really insolvent clients .
- The digonal shows a model-free classification: If 20 % of the clients are classified as insolvent, then the algorithm detects 20% of really insolvent clients.
- The steeper the curve, the better (with a single kink).

R: gains-package

% of detected insolvent clients.

optimal fraction of clients to be classified as insolvent.

ca. 50% of insolvent will be detected



% of clients classified as insolvent

none as insolvent (friendly bank)

all as insolvent (unfriendly bank)

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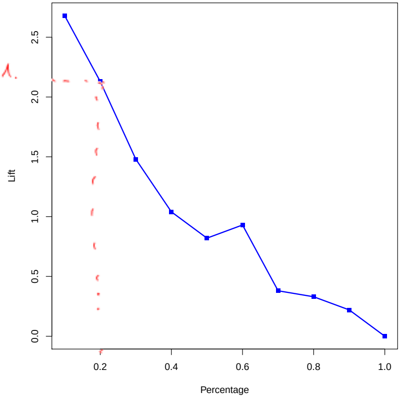
Lift curve: how much better is the predictive model compared to the model free classification?

- If lift is 2.1 for 20%, this implies that the model detection rate of insolvent clients is larger by 2.1 compared to model-free classification.

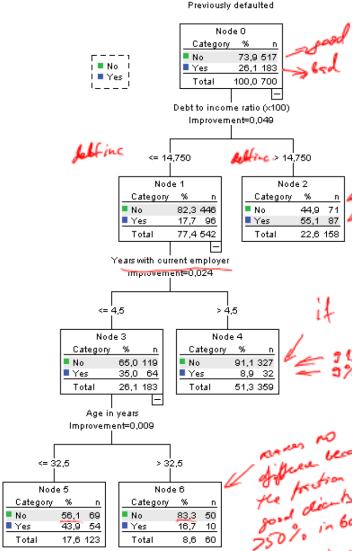
the larger the values on Y-Axis the better your model is compared to "tossing coins".

2.1.



The CART method can be applied to binary data: classification trees

CART ⇒ applied to a binary y.



44.9% of good
55.1% of bad
⇒ terminal node ⇒ every client with debtinc > 14,75 is classified as insolvent

if debtinc < 14,75 & employer > 4.5 ⇒ solvent

91% of solvent
9% of insolvent

no difference because the fraction of good clients > 50% in both
⇒ forecast is the same!

Classification

Observed	Predicted: No	Predicted: Yes	Predicted: Percent Correct
No	446	71	86,3%
Yes	96	87	47,5%
Overall Percentage	77,4%	22,6%	76,1%

Growing Method: CRT

Dependent Variable: Previously defaulted

worse than the logit

(but with threshold of 50%)

sens = $\frac{87}{96+87}$ < 0.45

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Modelling nominal data

Practical question:

- Choice of the political party depending of the characteristics of the voters;
- Choice of a product brand depending on the characteristics of the client;

4 categories (not ordered)

Example:

mode	–	"car", "air", "train", oder "bus"
choice	–	decision ⇒ the selected transport
wait	–	wait~~n~~ing time, 0 for "car"
vcost	–	variable costs
travel	–	time
gcost	–	total costs
income	–	income
size	–	number of persons

different values for different transportations modes

characterise the person, not the transport.
→ the same

fastest → (100)

longest → (417)

	individual	mode	choice	wait	vcost	travel	gcost	income	size
1	1	air	no	69	59	100	70	35	1
2	1	train	no	34	31	372	71	35	1
3	1	bus	no	35	25	417	70	35	1
4	1	car	yes	0	10	180	30	35	1
5	2	air	no	64	58	68	68	30	2
6	2	train	no	44	31	354	84	30	2

Multinomial logit model

logit.

For the simple logit model it holds:

same βs

$$P(Y=1|\boldsymbol{x}) = \frac{exp(\boldsymbol{x}'\boldsymbol{\beta})}{1+exp(\boldsymbol{x}'\boldsymbol{\beta})} = \frac{1}{1+e^{-z}} = \frac{e^{z}}{1+e^{z}}$$

$$\ln\left(\frac{P(Y=0|\boldsymbol{x})}{P(Y=1|\boldsymbol{x})}\right) = \boldsymbol{x}'\boldsymbol{\beta} = z.$$

→ linear combination of variables defines ln(odds)

For the k categories of Y we define:

category specific.

$k=4$
$r=1..3$

three such models/equations in our case.

$$\ln\left(\frac{P(Y=r|\boldsymbol{x})}{P(Y=k|\boldsymbol{x})}\right) = \boldsymbol{x}'\boldsymbol{\beta}_r, \qquad r=1,\ldots,k-1$$

→ last category is the reference category (air)

with

only / all categories ⇒ the model links all categories

$$P(Y=r|\boldsymbol{x}) \quad = \quad \frac{exp(\boldsymbol{x}'\boldsymbol{\beta}_r)}{1+\sum_{s=1}^{k-1} exp(\boldsymbol{x}'\boldsymbol{\beta}_s)}, \quad r=1,\ldots,k-1$$

$$P(Y=k|\boldsymbol{x}) \quad = \quad \frac{1}{1+\sum_{s=1}^{k-1} exp(\boldsymbol{x}'\boldsymbol{\beta}_s)}.$$

to guarantee that $\sum_{r=1}^{k} P(Y=r|x) = 1$

One category, i.e. the k-th, is the reference category.

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Note:

- Estimation via ML assuming independence of the observations. This is a questionable assumption:
  - similar categories; $\Rightarrow$ train → fast train, local train; air → lowcost, Lufthansa and Co.; bus → private (small), large. $\Rightarrow$ subcategories can be highly correlated; bus → red, non-red
  - odds do not depend on other categories, etc.
  - Solution: Hausmann/McFadden test
- Goodness-of-fit, tests as for logit.

$$\ln\left(\frac{P(Y=r|x)}{P(Y=k|x)}\right) = x'\beta_r$$

there is interrelation between categories

depends only on the r-th category!

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Global and category specific variables

$x'_r \alpha_r \Rightarrow$ non identifiable

$$\boldsymbol{x}'\boldsymbol{\beta}_r \quad \mapsto \quad \boldsymbol{x}'_{glob}\boldsymbol{\beta}^*_r + \boldsymbol{x}'_{spec,r}\boldsymbol{\alpha}$$

- Global variables (income, number of persons) do not depend on the categories and have individual parameters for each category: $\boldsymbol{x}'_{glob}\boldsymbol{\beta}^*_r$. $\Rightarrow$ income doesn't depend on category $\Rightarrow$ the parameter has to depend on the category.
  The sign of the parameters cannot be interpreted.
- The category specific variables (waiting time, costs) depend on the categories and are evaluated relatively to the reference category.
  $\Rightarrow$ waiting depends on its own on the category $\Rightarrow$ no sense to make the parameter depend on the category too!
  $$(\boldsymbol{x}_{spec,r} - \boldsymbol{x}_{spec,k})'\boldsymbol{\alpha} \quad \text{or} \quad \boldsymbol{x}'_{spec,r}\boldsymbol{\alpha}$$
  The sign of the parameters can be interpreted.

Let *gcost* and *wait* be category specific and *income* and *size* are global variables. The reference category is *air*.

global

```
> library("mlogit")
> mlogit(choice~wait+gcost|income+size, ...)
```

→ category specific

```
Coefficients :
                     Estimate Std. Error t-value  Pr(>|t|)
train:(intercept) -2.3115942  0.7525161 -3.0718 0.0021276 **
bus:(intercept)   -3.4504941  0.9064886 -3.8064 0.0001410 ***
car:(intercept)   -7.8913907  0.9880615 -7.9867 1.332e-15 ***
wait              -0.1013180  0.0112207 -9.0296 < 2.2e-16 ***
gcost             -0.0197064  0.0053844 -3.6599 0.0002523 ***
train:income      -0.0589804  0.0154532 -3.8167 0.0001352 ***
bus:income        -0.0277037  0.0169812 -1.6314 0.1027991
car:income        -0.0041153  0.0127301 -0.3233 0.7464866
train:size         1.3289497  0.3141683  4.2301 2.336e-05 ***
bus:size           1.0090796  0.3952899  2.5528 0.0106874 *
car:size           1.0392585  0.2665513  3.8989 9.663e-05 ***
---
```

single parameter for category specific variables

category specific parameters for global variables

```
Log-Likelihood: -176.77
McFadden R^2:  0.37705
Likelihood ratio test : chisq = 213.98 (p.value = < 2.22e-16)
```

→ the same goodness of fit measure as for Logit

With the estimated paremeters we can estimate the probabilities $P(Y_i = r|\boldsymbol{x}_i)$ for all r.

	air	train	bus	car
[1,]	0.2368302	0.00000000	0.24496423	0.5182056
[2,]	0.2083323	0.27785076	0.00000000	0.5138170
[3,]	0.0000000	0.12686485	0.23058033	0.6425548
[4,]	0.1151004	0.05063597	0.02141839	0.8128452
[5,]	0.3405917	0.20694648	0.05624436	0.3962174
[6,]	0.1316850	0.36965292	0.26144217	0.2372200

= 1

1 - sum of other

using formulas on the board

individual 1–5 ⇒ car
6 ⇒ train

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Ordered response models

Aim: Y can take M different ordered (!) values (credit ratings, grades, income classes, etc)

Using a single latent variable we can specify

$$\begin{aligned} Y_i^* &= \boldsymbol{x}_i'\boldsymbol{\beta} + u_i \\ Y_i &= j \quad \text{if} \quad \gamma_{j-1} < Y_i^* \leq \gamma_j \end{aligned}$$

for some unknown threshold values γ_j with $\gamma_0 = -\infty$ and $\gamma_M = \infty$.

Assuming logistic cdf for u_i we obtain ordered logit model and assuming normality we obtain ordered probit model.

Example: a (simplified) rating of companies - $Y = 1$ - lowest, $Y = 2$ - average, $Y = 3$ - highest

MARKET_VALUE	DIV_PER_SHR	TOTAL_DEBT	rating
-0.08911931	-0.08063048	-0.02276501	1
-0.09350059	-0.08148114	-0.14012151	3
-0.09452652	-0.08019333	-0.13869093	2
-0.09633656	-0.08090222	-0.13784721	2
-0.09254201	-0.08130392	-0.13822051	2
0.95192265	-0.06091170	13.33345544	3

$$
\begin{aligned}
Y^* &= \boldsymbol{x}'\boldsymbol{\beta} + u \\
Y &= 1 \quad \text{if} \quad y^* \leq \gamma_{1|2} \\
&= 2 \quad \text{if} \quad \gamma_{1|2} < y^* \leq \gamma_{2|3} \\
&= 3 \quad \text{if} \quad \gamma_{2|3} < y^*
\end{aligned}
$$

Logi

Assuming normal error terms we can state the corresponding prob's

$$
\begin{aligned}
P(Y_i \leq k|\boldsymbol{x}_i) &= P(Y_i^* \leq \gamma_{(k-1)|k}|\boldsymbol{x}_i) = \Phi(\gamma_{(k-1)|k} - \boldsymbol{x}_i'\boldsymbol{\beta}) \\
P(Y_i = 1|\boldsymbol{x}_i) &= P(Y_i^* \leq \gamma_{1|2}|\boldsymbol{x}_i) = \Phi(\gamma_{1|2} - \boldsymbol{x}_i'\boldsymbol{\beta}) \\
P(Y_i = 3|\boldsymbol{x}_i) &= P(Y_i^* > \gamma_{2|3}|\boldsymbol{x}_i) = 1 - \Phi(\gamma_{2|3} - \boldsymbol{x}_i'\boldsymbol{\beta}) \\
P(Y_i = 2|\boldsymbol{x}_i) &= P(\gamma_{1|2} < Y_i^* \leq \gamma_{2|3}|\boldsymbol{x}_i) = \Phi(\gamma_{2|3} - \boldsymbol{x}_i'\boldsymbol{\beta}) - \Phi(\gamma_{1|2} - \boldsymbol{x}_i'\boldsymbol{\beta})
\end{aligned}
$$

The log-likelihood function is then given by

$P(Y_1 = 1, Y_2 =$

$$
LL(\boldsymbol{\beta}|\boldsymbol{X}) = \sum_{i=1}^{N} \log P(Y_i = y_i|\boldsymbol{x}_i) \rightarrow max, \text{ w.r.t. } \boldsymbol{\beta}
$$

The inferences follow in a similar fashion as for the simple logit.

$P($

Example:

```
Coefficients:
                Value Std. Error t value
MARKET_VALUE  2.14118     0.6803  3.1474
DIV_PER_SHR  -0.70836     0.3189 -2.2212
TOTAL_DEBT    0.05553     0.1367  0.4063
Intercepts:
    Value   Std. Error t value
1|2 -0.0309  0.0789    -0.3916
2|3  1.0181  0.0879    11.5797
Residual Deviance: 1480.408
AIC: 1490.408
> ordlog.pred = predict(ordlog, type="probs")
> ordlog.pred
              1            2         3
   5.259901e-01 2.340771e-01 0.2399328
   5.298021e-01 2.330434e-01 0.2371545
   5.305567e-01 2.328364e-01 0.2366069
   5.313852e-01 2.326083e-01 0.2360065
   5.292958e-01 2.331818e-01 0.2375224
   5.453970e-02 8.685558e-02 0.8586047
```

γ
γ

>0.5

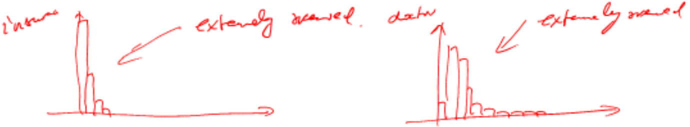
Modelling for count data

$Y = \{0,1,2,3\dots\}$ ⇒ a few values, but the set is not limited.
⇒ cannot apply ordered Logit ⇐ fixed number of categories

Practical questions:

- the number of claims by an insurance company per time period;
- the number of cosultations by a doctor per year ;
- the number of insolvent companies per time period;
- occurrences of a seldom disease per season; ⇒
- ....

Note: the modelling is particularly important for small values of the target variable (rare events) and the distribution is heavily skewed.



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Poisson distribution

The Poisson distribution is frequently used to model rare events

discrete

$$P(Y=y)=\begin{cases}\frac{\lambda^y}{y!}e^{-y}, & \text{for } y=0,1,2,\dots\\ 0, & \text{else,}\end{cases}$$

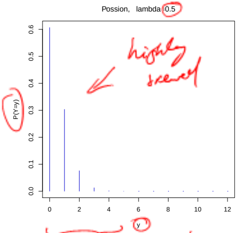
any natural number

with the intensity parameter λ. It fulfils the *equidispersion*-condition:

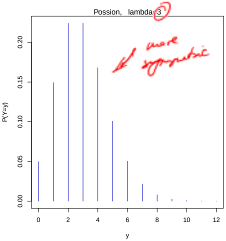
$$E(Y)=Var(Y)=\lambda$$

low intensity expected number of claims per year = 0.5

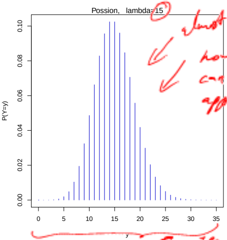
⇒ we expect 3 claims



highly skewed



more symmetric



almost symmetric normal distribution can be a good approximation

up to 5 different outcomes in the sample ⇒ Poisson

OLS for ≥ 5 possible outcomes.

LR: $y = \underline{\beta_0 + \beta_1 x_1 + \ldots + \beta_J x_J} + u. \Rightarrow E(Y|x) = \beta_0 + \beta_1 x_1 + \ldots + \beta_J x_J$

Poisson regression model

$E(Y) = \lambda$ plug x's into λ. $\lambda > 0$ - always $\Rightarrow \lambda = \beta_0 + \beta_1 x_1 + \ldots + \beta_J x_J$ - can be negative.

Let $Y_i, \boldsymbol{x}_i$ be independent realisations, while Y_i follows Poisson distribution with

link forces $x'\beta$ positive and guarantees positive λ's.

$$E(Y_i|\boldsymbol{x}_i) = h(\boldsymbol{x}_i'\boldsymbol{\beta}) = exp(\boldsymbol{x}_i'\boldsymbol{\beta}) = \lambda_i.$$

- The interpretation of the parameters follows as for the logit model.
- The parameters are estimated via ML:

$$LL(\boldsymbol{\beta}) = \sum_{i=1}^{n} y_i \ln(h(\boldsymbol{x}_i'\boldsymbol{\beta})) - h(\boldsymbol{x}_i'\boldsymbol{\beta}) - \ln(y_i!) \longrightarrow max, \text{ w.r.t. } \boldsymbol{\beta}$$

$P(Y_1 = 2, Y_2 = 1, Y_3 = 0) = P(Y_1 = 2) \cdot P(Y_2 = 1) P(Y_3 = 0) =$

$= \frac{(h(x_1'\beta))^2}{2!} e^{-h(x_1'\beta)} \cdot \frac{(h(x_2'\beta))^1}{1!} \cdot e^{-h(x_2'\beta)} \cdot \frac{(h(x_3'\beta))^0}{0!} e^{-h(x_3'\beta)} \rightarrow$

$\rightarrow$ max w.r.t β $x = (\text{age}, \text{job}, \ldots)$.

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Goodness of the model

To measure the goodness of the model we use deviance, i.e. the difference between the log-likelihood for the actual observations (perfect/saturiertes model) and the log-likelihood for the predicted values:

$$D = -2\sum_{i=1}^{n}[LL_i(\hat{Y}_i) - LL_i(Y_i)] = 2\sum_{i=1}^{n}\left[Y_i \ln(Y_i/\hat{\lambda}_i)\right] \sim \chi^2_{n-p}$$

likelihood of your model (max on the previous slide)

likelihood of the oracle model.

R^2 cannot be applied, as for logit.

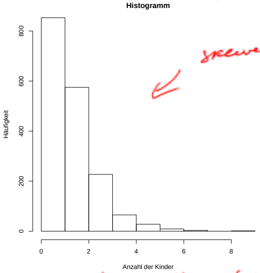
Example: number of children

child	-	number of children
age	-	age of the woman
dur	-	years at school/college
nation	-	nationality, 0 = german , 1 = else
god	-	trust in God: 1 = strong, ..., 6 = never thought about it
univ	-	university degree: 0 = no, 1 = yes ⇒ dummy.

```
mean(children$child)
[1] 1.57297
> var(children$child)
[1] 1.552769
```

equidispersion property is more or less fulfilled.

$E(Y) = Var(Y) = \lambda$ ← for a Poisson distribution



skewed.

⇒ less possible outcomes

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```
glm(formula = child ~ age + I(age^2) + I(age^3) + I(age^4) +
    dur + I(dur^2) + nation + god + univ, family = poisson(link = log),
    data = children)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.1514  -0.7559   0.0102   0.4832   3.6715

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.228e+01  1.484e+00  -8.277  < 2e-16 ***
age          9.359e-01  1.239e-01   7.553 4.26e-14 ***
I(age^2)    -2.490e-02  3.786e-03  -6.577 4.80e-11 ***
I(age^3)     2.842e-04  4.915e-05   5.781 7.42e-09 ***
I(age^4)    -1.180e-06  2.297e-07  -5.137 2.80e-07 ***
dur          1.118e-01  6.652e-02   1.680 0.092904 .
I(dur^2)    -8.328e-03  2.997e-03  -2.779 0.005454 **
nation1      5.686e-02  1.386e-01   0.410 0.681599
god2        -1.025e-01  5.903e-02  -1.736 0.082599 .
god3        -1.448e-01  6.780e-02  -2.136 0.032683 *
god4        -1.279e-01  7.088e-02  -1.805 0.071128 .
god5        -3.621e-02  6.695e-02  -0.541 0.588569
god6        -9.241e-02  7.505e-02  -1.231 0.218239
univ1        6.372e-01  1.729e-01   3.686 0.000228 ***
---

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2067.4  on 1760  degrees of freedom
Residual deviance: 1718.6  on 1747  degrees of freedom
AIC: 5196.8
```

age, age^2, age^3, age^4 ⇒ polynomial of order 4

⇒ education

⇒ minor significance.

⇒ education

$1 - \frac{1718}{2067} \approx$ 15–20% wrt dispersion

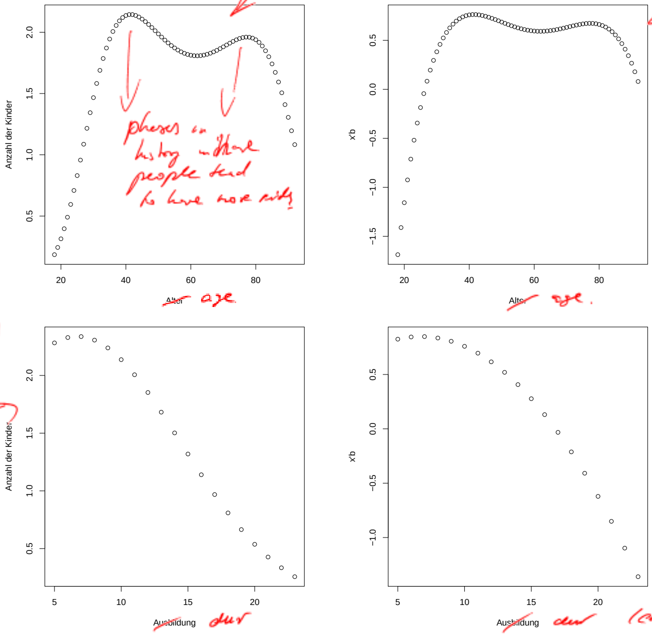
McFadden R^2

4th polynomial.

$\hat{y}$

logit

$x'\beta \to e^{x'\beta}$



$\hat{y}$

negative correlation between education & of kids.

(education)

Note: for the Poisson distribution it should hold $E(Y_i) = Var(Y_i) = \lambda_i$.

If this assumption is not fulfilled then we have *overdispersion/underdispersion*.

Solution: as an alternative we can use Quasi-Poisson- or the negative binomial distribution (negbin). Both distributions allow for different expectations and variances.

For negbin it holds:

$$P(Y_i|\boldsymbol{x}_i) = \frac{\Gamma(Y_i+\nu)}{\Gamma(\nu)\Gamma(Y_i+1)} \cdot \left(\frac{\lambda_i}{\lambda_i+\nu}\right)^{Y_i} \cdot \left(\frac{\nu}{\lambda_i+\nu}\right)^{\nu}$$

with $E(Y_i) = \lambda_i = exp(\boldsymbol{x}_i'\boldsymbol{\beta})$ and $Var(Y_i) = \lambda_i + \lambda_i^2/\nu$.

$\nu \to +\infty$

negbin $\to$ Poisson.

allow the E and Var to be different (in the sample) generalized linear model

```
glm(formula = child ~ age + I(age^2) + I(age^3) + I(age^4) +
    dur + I(dur^2) + nation + god + univ, family = negative.binomial(theta = 1,
    link = log), data = children)
```

link function. (or exp)

```
Deviance Residuals:
     Min        1Q    Median        3Q       Max
-1.56820  -0.50984  -0.01054   0.29990   1.90633

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.338e+01  1.267e+00 -10.555  < 2e-16 ***
age          1.022e+00  1.075e-01   9.502  < 2e-16 ***
I(age^2)    -2.730e-02  3.342e-03  -8.169 5.90e-16 ***
I(age^3)     3.126e-04  4.395e-05   7.113 1.65e-12 ***
I(age^4)    -1.302e-06  2.074e-07  -6.277 4.34e-10 ***
dur          1.269e-01  5.990e-02   2.118 0.034294 *
I(dur^2)    -9.577e-03  2.637e-03  -3.632 0.000289 ***
nation1      8.309e-02  1.349e-01   0.616 0.538128
god2        -1.186e-01  5.849e-02  -2.028 0.042743 *
god3        -1.681e-01  6.642e-02  -2.530 0.011483 *
god4        -1.563e-01  6.923e-02  -2.258 0.024075 *
god5        -3.273e-02  6.602e-02  -0.496 0.620135
god6        -1.205e-01  7.384e-02  -1.632 0.102848
univ1        7.749e-01  1.581e-01   4.900 1.04e-06 ***
---

(Dispersion parameter for Negative Binomial(1) family taken to be 0.3516262)

    Null deviance: 1023.1  on 1760  degrees of freedom
Residual deviance:  852.3  on 1747  degrees of freedom
AIC: 5911.9
```