DIGITAL COMMUNICATION

Through Simulations

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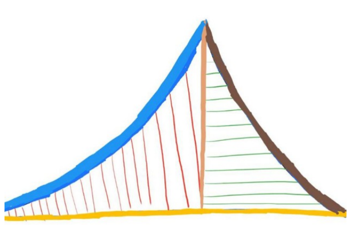

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Introduction

Chapter 2

Axioms

2.1. Examples

2.1.1 Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football game?

Solution: Let X be a random variable which takes the values 0 and 1.

$$X = \begin{cases} 1, & \text{if coin toss results in Head} \\ 0, & \text{if coin toss results in Tail} \end{cases} \tag{2.1}$$

From law of total probability,

$$\Pr(X = 0) + \Pr(X = 1) = 1 \tag{2.2}$$

Since there is only one head,

$$\Pr(X = 1) = \frac{1}{2} \tag{2.3}$$

Similarly,

$$\Pr(X=0) = 1 - \Pr(X=1) = \frac{1}{2} \tag{2.4}$$

Thus,

$$\Pr(X=0) = \Pr(X=1) \tag{2.5}$$

which is why tossing the coin is a fair way to decide.

2.1.2 Which of the following cannot be the probability of an event ?

(a) $\frac{2}{3}$

(b) -1.5

(c) 15%

(d) 0.7

Solution: From the axioms of probability,

$$0 \le \Pr(E) \le 1 \tag{2.6}$$

(a) $\Pr(E) = \frac{2}{3}$

$$\because 0 \le \frac{2}{3} \le 1 \tag{2.7}$$

from (2.6), it can be probability of an event.

(b) $\Pr(E) = -1.5$

$$\because -1.5 < 0 \tag{2.8}$$

from (2.6), it cannot be a probability of any event.

(c)

$$\Pr(E) = \frac{15}{100} \tag{2.9}$$

$$\because 0 \le \frac{15}{100} \le 1, \tag{2.10}$$

from (2.6), it can be probability of an event.

(d) $\Pr(E) = 0.7$

$$\because 0 \le 0.7 \le 1 \tag{2.11}$$

from (2.6), it can be a probability of an event.

2.1.3 If P(E) = 0.05, what is the probability of 'not E'?

Solution: The desired probability is

$$\Pr\left(E'\right) = 1 - \Pr(E) = 0.95 \tag{2.12}$$

2.1.4 A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

(a) an orange flavoured candy?

(b) a lemon flavoured candy?

Solution:

$$\Pr(O) = 0 \tag{2.13}$$

$$\Pr(L) = 1 \tag{2.14}$$

2.1.5 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: Let E be the event that no 2 students in a group of 3 share a birthday. Then

$$\Pr(E) = 0.992 \implies \Pr\left(E'\right) = 1 - \Pr(E) = 0.008 \tag{2.15}$$

2.1.6 A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is

(i) red ?

(ii) not red?

Solution: Let

$$X = \begin{cases} 1 & \text{if drawn ball is red} \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

(i) Probability that the drawn ball is red

$$\Pr(X = 1) = \frac{3}{8} \tag{2.17}$$

$$\tag{2.18}$$

(ii) Probability that the drawn ball is not red

$$\Pr(X = 0) = 1 - \frac{3}{8} = \frac{5}{8} \tag{2.19}$$

2.1.7 Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish . What is the probability that the fish taken out is a male fish?

Solution:

Let

$$X = \begin{cases} 1, & \text{if the chosen fish is male} \\ 0, & \text{if the chosen fish is female} \end{cases} \tag{2.20}$$

Then

$$\Pr(X = 1) = \frac{5}{13} \tag{2.21}$$

2.1.8 A box contains 12 balls, out of which x are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.

Solution: From Table 2.1,

Random Variable	Sample space	Value	Event	Probability
X_1	12	0	not choosing black ball	12-x/12
		1	choosing black ball	x/12
X_2	18	0	not choosing black ball	12-x/18
		1	choosing black ball	x+6/18

Table 2.1:

$$\Pr\left(X_1 = 1\right) = \frac{x}{12} \tag{2.22}$$

Since

$$\Pr\left(X_2 = 1\right) = 2\Pr\left(X_1 = 1\right), \tag{2.23}$$

$$\frac{x+6}{18} = 2\left(\frac{x}{12}\right) \tag{2.24}$$

$$\implies x = 3 \tag{2.25}$$

2.1.9 A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is

(a) a vowel

(b) a consonant

. **Solution:** The number of vowels is 6 and consonants is 7.

(a)

$$\Pr\left(X\right) = \frac{6}{13} \tag{2.26}$$

(b)

$$\Pr(Y) = \frac{7}{13} \tag{2.27}$$

2.1.10 In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prizes in the game? [Hint : order of the numbers is not important.]

Solution: The desired probability is given by

$$\frac{1}{{}^{20}C_6} = \frac{1}{38,760} = 0.0000258 \tag{2.28}$$

2.1.11 Check whether the following probabilities $\Pr(A)$ and $\Pr(B)$ are consistently defined

(a) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cap B) = 0.6$

(b) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cup B) = 0.8$

Solution: To check whether the given probabilities are consistently defined, we check whether the following property holds correctly with the probability axioms

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.29}$$

(a) Given that

$$\Pr(A) = 0.5, \ \Pr(B) = 0.7, \ \Pr(AB) = 0.6 \tag{2.30}$$

From (2.29),

$$\Pr(A+B) = 0.5 + 0.7 - 0.6 = 0.6 \tag{2.31}$$

From (2.31) we have

$$0 \le \Pr(A+B) \le 1 \tag{2.32}$$

Hence the given probabilities are consistently defined.

(b) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(A+B) = 0.8 \tag{2.33}$$

From (2.29) we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \tag{2.34}$$

$$= 0.4 \tag{2.35}$$

From (2.35) we have

$$0 \le \Pr(AB) \le 1 \tag{2.36}$$

Hence the given probabilities are consistently defined

2.1.12 Given $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$. Find $\Pr(A+B)$ if A and B are mutually exclusive events.

Solution: Since $AB = 0$,

$$\Pr(A+B) = \Pr(A) + \Pr(B) = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} \tag{2.37}$$

2.1.13 If E and F are events such that $\Pr(E) = \frac{1}{4}$, $\Pr(F) = \frac{1}{2}$ and $\Pr(EF) = \frac{1}{8}$, find

(a) $\Pr(E+F)$

(b) $\Pr(E'F')$

Solution:

(a)

$$\Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(EF) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8} \tag{2.38}$$

$$\tag{2.39}$$

(b)

$$(E'F') = (E+F)' \implies \Pr\left(E'F'\right) = \Pr\left((E+F)'\right) = 1 - \Pr(E+F) \tag{2.40}$$

$$\implies \Pr\left(E'F'\right) = 1 - \frac{5}{8} = \frac{3}{8} \tag{2.41}$$

2.1.14 Events E and F are such that $\Pr(E'+F') = 0.25$, state whether E and F are mutually exclusive.

Solution:

$$\Pr\left(E' + F'\right) = \Pr\left(EF\right)' = 1 - \Pr\left(EF\right) = 0.25 \tag{2.42}$$

$$\implies \Pr\left(EF\right) = 0.75 \tag{2.43}$$

$$\because \Pr\left(EF\right) \neq 0 \tag{2.44}$$

E and F are not mutually exclusive events.

2.1.15 A and B are events such that $\Pr\left(A\right) = 0.42, \Pr\left(B\right) = 0.48$ and $\Pr\left(\text{A and B}\right) = 0.16$. Determine

(a) $\Pr\left(\text{not A}\right)$

(b) $\Pr\left(\text{not B}\right)$

(c) $\Pr\left(\text{A or B}\right)$

Solution: Solution:

(a) $\Pr\left(\text{not A}\right)$

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right) \tag{2.45}$$

$$= 1 - 0.42 \tag{2.46}$$

$$= 0.58 \tag{2.47}$$

(b) $\Pr\left(\text{not B}\right)$

$$\Pr\left(B'\right) = 1 - \Pr\left(B\right) \tag{2.48}$$

$$= 1 - 0.48 \tag{2.49}$$

$$= 0.52 \tag{2.50}$$

(c) $\Pr(\text{A or B})$

$$\Pr(\text{A+B}) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.51}$$

$$= 0.42 + 0.48 - 0.16 \tag{2.52}$$

$$= 0.74 \tag{2.53}$$

2.1.16 In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution: The given information is summarised in Table 2.3. Thus,

Random Variable	**Subject**	**Probability**
M	Mathematics	$\Pr(M)$=0.4
B	Biology	$\Pr(B)$=0.3
M, B	Both	$\Pr(MB)$=0.10

Table 2.3:

$$\Pr(M + B) = \Pr(M) + \Pr(B) - \Pr(M, B) \tag{2.54}$$

$$= 0.6 \tag{2.55}$$

2.1.17 In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

Solution: Let

A : Probability of random student passing the first exam

B : Probability of random student passing the second exam

Given

$$\Pr(A) = 0.8, \Pr(B) = 0.7, \Pr(A + B) = 0.95. \tag{2.56}$$

$$\because \Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB), \tag{2.57}$$

$$\implies \Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B) \tag{2.58}$$

$$= 0.55 \tag{2.59}$$

2.1.18 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(a) The student opted for NCC or NSS.

(b) The student has opted neither NCC nor NSS.

(c) The student has opted NSS but not NCC.

Solution: Define random variables X and Y as shown in Tables 2.4 and 2.5. From

$X = 0$	Student does not opt for NCC.
$X = 1$	Student opts for NCC.

Table 2.4: Definition of X.

$Y = 0$	Student does not opt for NSS.
$Y = 1$	Student opts for NSS.

Table 2.5: Definition of Y.

the given data

$$\Pr(X=1) = \frac{30}{60} = \frac{1}{2} \tag{2.60}$$

$$\Pr(Y=1) = \frac{32}{60} = \frac{8}{15} \tag{2.61}$$

$$\Pr(X=1,\ Y=1) = \frac{24}{60} = \frac{2}{5} \tag{2.62}$$

Thus, we write

$$\Pr(X=1,\ Y=0) = \Pr(X=1) - \Pr(X=1,\ Y=1) = \frac{1}{10} \tag{2.63}$$

$$\Pr(X=0,\ Y=1) = \Pr(Y=1) - \Pr(X=1,\ Y=1) = \frac{2}{15} \tag{2.64}$$

$$\Pr(X=0,\ Y=0) = \Pr(Y=0) - \Pr(X=1,\ Y=0) \tag{2.65}$$

$$= 1 - \Pr(Y=1) - \Pr(X=1,\ Y=0) \tag{2.66}$$

$$= 1 - \frac{8}{15} - \frac{1}{10} = \frac{11}{30} \tag{2.67}$$

and form the joint pmf as in Table 2.6.

	$X=0$	$X=1$
$Y=0$	$\frac{11}{30}$	$\frac{1}{10}$
$Y=1$	$\frac{2}{15}$	$\frac{2}{5}$

Table 2.6: Joint pmf of X and Y.

(a) From Table 2.6,

$$\Pr(X+Y \geq 1) = 1 - \Pr(X+Y=0) \tag{2.68}$$

$$= \frac{19}{30} \tag{2.69}$$

(b) From Table 2.6,

$$\Pr(X = 0,\ Y = 0) = \frac{11}{30} \tag{2.70}$$

(c) From Table 2.6,

$$\Pr(X = 0,\ Y = 1) = \frac{2}{15} \tag{2.71}$$

2.1.19 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: The given information is summarized in the following table 2.7

RV	**Description**	**Probability**
$X = 1$	Die rolls to 1	$\frac{1}{3}$
$X = 2$	Die rolls to 2	$\frac{1}{2}$
$X = 3$	Die rolls to 3	$\frac{1}{6}$

Table 2.7: Random variable X

(a)

$$\Pr(X = 2) = \frac{1}{2} \tag{2.72}$$

(b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1, 3\} \tag{2.73}$$

$$X = 1 \text{ and } X = 3 \equiv X = \phi \tag{2.74}$$

$$\Pr(X \in \{1, 3\}) = \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \phi) \tag{2.75}$$

$$= \frac{1}{3} + \frac{1}{6} \tag{2.76}$$

$$= \frac{1}{2} \tag{2.77}$$

(c)

$$\Pr(X \neq 3) = 1 - \Pr(X = 3) \tag{2.78}$$

$$= 1 - \frac{1}{6} \tag{2.79}$$

$$= \frac{5}{6} \tag{2.80}$$

2.1.20 A and B are two events such that $\Pr(A) = 0.54$, $\Pr(B) = 0.69$ and $\Pr(AB) = 0.35$. Find

(a) $\Pr(A + B)$

(b) $\Pr(A'B')$

(c) $\Pr(AB')$

(d) $\Pr(BA')$

Solution:

(a)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.81}$$

$$= 0.54 + 0.69 - 0.35 = 0.88 \tag{2.82}$$

(b) By De Morgan's Law,

$$A'B' = (A+B)' \tag{2.83}$$

$$\implies \Pr\left(A'B'\right) = \Pr(A+B)' \tag{2.84}$$

$$= 1 - \Pr(A+B) \tag{2.85}$$

$$= 1 - 0.88 = 0.12 \tag{2.86}$$

(c) We know that,

$$B + B' = 1BB' = 0 \tag{2.87}$$

$$\implies A = A(B+B') = AB + AB' \tag{2.88}$$

$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) - \Pr\left(ABB'\right) \tag{2.89}$$

$$= \Pr(AB) + \Pr\left(AB'\right) \tag{2.90}$$

$$\implies \Pr\left(AB'\right) = \Pr(A) - \Pr(AB) \tag{2.91}$$

$$= 0.54 - 0.35 = 0.19 \tag{2.92}$$

(d) From (2.91),

$$\Pr\left(BA'\right) = \Pr(B) - \Pr(AB) = 0.69 - 0.35 = 0.34. \tag{2.93}$$

2.1.21 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.

Solution: Since the events A, B are independent, we have

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{2.94}$$

2.1.22 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event,'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

Solution: Let

$$X = \begin{cases} 0, & \text{if number is odd} \\ 1, & \text{if number is even} \end{cases} \tag{2.95}$$

$$Y = \begin{cases} 0, & \text{if number is green} \\ 1, & \text{if number is red} \end{cases} \tag{2.96}$$

From the given information,

$$\Pr(X = 1) = \frac{3}{6} = \frac{1}{2}, \Pr(Y = 1) = \frac{3}{6} = \frac{1}{2} \tag{2.97}$$

$$\Pr(X = 1, Y = 1) = \frac{1}{6} \tag{2.98}$$

Now,

$$\Pr(X = 1) \times \Pr(Y = 1) = \frac{1}{4} \tag{2.99}$$

$$\implies \Pr(X = 1, Y = 1) \neq \Pr(X = 1) \times \Pr(Y = 1) \tag{2.100}$$

Hence, A and B are not independent.

2.1.23 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50} \tag{2.101}$$
$$\Pr(EF) = \frac{1}{50} \tag{2.102}$$
$$\implies \Pr(EF) \neq P(E)P(F) \tag{2.103}$$

$\therefore E$ and F are not independent events.

2.1.24 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution: Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{2.104}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{2.105}$$
$$\therefore p = \frac{1}{10} \tag{2.106}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.107}$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.108}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.109}$$

$$\therefore p = \frac{1}{5} \tag{2.110}$$

2.1.25 Let and B be independent events with $\Pr(A) = 0.3$ and $\Pr(B) = 0.4$. Find

(a) $\Pr(AB)$

(b) $\Pr(A+B)$

(c) $\Pr(A|B)$

(d) $\Pr(B|A)$

Solution:

(a)

$$\Pr(AB) = \Pr(A) \times \Pr(B) = 0.3 \times 0.4 = 0.12 \tag{2.111}$$

(b)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) = 0.3 + 0.4 - 0.12 = 0.58 \tag{2.112}$$

(c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A) \times \Pr(B)}{\Pr(B)} = \Pr(A) = 0.3 \tag{2.113}$$

(d)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B) \times \Pr(A)}{\Pr(A)} = \Pr(B) = 0.4 \tag{2.114}$$

2.1.26 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr(\text{not A and not B})$.

Solution: Since,

$$A'B' = (A+B)' \tag{2.115}$$

$$\implies \Pr(A'B') = \Pr\left((A+B)'\right) \tag{2.116}$$

$$= 1 - \Pr((A+B)) \tag{2.117}$$

we also know that,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.118}$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} \tag{2.119}$$

$$= \frac{5}{8} \tag{2.120}$$

Hence, by substituting in (2.116) we get

$$\Pr(A'B') = 1 - \frac{5}{8} \tag{2.121}$$

$$= \frac{3}{8} \tag{2.122}$$

2.1.27 Events A and B are such that

$$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{7}{12} \text{and} \Pr\left(A' + B'\right) = \frac{1}{4}. \tag{2.123}$$

State whether A and B are independent.

Solution:

$$\Pr(AB) = 1 - \Pr\left(A' + B'\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{2.124}$$

$$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \tag{2.125}$$

$$\implies \Pr(AB) \neq \Pr(A)\Pr(B) \tag{2.126}$$

$\therefore$ A and B are not independent.

2.1.28 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

2.1.29 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even, ' and B be the event, 'the number is red'. Are A and B independent?

2.1.30 Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

2.1.31 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

2.1.32 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(\text{not } A \text{ and not } B)$

2.1.33 Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?

2.1.34 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(a) $P(A \text{ and } B)$

(b) $P(A \text{ and not } B)$

(c) $P(A \text{ or } B)$

(d) $P(\text{neither } A \text{ nor } B)$

2.1.35 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved

(b) exactly one of them solves the problem

Solution: Given that $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \frac{1}{3}$ A, B are independent so

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \tag{2.127}$$

(a) The probability of the problem being solved is

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.128}$$

$$= \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.129}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \tag{2.130}$$

(b) Probalbility that exactly one person solves problem is

$$\Pr\left(AB'\right) + \Pr\left(A'B\right) = \Pr(A)\Pr\left(B'\right) + \Pr\left(A'\right)\Pr(B) \tag{2.131}$$

$$= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} \tag{2.132}$$

2.1.36 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(a) E: 'the card drawn is spade'

F: 'the card drawn is an ace'

(b) E: 'the card drawn is black'

F : 'the card drawn is a king'

(c) E : 'the card drawn is a king or queen'

F : 'the card drawn is a queen or jack'

Solution:

(i) E denotes the event that the card drawn is spade

$$\Pr(E) = \frac{13}{52} = \frac{1}{4} \tag{2.133}$$

F denotes the event that card drawn is ace

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{2.134}$$

$$\Pr(EF) = \frac{1}{52} \tag{2.135}$$

$$\Pr(E)\Pr(F) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{2.136}$$

$$\therefore \; \Pr(EF) = \Pr(E)\Pr(F) \tag{2.137}$$

and the events are independent.

(ii) E denotes the event that the card drawn is black

$$\Pr(E) = \frac{26}{52} = \frac{1}{2} \tag{2.138}$$

F denotes the event that card drawn is a king

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{2.139}$$

$$\Pr(EF) = \frac{2}{52} = \frac{1}{26} \tag{2.140}$$

$$\Pr(E)\Pr(F) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \tag{2.141}$$

$$\therefore \quad \Pr(EF) = \Pr(E)\Pr(F) \tag{2.142}$$

and E and F are independent events.

(iii) E denotes the event that the card drawn is king or queen

$$\Pr(E) = \frac{8}{52} = \frac{2}{13} \tag{2.143}$$

F denotes the event that card drawn is a queen or jack

$$\Pr(F) = \frac{8}{52} = \frac{2}{13} \tag{2.144}$$

$$\Pr(EF) = \frac{4}{52} = \frac{1}{13} \tag{2.145}$$

$$\Pr(E)\Pr(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169} \tag{2.146}$$

$$\therefore \quad \Pr(EF) \neq \Pr(E)\Pr(F) \tag{2.147}$$

and E and F are not independent events.

Choose the correct answer in the following exercises

2.1.37 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

2.1.38 Two events A and B will be independent, if

(a) A and B are mutually exclusive

(b) P(not $A \cap$ not B) $= [1 - P(A)]\,[1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

Solution:

(a) When tossing a coin, the event of getting a head and tail are mutually exclusive and let them be denoted by A and B respectively.

$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A) \times \Pr(B) = \frac{1}{4} \tag{2.148}$$

$$\text{or, } \Pr(AB) = 0 \neq \Pr(A) \times \Pr(B) \tag{2.149}$$

Hence A and B are not independent.

(b)

$$\Pr\left(A'B'\right) = [1 - \Pr(A)][1 - \Pr(B)] \tag{2.150}$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.151}$$

$$\implies 1 - \Pr(A + B) = 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.152}$$

$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{2.153}$$

which implies that A and B are independent.

(c) For the same counter example given for option 2.1.38a, $\Pr(A) = \Pr(B)$, but A and B are not independent events.

(d) For the same counter example given for option 2.1.38a, $\Pr(A) + \Pr(B) = 1$, but A and B are not independent events.

2.1.39 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

2.1.40 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{0.32}{0.5} = 0.64 \tag{2.154}$$

2.1.41 A fair die is rolled. Consider events $E = 1, 3, 5$, $F = 2, 3$ and $G = 2, 3, 4, 5$. Find

(a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$

(b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$

(c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: See Table 2.8.

E = {1,3,5}	Pr(E) = $\frac{1}{2}$
F = {2,3}	Pr(F) = $\frac{1}{3}$
G = {2,3,4,5}	Pr(G) = $\frac{2}{3}$
EF = {3}	Pr(EF) = $\frac{1}{6}$
FG = {2,3}	Pr(FG) = $\frac{1}{3}$
EG = {3,5}	Pr(EG) = $\frac{1}{3}$
EFG = {3}	Pr(EFG) = $\frac{1}{6}$

Table 2.8: From given data

(a)

$$\Pr\left(E|F\right) = \frac{\Pr(EF)}{\Pr(F)} = \frac{1/6}{1/3} = 1/2 \tag{2.155}$$

(b)

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1/6}{1/2} = 1/3 \tag{2.156}$$

(c)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} = \frac{1/3}{2/3} = 1/2 \tag{2.157}$$

(d)

$$\Pr(G|E) = \frac{\Pr(EG)}{\Pr(E)} = \frac{1/3}{1/2} = 2/3 \tag{2.158}$$

(e) Since

$$\Pr\left((E+F)G\right) = \Pr\left((EG)+(FG)\right) = \Pr(EG)+\Pr(FG)-\Pr(EFG), \tag{2.159}$$

$$= \frac{1}{3}+\frac{1}{3}-\frac{1}{6}=\frac{1}{2} \tag{2.160}$$

$$\implies \Pr((E+F)|G) = \frac{\Pr((E+F)G)}{\Pr(G)} = \frac{1/2}{2/3} = \frac{3}{4} \tag{2.161}$$

(f)

$$\Pr((EF)|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1/6}{2/3} = \frac{1}{4} \tag{2.162}$$

2.1.42 State which of the following are not the probability distributions of a random variable. Give reasons for your answer.

Table 2.9:

(a)

X	0	1	2
P(X)	0.4	0.4	0.2

Table 2.10:

(b)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

Table 2.11:

(c)

Y	-1	0	1
P(Y)	0.6	0.1	0.2

Table 2.12:

(d)

X	0	1	2	3	4
P(Z)	0.3	0.2	0.4	0.1	0.05

Solution: From the axioms of probability,

$$0 < \Pr(X = i) < 1, i = 1, 2, 3...n. \tag{2.163}$$

$$\sum_{i=1}^{n} \Pr(X = i) = 1, i = 1, 2, 3...n. \tag{2.164}$$

(a)

$$\sum_{i=0}^{2} \Pr(X = i) = 0.4 + 0.4 + 0.2 = 1 \tag{2.165}$$

satisfies both (2.164) and (2.163), so it is a probability distribution.

(b)

$$\sum_{i=0}^{4} \Pr(X = i) = 0.1 + 0.5 + 0.2 - 0.1 + 0.3 = 1 \tag{2.166}$$

Satisfies (2.164) but does not satisfy (2.163) as $P(3) < 0$. Hence NOT a probability distribution.

(c)

$$\sum_{i=-1}^{1} \Pr(X = i) = 0.6 + 0.1 + 0.2 = 0.9 \tag{2.167}$$

(2.164) not satisfied, so it is NOT a probability distribution.

(d)

$$\sum_{i=0}^{4} \Pr(X = i) = 0.3 + 0.2 + 0.4 + 0.1 + 0.05 = 1.05 \tag{2.168}$$

(2.164) not satisfied, so it is NOT a probability distribution.

2.1.43 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

Solution: The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \tag{2.169}$$

The probability of one of the two remaining days being a Tuesday is $\frac{2}{7}$.

2.1.44 Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Solution: The given information is summarised in Table From the given information,

A:	Person with heat attack	$\Pr(A)$=0.40
E_1:	Person treated with meditation and yoga	$\Pr(E_1)$=0.50
E_2:	Person treated with drug	$\Pr(E_2)$=0.50

Table 2.13: Given Information

$$\Pr(A|E_1) = \Pr(A)(1 - (0.30)) = 0.40 \times 0.70 = 0.28 \tag{2.170}$$

and

$$\Pr\left(A|E_2\right) = \Pr\left(A\right)\left(1 - (0.25)\right) = 0.40 \times 0.75 = 0.30 \tag{2.171}$$

From (2.170) and (2.171),

$$\Pr\left(E_1|A\right) = \frac{\Pr\left(E_1\right)\Pr\left(A|E_1\right)}{\sum_{i=1}^{2}\Pr\left(E_i\right)\Pr\left(A|E_i\right)} = \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30} = \frac{14}{29} \tag{2.172}$$

2.1.45 An electronic assembley consists of two subsystems,say A and B.From previous testing procedures, the following probabilities are assumed to be known

$$\Pr\left(\text{A fails}\right) = 0.20 \tag{2.173}$$

$$\Pr\left(\text{B alone fails}\right) = 0.15 \tag{2.174}$$

$$\Pr\left(\text{A and B fails}\right) = 0.15 \tag{2.175}$$

Evaluate the following probabilities

(a) $\Pr\left(\text{A fails given B has failed}\right)$

(b) $\Pr\left(\text{A fails alone}\right)$

Solution: From the given information,

$$\Pr\left(A'\right) = 0.20 \tag{2.176}$$

$$\Pr\left(AB'\right) = 0.15 \tag{2.177}$$

$$\Pr\left(A'B'\right) = 0.15 \tag{2.178}$$

(a)

$$\Pr\left(A'|B'\right) = \frac{\Pr\left(A'B'\right)}{\Pr\left(B'\right)} \tag{2.179}$$

Since

$$B'(1) = B'(A + A') = B'A + B'A' \tag{2.180}$$

$$\Pr\left(B'\right) = \Pr\left(AB'\right) + \Pr\left(A'B'\right) \because ((B'A)(B'A')) = 0 \tag{2.181}$$

$$= 0.15 + 0.15 = 0.30 \tag{2.182}$$

Thus,

$$\Pr\left(A'|B'\right) = 0.15/0.30 == 0.50 \tag{2.183}$$

(b) Similiarly,

$$\Pr\left(A'\right) = \Pr\left(BA'\right) + \Pr\left(A'B'\right) \tag{2.184}$$

$$\implies \Pr\left(BA'\right) = \Pr\left(A'\right) - \Pr\left(A'B'\right) = 0.20 - 0.15 = 0.05 \tag{2.185}$$

2.1.46 If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

(a) $A \subset B$

(b) $B \subset A$

(c) $B = \phi$

(d) $A = \phi$

Solution:

$$\Pr(B|A) = 1 \implies \frac{\Pr(BA)}{\Pr(A)} = 1 \tag{2.186}$$

$$\implies \Pr(BA) = \Pr(A) \tag{2.187}$$

yielding

$$BA = A, \text{ or, } A \subset B \tag{2.188}$$

2.1.47 If $\Pr(A \mid B) > \Pr(A)$, then which of the following is correct

(a) $\Pr(B \mid A) < \Pr((B))$

(b) $\Pr(AB) < \Pr(A) \Pr(B)$

(c) $\Pr(B \mid A) > \Pr(B)$

(d) $\Pr(B \mid A) = \Pr(B)$

Solution:

$$\Pr(A \mid B) > \Pr(A) \implies \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \tag{2.189}$$

$$\implies \Pr(AB) > \Pr(A) \Pr(B) \implies \frac{\Pr(AB)}{\Pr(A)} > \Pr(B) \tag{2.190}$$

Since

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}, \Pr(B \mid A) > \Pr(B) \tag{2.191}$$

Hence, option 2.1.47c is correct.

2.1.48 If A and B are any two events such that $\Pr(A)$+$\Pr(B)$-$\Pr(AB)$=$\Pr(A)$, then choose the correct option

(a) $\Pr(B|A) = 1$

(b) $\Pr(A|B) = 1$

(c) $\Pr(B|A) = 0$

(d) $\Pr(A|B) = 0$

Solution: From the given information,

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A) \tag{2.192}$$

$$\implies \Pr(B) = \Pr(AB) \tag{2.193}$$

Hence,

(a)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B)}{\Pr(A)} \tag{2.194}$$

(b)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1 \tag{2.195}$$

Hence option 2.1.48b is correct.

2.2. Exercises

2.2.1 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very-complex, complex, routine, simple or very-simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

(a) complex or very-complex

(b) neither very-complex nor very simple

(c) routine or complex

(d) routine or simple

Solution: The given information is summarised in Table 2.15

Random Variables	**Difficulty Levels**	**Probability**
E_1	Very-Complex	$\Pr(E_1)$=0.15
E_2	Complex	$\Pr(E_2)$=0.2
E_3	Routine	$\Pr(E_3)$=0.31
E_4	Simple	$\Pr(E_4)$=0.26
E_5	Very-Simple	$\Pr(E_5)$=0.08

Table 2.15:

(a)

$$\Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2) \qquad \because E_1E_2 = 0 \tag{2.196}$$

$$= 0.15 + 0.20 = 0.35 \tag{2.197}$$

(b)

$$\Pr\left(E_1'E_5'\right) = \Pr\left((E_1+E_5)'\right) \tag{2.198}$$
$$= 1 - \Pr\left(E_1+E_5\right) \tag{2.199}$$
$$= 1 - \left[\Pr\left(E_1\right)+\Pr\left(E_5\right)\right] \qquad \because E_1E_5 = 0 \tag{2.200}$$
$$= 1 - \left[0.15+0.08\right] = 0.77 \tag{2.201}$$
$$\tag{2.202}$$

(c)

$$\Pr\left(E_3+E_2\right) = \Pr\left(E_3\right)+\Pr\left(E_2\right) \qquad \because E_3E_2 = 0 \tag{2.203}$$
$$= 0.31+0.20 = 0.51 \tag{2.204}$$

(d) To find the probabilities that a particular surgery will be rated routine or simple:

$$\Pr\left(E_3+E_4\right) = \Pr\left(E_3\right)+\Pr\left(E_4\right) \qquad \because E_3E_4 = 0 \tag{2.205}$$
$$= 0.31+0.26 = 0.57 \tag{2.206}$$
$$\tag{2.207}$$

2.2.2 The accompanying Venn diagram in Fig. 2.1 shows three events, A, B, and C, and also the probabilities of the various intersections (for instance, $\Pr(AB) = .07$). Determine

(a) $\Pr(A)$

(b) $\Pr(BC')$

(c) $\Pr(A+B)$

(d) $\Pr(AB')$

(e) $\Pr(BC)$

(f) Probability of exactly one of the three occurs.

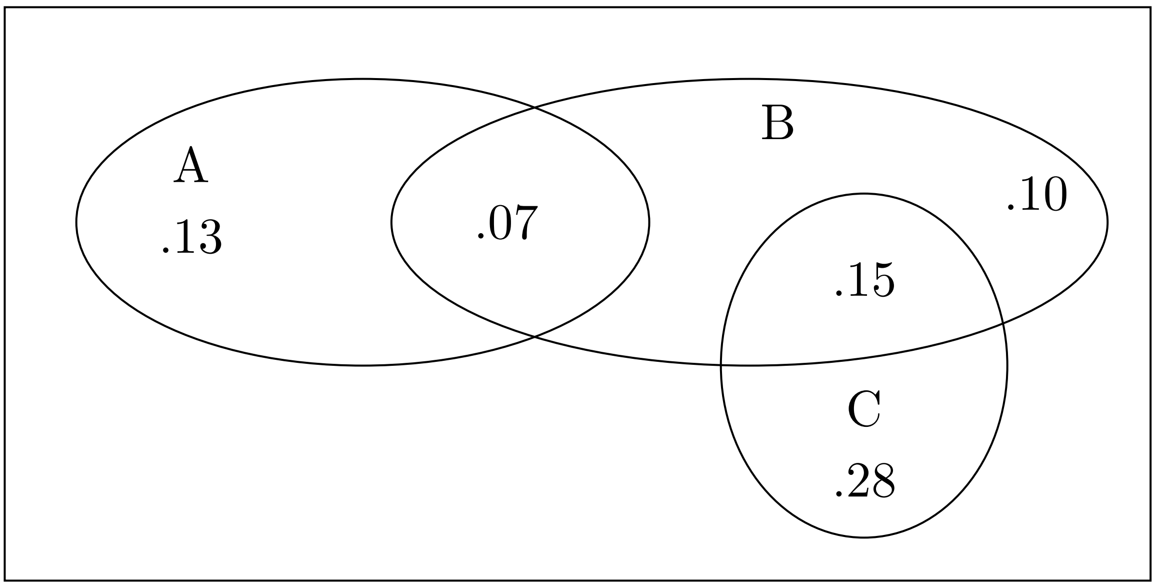

Figure 2.1: Question Figure

Solution: From Fig. 2.1,

$$\Pr(AB) = 0.07 \tag{2.208}$$

$$\Pr\left(AB'\right) = 0.13 \tag{2.209}$$

$$\Pr\left(A'B\right) = 0.25 \tag{2.210}$$

$$\Pr(BC) = 0.15 \tag{2.211}$$

$$\Pr\left(CB'\right) = 0.28 \tag{2.212}$$

$$\Pr\left(AB'C'\right) = 0.13 \tag{2.213}$$

$$\Pr\left(A'BC'\right) = 0.10 \tag{2.214}$$

$$\Pr\left(A'B'C\right) = 0.28 \tag{2.215}$$

(a)

$$A = A\left(B + B'\right) = AB + AB' \quad \left(\because B + B' = 1\right) \tag{2.216}$$

$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) \quad \left(\because BB' = 0\right) \tag{2.217}$$

$$= 0.13 + 0.07 = 0.20 \tag{2.218}$$

from (2.208) and (2.209).

(b) Similarly,

$$\Pr(B) = \Pr\left(A'B\right) + \Pr(AB) \tag{2.219}$$

$$= 0.25 + 0.07 = 0.32 \tag{2.220}$$

from (2.208) and (2.210). Using (2.217),

$$\Pr\left(BC'\right) = \Pr(B) - \Pr(BC) = 0.32 - 0.15 = 0.17 \tag{2.221}$$

(c)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.222}$$

$$= 0.20 + 0.32 - 0.07 = 0.45 \tag{2.223}$$

(d) Using (2.217)

$$\Pr\left(AB'\right) = \Pr(A) - \Pr(AB) = 0.20 - 0.07 = 0.13 \tag{2.224}$$

(e) From (2.211)

$$\Pr(BC) = 0.15 \tag{2.225}$$

(f)

$$X = AB'C' + A'B'C' + A'B'C \tag{2.226}$$

$$\Longrightarrow \Pr(X) = \Pr\left(AB'C'\right) + \Pr\left(A'BC'\right) + \Pr\left(A'B'C\right) \tag{2.227}$$

$$= 0.13 + 0.10 + 0.28 = 0.51 \tag{2.228}$$

from (2.213), (2.214) and (2.215).

Chapter 3

Random Variables

3.1. Examples

3.1.1 One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

(a) A king of red colour

(b) A face card

(c) A red face card

(d) The jack of hearts

(e) A spade

(f) The queen of diamonds

Solution: See Table 3.1. Consider 3 random variables X, Y and Z, which represent the Colour, Class and Value of each card respectively. The pmfs are

$$p_X(i) = \frac{1}{2} \;\forall\; i \in [0, 1] \tag{3.1}$$

$$p_Y(i) = \frac{1}{4} \;\forall\; i \in [1, 4] \tag{3.2}$$

$$p_Z(i) = \frac{1}{13} \;\forall\; i \in [1, 13] \tag{3.3}$$

Event	Value of X	Value of Y	Value of Z
Draw Red King	1	N/A	3
Draw Face Card	N/A	N/A	1,2 or 3
Draw Red Face Card	1	N/A	1,2 or 3
Draw Hearts Jack	N/A	3	1
Draw Spade	N/A	4	N/A
Draw Diamonds Queen	N/A	1	2

Table 3.1: Values of X,Y,Z for each event

and

$$\begin{aligned} F_Z(z) = \Pr(Z \le z) &= \sum_{i=1}^{z} \Pr(Z = i) \\ &= z \times \Pr(Z = 1) = \frac{z}{13} \end{aligned} \tag{3.4}$$

Also, the random variable pairs X,Z and Y,Z are independent.

(a) Probability of drawing a King of Red colour

$$\begin{aligned} \Pr(X = 1, Z = 3) &= \Pr(X = 1) \times \Pr(Z = 3) \\ &= \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \end{aligned} \tag{3.5}$$

(b) Probability of drawing a Face Card

$$\Pr(1 \le Z \le 3) = F_Z(3) = \frac{3}{13} \tag{3.6}$$

(c) Probability of drawing a Red Face Card

$$\Pr(X = 1, 1 \leq Z \leq 3) = \Pr(X = 1) \times F_Z(3) \tag{3.7}$$
$$= \frac{1}{2} \times \frac{3}{13} = \frac{3}{26} \tag{3.8}$$

(d) Probability of drawing the Jack of Hearts

$$\Pr(Y = 3, Z = 1) = \Pr(Y = 3) \times \Pr(Z = 1) \tag{3.9}$$
$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{3.10}$$

(e) Probability of drawing a Spade

$$\Pr(Y = 4) = \frac{1}{4} \tag{3.11}$$

(f) Probability of drawing the Queen of Diamonds:

$$\Pr(Y = 1, Z = 2) = \Pr(Y = 1) \times \Pr(Z = 2) \tag{3.12}$$
$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{3.13}$$

3.1.2 Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

(a) What is the probability that the card is the queen?

(b) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

Solution: See Table 3.3.

EVENT	DESCRIPTION
E	Event of picking a card.
S	Sample space of picking a card.
Q	Event of the card picked be Queen.
A	Event of the card picked be Ace.

Table 3.3:

(a)

$$\Pr(Q) = \frac{1}{5} \tag{3.14}$$

(b) i.

$$\Pr(A) = \frac{1}{4} \tag{3.15}$$

ii.

$$\Pr(Q) = 0 \tag{3.16}$$

3.1.3 A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that if a red ball, determine the number of blue balls in the bag.

Solution: The probability of drawing a red ball is

$$\Pr(R) = \frac{5}{5+x} \tag{3.17}$$

The probability of drawing a blue ball is

$$\Pr(B) = \frac{10}{5+x} \tag{3.18}$$

Thus,

$$\left(\frac{5}{5+x}\right)+2\left(\frac{5}{5+x}\right)=1 \tag{3.19}$$

$$\implies \frac{15}{5+x}=1 \tag{3.20}$$

$$\text{or, } x=10 \tag{3.21}$$

3.1.4 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution: See Table 3.4

Table 3.4: Random Variable and probability Table

Random variable	value of R.V	Probability
\$X\$	1,2	26/52
\$Y\$	1,2,3,4	13/52
\$Z\$	1 \le Z \le 13	1/13

(a)

$$\Pr\left(Y=1,Z=1\right)=\Pr\left(Y=1\right)\Pr\left(Z=1\right)=\left(\frac{1}{4}\right)\left(\frac{1}{13}\right)=\frac{1}{52} \tag{3.22}$$

(b) The probability when the card choosen is ,

i. an ace $(Z=1)$

$$\Pr\left(Z=1\right)=\frac{1}{13}. \tag{3.23}$$

ii. black card $(X = 1)$

$$\Pr(X = 1) = \frac{1}{2}. \tag{3.24}$$

3.1.5 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 3.6. yielding

RV	**Values**	**Description**
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 3.6: Random variables(RV) X,Y and X,Y

$$\Pr(00, 10, 20, 31) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \tag{3.25}$$

$$= \frac{286}{20285} \tag{3.26}$$

3.1.6 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 3.8 The total number of possible outcomes is ${}^{N}C_n$ and the total number of favourable outcomes is ${}^{q}C_n$ yielding the desired probability

$$\Pr(n) = \frac{{}^{q}C_n}{{}^{N}C_n} \tag{3.27}$$

Substituting numerical values,

Variable	Value	Description
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	{0,1,2,...,N}	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 3.8:

(a) For one ticket,

$$\Pr(1) = \frac{^{9990}C_1}{^{10000}C_1} = 0.9990 \tag{3.28}$$

(b) For two tickets,

$$\Pr(2) = \frac{^{9990}C_2}{^{10000}C_2} = 0.9980 \tag{3.29}$$

(c) For 10 tickets

$$\Pr(3) = \frac{^{9990}C_{10}}{^{10000}C_{10}} = 0.9901 \tag{3.30}$$

3.1.7 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 3.10 summarises the given information.

RV	Values	Description
X	{0,1}	0: sectioin1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{{}^{40}C_2}{{}^{100}C_2} + \frac{{}^{60}C_2}{{}^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \tag{3.31}$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \tag{3.32}$$

3.1.8 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution: Let

$$X_i = \begin{cases} 1, & \text{correct number choosen in } i^{th} \text{ wheel} \\ 0, & \text{otherwise} \end{cases} \tag{3.33}$$

and since repetition is not allowed, sample space for every next wheel will reduce by 1 unit. Therefore,

$$p_{X_i}(1) = \frac{1}{11 - i} \tag{3.34}$$

$$p_{X_i}(0) = 1 - \frac{1}{11 - i} \tag{3.35}$$

$$= \frac{10 - i}{11 - i} \tag{3.36}$$

Therefore, the desired probability is

$$\Pr(E) = \prod_{i=1}^{4} p_{X_i}(1) \tag{3.37}$$

$$= \frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} \times \frac{1}{7} \tag{3.38}$$

$$= \frac{1}{5040} \tag{3.39}$$

3.1.9 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 3.12 summarizes the various events Given that the cards are drawn

RV	Values	Description
X	{0,1}	number of cards drawn 2
Y	{0,1}	0: black card, 1: red card
XY	{00,10}	card drawn is black

Table 3.12:

at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00,10) = \frac{^{26}C_1}{^{52}C_1} \times \frac{^{25}C_1}{^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \tag{3.40}$$

3.1.10 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

3.1.11 Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(a) both balls are red.

(b) first ball is black and second is red.

(c) one of them is black and other is red.

3.1.12 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

3.1.13 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

Solution: Let X and Y be two random variables representing outcomes on both the die, See Table 3.13. Since both die rolls are independent,

$$\Pr\left(X = 2, Y = 2\right) = \Pr\left(X = 2\right)\Pr\left(Y = 2\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \tag{3.41}$$

$\Pr(X=2)$	The probability of occurence of 2 on die roll 1.
$\Pr(Y=2)$	The probability of occurence of 2 on die roll 2.
$\Pr(X=2, Y=2)$	The probability of occurence of 2 on both the die.

Table 3.13:

3.1.14 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution: See Table 3.14 Given,

Table 3.14: Random Variable Declaration

Random Variable	Value of the random variable	Event
B	0	selecting first bag
	1	selecting second bag
R	0	choosing white ball from the bag
	1	choosing red ball from the bag

$$\Pr(R=1|B=0) = \frac{4}{8} = \frac{1}{2} \tag{3.42}$$

$$\Pr(R=1|B=1) = \frac{2}{8} = \frac{1}{4} \tag{3.43}$$

$$\Pr(B=0) = \frac{1}{2} \tag{3.44}$$

$$\Pr(B=1) = \frac{1}{2} \tag{3.45}$$

$$\Pr(B=0|R=1) = \frac{\Pr(R=1|B=0)\Pr(B=0)}{\Pr(R=1|B=0)\Pr(B=0)+\Pr(R=1|B=1)\Pr(B=1)} \tag{3.46}$$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{2}} \tag{3.47}$$

$$= \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}} = \frac{2}{3} \tag{3.48}$$

3.1.15 Cards with numbers 2 to 101 are placed in a box. A card is selected at random.Find the probability that the card has

(i) an even number

(ii) a square number

Solution:

(i) an even number

$$X = \begin{cases} 1, & \text{if number is even} \\ 0, & \text{otherwise} \end{cases} \tag{3.49}$$

Then

$$p_X(1) = \frac{50}{100} \tag{3.50}$$

$$= \frac{1}{2} \tag{3.51}$$

(ii) a square number

$$Y = \begin{cases} 1, & \text{if square number} \\ 0, & \text{otherwise} \end{cases} \tag{3.52}$$

Then

$$p_Y(1) = \frac{9}{100} \tag{3.53}$$

3.2. Exercises

3.2.1 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die.

Solution: See Table 3.16 for the input parameters. Then,

Parameter	**Value**	**Description**
X	{1,2,3,4,5,6}	Number obtained on the die

Table 3.16: Parameters and their Description

$$p_X(k) = \begin{cases} 2p, & if\ k = 2m-1 \\ p, & if\ k = 2m \end{cases} \tag{3.54}$$

Since $1 \leq X \leq 6$,

$$\sum_{i=1}^{6} \Pr(X = i) = 1 \tag{3.55}$$

$$\implies 6p + 3p = 1 \tag{3.56}$$

$$\implies p = \frac{1}{9} \tag{3.57}$$

The CDF

$$F_X(k) = \begin{cases} \frac{3k+1}{18}, & if\ k = 2m - 1 \\ \frac{k}{6}, & if\ k = 2m \end{cases} \tag{3.58}$$

Thus,

$$\Pr(G) = \Pr(X > 3) = F_X(6) - F_X(3) \tag{3.59}$$

$$= 1 - \frac{3(3) + 1}{18} = \frac{4}{9} \tag{3.60}$$

Chapter 4

Conditional Probability

4.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{4.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{4.2}$$

4.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

4.3 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

4.4 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{4.3}$$

4.5 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A = 0.4)$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A + B)$

Solution:

4.6 If $\Pr(A) = \frac{6}{11}$, $\Pr(B) = \frac{5}{11}$ and $\Pr(A + B) = \frac{7}{11}$, find

(a) $\Pr(AB)$

(b) $\Pr(A \mid B)$

(c) $\Pr(B \mid A)$

Solution:

(a) Since

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B), \tag{4.4}$$

$$\Pr(AB) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11} \tag{4.5}$$

(b) Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.6}$$

$$\Pr(A \mid B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5} \tag{4.7}$$

(c) Since

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)}, \tag{4.8}$$

$$\Pr(B \mid A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3} \tag{4.9}$$

4.7 Mother, Father and Son line up at random for a family picture. Determine $\Pr(E \mid F)$ where E: Son on one end, F: Father in middle

Solution: The total ways of arranging Father, Son, Mother in the family chart is 3! = 6. The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \tag{4.10}$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \tag{4.11}$$

Thus,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \tag{4.12}$$

4.8 Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that

(a) The youngest is a girl

(b) At least one is a girl

Solution:

Variable	Description	Probability
$X_i = 1$	ith born child is a boy	$\Pr(X_i = 1) = 0.50$
$X_i = 0$	ith born child is a girl	$\Pr(X_i = 0) = 0.50$

Table 4.1: Random variable definitions.

(a)

$$\Pr\left((X_1 + X_2)' | X_2'\right) = \frac{\Pr\left((X_1' X_2') X_2'\right)}{\Pr\left(X_2'\right)} \tag{4.13}$$

$$= \frac{\Pr\left(X_1'\right) \Pr\left(X_2'\right)}{\Pr\left(X_2'\right)} \tag{4.14}$$

$$= \Pr\left(X_1'\right) = \frac{1}{2} \tag{4.15}$$

(b)

$$\Pr\left((X_1 + X_2)' | (X_1 X_2)'\right) = \frac{\Pr\left((X_1' X_2')(X_1' + X_2')\right)}{1 - \Pr\left(X_1 X_2\right)} \tag{4.16}$$

$$= \frac{\Pr\left(X_1' X_2'\right)}{1 - \Pr\left(X_1 X_2\right)} \tag{4.17}$$

$$= \frac{\Pr\left(X_1'\right) \Pr\left(X_2'\right)}{1 - \Pr\left(X_1\right) \Pr\left(X_2\right)} \tag{4.18}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 - (\frac{1}{2} \times \frac{1}{2})} = \frac{1}{3} \tag{4.19}$$

4.9 An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution: From the law of total probability,

Variable	**Event**
$X = 0$	Easy question
$X = 1$	Difficult question
$Y = 0$	True/False question
$Y = 1$	Multiple choice question

Table 4.2:

$$p_X(0) + p_X(1) = 1 \tag{4.20}$$

$$p_Y(0) + p_Y(1) = 1 \tag{4.21}$$

From Table 4.2,

$$p_X(0) = p_{X,Y}(0,0) + p_{X,Y}(0,1) \tag{4.22}$$

$$= \frac{300 + 500}{300 + 200 + 500 + 400} = \frac{4}{7} \tag{4.23}$$

$$p_Y(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0) \tag{4.24}$$

$$= \frac{300 + 200}{300 + 200 + 500 + 400} = \frac{5}{14} \tag{4.25}$$

From (4.20), (4.21) and (4.25),

$$p_X(1) = 1 - p_X(0) = \frac{3}{7} \tag{4.26}$$

$$p_Y(1) = 1 - p_Y(0) = \frac{9}{14} \tag{4.27}$$

$$p_{X,Y}(0,0) = \frac{300}{1400} = \frac{3}{14} \tag{4.28}$$

$$p_{X,Y}(1,1) = \frac{400}{1400} = \frac{2}{7} \tag{4.29}$$

From (4.22), (4.24), (4.25) and (4.29),

$$p_{X,Y}(0,1) = p_X(0) - p_{X,Y}(0,0) = \frac{5}{14} \tag{4.30}$$

$$p_{X,Y}(1,0) = p_Y(0) - p_{X,Y}(0,0) = \frac{1}{7} \tag{4.31}$$

By definition,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \tag{4.32}$$

From (4.28) and (4.31),

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} \tag{4.33}$$

$$= \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9} \tag{4.34}$$

4.10 If $\Pr(A) = \frac{1}{2}$, $\Pr(B) = 0$, then $\Pr(A \mid B)$ is

(a) 0

(b) $\frac{1}{2}$

(c) not defined

(d) 1

Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.35}$$

$$\Pr(A \mid B)\, \text{is not defined} \tag{4.36}$$

4.11 If A and B are events such that

$$\Pr(A|B) = \Pr(B|A) \tag{4.37}$$

then

(a) $A \subset B$ but $A \neq B$

(b) $A = B$

(c) $A \cap B = \phi$

(d) $\Pr(A) = \Pr(B)$

Solution: Using Bayes' Rule,

$$\Pr(AB) = \Pr(A)\Pr(B|A) \tag{4.38}$$
$$= \Pr(B)\Pr(A|B) \tag{4.39}$$

Using (4.37) in (4.38) and (4.39),

$$\Pr(A) = \Pr(B) \tag{4.40}$$

We consider the options one by one.

(a) If $A \subset B$ and $A \neq B$, then we can write $B = A + C$, where $AC = 0$ and $C \neq 0$. Thus,

$$\Pr(B) = \Pr(A + C) \tag{4.41}$$
$$= \Pr(A) + \Pr(C) - \Pr(AC) \tag{4.42}$$
$$= \Pr(A) + \Pr(C) > \Pr(A) \tag{4.43}$$

However, (4.43) contradicts (4.40).

(b) We give a counterexample to show this is wrong. Consider A as the event that an even number shows on rolling a fair die and B as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 4.3. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{2} \tag{4.44}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{2} \tag{4.45}$$

(c) The same example as before provides the required counterexample, as $\Pr(AB) = \frac{1}{6}$.

(d) This is the correct answer, as discussed above.

	A	$\bar{A}$
B	$\frac{1}{6}$	$\frac{1}{3}$
$\bar{B}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 4.3: Joint pmf for events A and B.

4.12 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Let X denote the die roll for the first trial. The pmf of X is

$$\Pr(X = k) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.46}$$

Let Y be the random variable denoting the outcome of the coin toss in the second trial. The pmf of Y is

$$\Pr(Y = k) = \begin{cases} \frac{1}{2} & 0 \leq i \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.47}$$

We are required to find $\Pr(Y = 1|X = 3)$. However, from the given data,

$$\Pr(Y = 1,\ X = k) = \begin{cases} \frac{1}{12} & k \in \{1, 2, 4, 5\} \\ 0 & \text{otherwise} \end{cases} \tag{4.48}$$

Therefore, from (4.48),

$$\Pr(Y = 1|X = 3) = \frac{\Pr(X = 3,\ Y = 1)}{\Pr(X = 3)} = 0 \tag{4.49}$$

4.13 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Let E_1 denote the event that the first card drawn is Black, E_2 denote the event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52}, \Pr(E_2 \mid E_1) = \frac{25}{51} \tag{4.50}$$

$$\implies \Pr(E_1 E_2) = \Pr(E_1)\Pr(E_2 \mid E_1) = \frac{25}{102} \tag{4.51}$$

4.14 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B)$

(d) $P(B|A)$

4.15 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution: The given information is summarized in Tables 4.4 and 4.5.

RV	**Values**	**Description**
X	$\{0,1\}$	1st draw - 0: Red, 1: Black
Y	$\{0,1\}$	2nd draw - 0: Red, 1: Black

Table 4.4: Random variables X,Y

Event	**Probability**
$\Pr(X=0)$	$\frac{5}{10}$
$\Pr(X=1)$	$\frac{5}{10}$
$\Pr(Y=1 \mid X=0)$	$\frac{7}{12}$
$\Pr(Y=1 \mid X=1)$	$\frac{5}{12}$

Table 4.5: Probabilities

The required probability is given by

$$\Pr(Y=0) = \Pr(X=0)\Pr(Y=0 \mid X=0) + \Pr(X=1)\Pr(Y=0 \mid X=1) \quad (4.52)$$

$$= \left(\frac{5}{10} \times \frac{7}{12}\right) + \left(\frac{5}{10} \times \frac{5}{12}\right) = \frac{1}{2} \quad (4.53)$$

4.16 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.17 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Solution:

Let

$$X = \begin{cases} 0, & \text{if student is resides in hostel} \\ 1, & \text{if student is a day scholar} \end{cases} \tag{4.54}$$

$$Y = \begin{cases} 0, & \text{if student does not attain A grade} \\ 1, & \text{if student attains A grade} \end{cases} \tag{4.55}$$

From the given data,

$$\Pr\left(X = 0\right) = \frac{3}{5} \tag{4.56}$$

$$\Pr\left(X = 1\right) = \frac{2}{5} \tag{4.57}$$

$$\Pr\left(Y = 1 \mid X = 0\right) = \frac{3}{10} \tag{4.58}$$

$$\Pr\left(Y = 1 \mid X = 1\right) = \frac{1}{5} \tag{4.59}$$

The desired probability is

$$\Pr\left(X = 0 \mid Y = 1\right) = \frac{\Pr\left(Y = 1 \mid X = 0\right) \times \Pr\left(X = 0\right)}{\sum_{k=0}^{1} \Pr\left(Y = 1 \mid X = k\right) \times \Pr\left(X = k\right)} \tag{4.60}$$

$$= \frac{\frac{3}{10} \times \frac{3}{5}}{\frac{3}{10} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5}} = \frac{9}{13} \tag{4.61}$$

4.18 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: See Tables 4.18 and 4.18

Random Variable	Description
$X = 0$	Student guesses the answer
$X = 1$	Student knows the answer
$Y = 0$	Answer is incorrect
$Y = 1$	Answer is correct

Table 4.6: Random Variable and their description

Pr(Event)	Value
Pr(Y=1 | X=0)	0.25
Pr(Y=1 | X=1)	1
Pr(X=0)	0.25
Pr(X=1)	0.75

Table 4.7: Probability of events

The probability that the student knows the answer and he answered it correctly is

$$\Pr(X = 1|Y = 1) = \frac{\Pr(Y = 1|X = 1)\Pr(X = 1)}{\sum_{i=0}^{i=1} Pr(Y = 1|X = i)Pr(X = i)} \quad (4.62)$$

$$= \frac{0.75}{0.25 \times 0.25 + 1 \times 0.75} = 0.92308 \quad (4.63)$$

4.19 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

Solution: See Table 4.19 for the given information.

$$\Pr(E_2) = 1 - \Pr(E_1) = 1 - 0.001 = 0.999 \tag{4.64}$$

A:	Person with positive blood test	$\Pr(A)$
E_1:	Person suffering from a disease	$\Pr(E_1)$=0.001
E_2:	Person not suffering from a disease	$\Pr(E_2)$=0.999
$A\|E_1$:	Event of positive blood test when person suffers from disease	$\Pr(A\|E_1)$=0.99
$A\|E_2$:	Event of positive blood test when person not suffers from disease	$\Pr(A\|E_2)$=0.005

Table 4.8: Given Information

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} \tag{4.65}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} = \frac{22}{133} \tag{4.66}$$

4.20 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Solution: Define the random variable X as in Table 4.9. Clearly, the pmf of X is

$X = 1$	Two-headed coin is selected.
$X = 2$	75% biased coin is selected.
$X = 3$	Fair coin is selected.

Table 4.9: Definition of X.

$$\Pr(X=k) = \begin{cases} \frac{1}{3} & 1 \le k \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{4.67}$$

Let the random variables Y_1, Y_2 and Y_3 (one for each coin) be defined as

$$Y_1 \sim \mathrm{Ber}(1) \tag{4.68}$$

$$Y_2 \sim \mathrm{Ber}\left(\frac{3}{4}\right) \tag{4.69}$$

$$Y_3 \sim \mathrm{Ber}\left(\frac{1}{2}\right) \tag{4.70}$$

Define Y as

$$Y \triangleq \sum_{i=1}^{3} \mathbf{1}_i(X)\, Y_i \tag{4.71}$$

where $\mathbf{1}$ denotes the indicator random variable, defined as

$$\mathbf{1}_i(X) = \begin{cases} 1 & \text{if } X = i \\ 0 & \text{otherwise} \end{cases} \tag{4.72}$$

We are required to find $\Pr(X=1|Y=1)$. However, from Bayes' Rule,

$$\Pr(X=1, Y=1) = \Pr(X=1)\Pr(Y=1|X=1) \tag{4.73}$$

$$= \Pr(Y=1)\Pr(X=1|Y=1) \tag{4.74}$$

Note from (4.71) that

$$X = 1 \implies Y = Y_1 \tag{4.75}$$

and also,

$$\Pr(Y=1) = \sum_{i=1}^{3} \Pr(X=i)\Pr(Y_i=1) \tag{4.76}$$

$$= \frac{1}{3}\left(1+\frac{3}{4}+\frac{1}{2}\right) = \frac{3}{4} \tag{4.77}$$

Thus, from (4.73), (4.74) and (4.77), we see that

$$\Pr(X=1|Y=1) = \frac{\Pr(X=1)\Pr(Y_1=1)}{\Pr(Y=1)} \tag{4.78}$$

$$= \frac{\frac{1}{3}}{\frac{3}{4}} = \frac{4}{9} \tag{4.79}$$

4.21 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

4.22 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

4.23 . Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution: The given information is listed in Tables 4.11 and 4.13

RV	**Values**	**Description**
X	{1,2}	1:Group1 ,2:Group2
Y	{0,1}	0:New product not introduced ,1:New product introduced

Table 4.11: Random variables(RV) X,Y

Event	**Probability**	**Description**
$\Pr(X = 1)$	0.6	First group winning
$\Pr(X = 2)$	0.4	Second group winning
$\Pr(Y = 1 \mid X = 1)$	0.7	Introducing 1 if 1 wins
$\Pr(Y = 1 \mid X = 2)$	0.3	Introducing 1 if 2 wins

Table 4.13: Probabilities

$$\Pr(X = 2 \mid Y = 1) = \frac{\Pr(2)\Pr(1 \mid 2)}{\Pr(1)\Pr(1 \mid 1) + \Pr(2)\Pr(1 \mid 2)} = \frac{2}{9} \tag{4.80}$$

4.24 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

4.25 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

4.26 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

4.27 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

Solution: Consider the random variables A, X as described in the table 4.14.

RV	**Values**	**Description**
A	$\{0, 1\}$	1: A speaks truth, 0: A lies
X	$\{0, 1\}$	1: Heads, 0: Tails

Table 4.14: Random variables A, X

The given information about probabilities is listed in table 4.15.

Event	**Probability**
$\Pr(A = 1)$	$\frac{4}{5}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(X = 1 \mid A = 1)$	$\frac{1}{2}$

Table 4.15: Probabilities

The required probability is given by

$$\Pr(A = 1 \mid X = 1) = \frac{\Pr(A = 1)\Pr(X = 1 \mid A = 1)}{\Pr(X = 1)} \tag{4.81}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} \tag{4.82}$$

$$= \frac{4}{5} \tag{4.83}$$

4.28 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A \mid B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A \mid B) < \Pr(A)$

(c) $\Pr(A \mid B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{4.84}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{4.85}$$

we know that

$$\Pr(B) \leq 1 \tag{4.86}$$

$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{4.87}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{4.88}$$

$$= \Pr(A \mid B) \tag{4.89}$$

from (4.85).

4.29 A and B are two events such that $\Pr(A) \neq 0$. Find $\Pr(B \mid A)$, if

(a) A is a subset of B

(b) $A \cap B = \phi$

Solution: We use

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)} \tag{4.90}$$

(a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \tag{4.91}$$

From (4.90),

$$\Pr(B \mid A) = 1 \tag{4.92}$$

(b) $A \cap B = \phi$. This implies

$$\Pr(BA) = 0 \tag{4.93}$$

From (4.90),

$$\Pr(B \mid A) = 0 \tag{4.94}$$

4.30 A couple has two children.

(a) Find the probability that both children are males, if it is known that at least one of the children is male.

(b) Find the probability that both children are females, if it is known that the elder child is a female.

Solution: Consider the random variables X, Y, which denotes the first child, second child gender respectively as described in table 4.16.

RV	Values	Description
X	$\{0, 1\}$	0: Male , 1: Female
Y	$\{0, 1\}$	0: Male, 1: Female

Table 4.16: Random variables X

The probabilities for the random variables X, Y is listed in table 4.17.

Event	Probability
$\Pr(X = 0)$	$\frac{1}{2}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(Y = 0)$	$\frac{1}{2}$
$\Pr(Y = 1)$	$\frac{1}{2}$
$\Pr(X + Y = 0)$	$\frac{1}{4}$
$\Pr(X + Y = 2)$	$\frac{1}{4}$
$\Pr(XY = 0)$	$\frac{3}{4}$

Table 4.17: Probabilities

The probability $\Pr(XY = 0)$ is given by

$$= \Pr(X = 0) + \Pr(Y = 0) - \Pr(X + Y = 0) \tag{4.95}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \tag{4.96}$$

$$= \frac{3}{4} \tag{4.97}$$

(a) The event of both children being Male is when $X + Y = 0$. The event of atleast one of the children being Male is when $XY = 0$.

$$\{X + Y = 0\} \cap \{XY = 0\} \equiv \{X + Y = 0\} \tag{4.98}$$

The required probability is given by,

$$\Pr(X + Y = 0 \mid XY = 0) \tag{4.99}$$

$$= \frac{\Pr(X + Y = 0)}{\Pr(XY = 0)} \tag{4.100}$$

$$= \frac{1}{3} \tag{4.101}$$

(b) The event of both children being Female is when $X + Y = 2$. The event of elder child being Female is when $X = 1$.

$$\{X + Y = 2\} \cap \{X = 1\} \equiv \{X + Y = 2\} \tag{4.102}$$

The required probability is given by,

$$\Pr(X+Y=2 \mid X=1) \tag{4.103}$$

$$= \frac{\Pr(X+Y=2)}{\Pr(X=1)} \tag{4.104}$$

$$= \frac{1}{2} \tag{4.105}$$

4.31 Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability that this person being male? Assume that there are equal number of males and females.

Solution: See Table 4.18. It is given that,

Variable	**Event**
$X=0$	Men
$X=1$	Women
$Y=0$	Non-grey hair
$Y=1$	grey hair

Table 4.18:

$$p_X(0) = p_X(1) = \frac{1}{2} \tag{4.106}$$

$$p_{Y|X}(1|0) = \frac{5}{100} = \frac{1}{20} \tag{4.107}$$

$$p_{Y|X}(1|1) = \frac{0.25}{100} = \frac{1}{400} \tag{4.108}$$

From the law of total probability,

$$p_Y(1) = p_{Y|X}(1|0) \times p_X(0) + p_{Y|X}(1|1) \times p_X(1) \tag{4.109}$$

$$= \frac{1}{20} \times \frac{1}{2} + \frac{1}{400} \times \frac{1}{2} = \frac{21}{800} \tag{4.110}$$

$$\therefore \ p_Y(1) = \frac{21}{800} \tag{4.111}$$

Thus,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \ p_{X,Y}(x,y) = p_{Y|X}(y|x)p_X(x) \tag{4.112}$$

yielding

$$p_{X,Y}(0,1) = p_{Y|X}(1|0) \times p_X(0) \tag{4.113}$$

$$= \frac{1}{20} \times \frac{1}{2} = \frac{1}{40} \tag{4.114}$$

$$\tag{4.115}$$

resulting in

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{1}{40}}{\frac{21}{800}} = \frac{20}{21} \tag{4.116}$$

4.32 Suppose we have four boxes A,B,C and D containing coloured marbles as given in Table 4.19. One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from

1) Box A ?

2) Box B ?

3) Box C ?

Box	Marble colour		
	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

Table 4.19: Question Table

Solution: See Table Table 4.20. Here,

Events	Definition
E	drawn marble is red
E_1	selected box is A
E_2	selected box is B
E_3	selected box is C
E_4	selected box is D

Table 4.20: Events Table

$$\Pr\left(E|E_1\right) = \frac{1}{10}, \Pr\left(E|E_2\right) = \frac{6}{10}, \Pr\left(E|E_3\right) = \frac{8}{10}, \Pr\left(E|E_4\right) = \frac{0}{10} \tag{4.117}$$

$$\Pr\left(E_i\right) = \frac{1}{4} \ \forall 1 \leq i \leq 4 \tag{4.118}$$

(a)

$$\Pr\left(E_1|E\right) = \frac{\Pr\left(E|E_1\right)\Pr\left(E_1\right)}{\sum_{i=1}^{i=4}\left(\Pr\left(E|E_i\right)\Pr\left(E_i\right)\right)} \tag{4.119}$$

$$= \frac{\frac{1}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{1}{15} \tag{4.120}$$

(b)

$$\Pr\left(E_2|E\right) = \frac{\Pr\left(E|E_2\right)\Pr\left(E_2\right)}{\sum_{i=1}^{i=4}(\Pr\left(E|E_i\right)\Pr\left(E_i\right))} \tag{4.121}$$

$$= \frac{\frac{6}{10}\times\frac{1}{4}}{\frac{1}{10}\times\frac{1}{4}+\frac{6}{10}\times\frac{1}{4}+\frac{8}{10}\times\frac{1}{4}+\frac{0}{10}\times\frac{1}{4}} = \frac{2}{5} \tag{4.122}$$

(c)

$$\Pr\left(E_3|E\right) = \frac{\Pr\left(E|E_3\right)\Pr\left(E_3\right)}{\sum_{i=1}^{i=4}(\Pr\left(E|E_i\right)\Pr\left(E_i\right))} \tag{4.123}$$

$$= \frac{\frac{8}{10}\times\frac{1}{4}}{\frac{1}{10}\times\frac{1}{4}+\frac{6}{10}\times\frac{1}{4}+\frac{8}{10}\times\frac{1}{4}+\frac{0}{10}\times\frac{1}{4}} = \frac{8}{15} \tag{4.124}$$

4.33 Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution: Let

$$X = \begin{cases} 1, & \text{if ball is being drawn from Bag I} \\ 2, & \text{if ball is being drawn from Bag I} \end{cases} \tag{4.125}$$

$$Y = \begin{cases} 1, & \text{if ball drawn is Red} \\ 2, & \text{if ball drawn is Black} \end{cases} \tag{4.126}$$

Then

$$\Pr(X=1, Y=1) = \frac{3}{7} \Pr(X=1, Y=2) = \frac{4}{7} \tag{4.127}$$

$$\implies \Pr(X=2, Y=1) = \Pr(X=1, Y=1) \times \frac{5}{10} + \Pr(X=1, Y=2) \times \frac{4}{10} = \frac{15}{70} + \frac{16}{70} = \frac{31}{70} \tag{4.128}$$

Consequently,

$$\Pr(X=2, Y=2) = 1 - \Pr(X=2, Y=1) = \frac{39}{70} \tag{4.129}$$

See Thus, the desired probability is

Red ball from Bag I:	$\Pr(X=1, Y=1)=\frac{3}{7}$
Black ball from Bag I:	$\Pr(X=1, Y=2)=\frac{4}{7}$
Red ball from Bag II:	$\Pr(X=2, Y=1)=\frac{31}{70}$
Black ball from Bag II:	$\Pr(X=2, Y=2)=\frac{39}{70}$

(4.130)

Table 4.21: Final probabilities of the events.

$$\begin{aligned}
&\Pr(X=1, Y=2 | X=2, Y=2) \\
&\qquad = \frac{\Pr(X=2, Y=1 | X=1, Y=1)}{\sum_{i=1}^{i=2} \Pr(X=2, Y=1 | X=1, Y=i) \Pr(X=1, Y=i)} \\
&\qquad\qquad = \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{4}{10} \times \frac{4}{7} + \frac{5}{10} \times \frac{3}{7}} = \frac{16}{31}
\end{aligned} \tag{4.131}$$

Chapter 5

Discrete Distributions

5.1. Bernoulli

5.1.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(a) She will buy it?

(b) She will not buy it?

Solution: We can model this situation using the random variable $X \sim \text{Ber}(p)$, where p is the probability of success, *i.e.* the pen is purchased. From the given data,

$$1 - p = \frac{20}{144} \implies p = \frac{67}{72} \tag{5.1}$$

(a) Probability that the pen is purchased is

$$\Pr(X = 1) = p = \frac{67}{72} \tag{5.2}$$

(b) Probability that the pen is not purchased is

$$\Pr(X = 0) = 1 - p = \frac{5}{72} \tag{5.3}$$

5.1.2 A bag contains slips numbered from 1 to 100. If Fatima chooses a slip at random from the bag, it will either be an odd number or an even number. Since this situation has only two possible outcomes, so, the probability of each is $\frac{1}{2}$.Justify.

Solution: Let

$$X = \begin{cases} 1, & \text{if number is even} \\ 0, & \text{if number is odd} \end{cases} \tag{5.4}$$

Then

$$p_X(1) = \frac{50}{100} \tag{5.5}$$
$$= \frac{1}{2} \tag{5.6}$$

Similarly

$$p_X(0) = \frac{50}{100} \tag{5.7}$$
$$= \frac{1}{2} \tag{5.8}$$

5.2. Multinomial

5.2.1 A box contains 5 red marbles, 8 white marbles and 4 green marbles.One marble is taken out of the box at random. What is the probability that the marble taken out will be

Solution:

(a) red ?

(b) white ?

(c) not green?

Solution: Let

$$\mathrm{N} = \mathrm{R} + \mathrm{W} + \mathrm{G} \tag{5.9}$$

$$\mathrm{n} = \mathrm{r} + \mathrm{w} + \mathrm{g} \tag{5.10}$$

where R,B,G and r, b, g represent the number of red, white and green marbles respectively within N and n. Then

$$\Pr(r, w, g) = \frac{{}^{R}C_{r}\,{}^{W}C_{w}\,{}^{G}C_{g}}{{}^{R+W+G}C_{r+w+g}} \tag{5.11}$$

(a) Probability that the marble taken out is red

$$\Pr(1, 0, 0) = \frac{{}^{5}C_{1}\,{}^{8}C_{0}\,{}^{4}C_{0}}{{}^{17}C_{1}} = \frac{5}{17} \approx 0.2941 \tag{5.12}$$

(b) Probability that the marble taken out is white

$$\Pr(0, 1, 0) = \frac{{}^{5}C_{0}\,{}^{8}C_{1}\,{}^{4}C_{0}}{{}^{17}C_{1}} = \frac{8}{17} \approx 0.4706 \tag{5.13}$$

(c) Probability that the marble taken out is not green

$$1 - \Pr(0, 0, 1) = 1 - \frac{{}^{5}C_{0}\,{}^{8}C_{0}\,{}^{4}C_{1}}{{}^{17}C_{1}} = 1 - \frac{4}{17} = \frac{13}{17} \approx 0.7647 \tag{5.14}$$

5.2.2 A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 5.1), and these are equally likely outcomes. What is the probability that it will point at:

(a) 8?

(b) an odd number?

(c) a number greater than 2?

(d) a number less than 9?

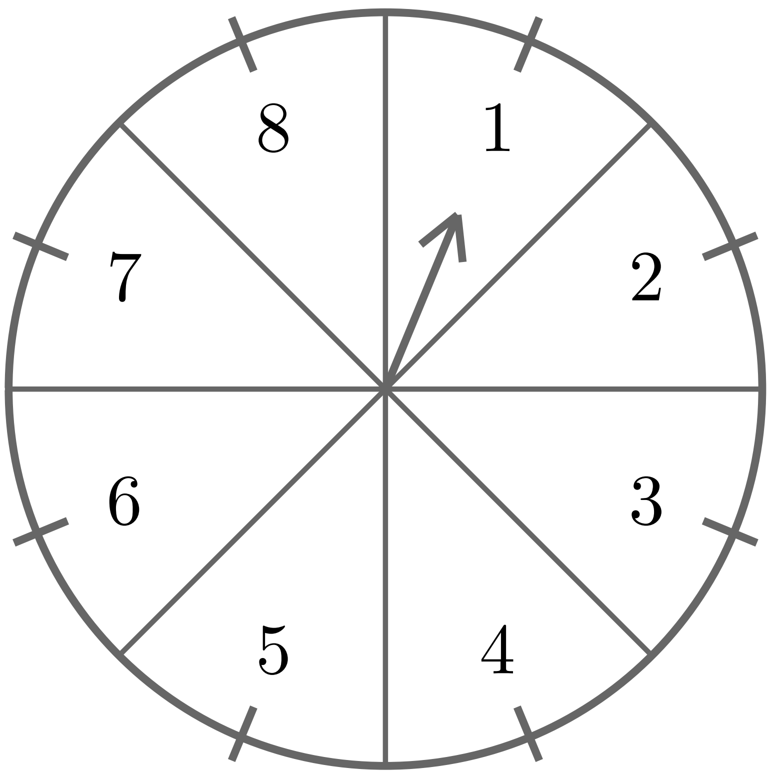

Figure 5.1: Spinner

Solution: Let X be a random variable defined as the value given by the pointer.

Then,

$$\Pr(X = i) = \frac{1}{8} \quad 1 \le i \le 8 \tag{5.15}$$

$$F_X(i) = \Pr(X \le i) \tag{5.16}$$

$$= \begin{cases} 0, & i \le 0 \\ \frac{i}{8} & 1 \le i \le 8 \\ 1, & i \ge 9 \end{cases} \tag{5.17}$$

which are plotted in Fig. 5.2 and Fig. 5.3 respectively.

(a)

$$\Pr(X = 8) = \frac{1}{8} = 0.125 \tag{5.18}$$

(b) For i being odd,

$$\Pr(X = \{1, 3, 5, 7\}) = \frac{4}{8} = 0.5 \tag{5.19}$$

(c)

$$\Pr(X > 2) = 1 - \Pr(X \le 2) \tag{5.20}$$

$$= 1 - (F_X(2) - F_X(0)) \tag{5.21}$$

$$= \frac{6}{8} \tag{5.22}$$

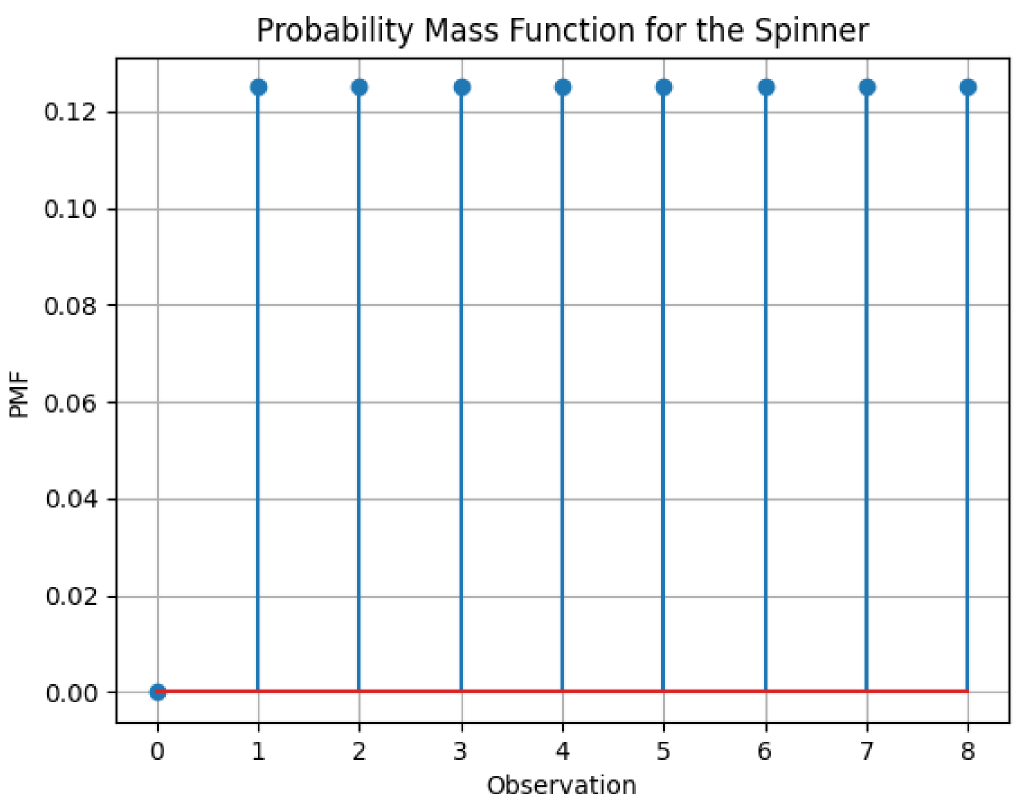

Figure 5.2: Plot of Probability Mass Function

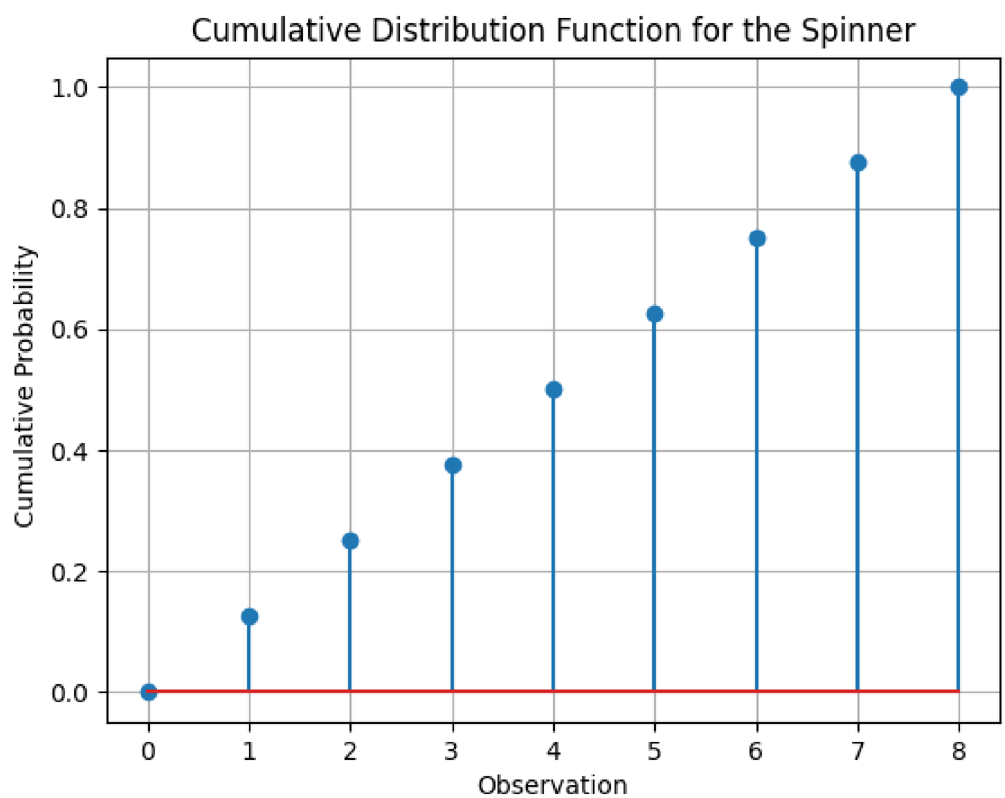

Figure 5.3: Plot of Cumulative Distribution Function

(d)

$$\Pr\left(1 \le X < 9\right) = F_X\left(8\right) - F_X\left(0\right) = 1 \tag{5.23}$$

5.2.3 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

(a) all will be blue?

(b) atleast one will be green?

Solution: See (**??**). In this question,

$$N = 60,\ R = 10,\ B = 20,\ G = 30,\ n = 5 \tag{5.24}$$

(a) From (**??**),

$$\Pr\left(0, 5, 0\right) = \frac{{}^{20}C_5}{{}^{60}C_5} \tag{5.25}$$

(b) Since

$$\Pr\left(r, b, 0\right) = \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} \tag{5.26}$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} \Pr\left(r, b, 0\right) = 1 - \sum_{r+b=n} \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} = 1 - \frac{{}^{R+B}C_n}{{}^{R+B+G}C_n} \tag{5.27}$$

from (**??**). Substituting numerical values, the desired probability is

$$1 - \frac{{}^{30}C_5}{{}^{60}C_5} \tag{5.28}$$

5.2.4 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution: Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \tag{5.29}$$

in (**??**) the desired probability is

$$\Pr(3, 0, 0) = \frac{{}^{12}C_3}{{}^{15}C_3} = \frac{44}{91} \tag{5.30}$$

5.3. Uniform

5.3.1 A die is thrown, find the probability of following events:

(a) A prime number will appear

(b) A number greater than or equal to 3 will appear

(c) A number less than or equal to one will appear

(d) A number more than 6 will appear

(e) A number less than 6 will appear

Solution: The CDF of the random variable X representing the roll of a dice, is available in (**??**).

(a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr\left(X \in \{2,3,5\}\right) = p_X\left(2\right) + p_X\left(3\right) + p_X\left(5\right) \tag{5.31}$$

$$= \frac{1}{2} \tag{5.32}$$

(b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr\left(X \geq 3\right) = 1 - \Pr\left(X \leq 2\right) \tag{5.33}$$

$$= 1 - F_X\left(2\right) \tag{5.34}$$

$$= \frac{2}{3} \tag{5.35}$$

(c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr\left(X \leq 1\right) = F_X\left(1\right) \tag{5.36}$$

$$= \frac{1}{6} \tag{5.37}$$

(d) The probability that a number greater than 6 will appear is given by

$$\Pr\left(X > 6\right) = 1 - \Pr\left(X \leq 6\right) \tag{5.38}$$

$$= 1 - F_X\left(6\right) \tag{5.39}$$

$$= 0 \tag{5.40}$$

(e) The probability that a number less than 6 will appear is given by

$$\begin{aligned}
\Pr(X < 6) &= \Pr(X \le 5) && (5.41)\\
&= F_X(5) && (5.42)\\
&= \frac{5}{6} && (5.43)
\end{aligned}$$

5.4. Binomial

5.4.1 A die is thrown twice. What is the probability that

(a) 5 will not come up either time?

(b) 5 will come up at least once?

Solution: 5.2 From Table 5.2, the PMF of X is

Parameters	**Value**	**Description**
n	2	Number of trials in an Experiment
p	1/6	Probability of Success
q	5/6	Probability of Failure

Table 5.2:

$$\begin{aligned}
\Pr(X = k) &= {}^nC_k p^k q^{n-k} && (5.44)\\
&= {}^2C_k \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{2-k} \qquad \forall k = 0, 1, 2 && (5.45)
\end{aligned}$$

and the CDF is

$$F_X(k) = \Pr(X \le k) = \sum_{i=0}^{k} {}^nC_i p^i q^{n-i} \tag{5.46}$$

(a)

$$\Pr\left(X=0\right) = {}^{2}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^2 = \frac{25}{36} \tag{5.47}$$

(b)

$$\Pr\left(X \geq 1\right) = 1 - \Pr\left(X \leq 0\right) = 1 - F_X(0) \tag{5.48}$$

$$= 1 - \frac{25}{36} = \frac{11}{36} \tag{5.49}$$

5.4.2 Three coins are tossed once. Find the probability of getting

(a) 3 heads

(b) 2 heads

(c) atleast 2 heads

(d) atmost 2 heads

(e) no head

(f) 3 tails

(g) exactly two tails

(h) no tail

(i) atmost two tails

Solution: Let the random variable X denote one single coin toss, where obtaining a head is considered a success. Then,

$$X \sim \mathrm{Ber}\left(p\right) \tag{5.50}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n tosses. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{5.51}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \text{Bin}\left(n, p\right) \tag{5.52}$$

The cdf of Y is given by

$$F_Y\left(k\right) = \Pr\left(Y \leq k\right) \tag{5.53}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ \\ i \end{pmatrix} p^i \left(1-p\right)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{5.54}$$

In this case,

$$p = \frac{1}{2}, \ n = 3 \tag{5.55}$$

(a) We require $\Pr\left(Y = 3\right)$. Thus, from (5.52),

$$\Pr\left(Y = 3\right) = \begin{pmatrix} n \\ \\ 3 \end{pmatrix} p^3 \left(1-p\right)^{n-3} \tag{5.56}$$

$$= \frac{1}{8} \tag{5.57}$$

(b) We require $\Pr(Y = 2)$. Thus, from (5.52),

$$\Pr(Y = 2) = \begin{pmatrix} n \\ 2 \end{pmatrix} p^2 (1-p)^{n-2} \tag{5.58}$$

$$= \frac{3}{8} \tag{5.59}$$

(c) We require $\Pr(Y \geq 2)$. Since $n = 3$ in (5.54),

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{5.60}$$

$$= F_Y(3) - F_Y(1) \tag{5.61}$$

$$= \sum_{k=2}^{3} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{5.62}$$

$$= \frac{1}{2} \tag{5.63}$$

(d) We require $\Pr(Y \leq 2)$. Thus, from (5.54),

$$\Pr(Y \leq 2) = \sum_{k=0}^{2} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{5.64}$$

$$= \frac{7}{8} \tag{5.65}$$

(e) We require $\Pr(Y = 0)$. Thus, from (5.52),

$$\Pr(Y = 0) = \begin{pmatrix} n \\ 0 \end{pmatrix} p^0 (1-p)^n \tag{5.66}$$

$$= \frac{1}{8} \tag{5.67}$$

(f) Obtaining 3 tails is the same as obtaining no heads. Hence, from (5.67), we require $\Pr(Y=0) = \frac{1}{8}$.

(g) We require $\Pr(Y=1)$ (since only one head is obtained). Thus, from (5.52),

$$\Pr(Y=1) = \binom{n}{1} p^1 (1-p)^{n-1} \tag{5.68}$$

$$= \frac{3}{8} \tag{5.69}$$

(h) We require $\Pr(Y=3) = \frac{1}{8}$ from (5.57).

(i) We require $\Pr(Y \geq 1)$ (since at least one head is obtained). Thus, from (5.54) and (5.67),

$$\Pr(Y \geq 1) = 1 - \Pr(Y < 1) \tag{5.70}$$

$$= 1 - F_Y(0) \tag{5.71}$$

$$= 1 - \Pr(Y = 0) \tag{5.72}$$

$$= \frac{7}{8} \tag{5.73}$$

5.4.3 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game

5.4.4 A coin is tossed three times, where. Determine $\Pr(E \mid F)$ where

(a) E : head on third toss, F : heads on first two tosses

(b) E : at least two heads, F : at most two heads

(c) E : at most two tails, F : at least one tail

Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 5.3.

RV	**Values**	**Description**
X	$\{0, 1, 2, 3\}$	Number of heads in 3 tosses
X_1	$\{0, 1\}$	0: Heads , 1: Tails
X_2	$\{0, 1\}$	0: Heads , 1: Tails
X_3	$\{0, 1\}$	0: Heads , 1: Tails

Table 5.3: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \tag{5.74}$$

The PMF of the random variable X is given by,

$$P_X(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \tag{5.75}$$

Here we have

$$N = 3, \, p = \frac{1}{2} \tag{5.76}$$

The CDF of the random variable X is given by,

$$F_X(n) = \Pr(X \leq n) = \sum_{i=0}^{n} {}^N C_i \, p^i \, (1-p)^{N-i} \tag{5.77}$$

(a) The events E, F can be described by the RV as

$$E : X_3 = 0 \tag{5.78}$$

$$F : X_1 + X_2 = 0 \tag{5.79}$$

Y is another random variable which represents the number of heads in first two tosses.

$$Y = X_1 + X_2 \tag{5.80}$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \tag{5.81}$$

Here we have

$$N = 2, \, p = \frac{1}{2} \tag{5.82}$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \tag{5.83}$$

$$\triangleq X_1 + X_2 + X_3 = 0 \tag{5.84}$$

$$\implies X = 0 \tag{5.85}$$

The required probability is given by,

$$\Pr(X_3 = 0 \mid Y = 0) \tag{5.86}$$

$$= \frac{\Pr(X=0)}{\Pr(Y=0)} \tag{5.87}$$

$$= \frac{1}{2} \tag{5.88}$$

(b) The events E, F, F' can be described by the RV as

$$E : X \leq 1 \tag{5.89}$$

$$F : X \geq 1 \tag{5.90}$$

$$F' : X = 0 \tag{5.91}$$

The required probability is given by,

$$= \frac{\Pr(EF)}{1 - \Pr(F')} \tag{5.92}$$

The event EF can be expressed as,

$$X \leq 1 \cap X \geq 1 \tag{5.93}$$

$$\implies X = 1 \tag{5.94}$$

Hence, the required probability is given by,

$$= \frac{\Pr(X=1)}{1 - \Pr(X=0)} \tag{5.95}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \tag{5.96}$$

$$= \frac{3}{7} \tag{5.97}$$

(c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \tag{5.98}$$

$$F' : X = 0 \tag{5.99}$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{5.100}$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \tag{5.101}$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \tag{5.102}$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3) - \Pr(\phi))}{1 - \Pr(X = 0)} \tag{5.103}$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8} - 0\right)}{1 - \frac{1}{8}} \tag{5.104}$$

$$= \frac{6}{7} \tag{5.105}$$

5.4.5 A die is tossed thrice. Find the probability of getting an odd number at least once.

Solution: The parameters for the equivalent binomial distribution is

$$p = \frac{1}{2},\ n = 3 \tag{5.106}$$

The CDF is given by

$$F_X(k) = \sum_{0}^{k} {}^{n}C_k p^k (1-p)^{(n-k)} \tag{5.107}$$

and the required probability is

$$\Pr(1 \leq X \leq 3) = F_X(3) - F_X(0) = \frac{7}{8} \tag{5.108}$$

5.4.6 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

Solution: Table 5.5 summarises the given information.

Variable	**Value**	**Description**
n	$\{2, 3, 4\}$	Number of trials in 2,3,4 tosses of a coin
p	$\frac{1}{2}$	Probability of getting a head
q	$1 - p$	Probability of not getting a head
X_1	$\{0, 1, 2\}$	Number of heads in 2 tosses of a coin
X_2	$\{0, 1, 2, 3\}$	Number of tails in 3 tosses of a coin
X_3	$\{0, 1, 2, 3, 4\}$	Number of heads in 4 tosses of a coin

Table 5.5: Variable Description

(a) Number of heads in two tosses of a coin.

$$p_{X_1}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 2, n = 2 \tag{5.109}$$

(b) Number of tails in the simultaneous tosses of three coins.

$$p_{X_2}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 3, n = 3 \tag{5.110}$$

(c) Number of heads in four tosses of a coin.

$$p_{X_3}(k) = {}^{n}C_k p^k q^{n-k}, 0 \le k \le 4, n = 4 \tag{5.111}$$

5.4.7 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(a) number greater than 4

(b) six appears on at least one die

Solution: Let X be a random variable denoting the outcome of a die toss.

(a)

$$\Pr(X > 4) = 1 - F_X(3) = \frac{1}{3} \tag{5.112}$$

Let Y be the random variable denoting number of successes. Then,

$$Y \sim \text{Bin}(n, p) \tag{5.113}$$

where

$$n = 2, p = \frac{1}{3}. \tag{5.114}$$

Thus,

$$\therefore \Pr(Y = i) = {}^{2}C_i (1-p)^{2-i} p^i \tag{5.115}$$

and the desired distribution is

$$p_Y(k) = \begin{cases} \frac{4}{9}, & k = 0 \\ \frac{4}{9}, & k = 1 \\ \frac{1}{9}, & k = 2 \\ 0, & \text{otherwise} \end{cases} \tag{5.116}$$

(b) In this case, the binomial distribution has parameters

$$n = 2, p = \frac{1}{6} \tag{5.117}$$

yielding

$$p_Z(k) = \begin{cases} \frac{25}{36}, & k = 0 \\ \frac{11}{36}, & k = 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.118}$$

5.4.8 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution: The parameters of the corresponding Binomial distribution are

$$p = 0.05,\ q = 1 - p = 0.95, n = 10 \tag{5.119}$$

The CDF of is given by

$$F_X(n) = \Pr(X \leq n) \tag{5.120}$$

$$= \begin{cases} 0 & n < 0 \\ \sum_{k=0}^{n} \binom{10}{k} q^{10-k} p^k & 0 \leq n \leq 10 \\ 1 & \text{otherwise} \end{cases} \tag{5.121}$$

The desired probability is $F_X(1)$.

5.4.9 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution: A deck has 52 cards among which 13 are spades. Since we are replacing drawn cards, the probability of getting spade on any draw is

$$p = \frac{13}{52} = \frac{1}{4} \tag{5.122}$$

This is a binomial distribution where getting a card of spades is considered success. The pmf is given by

$$\Pr(X = r) = {}^nC_r p^r (1-p)^{n-r}, \quad p = \frac{1}{4}, n = 5 \tag{5.123}$$

The desired probabilities are then obtained as

(i)

$$\Pr(X=5) = {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0 \tag{5.124}$$

$$= \frac{1}{1024} \approx 0.00098 \tag{5.125}$$

(ii)

$$\Pr(X=3) = {}^5C_5 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \tag{5.126}$$

$$= \frac{45}{512} \approx 0.08789 \tag{5.127}$$

(iii)

$$\Pr(X=0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5 \tag{5.128}$$

$$= \frac{243}{1024} \approx 0.23730 \tag{5.129}$$

5.4.10 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(a) none

(b) not more than one

(c) more than one

(d) at least one

will fuse after 150 days of use.

Solution: The binomial distribution parameters are

$$n = 5,\ p = 0.05,\ q = 1 - p = 0.95. \tag{5.130}$$

The pmf and CDF are

$$p_X(i) = {}^5C_i p^i q^{5-i} \tag{5.131}$$

$$F_X(i) = \Pr(X \leq i) = \sum_{r=0}^{i} {}^5C_r p^r q^{5-r} \tag{5.132}$$

(a) Probability that none of the 5 bulbs fuses is

$$\Pr(X = 0) = F_X(0) = 0.95^5 \tag{5.133}$$

(b) Probability that not more than one bulb fuses is

$$\Pr(X \leq 1) = F_X(1) = 0.9774075 \tag{5.134}$$

(c) Probability that more than one bulb will fuse will be

$$\Pr(1 < X \leq 5) = F_X(5) - F_X(1) = 0.0225925 \tag{5.135}$$

(d) Probability that at least one bulb is fused is

$$\Pr(1 \leq X \leq 5) = F_X(5) - F_X(0) = 1 - (0.95)^5 \tag{5.136}$$

5.4.11 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Let X denote the number of correct answers out of 20 questions. Then X is binomial with

$$n = 20,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2} \tag{5.137}$$

The desired probability is then given by

$$\Pr(X >= 12) = 1 - F_X(11) = 0.2517 \tag{5.138}$$

5.4.12 Find the probability of getting 5 twice in 7 throws of a dice.

Solution: The Binomial r.v. parameters are

$$\therefore\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}, n = 7 \tag{5.139}$$

with pmf

$$\therefore\ \Pr(X = k) = {}^{n}C_k \times q^{n-k} \times p^k = {}^{7}C_k \times \left(\frac{5}{6}\right)^{(7-k)} \times \left(\frac{1}{6}\right)^k \tag{5.140}$$

The desired probability is

$$\Pr(X = 2) = {}^{7}C_2 \times \left(\frac{5}{6}\right)^{(7-2)} \times \left(\frac{1}{6}\right)^2 = \left(\frac{7}{12}\right) \times \left(\frac{5}{6}\right)^5 \tag{5.141}$$

5.4.13 On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution: See

Parameter	Value	Description
X	$bin(n,p)$	no of correct answers that candidate gets by guessing
n	5	total no of questions
p	$\frac{1}{3}$	probability of getting correct answer by guessing

Table 5.6: The binomial random variable,it's parameters and their values

The pmf and CDF are

$$p_X(k) = {}^5C_k \times \left(\frac{2}{3}\right)^5 \times \frac{1}{2^k} \tag{5.142}$$

$$F_X(k) = \left(\frac{2}{3}\right)^5 \times \left(\sum_{i=0}^{k} {}^5C_i \times \frac{1}{2^i}\right) \tag{5.143}$$

The desired probability is

$$\Pr\left(X \geq 4\right) = 1 - F_X(3) = \frac{11}{243} \tag{5.144}$$

5.4.14 A bag consists of 10 balls each marked with one of the digits 0 to 9. If 4 balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution: Let, p_X be the sequence of independent Bernoulli random varibles.

$$X = \begin{cases} 0, & \text{non-zero marked ball} \\ 1, & \text{zero marked ball} \end{cases} \tag{5.145}$$

which means

$$p_X\left(0\right) = \frac{9}{10} \tag{5.146}$$

$$p_X\left(1\right) = \frac{1}{10} \tag{5.147}$$

Let, the total number of trials be n and Z be the random variable that represents the number of balls marked zero in n trials which is given by:

$$p_X\left(Z = k\right) = {}^{n}C_k p^{n-k} q^k \tag{5.148}$$

where,

$$Z = X_1 + X_2 + ... + X_n \tag{5.149}$$

For only non-zero marked balls in 4 trials,

$$p_X(Z = 0) = {}^{4}C_0 \left(\frac{9}{10}\right)^{4-0} \left(\frac{1}{10}\right)^{0} \tag{5.150}$$

$$= (1) \left(\frac{9}{10}\right)^{4} (1) \tag{5.151}$$

$$= \left(\frac{9}{10}\right)^{4} \tag{5.152}$$

$$= 0.6561 \tag{5.153}$$