

Assignment

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Question : Consider communication over a memoryless binary symmetric channel using a (7, 4) Hamming code. Each transmitted bit is received correctly with probability $(1-\epsilon)$, and flipped with probability ϵ . For each codeword transmission, the receiver performs minimum Hamming distance decoding, and correctly decodes the message bits if and only if the channel introduces at most one bit error.

For $\epsilon = 0.1$, the probability that a transmitted codeword is decoded correctly is _____ (rounded off to two decimal places). (rounded off to two decimal places).

Solution: Given that, Let X be a random variable defined in the Table I;

RV	Value	Description
n (or) p	7	The total number of bits
ϵ	0.1	Probability of error in transimitted bit
X	$0 \leq X \leq 7$	The number of bit errors in transmission

TABLE I
RANDOM VARIABLE X DECLARATION

Then, $X \sim \text{Bin}(n, p)$ where

$$n = 7 \quad p = \epsilon = 0.1 \quad (1)$$

the pmf of X is given by

$$p_X(k) = {}^7C_k(\epsilon)^k(1-\epsilon)^{7-k} \quad (2)$$

the cdf of X is given by

$$F_X(k) = \sum_{i=0}^k {}^7C_i(\epsilon)^i(1-\epsilon)^{7-i} \quad (3)$$

From equation (3), the probability of getting one or less error is given by

$$F_X(1) = \sum_{i=0}^1 {}^7C_i(\epsilon)^i(1-\epsilon)^{7-i} \quad (4)$$

$$= {}^7C_0(\epsilon)^0(1-\epsilon)^7 + {}^7C_1(\epsilon)^1(1-\epsilon)^6 \quad (5)$$

$$= (1-\epsilon)^7 + 7(\epsilon)^1(1-\epsilon)^6 \quad (6)$$

From (1) and (6),

$$F_X(1) = (1-0.1)^7 + 7(0.1)^1(1-0.1)^6 \quad (7)$$

$$= 0.85 \quad (8)$$

\therefore the probability that a transmitted codeword is decoded correctly is 0.85.

Gaussian

Let parameters be defined in the Table II;

RV	Value	Description
$\mu = np$	0.7	Mean of Binomial distribution
$\sigma^2 = npq$	0.63	Variance of Binomial distribution

TABLE II
PARAMETERS

Let Y is the Gaussian obtained by approximating binomial with parameters n, p then By Central limit theroem,

$$X \approx Y \sim \mathcal{N}(np, npq) \quad (9)$$

We need to find

$$\Pr(Y \leq 1) = F_Y(1) \quad (10)$$

After corrections to make the vlaues more accurate, we need to find

$$\Pr(Y \leq 1.33) = F_Y(1.33) \quad (11)$$

then CDF of Y is:

$$F_Y(x) = \Pr(Y \leq x) \quad (12)$$

$$= \Pr(Y - \mu \leq x - \mu) \quad (13)$$

$$= \Pr\left(\frac{Y - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \quad (14)$$

Since,

$$\frac{Y - \mu}{\sigma} \sim \mathcal{N}(0, 1) \quad (15)$$

Q function is defined

$$Q(x) = \Pr(Y > x) \quad \forall x \in Y \sim \mathcal{N}(0, 1) \quad (16)$$

From (14) and (16),

$$F_Y(x) = 1 - \Pr\left(\frac{Y - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \quad (17)$$

$$= \begin{cases} 1 - Q\left(\frac{x - \mu}{\sigma}\right), & x > \mu \\ Q\left(\frac{\mu - x}{\sigma}\right), & x < \mu \end{cases} \quad (18)$$

From (18) and Table II,

$$F_Y(1.33) = 1 - Q\left(\frac{1.33 - 0.7}{0.63}\right) \quad (19)$$

$$= 1 - Q(1) \quad (20)$$

$$= 0.8413 \quad (21)$$

\therefore the probability that a transmitted codeword is decoded correctly is 0.8413.

The Binomial CDF vs. Guassian CDF plot is given in fig1

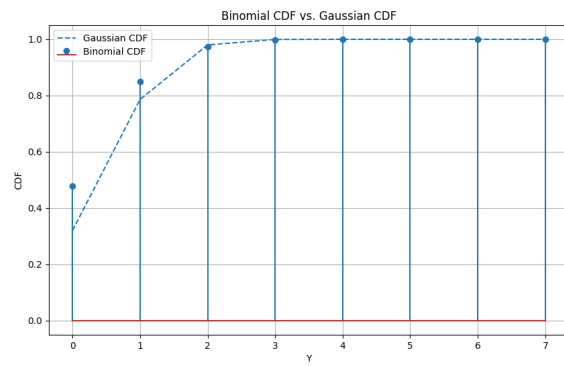


Fig. 1. Binomial CDF vs. Gaussian CDF