DIGITAL COMMUNICATION

Through Simulations

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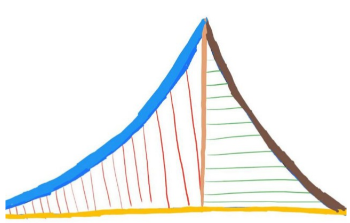

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This book introduces digital communication through probability.

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Introduction

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Axioms

2.1. Examples

2.1.1 Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football game?

Solution: Let X be a random variable which takes the values 0 and 1.

$$X = \begin{cases} 1, & \text{if coin toss results in Head} \\ 0, & \text{if coin toss results in Tail} \end{cases} \tag{2.1}$$

From law of total probability,

$$\Pr(X = 0) + \Pr(X = 1) = 1 \tag{2.2}$$

Since there is only one head,

$$\Pr(X = 1) = \frac{1}{2} \tag{2.3}$$

Similarly,

$$\Pr(X=0) = 1 - \Pr(X=1) = \frac{1}{2} \tag{2.4}$$

Thus,

$$\Pr(X=0) = \Pr(X=1) \tag{2.5}$$

which is why tossing the coin is a fair way to decide.

2.1.2 Which of the following cannot be the probability of an event ?

(a) $\frac{2}{3}$

(b) -1.5

(c) 15%

(d) 0.7

Solution: From the axioms of probability,

$$0 \le \Pr(E) \le 1 \tag{2.6}$$

(a) $\Pr(E) = \frac{2}{3}$

$$\because 0 \le \frac{2}{3} \le 1 \tag{2.7}$$

from (2.6), it can be probability of an event.

(b) $\Pr(E) = -1.5$

$$\because -1.5 < 0 \tag{2.8}$$

from (2.6), it cannot be a probability of any event.

(c)

$$\Pr(E) = \frac{15}{100} \tag{2.9}$$

$$\because 0 \le \frac{15}{100} \le 1, \tag{2.10}$$

from (2.6), it can be probability of an event.

(d) $\Pr(E) = 0.7$

$$\because 0 \le 0.7 \le 1 \tag{2.11}$$

from (2.6), it can be a probability of an event.

2.1.3 If P(E) = 0.05, what is the probability of 'not E'?

Solution: The desired probability is

$$\Pr\left(E'\right) = 1 - \Pr(E) = 0.95 \tag{2.12}$$

2.1.4 A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

(a) an orange flavoured candy?

(b) a lemon flavoured candy?

Solution:

$$\Pr(O) = 0 \tag{2.13}$$

$$\Pr(L) = 1 \tag{2.14}$$

2.1.5 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: Let E be the event that no 2 students in a group of 3 share a birthday. Then

$$\Pr(E) = 0.992 \implies \Pr\left(E'\right) = 1 - \Pr(E) = 0.008 \tag{2.15}$$

2.1.6 A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is

(i) red ?

(ii) not red?

Solution: Let

$$X = \begin{cases} 1 & \text{if drawn ball is red} \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

(i) Probability that the drawn ball is red

$$\Pr(X=1) = \frac{3}{8} \tag{2.17}$$

$$\tag{2.18}$$

(ii) Probability that the drawn ball is not red

$$\Pr(X=0) = 1 - \frac{3}{8} = \frac{5}{8} \tag{2.19}$$

2.1.7 Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish . What is the probability that the fish taken out is a male fish?

Solution:

Let

$$X = \begin{cases} 1, & \text{if the chosen fish is male} \\ 0, & \text{if the chosen fish is female} \end{cases} \tag{2.20}$$

Then

$$\Pr(X=1) = \frac{5}{13} \tag{2.21}$$

2.1.8 A box contains 12 balls, out of which x are black. If one ball is drawn at random from the box, what is the probability that it will be a black ball? If 6 more black balls are put in the box, the probability of drawing a black ball is now double of what it was before. Find x.

Solution: From Table 2.1,

Random Variable	Sample space	Value	Event	Probability
X_1	12	0	not choosing black ball	12-x/12
		1	choosing black ball	x/12
X_2	18	0	not choosing black ball	12-x/18
		1	choosing black ball	x+6/18

Table 2.1:

$$\Pr\left(X_1 = 1\right) = \frac{x}{12} \tag{2.22}$$

Since

$$\Pr\left(X_2 = 1\right) = 2\Pr\left(X_1 = 1\right), \tag{2.23}$$

$$\frac{x+6}{18} = 2\left(\frac{x}{12}\right) \tag{2.24}$$

$$\implies x = 3 \tag{2.25}$$

2.1.9 A letter is chosen at random from the word ‘ASSASSINATION’. Find the probability that letter is

(a) a vowel

(b) a consonant

. **Solution:** The number of vowels is 6 and consonants is 7.

(a)

$$\Pr\left(X\right) = \frac{6}{13} \tag{2.26}$$

(b)

$$\Pr(Y) = \frac{7}{13} \tag{2.27}$$

2.1.10 In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prizes in the game? [Hint : order of the numbers is not important.]

Solution: The desired probability is given by

$$\frac{1}{{}^{20}C_6} = \frac{1}{38,760} = 0.0000258 \tag{2.28}$$

2.1.11 Check whether the following probabilities $\Pr(A)$ and $\Pr(B)$ are consistently defined

(a) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cap B) = 0.6$

(b) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cup B) = 0.8$

Solution: To check whether the given probabilities are consistently defined, we check whether the following property holds correctly with the probability axioms

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.29}$$

(a) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(AB) = 0.6 \tag{2.30}$$

From (2.29),

$$\Pr(A+B) = 0.5 + 0.7 - 0.6 = 0.6 \tag{2.31}$$

From (2.31) we have

$$0 \le \Pr(A+B) \le 1 \tag{2.32}$$

Hence the given probabilities are consistently defined.

(b) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(A+B) = 0.8 \tag{2.33}$$

From (2.29) we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \tag{2.34}$$

$$= 0.4 \tag{2.35}$$

From (2.35) we have

$$0 \le \Pr(AB) \le 1 \tag{2.36}$$

Hence the given probabilities are consistently defined

2.1.12 Given $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$. Find $\Pr(A+B)$ if A and B are mutually exclusive events.

Solution: Since $AB = 0$,

$$\Pr(A+B) = \Pr(A) + \Pr(B) = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} \tag{2.37}$$

2.1.13 If E and F are events such that $\Pr(E) = \frac{1}{4}$, $\Pr(F) = \frac{1}{2}$ and $\Pr(EF) = \frac{1}{8}$, find

(a) $\Pr(E+F)$

(b) $\Pr(E'F')$

Solution:

(a)

$$\Pr(E+F) = \Pr(E) + \Pr(F) - \Pr(EF) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8} \tag{2.38}$$

$$\tag{2.39}$$

(b)

$$(E'F') = (E+F)' \implies \Pr\left(E'F'\right) = \Pr\left((E+F)'\right) = 1 - \Pr(E+F) \tag{2.40}$$

$$\implies \Pr\left(E'F'\right) = 1 - \frac{5}{8} = \frac{3}{8} \tag{2.41}$$

2.1.14 Events E and F are such that $\Pr(E' + F') = 0.25$, state whether E and F are mutually exclusive.

Solution:

$$\Pr\left(E' + F'\right) = \Pr\left(EF\right)' = 1 - \Pr\left(EF\right) = 0.25 \tag{2.42}$$

$$\implies \Pr\left(EF\right) = 0.75 \tag{2.43}$$

$$\because \Pr\left(EF\right) \neq 0 \tag{2.44}$$

E and F are not mutually exclusive events.

2.1.15 A and B are events such that $\Pr\left(A\right) = 0.42, \Pr\left(B\right) = 0.48$ and $\Pr\left(\text{A and B}\right) = 0.16$.

Determine

(a) $\Pr\left(\text{not A}\right)$

(b) $\Pr\left(\text{not B}\right)$

(c) $\Pr\left(\text{A or B}\right)$

Solution: Solution:

(a) $\Pr\left(\text{not A}\right)$

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right) \tag{2.45}$$

$$= 1 - 0.42 \tag{2.46}$$

$$= 0.58 \tag{2.47}$$

(b) $\Pr\left(\text{not B}\right)$

$$\Pr\left(B'\right) = 1 - \Pr\left(B\right) \tag{2.48}$$

$$= 1 - 0.48 \tag{2.49}$$

$$= 0.52 \tag{2.50}$$

(c) $\Pr(\text{A or B})$

$$\Pr(\text{A+B}) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.51}$$

$$= 0.42 + 0.48 - 0.16 \tag{2.52}$$

$$= 0.74 \tag{2.53}$$

2.1.16 In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution: The given information is summarised in Table 2.3. Thus,

Random Variable	**Subject**	**Probability**
M	Mathematics	$\Pr(M)$=0.4
B	Biology	$\Pr(B)$=0.3
M, B	Both	$\Pr(MB)$=0.10

Table 2.3:

$$\Pr(M + B) = \Pr(M) + \Pr(B) - \Pr(M, B) \tag{2.54}$$

$$= 0.6 \tag{2.55}$$

2.1.17 In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

Solution: Let

A : Probability of random student passing the first exam

B : Probability of random student passing the second exam

Given

$$\Pr(A) = 0.8,\ \Pr(B) = 0.7,\ \Pr(A+B) = 0.95. \tag{2.56}$$

$$\because \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB), \tag{2.57}$$

$$\implies \Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) \tag{2.58}$$

$$= 0.55 \tag{2.59}$$

2.1.18 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(a) The student opted for NCC or NSS.

(b) The student has opted neither NCC nor NSS.

(c) The student has opted NSS but not NCC.

Solution: Define random variables X and Y as shown in Tables 2.4 and 2.5. From

$X = 0$	Student does not opt for NCC.
$X = 1$	Student opts for NCC.

Table 2.4: Definition of X.

$Y = 0$	Student does not opt for NSS.
$Y = 1$	Student opts for NSS.

Table 2.5: Definition of Y.

the given data

$$\Pr(X=1) = \frac{30}{60} = \frac{1}{2} \tag{2.60}$$

$$\Pr(Y=1) = \frac{32}{60} = \frac{8}{15} \tag{2.61}$$

$$\Pr(X=1,\ Y=1) = \frac{24}{60} = \frac{2}{5} \tag{2.62}$$

Thus, we write

$$\Pr(X=1,\ Y=0) = \Pr(X=1) - \Pr(X=1,\ Y=1) = \frac{1}{10} \tag{2.63}$$

$$\Pr(X=0,\ Y=1) = \Pr(Y=1) - \Pr(X=1,\ Y=1) = \frac{2}{15} \tag{2.64}$$

$$\Pr(X=0,\ Y=0) = \Pr(Y=0) - \Pr(X=1,\ Y=0) \tag{2.65}$$

$$= 1 - \Pr(Y=1) - \Pr(X=1,\ Y=0) \tag{2.66}$$

$$= 1 - \frac{8}{15} - \frac{1}{10} = \frac{11}{30} \tag{2.67}$$

and form the joint pmf as in Table 2.6.

	$X=0$	$X=1$
$Y=0$	$\frac{11}{30}$	$\frac{1}{10}$
$Y=1$	$\frac{2}{15}$	$\frac{2}{5}$

Table 2.6: Joint pmf of X and Y.

(a) From Table 2.6,

$$\Pr(X+Y \geq 1) = 1 - \Pr(X+Y=0) \tag{2.68}$$

$$= \frac{19}{30} \tag{2.69}$$

(b) From Table 2.6,

$$\Pr(X = 0,\ Y = 0) = \frac{11}{30} \tag{2.70}$$

(c) From Table 2.6,

$$\Pr(X = 0,\ Y = 1) = \frac{2}{15} \tag{2.71}$$

2.1.19 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: The given information is summarized in the following table 2.7

RV	**Description**	**Probability**
$X = 1$	Die rolls to 1	$\frac{1}{3}$
$X = 2$	Die rolls to 2	$\frac{1}{2}$
$X = 3$	Die rolls to 3	$\frac{1}{6}$

Table 2.7: Random variable X

(a)

$$\Pr(X = 2) = \frac{1}{2} \tag{2.72}$$

(b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1, 3\} \tag{2.73}$$

$$X = 1 \text{ and } X = 3 \equiv X = \phi \tag{2.74}$$

$$\begin{aligned} \Pr(X \in \{1, 3\}) &= \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \phi) && (2.75) \\ &= \frac{1}{3} + \frac{1}{6} && (2.76) \\ &= \frac{1}{2} && (2.77) \end{aligned}$$

(c)

$$\begin{aligned} \Pr(X \neq 3) &= 1 - \Pr(X = 3) && (2.78) \\ &= 1 - \frac{1}{6} && (2.79) \\ &= \frac{5}{6} && (2.80) \end{aligned}$$

2.1.20 A and B are two events such that $\Pr(A) = 0.54$, $\Pr(B) = 0.69$ and $\Pr(AB) = 0.35$. Find

(a) $\Pr(A + B)$

(b) $\Pr(A'B')$

(c) $\Pr(AB')$

(d) $\Pr(BA')$

Solution:

(a)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.81}$$

$$= 0.54 + 0.69 - 0.35 = 0.88 \tag{2.82}$$

(b) By De Morgan's Law,

$$A'B' = (A+B)' \tag{2.83}$$

$$\implies \Pr\left(A'B'\right) = \Pr(A+B)' \tag{2.84}$$

$$= 1 - \Pr(A+B) \tag{2.85}$$

$$= 1 - 0.88 = 0.12 \tag{2.86}$$

(c) We know that,

$$B + B' = 1BB' = 0 \tag{2.87}$$

$$\implies A = A(B+B') = AB + AB' \tag{2.88}$$

$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) - \Pr\left(ABB'\right) \tag{2.89}$$

$$= \Pr(AB) + \Pr\left(AB'\right) \tag{2.90}$$

$$\implies \Pr\left(AB'\right) = \Pr(A) - \Pr(AB) \tag{2.91}$$

$$= 0.54 - 0.35 = 0.19 \tag{2.92}$$

(d) From (2.91),

$$\Pr\left(BA'\right) = \Pr(B) - \Pr(AB) = 0.69 - 0.35 = 0.34. \tag{2.93}$$

2.1.21 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.

Solution: Since the events A, B are independent, we have

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{2.94}$$

2.1.22 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event,'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

Solution: Let

$$X = \begin{cases} 0, & \text{if number is odd} \\ 1, & \text{if number is even} \end{cases} \tag{2.95}$$

$$Y = \begin{cases} 0, & \text{if number is green} \\ 1, & \text{if number is red} \end{cases} \tag{2.96}$$

From the given information,

$$\Pr(X = 1) = \frac{3}{6} = \frac{1}{2}, \Pr(Y = 1) = \frac{3}{6} = \frac{1}{2} \tag{2.97}$$

$$\Pr(X = 1, Y = 1) = \frac{1}{6} \tag{2.98}$$

Now,

$$\Pr(X = 1) \times \Pr(Y = 1) = \frac{1}{4} \tag{2.99}$$

$$\implies \Pr(X = 1, Y = 1) \neq \Pr(X = 1) \times \Pr(Y = 1) \tag{2.100}$$

Hence, A and B are not independent.

2.1.23 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50} \tag{2.101}$$

$$\Pr(EF) = \frac{1}{50} \tag{2.102}$$

$$\implies \Pr(EF) \neq P(E)P(F) \tag{2.103}$$

$\therefore$ E and F are not independent events.

2.1.24 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution: Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{2.104}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{2.105}$$

$$\therefore p = \frac{1}{10} \tag{2.106}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.107}$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.108}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.109}$$

$$\therefore p = \frac{1}{5} \tag{2.110}$$

2.1.25 Let and B be independent events with $\Pr(A) = 0.3$ and $\Pr(B) = 0.4$. Find

(a) $\Pr(AB)$

(b) $\Pr(A+B)$

(c) $\Pr(A|B)$

(d) $\Pr(B|A)$

Solution:

(a)

$$\Pr(AB) = \Pr(A) \times \Pr(B) = 0.3 \times 0.4 = 0.12 \tag{2.111}$$

(b)

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) = 0.3 + 0.4 - 0.12 = 0.58 \tag{2.112}$$

(c)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A) \times \Pr(B)}{\Pr(B)} = \Pr(A) = 0.3 \tag{2.113}$$

(d)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B) \times \Pr(A)}{\Pr(A)} = \Pr(B) = 0.4 \tag{2.114}$$

2.1.26 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr$(not A and not B).

Solution: Since

$$A'B' = (A+B)', \tag{2.115}$$

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{2.116}$$

$$= 1 - \Pr(A+B) \tag{2.117}$$

Thus,

$$\Pr\left(A'B'\right) = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \tag{2.118}$$

$$= \frac{3}{8} \tag{2.119}$$

2.1.27 Events A and B are such that

$$\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{7}{12} \text{and} \Pr\left(A' + B'\right) = \frac{1}{4}. \tag{2.120}$$

State whether A and B are independent.

Solution:

$$\Pr(AB) = 1 - \Pr\left(A' + B'\right) = 1 - \frac{1}{4} = \frac{3}{4} \tag{2.121}$$

$$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \tag{2.122}$$

$$\implies \Pr(AB) \neq \Pr(A)\Pr(B) \tag{2.123}$$

$\therefore$ A and B are not independent.

2.1.28 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

2.1.29 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even, ' and B be the event, 'the number is red'. Are A and B independent?

2.1.30 Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

2.1.31 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

2.1.32 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(\text{not } A \text{ and not } B)$

2.1.33 Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?

2.1.34 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(a) $P(A \text{ and } B)$

(b) $P(A \text{ and not } B)$

(c) $P(A \text{ or } B)$

(d) $P(\text{neither } A \text{ nor } B)$

2.1.35 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved

(b) exactly one of them solves the problem

Solution: Given that $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \frac{1}{3}$ A, B are independent so

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \tag{2.124}$$

(a) The probability of the problem being solved is

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.125}$$

$$= \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.126}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \tag{2.127}$$

(b) Probalbility that exactly one person solves problem is

$$\Pr\left(AB'\right) + \Pr\left(A'B\right) = \Pr(A)\Pr\left(B'\right) + \Pr\left(A'\right)\Pr(B) \tag{2.128}$$

$$= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} \tag{2.129}$$

2.1.36 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(a) E: 'the card drawn is spade'

F: 'the card drawn is an ace'

(b) E: 'the card drawn is black'

F : 'the card drawn is a king'

(c) E : 'the card drawn is a king or queen'

F : 'the card drawn is a queen or jack'

Solution:

(i) E denotes the event that the card drawn is spade

$$\Pr(E) = \frac{13}{52} = \frac{1}{4} \tag{2.130}$$

F denotes the event that card drawn is ace

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{2.131}$$

$$\Pr(EF) = \frac{1}{52} \tag{2.132}$$

$$\Pr(E)\Pr(F) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{2.133}$$

$$\therefore \quad \Pr(EF) = \Pr(E)\Pr(F) \tag{2.134}$$

and the events are independent.

(ii) E denotes the event that the card drawn is black

$$\Pr(E) = \frac{26}{52} = \frac{1}{2} \tag{2.135}$$

F denotes the event that card drawn is a king

$$\Pr(F) = \frac{4}{52} = \frac{1}{13} \tag{2.136}$$
$$\Pr(EF) = \frac{2}{52} = \frac{1}{26} \tag{2.137}$$
$$\Pr(E)\Pr(F) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \tag{2.138}$$
$$\therefore \ \Pr(EF) = \Pr(E)\Pr(F) \tag{2.139}$$

and E and F are independent events.

(iii) E denotes the event that the card drawn is king or queen

$$\Pr(E) = \frac{8}{52} = \frac{2}{13} \tag{2.140}$$

F denotes the event that card drawn is a queen or jack

$$\Pr(F) = \frac{8}{52} = \frac{2}{13} \tag{2.141}$$
$$\Pr(EF) = \frac{4}{52} = \frac{1}{13} \tag{2.142}$$
$$\Pr(E)\Pr(F) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169} \tag{2.143}$$
$$\therefore \ \Pr(EF) \neq \Pr(E)\Pr(F) \tag{2.144}$$

and E and F are not independent events.

Choose the correct answer in the following exercises

2.1.37 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

2.1.38 Two events A and B will be independent, if

(a) A and B are mutually exclusive

(b) P(not $A \cap$ not B) $=$ $[1 - P(A)]\ [1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

Solution:

(a) When tossing a coin, the event of getting a head and tail are mutually exclusive and let them be denoted by A and B respectively.

$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A) \times \Pr(B) = \frac{1}{4} \tag{2.145}$$

$$\text{or, } \Pr(AB) = 0 \neq \Pr(A) \times \Pr(B) \tag{2.146}$$

Hence A and B are not independent.

(b)

$$\Pr\left(A'B'\right) = [1 - \Pr(A)][1 - \Pr(B)] \tag{2.147}$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.148}$$

$$\implies 1 - \Pr(A + B) = 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.149}$$

$$\implies \Pr(AB) = \Pr(A)\Pr(B) \tag{2.150}$$

which implies that A and B are independent.

(c) For the same counter example given for option 2.1.38a, $\Pr(A) = \Pr(B)$, but A and B are not independent events.

(d) For the same counter example given for option 2.1.38a, $\Pr(A) + \Pr(B) = 1$, but A and B are not independent events.

2.1.39 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

2.1.40 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution:

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{0.32}{0.5} = 0.64 \tag{2.151}$$

2.1.41 A fair die is rolled. Consider events $E = 1,3,5$, $F = 2,3$ and $G = 2,3,4,5$. Find

(a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$

(b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$

(c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: See Table 2.8.

E = {1,3,5}	Pr(E) = $\frac{1}{2}$
F = {2,3}	Pr(F) = $\frac{1}{3}$
G = {2,3,4,5}	Pr(G) = $\frac{2}{3}$
EF = {3}	Pr(EF) = $\frac{1}{6}$
FG = {2,3}	Pr(FG) = $\frac{1}{3}$
EG = {3,5}	Pr(EG) = $\frac{1}{3}$
EFG = {3}	Pr(EFG) = $\frac{1}{6}$

Table 2.8: From given data

(a)

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{1/6}{1/3} = 1/2 \tag{2.152}$$

(b)

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{1/6}{1/2} = 1/3 \tag{2.153}$$

(c)

$$\Pr(E|G) = \frac{\Pr(EG)}{\Pr(G)} = \frac{1/3}{2/3} = 1/2 \tag{2.154}$$

(d)

$$\Pr(G|E) = \frac{\Pr(EG)}{\Pr(E)} = \frac{1/3}{1/2} = 2/3 \tag{2.155}$$

(e) Since

$$\Pr((E+F)G) = \Pr((EG)+(FG)) = \Pr(EG) + \Pr(FG) - \Pr(EFG), \tag{2.156}$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{6} = \frac{1}{2} \tag{2.157}$$

$$\implies \Pr((E+F)|G) = \frac{\Pr((E+F)G)}{\Pr(G)} = \frac{1/2}{2/3} = \frac{3}{4} \tag{2.158}$$

(f)

$$\Pr((EF)|G) = \frac{\Pr(EFG)}{\Pr(G)} = \frac{1/6}{2/3} = \frac{1}{4} \tag{2.159}$$

2.1.42 State which of the following are not the probability distributions of a random variable.

Give reasons for your answer.

Table 2.9:

(a)

X	0	1	2
P(X)	0.4	0.4	0.2

Table 2.10:

(b)

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

Table 2.11:

(c)

Y	-1	0	1
P(Y)	0.6	0.1	0.2

Solution: From the axioms of probability,

$$0 < \Pr(X = i) < 1, i = 1, 2, 3...n. \tag{2.160}$$

$$\sum_{i=1}^{n} \Pr(X = i) = 1, i = 1, 2, 3...n. \tag{2.161}$$

(a)

$$\sum_{i=0}^{2} \Pr(X = i) = 0.4 + 0.4 + 0.2 = 1 \tag{2.162}$$

satisfies both (2.161) and (2.160), so it is a probability distribution.

(b)

$$\sum_{i=0}^{4} \Pr(X = i) = 0.1 + 0.5 + 0.2 - 0.1 + 0.3 = 1 \tag{2.163}$$

Table 2.12:

(d)

X	0	1	2	3	4
P(Z)	0.3	0.2	0.4	0.1	0.05

Satisfies (2.161) but does not satisfy (2.160) as $P(3) < 0$. Hence NOT a probability distribution.

(c)

$$\sum_{i=-1}^{1} \Pr(X = i) = 0.6 + 0.1 + 0.2 = 0.9 \tag{2.164}$$

(2.161) not satisfied, so it is NOT a probability distribution.

(d)

$$\sum_{i=0}^{4} \Pr(X = i) = 0.3 + 0.2 + 0.4 + 0.1 + 0.05 = 1.05 \tag{2.165}$$

(2.161) not satisfied, so it is NOT a probability distribution.

2.1.43 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

Solution: The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \tag{2.166}$$

The probability of one of the two remaining days being a Tuesday is $\frac{2}{7}$.

2.1.44 Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Solution: The given information is summarised in Table From the given information,

A:	Person with heat attack	$\Pr(A)$=0.40
E_1:	Person treated with meditation and yoga	$\Pr(E_1)$=0.50
E_2:	Person treated with drug	$\Pr(E_2)$=0.50

Table 2.13: Given Information

$$\Pr(A|E_1) = \Pr(A)(1-(0.30)) = 0.40 \times 0.70 = 0.28 \tag{2.167}$$

and

$$\Pr(A|E_2) = \Pr(A)(1-(0.25)) = 0.40 \times 0.75 = 0.30 \tag{2.168}$$

From (2.167) and (2.168),

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} = \frac{\frac{1}{2}\times 0.28}{\frac{1}{2}\times 0.28+\frac{1}{2}\times 0.30} = \frac{14}{29} \tag{2.169}$$

2.1.45 An electronic assembley consists of two subsystems,say A and B.From previous testing procedures, the following probabilities are assumed to be known

$$\Pr(\text{A fails}) = 0.20 \tag{2.170}$$

$$\Pr(\text{B alone fails}) = 0.15 \tag{2.171}$$

$$\Pr(\text{A and B fails}) = 0.15 \tag{2.172}$$

Evaluate the following probabilities

(a) Pr (A fails given B has failed)

(b) Pr (A fails alone)

Solution: From the given information,

$$\Pr\left(A'\right) = 0.20 \tag{2.173}$$

$$\Pr\left(AB'\right) = 0.15 \tag{2.174}$$

$$\Pr\left(A'B'\right) = 0.15 \tag{2.175}$$

(a)

$$\Pr\left(A'|B'\right) = \frac{\Pr\left(A'B'\right)}{\Pr\left(B'\right)} \tag{2.176}$$

Since

$$B'(1) = B'(A + A') = B'A + B'A' \tag{2.177}$$

$$\Pr\left(B'\right) = \Pr\left(AB'\right) + \Pr\left(A'B'\right) \because ((B'A)(B'A')) = 0 \tag{2.178}$$

$$= 0.15 + 0.15 = 0.30 \tag{2.179}$$

Thus,

$$\Pr\left(A'|B'\right) = 0.15/0.30 == 0.50 \tag{2.180}$$

(b) Similiarly,

$$\Pr\left(A'\right) = \Pr\left(BA'\right) + \Pr\left(A'B'\right) \tag{2.181}$$

$$\implies \Pr\left(BA'\right) = \Pr\left(A'\right) - \Pr\left(A'B'\right) = 0.20 - 0.15 = 0.05 \tag{2.182}$$

2.1.46 If A and B are two events such that $P(A) \neq 0$ and $P(B \mid A) = 1$, then

(a) $A \subset B$

(b) $B \subset A$

(c) $B = \phi$

(d) $A = \phi$

Solution:

$$\Pr(B|A) = 1 \implies \frac{\Pr(BA)}{\Pr(A)} = 1 \tag{2.183}$$

$$\implies \Pr(BA) = \Pr(A) \tag{2.184}$$

yielding

$$BA = A, \text{ or, } A \subset B \tag{2.185}$$

2.1.47 If $\Pr(A \mid B) > \Pr(A)$, then which of the following is correct

(a) $\Pr(B \mid A) < \Pr((B))$

(b) $\Pr(AB) < \Pr(A) \Pr(B)$

(c) $\Pr(B \mid A) > \Pr(B)$

(d) $\Pr(B \mid A) = \Pr(B)$

Solution:

$$\Pr(A \mid B) > \Pr(A) \implies \frac{\Pr(AB)}{\Pr(B)} > \Pr(A) \tag{2.186}$$

$$\implies \Pr(AB) > \Pr(A)\Pr(B) \implies \frac{\Pr(AB)}{\Pr(A)} > \Pr(B) \tag{2.187}$$

Since

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}, \Pr(B \mid A) > \Pr(B) \tag{2.188}$$

Hence, option 2.1.47c is correct.

2.1.48 If A and B are any two events such that $\Pr(A)$+$\Pr(B)$-$\Pr(AB)$=$\Pr(A)$, then choose the correct option

(a) $\Pr(B|A) = 1$

(b) $\Pr(A|B) = 1$

(c) $\Pr(B|A) = 0$

(d) $\Pr(A|B) = 0$

Solution: From the given information,

$$\Pr(A) + \Pr(B) - \Pr(AB) = \Pr(A) \tag{2.189}$$

$$\implies \Pr(B) = \Pr(AB) \tag{2.190}$$

Hence,

(a)

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(B)}{\Pr(A)} \tag{2.191}$$

(b)

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1 \tag{2.192}$$

Hence option 2.1.48b is correct.

2.2. Exercises

2.2.1 A team of medical students doing their internship have to assist during surgeries at a city hospital. The probabilities of surgeries rated as very-complex, complex, routine, simple or very-simple are respectively, 0.15, 0.20, 0.31, 0.26, .08. Find the probabilities that a particular surgery will be rated

(a) complex or very-complex

(b) neither very-complex nor very simple

(c) routine or complex

(d) routine or simple

Solution: The given information is summarised in Table 2.15

Random Variables	Difficulty Levels	Probability
E_1	Very-Complex	$\Pr(E_1)$=0.15
E_2	Complex	$\Pr(E_2)$=0.2
E_3	Routine	$\Pr(E_3)$=0.31
E_4	Simple	$\Pr(E_4)$=0.26
E_5	Very-Simple	$\Pr(E_5)$=0.08

Table 2.15:

(a)

$$\Pr(E_1 + E_2) = \Pr(E_1) + \Pr(E_2) \qquad \because E_1E_2 = 0 \tag{2.193}$$

$$= 0.15 + 0.20 = 0.35 \tag{2.194}$$

(b)

$$\Pr\left(E_1'E_5'\right) = \Pr\left((E_1 + E_5)'\right) \tag{2.195}$$

$$= 1 - \Pr(E_1 + E_5) \tag{2.196}$$

$$= 1 - \left[\Pr(E_1) + \Pr(E_5)\right] \qquad \because E_1E_5 = 0 \tag{2.197}$$

$$= 1 - \left[0.15 + 0.08\right] = 0.77 \tag{2.198}$$

$$\tag{2.199}$$

(c)

$$\Pr(E_3 + E_2) = \Pr(E_3) + \Pr(E_2) \qquad \because E_3E_2 = 0 \tag{2.200}$$

$$= 0.31 + 0.20 = 0.51 \tag{2.201}$$

(d) To find the probabilities that a particular surgery will be rated routine or simple:

$$\Pr(E_3 + E_4) = \Pr(E_3) + \Pr(E_4) \qquad \because E_3E_4 = 0 \tag{2.202}$$

$$= 0.31 + 0.26 = 0.57 \tag{2.203}$$

$$\tag{2.204}$$

2.2.2 The accompanying Venn diagram in Fig. 2.1 shows three events, A, B, and C, and also the probabilities of the various intersections (for instance, $\Pr(AB) = .07$). Determine

(a) $\Pr(A)$

(b) $\Pr(BC')$

(c) $\Pr(A + B)$

(d) $\Pr(AB')$

(e) $\Pr(BC)$

(f) Probability of exactly one of the three occurs.

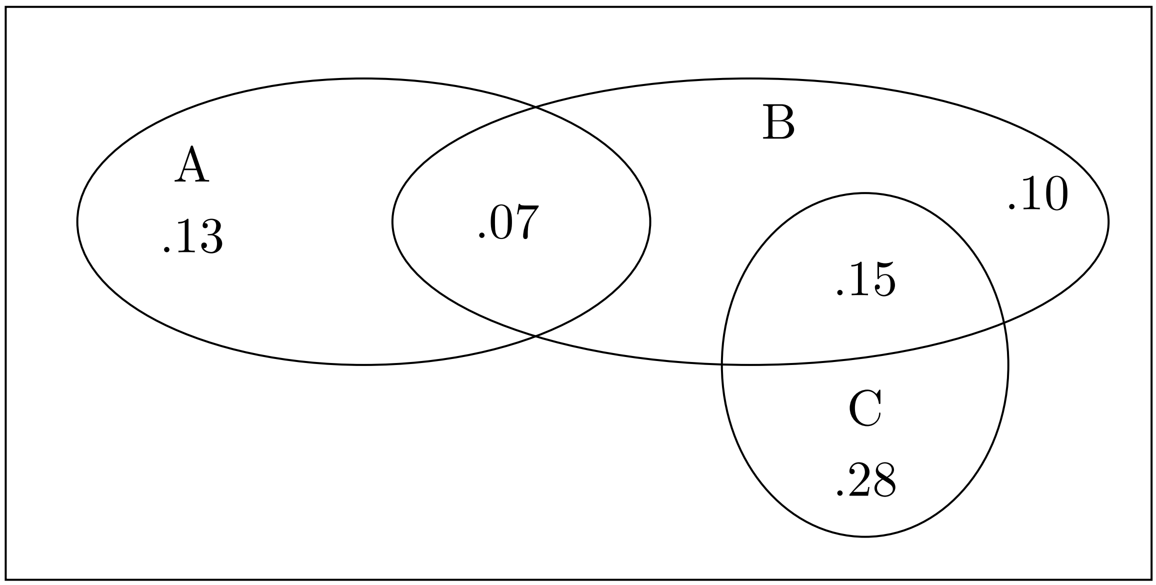

Figure 2.1: Question Figure

Solution: From Fig. 2.1,

$$\Pr(AB) = 0.07 \tag{2.205}$$

$$\Pr\left(AB'\right) = 0.13 \tag{2.206}$$

$$\Pr\left(A'B\right) = 0.25 \tag{2.207}$$

$$\Pr(BC) = 0.15 \tag{2.208}$$

$$\Pr\left(CB'\right) = 0.28 \tag{2.209}$$

$$\Pr\left(AB'C'\right) = 0.13 \tag{2.210}$$

$$\Pr\left(A'BC'\right) = 0.10 \tag{2.211}$$

$$\Pr\left(A'B'C\right) = 0.28 \tag{2.212}$$

(a)

$$A = A\left(B + B'\right) = AB + AB' \quad \left(\because B + B' = 1\right) \tag{2.213}$$

$$\implies \Pr(A) = \Pr(AB) + \Pr\left(AB'\right) \quad \left(\because BB' = 0\right) \tag{2.214}$$

$$= 0.13 + 0.07 = 0.20 \tag{2.215}$$

from (2.205) and (2.206).

(b) Similarly,

$$\Pr(B) = \Pr\left(A'B\right) + \Pr(AB) \tag{2.216}$$

$$= 0.25 + 0.07 = 0.32 \tag{2.217}$$

from (2.205) and (2.207). Using (2.214),

$$\Pr\left(BC'\right) = \Pr(B) - \Pr(BC) = 0.32 - 0.15 = 0.17 \tag{2.218}$$

(c)

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.219}$$

$$= 0.20 + 0.32 - 0.07 = 0.45 \tag{2.220}$$

(d) Using (2.214)

$$\Pr\left(AB'\right) = \Pr(A) - \Pr(AB) = 0.20 - 0.07 = 0.13 \tag{2.221}$$

(e) From (2.208)

$$\Pr(BC) = 0.15 \tag{2.222}$$

(f)

$$X = AB'C' + A'B'C' + A'B'C \tag{2.223}$$

$$\Longrightarrow \Pr(X) = \Pr\left(AB'C'\right) + \Pr\left(A'BC'\right) + \Pr\left(A'B'C\right) \tag{2.224}$$

$$= 0.13 + 0.10 + 0.28 = 0.51 \tag{2.225}$$

from (2.210), (2.211) and (2.212).

Chapter 3

Random Variables

3.1. Examples

3.1.1 One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

(a) A king of red colour

(b) A face card

(c) A red face card

(d) The jack of hearts

(e) A spade

(f) The queen of diamonds

Solution: See Table 3.1. Consider 3 random variables X, Y and Z, which represent the Colour, Class and Value of each card respectively. The pmfs are

$$p_X(i) = \frac{1}{2} \ \forall \ i \in [0, 1] \tag{3.1}$$

$$p_Y(i) = \frac{1}{4} \ \forall \ i \in [1, 4] \tag{3.2}$$

$$p_Z(i) = \frac{1}{13} \ \forall \ i \in [1, 13] \tag{3.3}$$

Event	Value of X	Value of Y	Value of Z
Draw Red King	1	N/A	3
Draw Face Card	N/A	N/A	1,2 or 3
Draw Red Face Card	1	N/A	1,2 or 3
Draw Hearts Jack	N/A	3	1
Draw Spade	N/A	4	N/A
Draw Diamonds Queen	N/A	1	2

Table 3.1: Values of X,Y,Z for each event

and

$$\begin{aligned} F_Z(z) = \Pr(Z \leq z) &= \sum_{i=1}^{z} \Pr(Z = i) \\ &= z \times \Pr(Z = 1) = \frac{z}{13} \end{aligned} \tag{3.4}$$

Also, the random variable pairs X,Z and Y,Z are independent.

(a) Probability of drawing a King of Red colour

$$\begin{aligned} \Pr(X = 1, Z = 3) &= \Pr(X = 1) \times \Pr(Z = 3) \\ &= \frac{1}{2} \times \frac{1}{13} = \frac{1}{26} \end{aligned} \tag{3.5}$$

(b) Probability of drawing a Face Card

$$\Pr(1 \leq Z \leq 3) = F_Z(3) = \frac{3}{13} \tag{3.6}$$

(c) Probability of drawing a Red Face Card

$$\Pr(X=1, 1 \le Z \le 3) = \Pr(X=1) \times F_Z(3) \tag{3.7}$$
$$= \frac{1}{2} \times \frac{3}{13} = \frac{3}{26} \tag{3.8}$$

(d) Probability of drawing the Jack of Hearts

$$\Pr(Y=3, Z=1) = \Pr(Y=3) \times \Pr(Z=1) \tag{3.9}$$
$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{3.10}$$

(e) Probability of drawing a Spade

$$\Pr(Y=4) = \frac{1}{4} \tag{3.11}$$

(f) Probability of drawing the Queen of Diamonds:

$$\Pr(Y=1, Z=2) = \Pr(Y=1) \times \Pr(Z=2) \tag{3.12}$$
$$= \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} \tag{3.13}$$

3.1.2 Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

(a) What is the probability that the card is the queen?

(b) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

Solution: See Table 3.3.

EVENT	DESCRIPTION
E	Event of picking a card.
S	Sample space of picking a card.
Q	Event of the card picked be Queen.
A	Event of the card picked be Ace.

Table 3.3:

(a)

$$\Pr(Q) = \frac{1}{5} \tag{3.14}$$

(b) i.

$$\Pr(A) = \frac{1}{4} \tag{3.15}$$

ii.

$$\Pr(Q) = 0 \tag{3.16}$$

3.1.3 A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball is double that if a red ball, determine the number of blue balls in the bag.

Solution: The probability of drawing a red ball is

$$\Pr(R) = \frac{5}{5+x} \tag{3.17}$$

The probability of drawing a blue ball is

$$\Pr(B) = \frac{10}{5+x} \tag{3.18}$$

Thus,

$$\left(\frac{5}{5+x}\right)+2\left(\frac{5}{5+x}\right)=1 \tag{3.19}$$

$$\Longrightarrow \frac{15}{5+x}=1 \tag{3.20}$$

$$\text{or, } x=10 \tag{3.21}$$

3.1.4 A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace and (ii) black card.

Solution: See Table 3.4

Table 3.4: Random Variable and probability Table

Random variable	value of R.V	Probability
\$X\$	1,2	26/52
\$Y\$	1,2,3,4	13/52
\$Z\$	1 \le Z \le 13	1/13

(a)

$$\Pr(Y=1, Z=1)=\Pr(Y=1)\Pr(Z=1)=\left(\frac{1}{4}\right)\left(\frac{1}{13}\right)=\frac{1}{52} \tag{3.22}$$

(b) The probability when the card choosen is ,

i. an ace $(Z=1)$

$$\Pr(Z=1)=\frac{1}{13}. \tag{3.23}$$

ii. black card $(X = 1)$

$$\Pr(X = 1) = \frac{1}{2}. \tag{3.24}$$

3.1.5 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 3.6. yielding

RV	**Values**	**Description**
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 3.6: Random variables(RV) X,Y and X,Y

$$\Pr(00, 10, 20, 31) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \tag{3.25}$$

$$= \frac{286}{20285} \tag{3.26}$$

3.1.6 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 3.8 The total number of possible outcomes is ${}^{N}C_n$ and the total number of favourable outcomes is ${}^{q}C_n$ yielding the desired probability

$$\Pr(n) = \frac{{}^{q}C_n}{{}^{N}C_n} \tag{3.27}$$

Substituting numerical values,

Variable	Value	Description
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	{0,1,2,...,N}	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 3.8:

(a) For one ticket,

$$\Pr(1) = \frac{^{9990}C_1}{^{10000}C_1} = 0.9990 \tag{3.28}$$

(b) For two tickets,

$$\Pr(2) = \frac{^{9990}C_2}{^{10000}C_2} = 0.9980 \tag{3.29}$$

(c) For 10 tickets

$$\Pr(3) = \frac{^{9990}C_{10}}{^{10000}C_{10}} = 0.9901 \tag{3.30}$$

3.1.7 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 3.10 summarises the given information.

RV	Values	Description
X	{0,1}	0: sectioin1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{^{40}C_2}{^{100}C_2} + \frac{^{60}C_2}{^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \tag{3.31}$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \tag{3.32}$$

3.1.8 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution: Let

$$X_i = \begin{cases} 1, & \text{correct number choosen in } i^{th} \text{ wheel} \\ 0, & \text{otherwise} \end{cases} \tag{3.33}$$

and since repetition is not allowed, sample space for every next wheel will reduce by 1 unit. Therefore,

$$p_{X_i}(1) = \frac{1}{11 - i} \tag{3.34}$$

$$p_{X_i}(0) = 1 - \frac{1}{11 - i} \tag{3.35}$$

$$= \frac{10 - i}{11 - i} \tag{3.36}$$

Therefore, the desired probability is

$$\Pr(E) = \prod_{i=1}^{4} p_{X_i}(1) \tag{3.37}$$

$$= \frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} \times \frac{1}{7} \tag{3.38}$$

$$= \frac{1}{5040} \tag{3.39}$$

3.1.9 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 3.12 summarizes the various events Given that the cards are drawn

RV	Values	Description
X	{0,1}	number of cards drawn 2
Y	{0,1}	0: black card, 1: red card
XY	{00,10}	card drawn is black

Table 3.12:

at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00,10) = \frac{{}^{26}C_1}{{}^{52}C_1} \times \frac{{}^{25}C_1}{{}^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \tag{3.40}$$

3.1.10 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

3.1.11 Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(a) both balls are red.

(b) first ball is black and second is red.

(c) one of them is black and other is red.

3.1.12 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

3.1.13 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

Solution: Let X and Y be two random variables representing outcomes on both the die, See Table 3.13. Since both die rolls are independent,

$$\Pr\left(X=2, Y=2\right) = \Pr\left(X=2\right)\Pr\left(Y=2\right) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \tag{3.41}$$

$\Pr(X=2)$	The probability of occurence of 2 on die roll 1.
$\Pr(Y=2)$	The probability of occurence of 2 on die roll 2.
$\Pr(X=2, Y=2)$	The probability of occurence of 2 on both the die.

Table 3.13:

3.1.14 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution: See Table 3.14 Given,

Table 3.14: Random Variable Declaration

Random Variable	Value of the random variable	Event
B	0	selecting first bag
	1	selecting second bag
R	0	choosing white ball from the bag
	1	choosing red ball from the bag

$$\Pr(R=1|B=0) = \frac{4}{8} = \frac{1}{2} \tag{3.42}$$

$$\Pr(R=1|B=1) = \frac{2}{8} = \frac{1}{4} \tag{3.43}$$

$$\Pr(B=0) = \frac{1}{2} \tag{3.44}$$

$$\Pr(B=1) = \frac{1}{2} \tag{3.45}$$

$$\Pr(B=0|R=1) = \frac{\Pr(R=1|B=0)\Pr(B=0)}{\Pr(R=1|B=0)\Pr(B=0)+\Pr(R=1|B=1)\Pr(B=1)} \tag{3.46}$$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{2}} \tag{3.47}$$

$$= \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}} = \frac{2}{3} \tag{3.48}$$

3.2. Exercises

3.2.1 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where G is the event that a number greater than 3 occurs on a single roll of the die.

Solution: See Table 3.16 for the input parameters. Then,

Parameter	Value	Description
X	{1,2,3,4,5,6}	Number obtained on the die

Table 3.16: Parameters and their Description

$$p_X(k) = \begin{cases} 2p, & if\ k = 2m-1 \\ p, & if\ k = 2m \end{cases} \tag{3.49}$$

Since $1 \leq X \leq 6$,

$$\sum_{i=1}^{6} \Pr(X=i) = 1 \tag{3.50}$$

$$\implies 6p + 3p = 1 \tag{3.51}$$

$$\implies p = \frac{1}{9} \tag{3.52}$$

The CDF

$$F_X(k) = \begin{cases} \frac{3k+1}{18}, & if\ k = 2m-1 \\ \\ \frac{k}{6}, & if\ k = 2m \end{cases} \tag{3.53}$$

Thus,

$$\Pr(G) = \Pr(X > 3) = F_X(6) - F_X(3) \tag{3.54}$$

$$= 1 - \frac{3(3)+1}{18} = \frac{4}{9} \tag{3.55}$$

Chapter 4

Conditional Probability

4.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{4.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{4.2}$$

4.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

4.3 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

4.4 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{4.3}$$

4.5 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A = 0.4)$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A + B)$

Solution:

4.6 If $\Pr(A) = \frac{6}{11}$, $\Pr(B) = \frac{5}{11}$ and $\Pr(A + B) = \frac{7}{11}$, find

(a) $\Pr(AB)$

(b) $\Pr(A \mid B)$

(c) $\Pr(B \mid A)$

Solution:

(a) Since

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A + B), \tag{4.4}$$

$$\Pr(AB) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11} \tag{4.5}$$

(b) Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.6}$$

$$\Pr(A \mid B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5} \tag{4.7}$$

(c) Since

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)}, \tag{4.8}$$

$$\Pr(B \mid A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3} \tag{4.9}$$

4.7 Mother, Father and Son line up at random for a family picture. Determine $\Pr(E \mid F)$ where E: Son on one end, F: Father in middle

Solution: The total ways of arranging Father, Son, Mother in the family chart is 3! = 6. The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \tag{4.10}$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \tag{4.11}$$

Thus,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \tag{4.12}$$

4.8 Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that

(a) The youngest is a girl

(b) At least one is a girl

Solution:

Variable	Description	Probability
$X_i = 1$	ith born child is a boy	$\Pr(X_i = 1) = 0.50$
$X_i = 0$	ith born child is a girl	$\Pr(X_i = 0) = 0.50$

Table 4.1: Random variable definitions.

(a)

$$\Pr\left((X_1 + X_2)' | X_2'\right) = \frac{\Pr\left((X_1' X_2') X_2'\right)}{\Pr(X_2')} \tag{4.13}$$

$$= \frac{\Pr(X_1')\Pr(X_2')}{\Pr(X_2')} \tag{4.14}$$

$$= \Pr\left(X_1'\right) = \frac{1}{2} \tag{4.15}$$

(b)

$$\Pr\left((X_1 + X_2)' | (X_1 X_2)'\right) = \frac{\Pr\left((X_1' X_2')(X_1' + X_2')\right)}{1 - \Pr(X_1 X_2)} \tag{4.16}$$

$$= \frac{\Pr(X_1' X_2')}{1 - \Pr(X_1 X_2)} \tag{4.17}$$

$$= \frac{\Pr(X_1')\Pr(X_2')}{1 - \Pr(X_1)\Pr(X_2)} \tag{4.18}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1 - (\frac{1}{2} \times \frac{1}{2})} = \frac{1}{3} \tag{4.19}$$

4.9 An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution: From the law of total probability,

Variable	**Event**
$X = 0$	Easy question
$X = 1$	Difficult question
$Y = 0$	True/False question
$Y = 1$	Multiple choice question

Table 4.2:

$$p_X(0) + p_X(1) = 1 \tag{4.20}$$

$$p_Y(0) + p_Y(1) = 1 \tag{4.21}$$

From Table 4.2,

$$p_X(0) = p_{X,Y}(0,0) + p_{X,Y}(0,1) \tag{4.22}$$

$$= \frac{300 + 500}{300 + 200 + 500 + 400} = \frac{4}{7} \tag{4.23}$$

$$p_Y(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0) \tag{4.24}$$

$$= \frac{300 + 200}{300 + 200 + 500 + 400} = \frac{5}{14} \tag{4.25}$$

From (4.20), (4.21) and (4.25),

$$p_X(1) = 1 - p_X(0) = \frac{3}{7} \tag{4.26}$$

$$p_Y(1) = 1 - p_Y(0) = \frac{9}{14} \tag{4.27}$$

$$p_{X,Y}(0,0) = \frac{300}{1400} = \frac{3}{14} \tag{4.28}$$

$$p_{X,Y}(1,1) = \frac{400}{1400} = \frac{2}{7} \tag{4.29}$$

From (4.22), (4.24), (4.25) and (4.29),

$$p_{X,Y}(0,1) = p_X(0) - p_{X,Y}(0,0) = \frac{5}{14} \tag{4.30}$$

$$p_{X,Y}(1,0) = p_Y(0) - p_{X,Y}(0,0) = \frac{1}{7} \tag{4.31}$$

By definition,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \tag{4.32}$$

From (4.28) and (4.31),

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} \tag{4.33}$$

$$= \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9} \tag{4.34}$$

4.10 If $\Pr(A) = \frac{1}{2}$, $\Pr(B) = 0$, then $\Pr(A \mid B)$ is

(a) 0

(b) $\frac{1}{2}$

(c) not defined

(d) 1

Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.35}$$

$$\Pr(A \mid B) \text{ is not defined} \tag{4.36}$$

4.11 If A and B are events such that

$$\Pr(A|B) = \Pr(B|A) \tag{4.37}$$

then

(a) $A \subset B$ but $A \neq B$

(b) $A = B$

(c) $A \cap B = \phi$

(d) $\Pr(A) = \Pr(B)$

Solution: Using Bayes' Rule,

$$\Pr(AB) = \Pr(A)\Pr(B|A) \tag{4.38}$$

$$= \Pr(B)\Pr(A|B) \tag{4.39}$$

Using (4.37) in (4.38) and (4.39),

$$\Pr(A) = \Pr(B) \tag{4.40}$$

We consider the options one by one.

(a) If $A \subset B$ and $A \neq B$, then we can write $B = A + C$, where $AC = 0$ and $C \neq 0$. Thus,

$$\Pr(B) = \Pr(A + C) \tag{4.41}$$

$$= \Pr(A) + \Pr(C) - \Pr(AC) \tag{4.42}$$

$$= \Pr(A) + \Pr(C) > \Pr(A) \tag{4.43}$$

However, (4.43) contradicts (4.40).

(b) We give a counterexample to show this is wrong. Consider A as the event that an even number shows on rolling a fair die and B as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 4.3. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{4.44}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{2} \tag{4.45}$$

(c) The same example as before provides the required counterexample, as $\Pr(AB) = \frac{1}{6}$.

(d) This is the correct answer, as discussed above.

	A	$\bar{A}$
B	$\frac{1}{6}$	$\frac{1}{3}$
$\bar{B}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 4.3: Joint pmf for events A and B.

4.12 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Let X denote the die roll for the first trial. The pmf of X is

$$\Pr\left(X = k\right) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.46}$$

Let Y be the random variable denoting the outcome of the coin toss in the second trial. The pmf of Y is

$$\Pr\left(Y = k\right) = \begin{cases} \frac{1}{2} & 0 \le i \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.47}$$

We are required to find $\Pr\left(Y = 1|X = 3\right)$. However, from the given data,

$$\Pr\left(Y = 1,\ X = k\right) = \begin{cases} \frac{1}{12} & k \in \{1, 2, 4, 5\} \\ 0 & \text{otherwise} \end{cases} \tag{4.48}$$

Therefore, from (4.48),

$$\Pr\left(Y = 1|X = 3\right) = \frac{\Pr\left(X = 3,\ Y = 1\right)}{\Pr\left(X = 3\right)} = 0 \tag{4.49}$$

4.13 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Let E_1 denote the event that the first card drawn is Black, E_2 denote the event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52}, \Pr(E_2 \mid E_1) = \frac{25}{51} \tag{4.50}$$

$$\implies \Pr(E_1E_2) = \Pr(E_1)\Pr(E_2 \mid E_1) = \frac{25}{102} \tag{4.51}$$

4.14 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B)$

(d) $P(B|A)$

4.15 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution: The given information is summarized in Tables 4.4 and 4.5.

RV	**Values**	**Description**
X	$\{0,1\}$	1st draw - 0: Red, 1: Black
Y	$\{0,1\}$	2nd draw - 0: Red, 1: Black

Table 4.4: Random variables X,Y

The required probability is given by

$$\Pr(Y=0) = \Pr(X=0)\Pr(Y=0 \mid X=0) + \Pr(X=1)\Pr(Y=0 \mid X=1) \tag{4.52}$$

$$= \left(\frac{5}{10} \times \frac{7}{12}\right) + \left(\frac{5}{10} \times \frac{5}{12}\right) = \frac{1}{2} \tag{4.53}$$

Event	Probability
$\Pr(X=0)$	$\frac{5}{10}$
$\Pr(X=1)$	$\frac{5}{10}$
$\Pr(Y=1 \mid X=0)$	$\frac{7}{12}$
$\Pr(Y=1 \mid X=1)$	$\frac{5}{12}$

Table 4.5: Probabilities

4.16 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.17 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Solution:

Let

$$X = \begin{cases} 0, & \text{if student is resides in hostel} \\ 1, & \text{if student is a day scholar} \end{cases} \tag{4.54}$$

$$Y = \begin{cases} 0, & \text{if student does not attain A grade} \\ 1, & \text{if student attains A grade} \end{cases} \tag{4.55}$$

From the given data,

$$\Pr(X = 0) = \frac{3}{5} \tag{4.56}$$

$$\Pr(X = 1) = \frac{2}{5} \tag{4.57}$$

$$\Pr(Y = 1 \mid X = 0) = \frac{3}{10} \tag{4.58}$$

$$\Pr(Y = 1 \mid X = 1) = \frac{1}{5} \tag{4.59}$$

The desired probability is

$$\Pr(X = 0 \mid Y = 1) = \frac{\Pr(Y = 1 \mid X = 0) \times \Pr(X = 0)}{\sum_{k=0}^{1} \Pr(Y = 1 \mid X = k) \times \Pr(X = k)} \tag{4.60}$$

$$= \frac{\frac{3}{10} \times \frac{3}{5}}{\frac{3}{10} \times \frac{3}{5} + \frac{1}{5} \times \frac{2}{5}} = \frac{9}{13} \tag{4.61}$$

4.18 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: See Tables 4.18 and 4.18

Random Variable	Description
$X = 0$	Student guesses the answer
$X = 1$	Student knows the answer
$Y = 0$	Answer is incorrect
$Y = 1$	Answer is correct

Table 4.6: Random Variable and their description

Pr(Event)	Value
$\Pr(Y{=}1 \mid X{=}0)$	0.25
$\Pr(Y{=}1 \mid X{=}1)$	1
$\Pr(X{=}0)$	0.25
$\Pr(X{=}1)$	0.75

Table 4.7: Probability of events

The probability that the student knows the answer and he answered it correctly is

$$\Pr(X=1|Y=1) = \frac{\Pr(Y=1|X=1)\Pr(X=1)}{\sum_{i=0}^{i=1} Pr(Y=1|X=i)Pr(X=i)} \tag{4.62}$$

$$= \frac{0.75}{0.25 \times 0.25 + 1 \times 0.75} = 0.92308 \tag{4.63}$$

4.19 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

Solution: See Table 4.19 for the given information.

$$\Pr(E_2) = 1 - \Pr(E_1) = 1 - 0.001 = 0.999 \tag{4.64}$$

A:	Person with positive blood test	$\Pr(A)$
E_1:	Person suffering from a disease	$\Pr(E_1)$=0.001
E_2:	Person not suffering from a disease	$\Pr(E_2)$=0.999
$A\|E_1$:	Event of positive blood test when person suffers from disease	$\Pr(A\|E_1)$=0.99
$A\|E_2$:	Event of positive blood test when person not suffers from disease	$\Pr(A\|E_2)$=0.005

Table 4.8: Given Information

$$\Pr(E_1|A) = \frac{\Pr(E_1)\Pr(A|E_1)}{\sum_{i=1}^{2}\Pr(E_i)\Pr(A|E_i)} \tag{4.65}$$

$$= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} = \frac{22}{133} \tag{4.66}$$

4.20 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Solution: Define the random variable X as in Table 4.9. Clearly, the pmf of X is

$X = 1$	Two-headed coin is selected.
$X = 2$	75% biased coin is selected.
$X = 3$	Fair coin is selected.

Table 4.9: Definition of X.

$$\Pr(X = k) = \begin{cases} \frac{1}{3} & 1 \le k \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{4.67}$$

Let the random variables Y_1, Y_2 and Y_3 (one for each coin) be defined as

$$Y_1 \sim \text{Ber}(1) \tag{4.68}$$

$$Y_2 \sim \text{Ber}\left(\frac{3}{4}\right) \tag{4.69}$$

$$Y_3 \sim \text{Ber}\left(\frac{1}{2}\right) \tag{4.70}$$

Define Y as

$$Y \triangleq \sum_{i=1}^{3} \mathbf{1}_i(X)\, Y_i \tag{4.71}$$

where $\mathbf{1}$ denotes the indicator random variable, defined as

$$\mathbf{1}_i(X) = \begin{cases} 1 & \text{if } X = i \\ 0 & \text{otherwise} \end{cases} \tag{4.72}$$

We are required to find $\Pr(X = 1|Y = 1)$. However, from Bayes' Rule,

$$\Pr(X = 1, Y = 1) = \Pr(X = 1)\Pr(Y = 1|X = 1) \tag{4.73}$$

$$= \Pr(Y = 1)\Pr(X = 1|Y = 1) \tag{4.74}$$

Note from (4.71) that

$$X = 1 \implies Y = Y_1 \tag{4.75}$$

and also,

$$\Pr(Y = 1) = \sum_{i=1}^{3} \Pr(X = i)\Pr(Y_i = 1) \tag{4.76}$$

$$= \frac{1}{3}\left(1 + \frac{3}{4} + \frac{1}{2}\right) = \frac{3}{4} \tag{4.77}$$

Thus, from (4.73), (4.74) and (4.77), we see that

$$\Pr(X = 1|Y = 1) = \frac{\Pr(X = 1)\Pr(Y_1 = 1)}{\Pr(Y = 1)} \tag{4.78}$$

$$= \frac{\frac{1}{3}}{\frac{3}{4}} = \frac{4}{9} \tag{4.79}$$

4.21 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

4.22 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

4.23 . Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution: The given information is listed in Tables 4.11 and 4.13

RV	Values	Description
X	{1,2}	1:Group1 ,2:Group2
Y	{0,1}	0:New product not introduced ,1:New product introduced

Table 4.11: Random variables(RV) X,Y

Event	Probability	Description
$\Pr(X=1)$	0.6	First group winning
$\Pr(X=2)$	0.4	Second group winning
$\Pr(Y=1 \mid X=1)$	0.7	Introducing 1 if 1 wins
$\Pr(Y=1 \mid X=2)$	0.3	Introducing 1 if 2 wins

Table 4.13: Probabilities

$$\Pr(X=2 \mid Y=1) = \frac{\Pr(2)\Pr(1 \mid 2)}{\Pr(1)\Pr(1 \mid 1) + \Pr(2)\Pr(1 \mid 2)} = \frac{2}{9} \tag{4.80}$$

4.24 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

4.25 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

4.26 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

4.27 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

Solution: Consider the random variables A, X as described in the table 4.14.

RV	**Values**	**Description**
A	$\{0,1\}$	1: A speaks truth, 0: A lies
X	$\{0,1\}$	1: Heads, 0: Tails

Table 4.14: Random variables A, X

The given information about probabilities is listed in table 4.15.

Event	**Probability**
$\Pr(A=1)$	$\frac{4}{5}$
$\Pr(X=1)$	$\frac{1}{2}$
$\Pr(X=1 \mid A=1)$	$\frac{1}{2}$

Table 4.15: Probabilities

The required probability is given by

$$\Pr(A=1 \mid X=1) = \frac{\Pr(A=1)\Pr(X=1 \mid A=1)}{\Pr(X=1)} \tag{4.81}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} \tag{4.82}$$

$$= \frac{4}{5} \tag{4.83}$$

4.28 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A \mid B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A \mid B) < \Pr(A)$

(c) $\Pr(A \mid B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{4.84}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{4.85}$$

we know that

$$\Pr(B) \leq 1 \tag{4.86}$$

$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{4.87}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{4.88}$$

$$= \Pr(A \mid B) \tag{4.89}$$

from (4.85).

4.29 A and B are two events such that $\Pr(A) \neq 0$. Find $\Pr(B \mid A)$, if

(a) A is a subset of B

(b) $A \cap B = \phi$

Solution: We use

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)} \tag{4.90}$$

(a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \tag{4.91}$$

From (4.90),

$$\Pr(B \mid A) = 1 \tag{4.92}$$

(b) $A \cap B = \phi$. This implies

$$\Pr(BA) = 0 \tag{4.93}$$

From (4.90),

$$\Pr(B \mid A) = 0 \tag{4.94}$$

4.30 A couple has two children.

(a) Find the probability that both children are males, if it is known that at least one of the children is male.

(b) Find the probability that both children are females, if it is known that the elder child is a female.

Solution: Consider the random variables X, Y, which denotes the first child, second child gender respectively as described in table 4.16.

RV	Values	Description
X	$\{0, 1\}$	0: Male , 1: Female
Y	$\{0, 1\}$	0: Male, 1: Female

Table 4.16: Random variables X

The probabilities for the random variables X, Y is listed in table 4.17.

Event	Probability
$\Pr(X = 0)$	$\frac{1}{2}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(Y = 0)$	$\frac{1}{2}$
$\Pr(Y = 1)$	$\frac{1}{2}$
$\Pr(X + Y = 0)$	$\frac{1}{4}$
$\Pr(X + Y = 2)$	$\frac{1}{4}$
$\Pr(XY = 0)$	$\frac{3}{4}$

Table 4.17: Probabilities

The probability $\Pr(XY = 0)$ is given by

$$= \Pr(X = 0) + \Pr(Y = 0) - \Pr(X + Y = 0) \tag{4.95}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \tag{4.96}$$

$$= \frac{3}{4} \tag{4.97}$$

(a) The event of both children being Male is when $X + Y = 0$. The event of atleast one of the children being Male is when $XY = 0$.

$$\{X + Y = 0\} \cap \{XY = 0\} \equiv \{X + Y = 0\} \tag{4.98}$$

The required probability is given by,

$$\Pr(X + Y = 0 \mid XY = 0) \tag{4.99}$$

$$= \frac{\Pr(X + Y = 0)}{\Pr(XY = 0)} \tag{4.100}$$

$$= \frac{1}{3} \tag{4.101}$$

(b) The event of both children being Female is when $X + Y = 2$. The event of elder child being Female is when $X = 1$.

$$\{X + Y = 2\} \cap \{X = 1\} \equiv \{X + Y = 2\} \tag{4.102}$$

The required probability is given by,

$$\Pr(X + Y = 2 \mid X = 1) \tag{4.103}$$

$$= \frac{\Pr(X + Y = 2)}{\Pr(X = 1)} \tag{4.104}$$

$$= \frac{1}{2} \tag{4.105}$$

4.31 Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability that this person being male? Assume that there are equal number of males and females.

Solution: See Table 4.18. It is given that,

Variable	**Event**
$X = 0$	Men
$X = 1$	Women
$Y = 0$	Non-grey hair
$Y = 1$	grey hair

Table 4.18:

$$p_X(0) = p_X(1) = \frac{1}{2} \tag{4.106}$$

$$p_{Y|X}(1|0) = \frac{5}{100} = \frac{1}{20} \tag{4.107}$$

$$p_{Y|X}(1|1) = \frac{0.25}{100} = \frac{1}{400} \tag{4.108}$$

From the law of total probability,

$$p_Y(1) = p_{Y|X}(1|0) \times p_X(0) + p_{Y|X}(1|1) \times p_X(1) \tag{4.109}$$

$$= \frac{1}{20} \times \frac{1}{2} + \frac{1}{400} \times \frac{1}{2} = \frac{21}{800} \tag{4.110}$$

$$\therefore \ p_Y(1) = \frac{21}{800} \tag{4.111}$$

Thus,

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \ p_{X,Y}(x,y) = p_{Y|X}(y|x) p_X(x) \tag{4.112}$$

yielding

$$p_{X,Y}(0,1) = p_{Y|X}(1|0) \times p_X(0) \tag{4.113}$$

$$= \frac{1}{20} \times \frac{1}{2} = \frac{1}{40} \tag{4.114}$$

$$\tag{4.115}$$

resulting in

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{1}{40}}{\frac{21}{800}} = \frac{20}{21} \tag{4.116}$$

4.32 Suppose we have four boxes A,B,C and D containing coloured marbles as given in Table 4.19. One of the boxes has been selected at random and a single marble is

Box	Marble colour		
	Red	White	Black
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

Table 4.19: Question Table

drawn from it. If the marble is red, what is the probability that it was drawn from

1) Box A ?

2) Box B ?

3) Box C ?

Solution: See Table Table 4.20. Here,

Events	Definition
E	drawn marble is red
E_1	selected box is A
E_2	selected box is B
E_3	selected box is C
E_4	selected box is D

Table 4.20: Events Table

$$\Pr(E|E_1) = \frac{1}{10}, \Pr(E|E_2) = \frac{6}{10}, \Pr(E|E_3) = \frac{8}{10}, \Pr(E|E_4) = \frac{0}{10} \tag{4.117}$$

$$\Pr(E_i) = \frac{1}{4} \ \forall 1 \leq i \leq 4 \tag{4.118}$$

(a)

$$\Pr(E_1|E) = \frac{\Pr(E|E_1)\Pr(E_1)}{\sum_{i=1}^{i=4}(\Pr(E|E_i)\Pr(E_i))} \tag{4.119}$$

$$= \frac{\frac{1}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{1}{15} \tag{4.120}$$

(b)

$$\Pr(E_2|E) = \frac{\Pr(E|E_2)\Pr(E_2)}{\sum_{i=1}^{i=4}(\Pr(E|E_i)\Pr(E_i))} \tag{4.121}$$

$$= \frac{\frac{6}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{2}{5} \tag{4.122}$$

(c)

$$\Pr(E_3|E) = \frac{\Pr(E|E_3)\Pr(E_3)}{\sum_{i=1}^{i=4}(\Pr(E|E_i)\Pr(E_i))} \tag{4.123}$$

$$= \frac{\frac{8}{10} \times \frac{1}{4}}{\frac{1}{10} \times \frac{1}{4} + \frac{6}{10} \times \frac{1}{4} + \frac{8}{10} \times \frac{1}{4} + \frac{0}{10} \times \frac{1}{4}} = \frac{8}{15} \tag{4.124}$$

4.33 Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls.

One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Solution: Let

$$X = \begin{cases} 1, & \text{if ball is being drawn from Bag I} \\ 2, & \text{if ball is being drawn from Bag I} \end{cases} \tag{4.125}$$

$$Y = \begin{cases} 1, & \text{if ball drawn is Red} \\ 2, & \text{if ball drawn is Black} \end{cases} \tag{4.126}$$

Then

$$\Pr(X = 1, Y = 1) = \frac{3}{7} \Pr(X = 1, Y = 2) = \frac{4}{7} \tag{4.127}$$

$$\implies \Pr(X = 2, Y = 1) = \Pr(X = 1, Y = 1) \times \frac{5}{10} + \Pr(X = 1, Y = 2) \times \frac{4}{10} = \frac{15}{70} + \frac{16}{70} = \frac{31}{70} \tag{4.128}$$

Consequently,

$$\Pr(X = 2, Y = 2) = 1 - \Pr(X = 2, Y = 1) = \frac{39}{70} \tag{4.129}$$

See Thus, the desired probability is

Red ball from Bag I:	$\Pr(X = 1, Y = 1) = \frac{3}{7}$
Black ball from Bag I:	$\Pr(X = 1, Y = 2) = \frac{4}{7}$
Red ball from Bag II:	$\Pr(X = 2, Y = 1) = \frac{31}{70}$
Black ball from Bag II:	$\Pr(X = 2, Y = 2) = \frac{39}{70}$

(4.130)

Table 4.21: Final probabilities of the events.

$$
\begin{aligned}
&\Pr\left(X=1, Y=2 | X=2, Y=2\right) \\
&\qquad = \frac{\Pr\left(X=2, Y=1 | X=1, Y=1\right)}{\sum_{i=1}^{i=2} \Pr\left(X=2, Y=1 | X=1, Y=i\right) \Pr\left(X=1, Y=i\right)} \\
&\qquad\qquad = \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{4}{10} \times \frac{4}{7} + \frac{5}{10} \times \frac{3}{7}} = \frac{16}{31} \qquad (4.131)
\end{aligned}
$$

Chapter 5

Discrete Distributions

5.1. Bernoulli

5.1.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(a) She will buy it?

(b) She will not buy it?

Solution: We can model this situation using the random variable $X \sim \text{Ber}(p)$, where p is the probability of success, *i.e.* the pen is purchased. From the given data,

$$1 - p = \frac{20}{144} \implies p = \frac{67}{72} \tag{5.1}$$

(a) Probability that the pen is purchased is

$$\Pr(X = 1) = p = \frac{67}{72} \tag{5.2}$$

(b) Probability that the pen is not purchased is

$$\Pr(X = 0) = 1 - p = \frac{5}{72} \tag{5.3}$$

5.2. Multinomial

5.2.1 A box contains 5 red marbles, 8 white marbles and 4 green marbles.One marble is taken out of the box at random. What is the probability that the marble taken out will be

Solution:

(a) red ?

(b) white ?

(c) not green?

Solution: Let

$$\text{N} = \text{R} + \text{W} + \text{G} \tag{5.4}$$

$$\text{n} = \text{r} + \text{w} + \text{g} \tag{5.5}$$

where R,B,G and r, b, g represent the number of red, white and green marbles respectively within N and n. Then

$$\Pr(r, w, g) = \frac{{}^{R}C_r \, {}^{W}C_w \, {}^{G}C_g}{{}^{R+W+G}C_{r+w+g}} \tag{5.6}$$

(a) Probability that the marble taken out is red

$$\Pr(1,0,0) = \frac{{}^5C_1\,{}^8C_0\,{}^4C_0}{{}^{17}C_1} = \frac{5}{17} \approx 0.2941 \tag{5.7}$$

(b) Probability that the marble taken out is white

$$\Pr(0,1,0) = \frac{{}^5C_0\,{}^8C_1\,{}^4C_0}{{}^{17}C_1} = \frac{8}{17} \approx 0.4706 \tag{5.8}$$

(c) Probability that the marble taken out is not green

$$1 - \Pr(0,0,1) = 1 - \frac{{}^5C_0\,{}^8C_0\,{}^4C_1}{{}^{17}C_1} = 1 - \frac{4}{17} = \frac{13}{17} \approx 0.7647 \tag{5.9}$$

5.2.2 A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 5.1), and these are equally likely outcomes. What is the probability that it will point at:

(a) 8?

(b) an odd number?

(c) a number greater than 2?

(d) a number less than 9?

Solution: Let X be a random variable defined as the value given by the pointer.

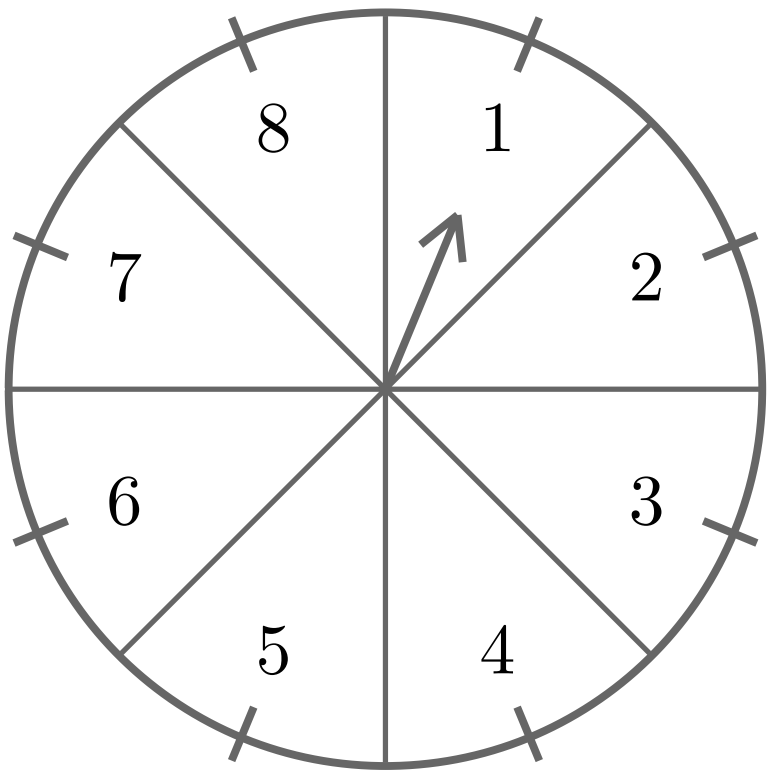

Figure 5.1: Spinner

Then,

$$\Pr\left(X = i\right) = \frac{1}{8} \quad 1 \le i \le 8 \tag{5.10}$$

$$F_X\left(i\right) = \Pr\left(X \le i\right) \tag{5.11}$$

$$= \begin{cases} 0, & i \le 0 \\ \frac{i}{8} & 1 \le i \le 8 \\ 1, & i \ge 9 \end{cases} \tag{5.12}$$

which are plotted in Fig. 5.2 and Fig. 5.3 respectively.

(a)

$$\Pr\left(X = 8\right) = \frac{1}{8} = 0.125 \tag{5.13}$$

(b) For i being odd,

$$\Pr\left(X = \{1, 3, 5, 7\}\right) = \frac{4}{8} = 0.5 \tag{5.14}$$

(c)

$$\Pr\left(X > 2\right) = 1 - \Pr\left(X \le 2\right) \tag{5.15}$$

$$= 1 - \left(F_X\left(2\right) - F_X\left(0\right)\right) \tag{5.16}$$

$$= \frac{6}{8} \tag{5.17}$$

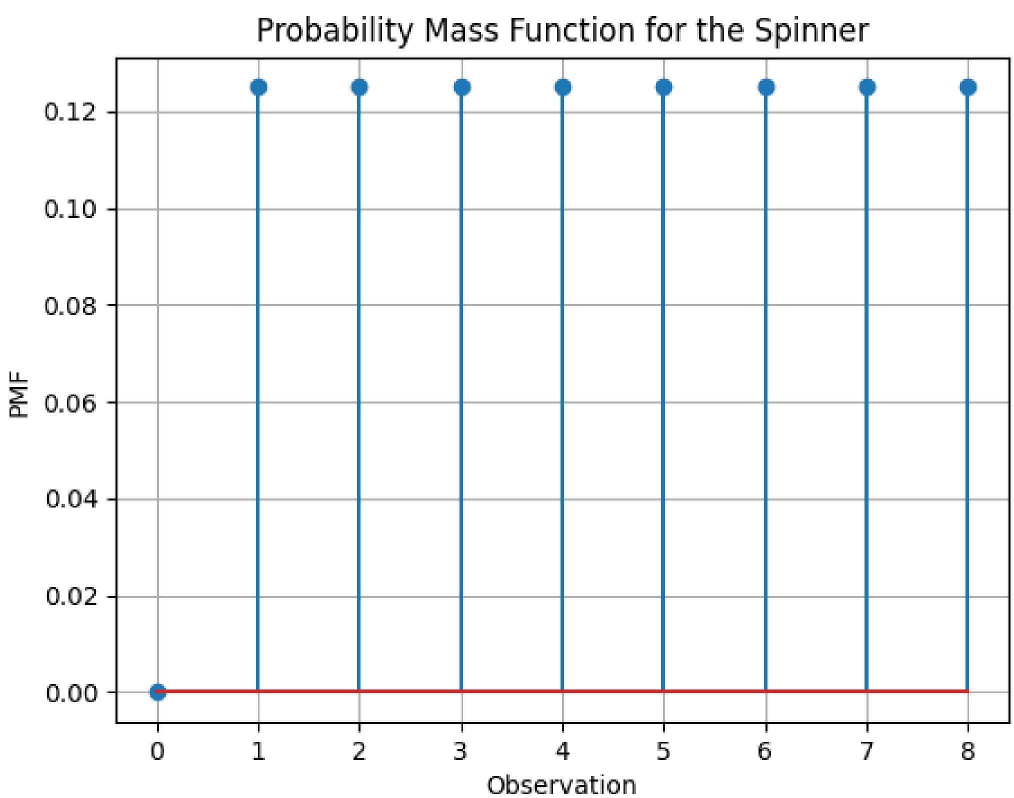

Figure 5.2: Plot of Probability Mass Function

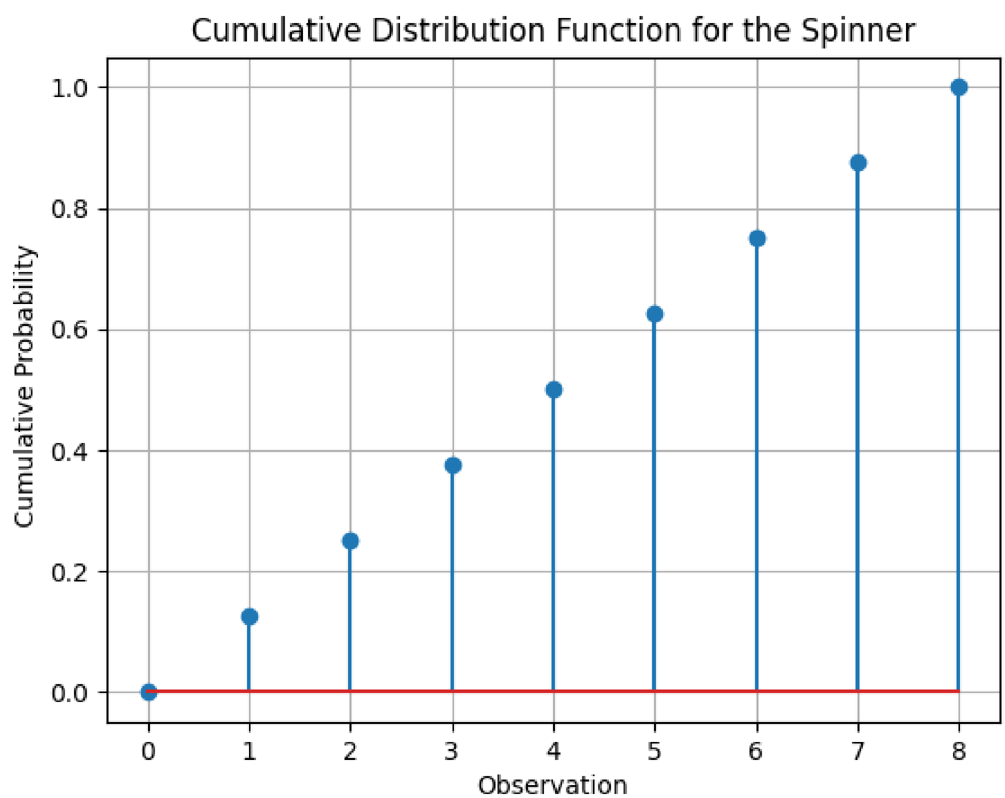

Figure 5.3: Plot of Cumulative Distribution Function

(d)

$$\Pr\left(1 \le X < 9\right) = F_X\left(8\right) - F_X\left(0\right) = 1 \tag{5.18}$$

5.2.3 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

(a) all will be blue?

(b) atleast one will be green?

Solution: See (E.1.2). In this question,

$$N = 60,\ R = 10,\ B = 20,\ G = 30,\ n = 5 \tag{5.19}$$

(a) From (E.1.2),

$$\Pr\left(0, 5, 0\right) = \frac{{}^{20}C_5}{{}^{60}C_5} \tag{5.20}$$

(b) Since

$$\Pr\left(r, b, 0\right) = \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} \tag{5.21}$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} \Pr\left(r, b, 0\right) = 1 - \sum_{r+b=n} \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} = 1 - \frac{{}^{R+B}C_n}{{}^{R+B+G}C_n} \tag{5.22}$$

from (E.2.1). Substituting numerical values, the desired probability is

$$1 - \frac{{}^{30}C_5}{{}^{60}C_5} \tag{5.23}$$

5.2.4 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution: Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \tag{5.24}$$

in (E.1.2) the desired probability is

$$\Pr(3, 0, 0) = \frac{{}^{12}C_3}{{}^{15}C_3} = \frac{44}{91} \tag{5.25}$$

5.3. Uniform

5.3.1 A die is thrown, find the probability of following events:

(a) A prime number will appear

(b) A number greater than or equal to 3 will appear

(c) A number less than or equal to one will appear

(d) A number more than 6 will appear

(e) A number less than 6 will appear

Solution: The CDF of the random variable X representing the roll of a dice, is available in (C.3.3.1).

(a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr(X \in \{2,3,5\}) = p_X(2) + p_X(3) + p_X(5) \tag{5.26}$$
$$= \frac{1}{2} \tag{5.27}$$

(b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr(X \geq 3) = 1 - \Pr(X \leq 2) \tag{5.28}$$
$$= 1 - F_X(2) \tag{5.29}$$
$$= \frac{2}{3} \tag{5.30}$$

(c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr(X \leq 1) = F_X(1) \tag{5.31}$$
$$= \frac{1}{6} \tag{5.32}$$

(d) The probability that a number greater than 6 will appear is given by

$$\Pr(X > 6) = 1 - \Pr(X \leq 6) \tag{5.33}$$
$$= 1 - F_X(6) \tag{5.34}$$
$$= 0 \tag{5.35}$$

(e) The probability that a number less than 6 will appear is given by

$$\Pr(X < 6) = \Pr(X \leq 5) \tag{5.36}$$
$$= F_X(5) \tag{5.37}$$
$$= \frac{5}{6} \tag{5.38}$$

5.4. Binomial

5.4.1 A die is thrown twice. What is the probability that

(a) 5 will not come up either time?

(b) 5 will come up at least once?

Solution: 5.2 From Table 5.2, the PMF of X is

Parameters	**Value**	**Description**
n	2	Number of trials in an Experiment
p	1/6	Probability of Success
q	5/6	Probability of Failure

Table 5.2:

$$\Pr(X = k) = {}^nC_k p^k q^{n-k} \tag{5.39}$$
$$= {}^2C_k \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{2-k} \qquad \forall k = 0, 1, 2 \tag{5.40}$$

and the CDF is

$$F_X(k) = \Pr(X \leq k) = \sum_{i=0}^{k} {}^nC_i p^i q^{n-i} \tag{5.41}$$

(a)

$$\Pr(X=0) = {}^2C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^2 = \frac{25}{36} \tag{5.42}$$

(b)

$$\Pr(X \geq 1) = 1 - \Pr(X \leq 0) = 1 - F_X(0) \tag{5.43}$$

$$= 1 - \frac{25}{36} = \frac{11}{36} \tag{5.44}$$

5.4.2 Three coins are tossed once. Find the probability of getting

(a) 3 heads

(b) 2 heads

(c) atleast 2 heads

(d) atmost 2 heads

(e) no head

(f) 3 tails

(g) exactly two tails

(h) no tail

(i) atmost two tails

Solution: Let the random variable X denote one single coin toss, where obtaining a head is considered a success. Then,

$$X \sim \text{Ber}(p) \tag{5.45}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n tosses. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{5.46}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \mathrm{Bin}\,(n, p) \tag{5.47}$$

The cdf of Y is given by

$$F_Y\,(k) = \Pr\,(Y \leq k) \tag{5.48}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ i \end{pmatrix} p^i\,(1-p)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{5.49}$$

In this case,

$$p = \frac{1}{2},\ n = 3 \tag{5.50}$$

(a) We require $\Pr\,(Y = 3)$. Thus, from (5.47),

$$\Pr\,(Y = 3) = \begin{pmatrix} n \\ 3 \end{pmatrix} p^3\,(1-p)^{n-3} \tag{5.51}$$

$$= \frac{1}{8} \tag{5.52}$$

(b) We require $\Pr(Y = 2)$. Thus, from (5.47),

$$\Pr(Y = 2) = \binom{n}{2} p^2 (1-p)^{n-2} \tag{5.53}$$

$$= \frac{3}{8} \tag{5.54}$$

(c) We require $\Pr(Y \geq 2)$. Since $n = 3$ in (5.49),

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{5.55}$$

$$= F_Y(3) - F_Y(1) \tag{5.56}$$

$$= \sum_{k=2}^{3} \binom{n}{k} p^k (1-p)^{n-k} \tag{5.57}$$

$$= \frac{1}{2} \tag{5.58}$$

(d) We require $\Pr(Y \leq 2)$. Thus, from (5.49),

$$\Pr(Y \leq 2) = \sum_{k=0}^{2} \binom{n}{k} p^k (1-p)^{n-k} \tag{5.59}$$

$$= \frac{7}{8} \tag{5.60}$$

(e) We require $\Pr(Y = 0)$. Thus, from (5.47),

$$\Pr(Y = 0) = \binom{n}{0} p^0 (1-p)^n \tag{5.61}$$

$$= \frac{1}{8} \tag{5.62}$$

(f) Obtaining 3 tails is the same as obtaining no heads. Hence, from (5.62), we require $\Pr(Y=0)=\frac{1}{8}$.

(g) We require $\Pr(Y=1)$ (since only one head is obtained). Thus, from (5.47),

$$\Pr(Y=1)=\binom{n}{1}p^1(1-p)^{n-1} \tag{5.63}$$

$$=\frac{3}{8} \tag{5.64}$$

(h) We require $\Pr(Y=3)=\frac{1}{8}$ from (5.52).

(i) We require $\Pr(Y\geq 1)$ (since at least one head is obtained). Thus, from (5.49) and (5.62),

$$\Pr(Y\geq 1)=1-\Pr(Y<1) \tag{5.65}$$

$$=1-F_Y(0) \tag{5.66}$$

$$=1-\Pr(Y=0) \tag{5.67}$$

$$=\frac{7}{8} \tag{5.68}$$

5.4.3 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game

5.4.4 A coin is tossed three times, where. Determine $\Pr(E\mid F)$ where

(a) E : head on third toss, F : heads on first two tosses

(b) E : at least two heads, F : at most two heads

(c) E : at most two tails, F : at least one tail

Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 5.3.

RV	**Values**	**Description**
X	$\{0,1,2,3\}$	Number of heads in 3 tosses
X_1	$\{0,1\}$	0: Heads , 1: Tails
X_2	$\{0,1\}$	0: Heads , 1: Tails
X_3	$\{0,1\}$	0: Heads , 1: Tails

Table 5.3: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \tag{5.69}$$

The PMF of the random variable X is given by,

$$P_X(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \tag{5.70}$$

Here we have

$$N = 3, \, p = \frac{1}{2} \tag{5.71}$$

The CDF of the random variable X is given by,

$$F_X(n) = \Pr(X \le n) = \sum_{i=0}^{n} {}^N C_i \, p^i \, (1-p)^{N-i} \tag{5.72}$$

(a) The events E, F can be described by the RV as

$$E : X_3 = 0 \tag{5.73}$$

$$F : X_1 + X_2 = 0 \tag{5.74}$$

Y is another random variable which represents the number of heads in first two tosses.

$$Y = X_1 + X_2 \tag{5.75}$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \tag{5.76}$$

Here we have

$$N = 2, \, p = \frac{1}{2} \tag{5.77}$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \tag{5.78}$$

$$\triangleq X_1 + X_2 + X_3 = 0 \tag{5.79}$$

$$\implies X = 0 \tag{5.80}$$

The required probability is given by,

$$\Pr(X_3 = 0 \mid Y = 0) \tag{5.81}$$

$$= \frac{\Pr(X = 0)}{\Pr(Y = 0)} \tag{5.82}$$

$$= \frac{1}{2} \tag{5.83}$$

(b) The events E, F, F' can be described by the RV as

$$E : X \le 1 \tag{5.84}$$

$$F : X \ge 1 \tag{5.85}$$

$$F' : X = 0 \tag{5.86}$$

The required probability is given by,

$$= \frac{\Pr(EF)}{1 - \Pr(F')} \tag{5.87}$$

The event EF can be expressed as,

$$X \le 1 \cap X \ge 1 \tag{5.88}$$

$$\implies X = 1 \tag{5.89}$$

Hence, the required probability is given by,

$$= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \tag{5.90}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \tag{5.91}$$

$$= \frac{3}{7} \tag{5.92}$$

(c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \tag{5.93}$$

$$F' : X = 0 \tag{5.94}$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{5.95}$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \tag{5.96}$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \tag{5.97}$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3) - \Pr(\phi))}{1 - \Pr(X = 0)} \tag{5.98}$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8} - 0\right)}{1 - \frac{1}{8}} \tag{5.99}$$

$$= \frac{6}{7} \tag{5.100}$$

5.4.5 A die is tossed thrice. Find the probability of getting an odd number at least once.

Solution: The parameters for the equivalent binomial distribution is

$$p = \frac{1}{2}, \; n = 3 \tag{5.101}$$

The CDF is given by

$$F_X(k) = \sum_{0}^{k} {}^nC_k p^k (1-p)^{(n-k)} \tag{5.102}$$

and the required probability is

$$\Pr(1 \leq X \leq 3) = F_X(3) - F_X(0) = \frac{7}{8} \tag{5.103}$$

5.4.6 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

Solution: Table 5.5 summarises the given information.

Variable	**Value**	**Description**
n	$\{2, 3, 4\}$	Number of trials in 2,3,4 tosses of a coin
p	$\frac{1}{2}$	Probability of getting a head
q	$1 - p$	Probability of not getting a head
X_1	$\{0, 1, 2\}$	Number of heads in 2 tosses of a coin
X_2	$\{0, 1, 2, 3\}$	Number of tails in 3 tosses of a coin
X_3	$\{0, 1, 2, 3, 4\}$	Number of heads in 4 tosses of a coin

Table 5.5: Variable Description

(a) Number of heads in two tosses of a coin.

$$p_{X_1}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 2, n = 2 \tag{5.104}$$

(b) Number of tails in the simultaneous tosses of three coins.

$$p_{X_2}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 3, n = 3 \tag{5.105}$$

(c) Number of heads in four tosses of a coin.

$$p_{X_3}(k) = {}^nC_k p^k q^{n-k}, 0 \le k \le 4, n = 4 \tag{5.106}$$

5.4.7 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(a) number greater than 4

(b) six appears on at least one die

Solution: Let X be a random variable denoting the outcome of a die toss.

(a)

$$\Pr(X > 4) = 1 - F_X(3) = \frac{1}{3} \tag{5.107}$$

Let Y be the random variable denoting number of successes. Then,

$$Y \sim \text{Bin}(n, p) \tag{5.108}$$

where

$$n = 2, p = \frac{1}{3}. \tag{5.109}$$

Thus,

$$\therefore \Pr(Y = i) = {}^2C_i (1-p)^{2-i} p^i \tag{5.110}$$

and the desired distribution is

$$p_Y(k) = \begin{cases} \frac{4}{9}, & k = 0 \\ \frac{4}{9}, & k = 1 \\ \frac{1}{9}, & k = 2 \\ 0, & \text{otherwise} \end{cases} \tag{5.111}$$

(b) In this case, the binomial distribution has parameters

$$n = 2, p = \frac{1}{6} \tag{5.112}$$

yielding

$$p_Z(k) = \begin{cases} \frac{25}{36}, & k = 0 \\ \frac{11}{36}, & k = 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.113}$$

5.4.8 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution: The parameters of the corresponding Binomial distribution are

$$p = 0.05,\ q = 1 - p = 0.95, n = 10 \tag{5.114}$$

The CDF of is given by

$$F_X(n) = \Pr(X \leq n) \tag{5.115}$$

$$= \begin{cases} 0 & n < 0 \\ \sum_{k=0}^{n} \binom{10}{k} q^{10-k} p^k & 0 \leq n \leq 10 \\ 1 & \text{otherwise} \end{cases} \tag{5.116}$$

The desired probability is $F_X(1)$.

5.4.9 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution: A deck has 52 cards among which 13 are spades. Since we are replacing drawn cards, the probability of getting spade on any draw is

$$p = \frac{13}{52} = \frac{1}{4} \tag{5.117}$$

This is a binomial distribution where getting a card of spades is considered success. The pmf is given by

$$\Pr(X = r) = {}^nC_r p^r (1-p)^{n-r}, \quad p = \frac{1}{4}, n = 5 \tag{5.118}$$

The desired probabilities are then obtained as

(i)

$$\Pr(X=5) = {}^{5}C_{5}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{0} \tag{5.119}$$

$$= \frac{1}{1024} \approx 0.00098 \tag{5.120}$$

(ii)

$$\Pr(X=3) = {}^{5}C_{5}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{2} \tag{5.121}$$

$$= \frac{45}{512} \approx 0.08789 \tag{5.122}$$

(iii)

$$\Pr(X=0) = {}^{5}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{5} \tag{5.123}$$

$$= \frac{243}{1024} \approx 0.23730 \tag{5.124}$$

5.4.10 The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs

(a) none

(b) not more than one

(c) more than one

(d) at least one

will fuse after 150 days of use.

Solution: The binomial distribution parameters are

$$n = 5,\ p = 0.05,\ q = 1 - p = 0.95. \tag{5.125}$$

The pmf and CDF are

$$p_X(i) = {}^5C_i p^i q^{5-i} \tag{5.126}$$

$$F_X(i) = \Pr(X \le i) = \sum_{r=0}^{i} {}^5C_r p^r q^{5-r} \tag{5.127}$$

(a) Probability that none of the 5 bulbs fuses is

$$\Pr(X = 0) = F_X(0) = 0.95^5 \tag{5.128}$$

(b) Probability that not more than one bulb fuses is

$$\Pr(X \le 1) = F_X(1) = 0.9774075 \tag{5.129}$$

(c) Probability that more than one bulb will fuse will be

$$\Pr(1 < X \le 5) = F_X(5) - F_X(1) = 0.0225925 \tag{5.130}$$

(d) Probability that at least one bulb is fused is

$$\Pr(1 \le X \le 5) = F_X(5) - F_X(0) = 1 - (0.95)^5 \tag{5.131}$$

5.4.11 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution: Let X denote the number of correct answers out of 20 questions. Then X is binomial with

$$n = 20,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2} \tag{5.132}$$

The desired probability is then given by

$$\Pr(X >= 12) = 1 - F_X(11) = 0.2517 \tag{5.133}$$

5.4.12 Find the probability of getting 5 twice in 7 throws of a dice.

Solution: The Binomial r.v. parameters are

$$\therefore\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}, n = 7 \tag{5.134}$$

with pmf

$$\therefore\ \Pr(X = k) = {}^{n}C_k \times q^{n-k} \times p^k = {}^{7}C_k \times \left(\frac{5}{6}\right)^{(7-k)} \times \left(\frac{1}{6}\right)^k \tag{5.135}$$

The desired probability is

$$\Pr(X = 2) = {}^{7}C_2 \times \left(\frac{5}{6}\right)^{(7-2)} \times \left(\frac{1}{6}\right)^2 = \left(\frac{7}{12}\right) \times \left(\frac{5}{6}\right)^5 \tag{5.136}$$

5.4.13 Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution: The binomial distribution parameters are

$$n = 6,\, p = \frac{1}{6},\, q = 1 - p = \frac{5}{6} \tag{5.137}$$

with pmf

$$Pr\left(X = r\right) = {}^{n}C_{r}\left(p\right)^{r}\left(q\right)^{n-r} \tag{5.138}$$

and CDF

$$F_X(r) = \sum_{i=0}^{r} {}^{n}C_{i} p^{i} q^{n-i} \tag{5.139}$$

The desired probability is

$$F_X\left(2\right) = \frac{21875}{23328} \tag{5.140}$$

5.4.14 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution: Let

$$X = \mathrm{Bin}(n, p),\, n = 10,\, p = 0.9. \tag{5.141}$$

Then,

$$p_X\left(k\right) = {}^{n}C_{k} p^{k} q^{n-k} \tag{5.142}$$

$$F_X\left(k\right) = \Pr\left(X \le k\right) = \sum_{t=0}^{k} {}^{n}C_{t} p^{t} q^{n-t} \tag{5.143}$$

Hence, the desired probability is

$$\Pr(X \leq 6) = F_X(6) = 0.012 \tag{5.144}$$

5.4.15 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

(a) all will bear 'X' mark.

(b) not more than 2 will bear 'Y' mark.

(c) at least one ball will bear 'Y' mark.

(d) the number of balls with 'X' mark and 'Y' mark will be equal.

Solution: The given information is listed in Table 5.6 The pmf and CDF are given

Variables	Definition	values
N	Balls in the urn	25
N_X	Balls marked with X	10
N_Y	Balls marked with Y	15
n	No. Of trials	6
k	No. Of balls marked X in n trials	
p	$\Pr(X)$	0.4
q	$\Pr(Y) = 1 - \Pr(X)$	0.6

Table 5.6: Given Information

by

$$\Pr(Z = i) = {}^6C_i p^i q^{6-i} \tag{5.145}$$

$$\therefore F_Z(i) = \sum_{r=0}^{i} {}^6C_r p^r q^{6-r} \tag{5.146}$$

(a)

$$\Pr(Z = 6) = {}^{6}C_{6}\,(0.4)^{6}\,(0.6)^{0} = 0.004096 \tag{5.147}$$

(b)

$$\Pr(Z \geq 4) = 1 - F_Z(3) = 0.1792 \tag{5.148}$$

(c)

$$\Pr(Z < 6) = F_Z(5) = 0.995904 \tag{5.149}$$

(d)

$$\Pr(Z = 3) = {}^{6}C_{3}\,(0.4)^{3}\,(0.6)^{3} = 0.13824 \tag{5.150}$$

5.4.16 An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

5.4.17 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

5.4.18 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

(a) number greater than 4

(b) six appears on at least one die

5.4.19 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

5.4.20 A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

5.4.21 A coin is tossed twice, what is the probability that atleast one tail occurs?

Solution: By using binomial distribution, the desired probability is given by

$$\Pr(Y \geq 1) = \sum_{k=1}^{2} \binom{n}{k} p^k (1-p)^{n-k} = \frac{3}{4} \tag{5.151}$$

upon substituting $p = \frac{1}{2}$.

5.4.22 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

5.5. Triangular

5.5.1 Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

5.6. Miscellaneous

5.6.1 The random variable X has a probability distribution $\Pr(X)$ of the following form, where k is some number

$$\Pr(X) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \\ 0, & \text{otherwise} \end{cases} \tag{5.152}$$

(a) Determine the value of k

(b) Find $\Pr(X < 2)$,$\Pr(X \le 2)$,$\Pr(X \ge 2)$

Solution:

(a) Using the axioms of probability,

$$k + 2k + 3k = 1 \implies k = \frac{1}{6} \tag{5.153}$$

(b) The CDF is given by

$$F_X(k) = \begin{cases} 0, & x < 0 \\ \frac{1}{6}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases} \tag{5.154}$$

Thus,

i.

$$\Pr\left(X < 2\right) = F(1) = \frac{1}{2} \tag{5.155}$$

ii.

$$\Pr\left(X \leq 2\right) = F(2) = 1 \tag{5.156}$$

iii.

$$\Pr\left(X \geq 2\right) = 1 - \Pr\left(X < 2\right) = 1 - F(1) = \frac{1}{2} \tag{5.157}$$

5.6.2 State which of the following are not the probability distributions of a random variable. Give reasons for your answer

X	0	1	2
P(X)	0.4	0.4	0.2

i

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

ii

Y	-1	0	1
P(Y)	0.6	0.1	0.2

iii

Z	3	2	1	0	-1
P(Z)	0.3	0.2	0.4	0.1	-0.05

iv

5.6.3 A random variable X has the following probability distribution

Determine

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

i k

ii P($X < 3$)

iii P($X > 6$)

iv P($0 < X < 3$)

5.6.4 The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & otherwise \end{cases}$$

i Determine the value of k.

ii Find P ($X < 2$), P ($X \leq 2$), P($X \geq 2$)

5.7. Gaussian

5.7.1 There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

5.7.2 Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?

(b) only 3 cards are spades?

(c) none is a spade?

Solution:

5.7.3 In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer true; if it falls tails, he answer false. Find the probability that he answers at least 12 questions correctly.

Solution:

5.7.4 It is known that 10 % of certain articles manufactured are defective. What is the probability that in a random sample space of 12 such articles,9 are defective?

Solution: From Table 5.13,

$$p_X(9) = {}^{12}C_9 \left(\frac{1}{10}\right)^9 \left(\frac{9}{10}\right)^3 \tag{5.158}$$

$$= 22\frac{9^3}{10^{11}} \tag{5.159}$$

Parameters	Value	Description
n	12	Number of Articles
p	0.1	Probability of Defective Articles
q	0.9	Probability of Non-Defective Articles

Table 5.13:

5.7.5 The probability that a student is not a swimmer is $\frac{1}{5}$ .Then the probability that out of five students, four are swimmers

(a) ${}^5C_4 \left(\frac{4}{5}\right)^4 \frac{1}{5}$

(b) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$

(c) ${}^5C_1 \frac{1}{5} \left(\frac{4}{5}\right)^4$

(d) None of these

Solution: See Table 5.14. The pmf of X is

Parameter	Value	Description
n	5	number of students
q	$\frac{1}{5}$	probability for not a swimmer
p	$\frac{4}{5}$	probability for a swimmer
k	4	number of swimmers

Table 5.14: Given Information

$$p_X(k) = {}^nC_k p^k q^{n-k} \tag{5.160}$$

and the desired probability is

$$p_X(4) = {}^5C_4 \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right)^{5-4} \tag{5.161}$$

Hence, option 5.7.5a is correct.

5.7.6 Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Solution:

Let,

$$Z = \frac{X - \mu}{\sigma}, \ \mu = np, \ \sigma^2 = npq \tag{5.162}$$

For large n,

$$Z \sim \mathcal{N}(0, 1) \tag{5.163}$$

$$\implies \Pr(Z < -2.63) = \Pr(-Z > 2.63) \equiv \Pr(Z > -2.63) = Q(2.63) \tag{5.164}$$

5.8. Exercises

5.8.1 A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find P(G), where G is the event that a number greater than 3 occurs on a single roll of the die. **Solution:**

Chapter 6

Moments

6.1 Find the mean number of heads in three tosses of a fair coin.

Solution: Substituting $n = 3, p = \frac{1}{2}$ in (C.2.3.1), the mean is $\frac{3}{2}$.

6.2 Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

6.3 Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

6.4 Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

6.5 A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

6.6 In a meeting, 70A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).

Choose the correct answer in each of the following:

6.7 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

(a) 1

(b) 2

(c) 5

(d) $\frac{8}{3}$

6.8 Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

(a) $\frac{37}{221}$

(b) $\frac{5}{13}$

(c) $\frac{1}{13}$

(d) $\frac{2}{13}$

Chapter 7

Random Algebra

7.1. Examples

7.1 Given that a fair coin is marked 1 on one face and 6 on the other and a fair die are tossed. Find the probability that the sum turns up to be 3 and 12.

Solution: Let the random variables X, Y denote the toss of a coin and roll of a dice respectively. Since,

$$M_X(z) = \frac{z^{-1}}{2}\left(1 + z^{-5}\right), \tag{7.1}$$

$$M_Y(z) = \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \tag{7.2}$$

$$M_Z(z) = \left[\frac{z^{-1} + z^{-6}}{2}\right]\left[\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right] \tag{7.3}$$

yielding

$$p_Z(n) = \begin{cases} \frac{1}{12} & 2 \leqslant n < 7, \\ \frac{1}{6} & n = 7, \\ \frac{1}{12} & 8 \leqslant n < 13 \end{cases} \tag{7.4}$$

See Fig. 7.1. Thus,

$$\Pr(Z = 3) = \frac{1}{12}, \Pr(Z = 12) = \frac{1}{12} \tag{7.5}$$

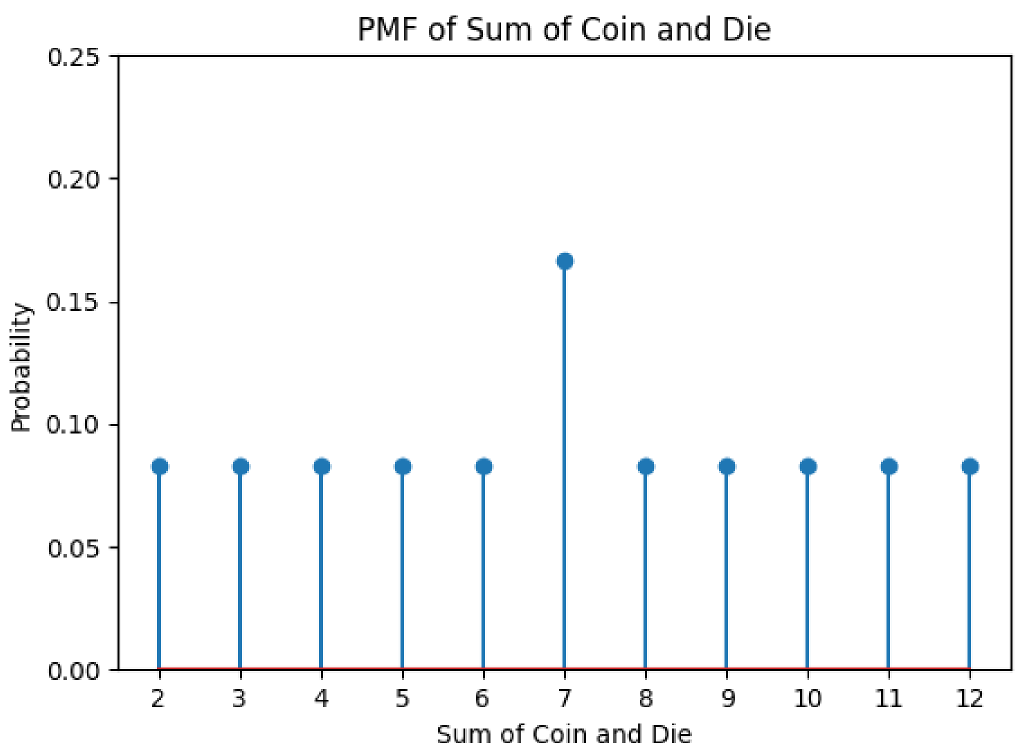

Figure 7.1: pmf of the sum when coin and die are rolled simultaneously

7.2 A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

Solution: The input parameters are listed in Table 7.1. The amount of money the

Variable	Description	Value
n	Number of tosses	4
A	Amount gained/lost	A
p	Profit when it is heads	Re 1
q	Loss when it is tails	Rs 1.5
X	Number of heads in n tosses	X

Table 7.1: Given Information

person will have after n tosses is

$$Y = (X \times 1) - ((n - X) \times 1.50) = 2.5X - 1.5n \tag{7.6}$$

The pmf and CDF of X are

$$p_X(k) = {}^nC_k(0.5)^k(0.5)^{n-k} = {}^nC_k(0.5)^n \tag{7.7}$$

$$F_X(k) = \Pr(X \le k) = \sum_{i=0}^{i=k} {}^nC_i \left(\frac{1}{2}\right)^n \tag{7.8}$$

The CDF of Y is

$$F_Y(k) = \Pr(A \le k) \tag{7.9}$$

$$= \Pr(2.5X - 1.5n \le k) \tag{7.10}$$

$$= \Pr\left(X \le \frac{k + 1.5n}{2.5}\right) \tag{7.11}$$

$$= F_X\left(\frac{k + 1.5n}{2.5}\right) = \sum_{i=0}^{i=\lfloor \frac{k+1.5n}{2.5} \rfloor} {}^nC_i \left(\frac{1}{2}\right)^n \tag{7.12}$$

from (7.8). Consequently,

$$p_Y(k) = \begin{cases} {}^nC_{\frac{k+1.5n}{2.5}} \left(\frac{1}{2}\right)^n, & \frac{k+1.5n}{2.5} \in I \text{ and } 0 \le \frac{k+1.5n}{2.5} \le n \\ 0, & \text{otherwise} \end{cases} \tag{7.13}$$

See Figs. 7.2, 7.3 and 7.4. for the distribution of Y.

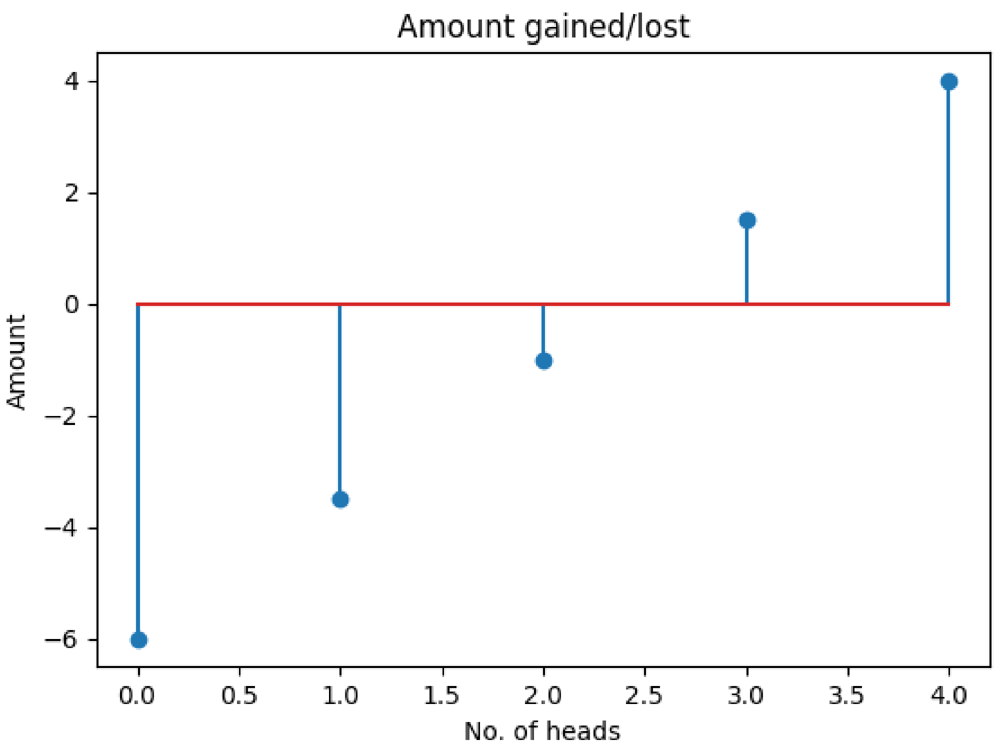

Figure 7.2: Plot of amount gained/lost

7.3 Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

7.4 A black and a red dice are rolled.

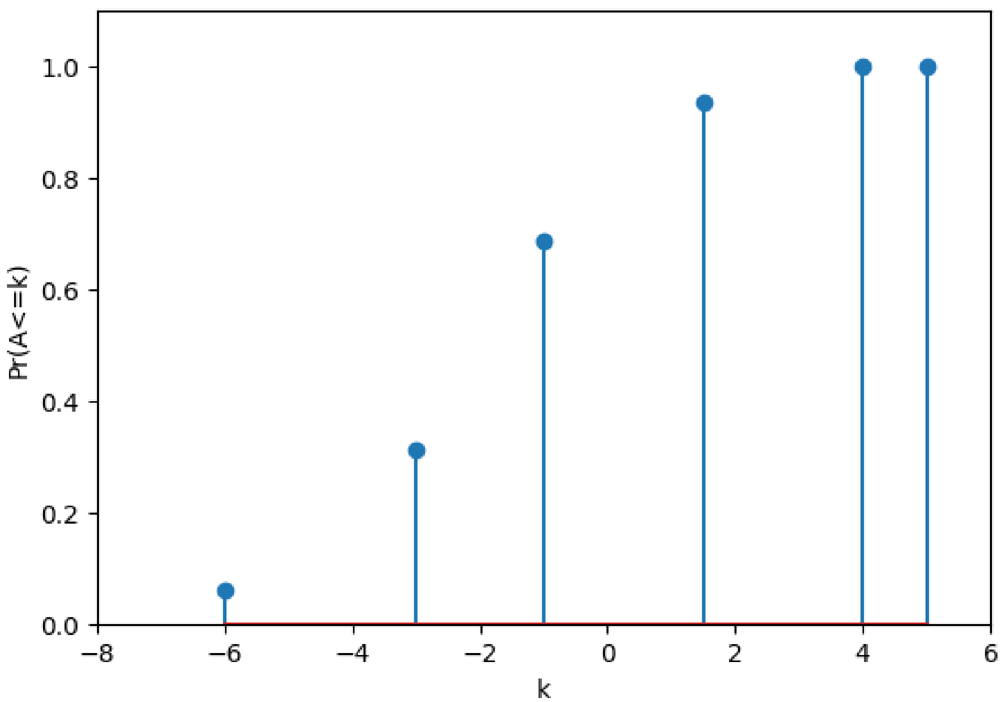

Figure 7.3: CDF of A

(a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution: Let X and Y be i.i.d, denoting the number which comes up on black and

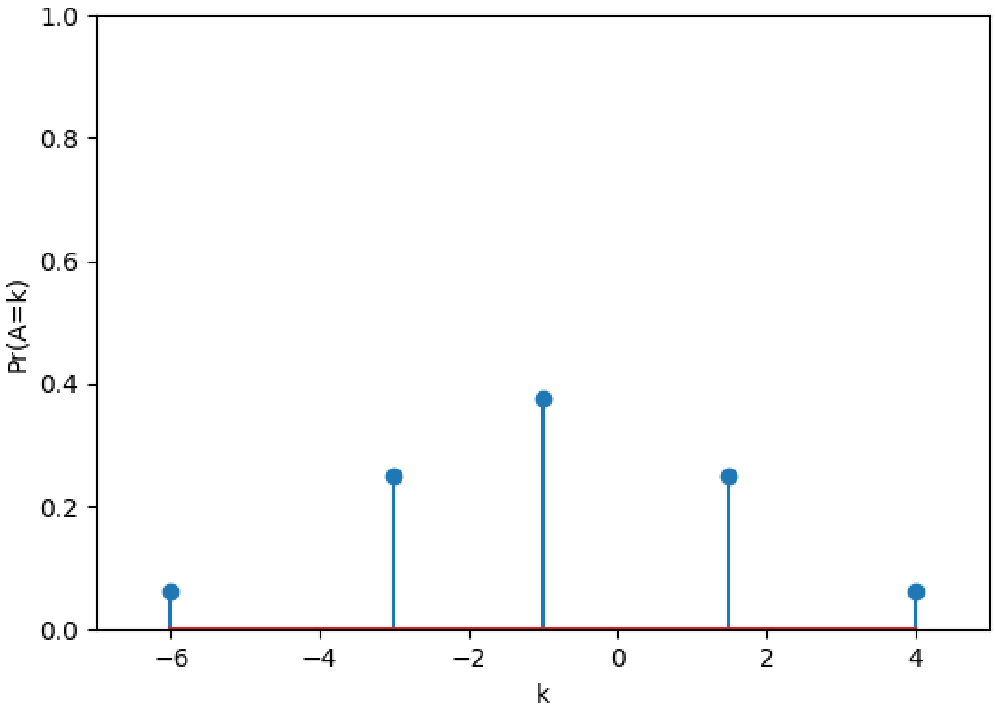

Figure 7.4: PMF of A

red die respectively.

$$F_X(i) = F_Y(i) = \begin{cases} 0 & i < 1 \\ \frac{i}{6} & 0 < i \leq 6 \\ 1 & i > 6 \end{cases} \tag{7.14}$$

Since X and Y are independent random variables.

$$\Pr(X = k, Y = r) = \Pr(X = k)\Pr(Y = r) \tag{7.15}$$

$$\implies p_{X,Y}(k,r) = \frac{1}{36} \tag{7.16}$$

(a)

$$\Pr(X+Y>9|X=5) = \frac{\Pr(X+Y>9, X=5)}{\Pr(X=5)} \tag{7.17}$$

$$= \frac{\Pr(Y>4, X=5)}{\Pr(X=5)} = \frac{\Pr(6 \geq Y > 4)\Pr(X=5)}{\Pr(X=5)} \tag{7.18}$$

$$= 1 - F_Y(4) = \frac{1}{3} \tag{7.19}$$

(b) From Fig. 7.5,

$$X+Y=8, Y<4 = (2,6)+(5,3) \tag{7.20}$$

Thus,

$$\Pr(X+Y=8, Y<4) = p_{X,Y}(2,6) + p_{X,Y}(5,3) = \frac{2}{36} \tag{7.21}$$

$$\implies \Pr(X+Y=8|Y<4) = \frac{\Pr(X+Y=8, Y<4)}{\Pr(Y<4)} \tag{7.22}$$

$$= \frac{\frac{2}{36}}{\frac{1}{2}} = \frac{1}{9} \tag{7.23}$$

7.5 Given that 2 numbers appearing on throwing two dice are different. Find the probability of the event ‘ the sum of numbers on the dice is 4’ .

Solution: See Tables 7.2 and 7.3.

Random Variable	Description
X	Number which comes up on Die1
Y	Number which comes up on Die2

Table 7.2: Random Variables for Die Rolls

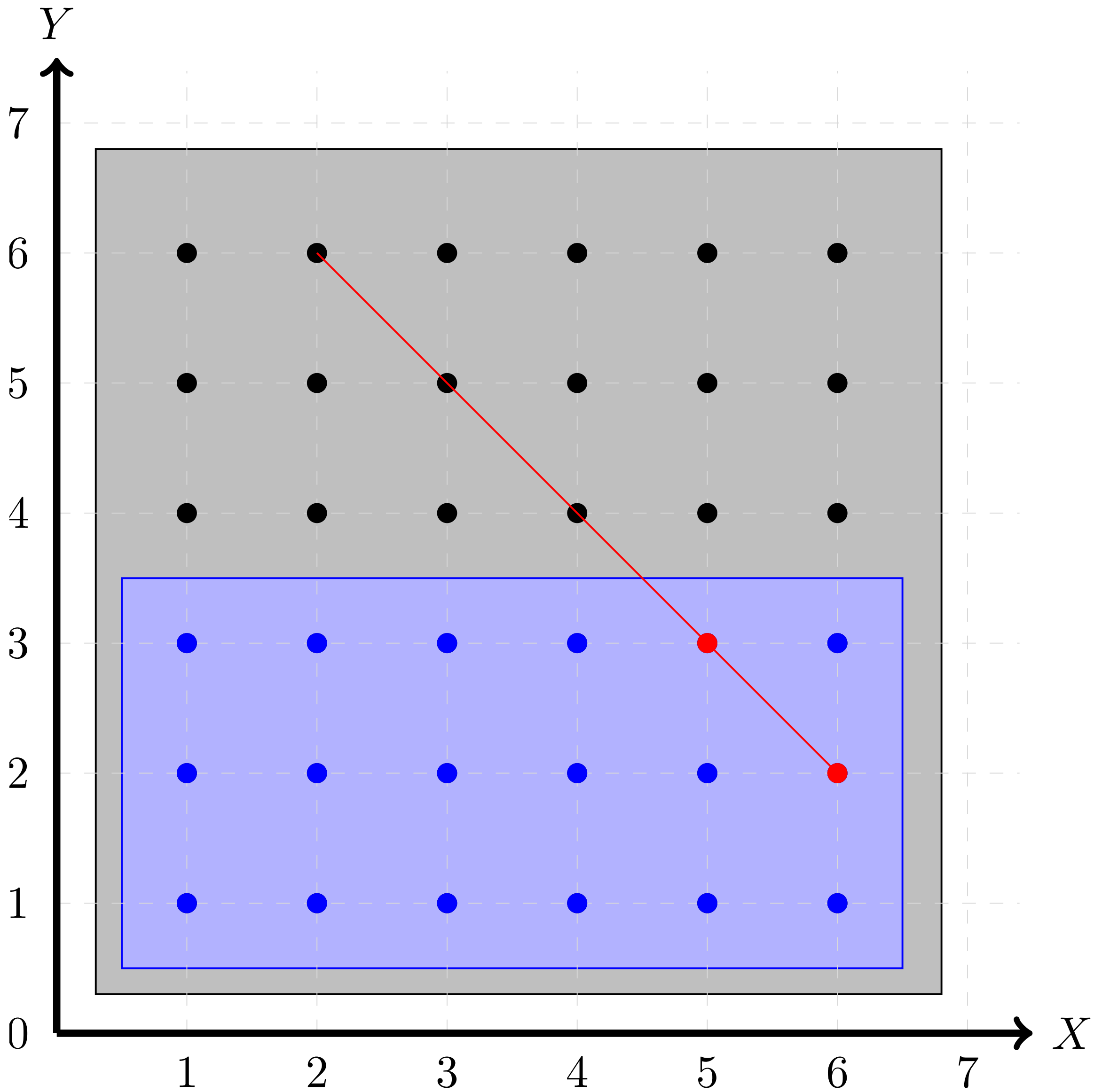

Figure 7.5: $X + Y = 8 | Y < 4$

$$\Pr\left(X + Y = 4 | X \neq Y\right) = \frac{\Pr\left(X + Y = 4, X \neq Y\right)}{\Pr\left(X \neq Y\right)} \tag{7.24}$$

Event	Description
A	$X+Y=4$
B	$X \neq Y$

Table 7.3: Events A and B

$$\Pr\left(X \neq Y\right) = 1 - \Pr\left(X = Y\right) \tag{7.25}$$
$$= 1 - \frac{6}{36} = \frac{5}{6} \tag{7.26}$$

$$\Pr\left(AB\right) = \Pr\left(A\right) - \Pr\left(AB'\right) \tag{7.27}$$

If

$$X = Y,\ X + Y = 4 \tag{7.28}$$
$$\implies X = Y = 2 \tag{7.29}$$
$$\therefore \Pr\left(AB'\right) = \Pr\left(X + Y = 4, X = Y\right) = \frac{1}{36} \tag{7.30}$$

Since,

$$\Pr\left(X+Y=n\right) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases}, \tag{7.31}$$
$$\Pr\left(A\right) = \Pr\left(X+Y=4\right) = \frac{4-1}{36} = \frac{1}{12} \tag{7.32}$$

and from (7.30)

$$\Pr(AB) = \Pr(X+Y=4, X \neq Y) = \frac{1}{12} - \frac{1}{36} \tag{7.33}$$

$$= \frac{1}{18} \tag{7.34}$$

Consequently, from (7.24) and (7.26)

$$\Pr(X+Y=4|X \neq Y) = \frac{\left(\frac{1}{18}\right)}{\left(\frac{5}{6}\right)} \tag{7.35}$$

$$\implies \Pr(X+Y=4|X \neq Y) = \frac{1}{15} \tag{7.36}$$

7.6 If each element of a 2×2 determinant is either zero or one. What is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently each value being assumed with probability $\frac{1}{2}$)

Solution: Let the matrix be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.37}$$

(a) The desired probability can be expressed as

$$\Pr(ad - bc > 0) = \Pr\left(a > \frac{bc}{d}\right) = 1 - \Pr\left(a \leq \frac{bc}{d}\right) \tag{7.38}$$

$$= 1 - F_A\left(\frac{bc}{d}\right) \tag{7.39}$$

(b) Since

$$F_A(x) = \begin{cases} 0 & x = 0, \\ \frac{1}{2} & 0 \leqslant x < 1, \\ 1 & 1 \leqslant x < \infty \end{cases}, \tag{7.40}$$

$$E_d\left(F_A\left(\frac{bc}{d}\right)\right) = \frac{1}{2}F_A(bc) + \frac{1}{2}F_A(\infty) = \frac{1}{2}F_A(bc) + \frac{1}{2} \tag{7.41}$$

(c) and

$$E_b\left(\frac{1}{2}F_A(bc) + \frac{1}{2}\right) = \frac{1}{2}E_b(F_A(bc)) + \frac{1}{2} = \frac{1}{2}\left(\frac{1}{2}F_A(0) + \frac{1}{2}F_A(c)\right) \tag{7.42}$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{4}F_A(c) = \frac{5}{8} + \frac{1}{4}F_A(c) \tag{7.43}$$

(d) yielding

$$E_c\left(\frac{5}{8} + \frac{1}{4}F_A(c)\right) = \frac{5}{8} + \frac{1}{4}E_c(F_A(c)) \tag{7.44}$$

$$= \frac{5}{8} + \frac{1}{4}\left(\frac{1}{2}F_A(0) + \frac{1}{2}F_A(1)\right) \tag{7.45}$$

$$= \frac{5}{8} + \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) = \frac{13}{16} \tag{7.46}$$

(e) Thus, the required probability is

$$\Pr\left(a > \frac{bc}{d}\right) = 1 - E_b, c, d\left(F_A\left(\frac{bc}{d}\right)\right) = 1 - \frac{13}{16} = \frac{3}{16} \tag{7.47}$$

7.2. Exercises

7.2.1 Two dice are thrown together. Find the probability that the product of the numbers on the top of the dice is

(a) 6

(b) 12

(c) 7

Solution:

Let X, Y be the outcome of the two dice and

$$Z = XY \tag{7.48}$$

$$p_X(k) = \begin{cases} 0 & k < 1 \\ \frac{1}{6} & 1 \le k \le 6 \\ 0 & k > 6 \end{cases} \tag{7.49}$$

$$F_X(k) = \begin{cases} 0 & k < 1 \\ \frac{k}{6} & 1 \le k \le 6 \\ 1 & k > 6 \end{cases} \tag{7.50}$$

$$F_Z(n) = E\left[F_Y\left(\frac{n}{k}\middle|k\right)\right] = \frac{1}{6}\sum_{k=1}^{6} F_Y\left(\frac{n}{k}\middle|k\right) \tag{7.51}$$

and

$$F\left(\frac{n}{k}\right) = \begin{cases} 1 & k < \frac{n}{6} \\ \frac{\left[\frac{n}{k}\right]}{6} & k \geq \frac{n}{6} \cap \frac{n}{k} \notin I \\ \frac{\left(\frac{n}{k}\right)}{6} & k \geq \frac{n}{6} \cap \frac{n}{k} \in I \end{cases} \tag{7.52}$$

Thus,

$$F_Z(n) = \frac{1}{6}\left[\sum_{k=1}^{\left[\frac{n}{6}\right]} F_Y\left(\frac{n}{k}\right) + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{7.53}$$

$$= \frac{1}{6}\left[\sum_{k=1}^{\left[\frac{n}{6}\right]} 1 + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{7.54}$$

$$= \frac{1}{6}\left[\left[\frac{n}{6}\right] + \sum_{k=\left[\frac{n}{6}\right]+1}^{6} F_Y\left(\frac{n}{k}\right)\right] \tag{7.55}$$

and

$$p_Z(n) = F_Z(n) - F_Z(n-1) \tag{7.56}$$

See Fig. 7.6 and Fig. 7.7 for the plots.

From (7.55),

(a)

$$p_Z(6) = F_Z(6) - F_Z(5) = \frac{1}{9} \tag{7.57}$$

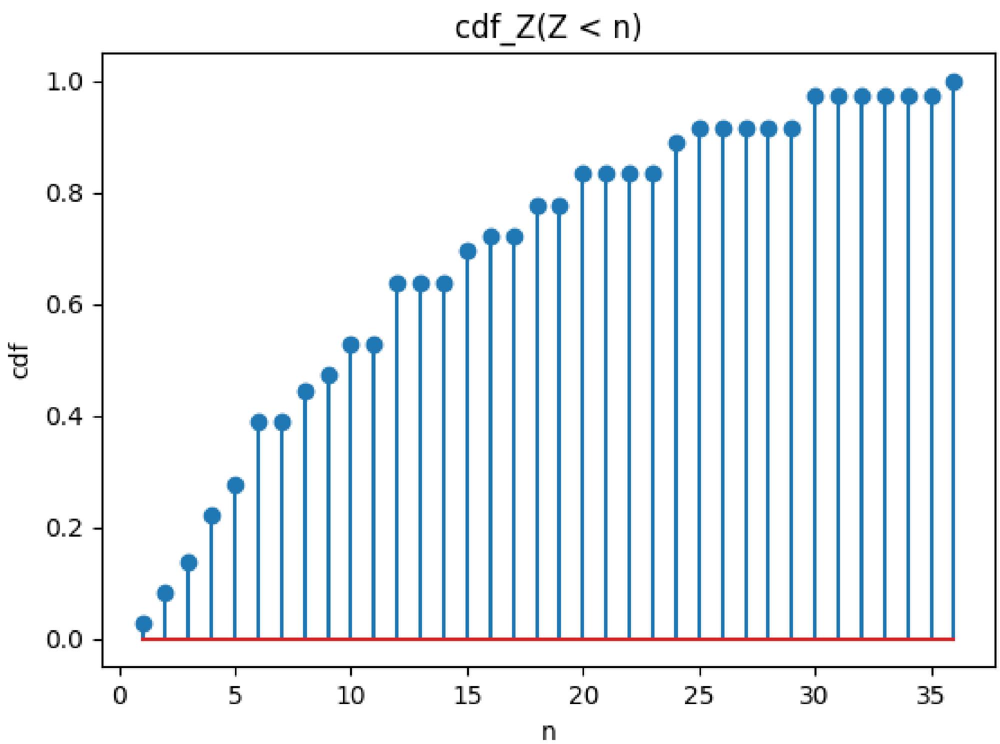

Figure 7.6: Plot of cummulative Distribution function

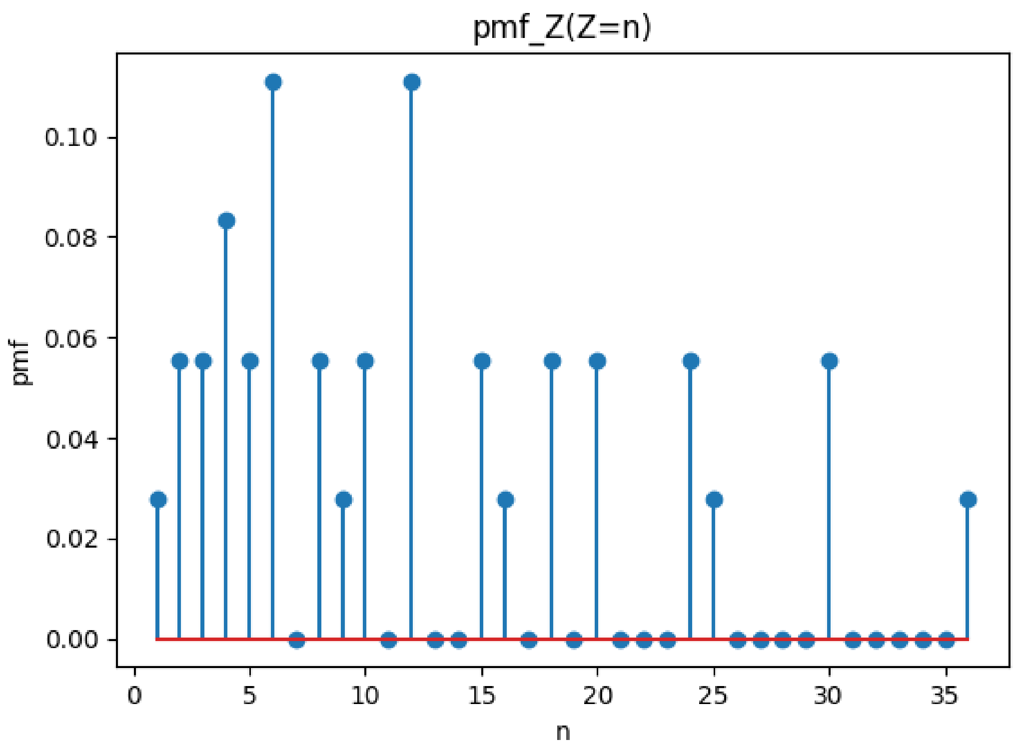

Figure 7.7: Plot of Probability Mass Function

(b)

$$p_Z(12) = F_Z(12) - F_Z(11) = \frac{1}{9} \tag{7.58}$$

(c)

$$p_Z(7) = 0 \tag{7.59}$$

Chapter 8

Markov Chain

8.1 Consider the experiment of throwing a die.

- If a multiple of 3 comes up, throw the die again
- If any other number comes, toss a coin.

Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

(a) See Table 8.1 and Fig. 8.1.

i	**State** (e_i)
0	$Y = 3$ OR $Y = 6$
1	$\sum (Y = k)\,; k \in \{1, 2, 4, 5\}$
2	Obtaining heads from coin toss
3	Obtaining tails from coin toss

Table 8.1: States in Markov Chain

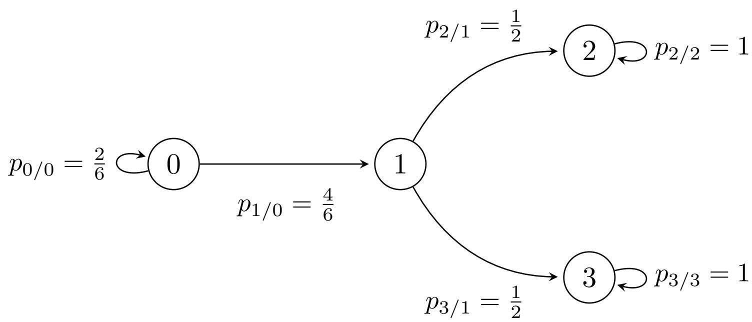

Figure 8.1: Graph of Markov Chain

(b) The state vector is,

$$\mathbf{Q}_n = \begin{pmatrix} p_0^{(n)} \\ p_1^{(n)} \\ p_2^{(n)} \\ p_3^{(n)} \end{pmatrix} \tag{8.1}$$

The probabilities after one step in time are

$$p_0^{(n+1)} = \frac{2}{6} \times p_0^{(n)} \tag{8.2}$$
$$p_1^{(n+1)} = \frac{4}{6} \times p_0^{(n)} \tag{8.3}$$
$$p_2^{(n+1)} = \frac{1}{2} \times p_1^{(n)} + 1 \times p_2^{(n)} \tag{8.4}$$
$$p_3^{(n+1)} = \frac{1}{2} \times p_1^{(n)} + 1 \times p_3^{(n)} \tag{8.5}$$

(c) The previous equations can be summarized as

$$\mathbf{Q}_{n+1} = \mathbf{P}\mathbf{Q}_n \tag{8.6}$$

Where $\mathbf{P}$ is the transition probability matrix. Its elements are values of $p_{i|j}$

$$\mathbf{P} = \begin{pmatrix} 2/6 & 0 & 0 & 0 \\ 4/6 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{pmatrix} \tag{8.7}$$

(d) The given condition is that ‘3 occurs at least once’. Let the first occurrence of 3 be the initial state $\mathbf{Q}_0$.

$$\mathbf{Q}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8.8}$$

Using (8.6), further states can be generated.

$$\mathbf{Q}_1 = \mathbf{P}\mathbf{Q}_0 = \begin{pmatrix} \frac{2}{6} \\ \frac{4}{6} \\ 0 \\ 0 \end{pmatrix} \tag{8.9}$$

$$\mathbf{Q}_2 = \mathbf{P}\mathbf{Q}_1 = \mathbf{P}^2\mathbf{Q}_0 = \begin{pmatrix} \frac{4}{9} \\ \frac{8}{9} \\ \frac{5}{24} \\ \frac{5}{12} \end{pmatrix} \tag{8.10}$$

$$\vdots \tag{8.11}$$

$$\mathbf{Q}_n = \mathbf{P}^n\mathbf{Q}_0 \tag{8.12}$$

(e) Now to find the eigen values,

$$\left|\mathbf{P} - \lambda\mathbf{I}\right| = 0 \tag{8.13}$$

$$\implies \begin{pmatrix} 2/6-\lambda & 0 & 0 & 0 \\ 4/6 & -\lambda & 0 & 0 \\ 0 & 1/2 & 1-\lambda & 0 \\ 0 & 1/2 & 0 & 1-\lambda \end{pmatrix} = 0 \tag{8.14}$$

$$\implies \lambda\left(\frac{2}{6} - \lambda\right)\left(1 - \lambda^2\right) = 0 \tag{8.15}$$

$$\text{or, } \lambda = \frac{2}{6}, 0, 1, 1 \tag{8.16}$$

The corresponding eigenvectors are

i. $\lambda = \frac{2}{6}$

$$\mathbf{X} = \begin{pmatrix} \frac{-2}{3} \\ \frac{-4}{3} \\ 1 \\ 1 \end{pmatrix} \tag{8.17}$$

ii. $\lambda = 0$

$$\mathbf{X} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \tag{8.18}$$

iii. $\lambda = 1$

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.19}$$

$$\tag{8.20}$$

resulting in the eigenvector matrix

$$\mathbf{S} = \begin{pmatrix} -2/3 & 0 & 0 & 0 \\ -4/3 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{8.21}$$

Thus,

$$\mathbf{P} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} \tag{8.22}$$

Where $\mathbf{D}$ is eigenvalue matrix

$$\mathbf{D} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.23}$$

(f)

$$\mathbf{P}^n = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\dots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) \tag{8.24}$$

$$\implies = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{8.25}$$

$$\implies \lim_{n\to\infty} \mathbf{P}^n = \lim_{n\to\infty} \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \tag{8.26}$$

From (8.6),

$$\mathbf{Q}_n = \mathbf{P}^n\mathbf{Q}_0 \tag{8.27}$$

and Now,

$$\lim_{\mathbf{n}\to\infty} \mathbf{D^n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.28}$$

$$\implies \lim_{\mathbf{n}\to\infty} \mathbf{SD^nS^{-1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{P^n} \tag{8.29}$$

$$\implies \mathbf{Q}_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{Q}_0 \tag{8.30}$$

$$\implies \mathbf{Q}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8.31}$$

(g) Probability of the coin showing tails, given that at least one die shows a 3,

$$\lim_{n\to\infty} p_3^{(n)} = 0 \tag{8.32}$$

Chapter 9

Continuous Distributions

9.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

9.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Execute the following C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Gaussian random numbers
```

```
gaussian("gau.dat", 1000000);


//Mean of uniform
//printf("%lf",mean("uni.dat"));
return 0;
}
```

```
//Function declaration
void uniform(char *str, int len);
void gaussian(char *str, int len);
double mean(char *str);
//End function declaration




//Defining the function for generating uniform random numbers
void uniform(char *str, int len)
{
int i;
FILE *fp;


fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
```

```
fprintf(fp,"%lf\n",(double)rand()/RAND_MAX);
}
fclose(fp);


}
//End function for generating uniform random numbers


//Defining the function for calculating the mean of random numbers
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;


fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
```

```
return temp;

}
//End function for calculating the mean of random numbers

//Defining the function for generating Gaussian random numbers
void gaussian(char *str, int len)
{
int i,j;
double temp;
FILE *fp;

fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
temp = 0;
for (j = 0; j < 12; j++)
{
temp += (double)rand()/RAND_MAX;
}
temp-=6;
fprintf(fp,"%lf\n",temp);
}
fclose(fp);
```

```
}
//End function for generating Gaussian random numbers
```

9.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{9.1}$$

Solution: The following code plots Fig. 9.1

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if




x = np.linspace(-4,4,30)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
#randvar = np.random.normal(0,1,simlen)
```

```
randvar = np.loadtxt('uni.dat',dtype='double')
#randvar = np.loadtxt('gau.dat',dtype='double')
for i in range(0,30):
        err_ind = np.nonzero(randvar < x[i]) #checking probability condition
        err_n = np.size(err_ind) #computing the probability
        err.append(err_n/simlen) #storing the probability values in a list


plt.plot(x.T,err)#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x$')
plt.ylabel('$F_X(x)$')

#if using termux
plt.savefig('../figs/uni_cdf.pdf')
plt.savefig('../figs/uni_cdf.eps')
subprocess.run(shlex.split("termux-open ../figs/uni_cdf.pdf"))
#if using termux
#plt.savefig('../figs/gauss_cdf.pdf')
#plt.savefig('../figs/gauss_cdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/gauss_cdf.pdf"))
#else
#plt.show() #opening the plot window
```

9.1.3 Find a theoretical expression for $F_U(x)$.

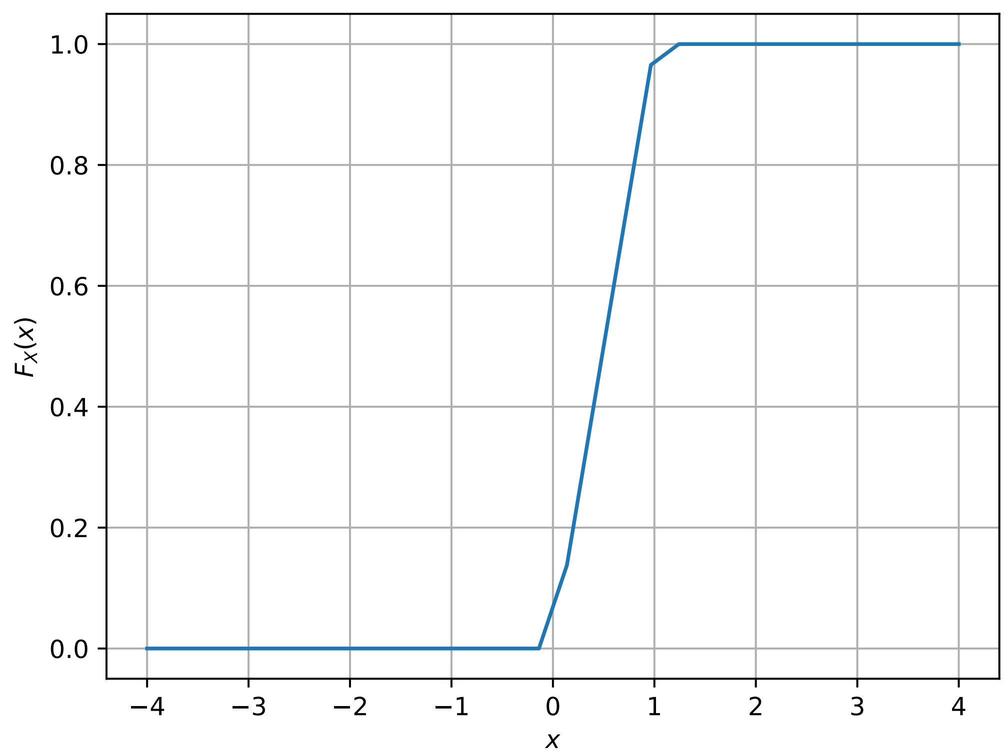

Figure 9.1: The CDF of U

9.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{9.2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{9.3}$$

Write a C program to find the mean and variance of U.

9.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{9.4}$$

9.2. Central Limit Theorem

9.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{9.5}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

9.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 9.2

9.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

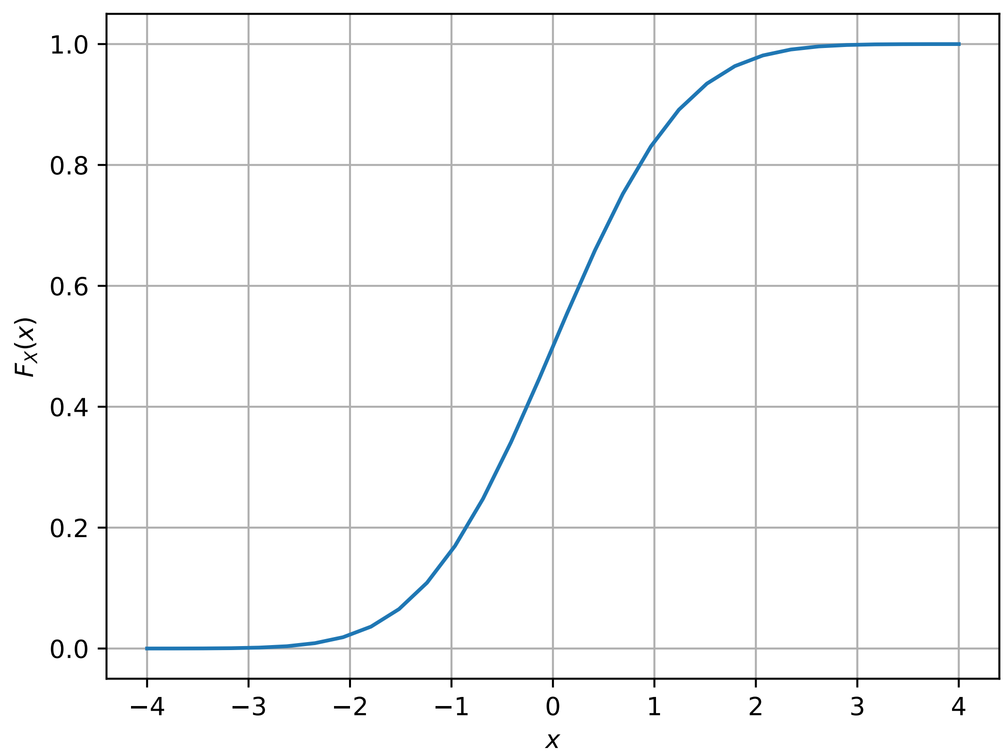

Figure 9.2: The CDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{9.6}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 9.3 using the code below

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import mpmath as mp
import scipy
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if



maxrange=50
maxlim=6.0
x = np.linspace(-maxlim,maxlim,maxrange)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
pdf = [] #declaring pdf list
h = 2*maxlim/(maxrange-1);
#randvar = np.random.normal(0,1,simlen)
```

```
#randvar = np.loadtxt('uni.dat',dtype='double')
randvar = np.loadtxt('gau.dat',dtype='double')

for i in range(0,maxrange):
	err_ind = np.nonzero(randvar < x[i]) #checking probability condition
	err_n = np.size(err_ind) #computing the probability
	err.append(err_n/simlen) #storing the probability values in a list



for i in range(0,maxrange-1):
	test = (err[i+1]-err[i])/(x[i+1]-x[i])
	pdf.append(test) #storing the pdf values in a list

def gauss_pdf(x):
	return 1/mp.sqrt(2*np.pi)*np.exp(-x**2/2.0)

vec_gauss_pdf = scipy.vectorize(gauss_pdf)

plt.plot(x[0:(maxrange-1)].T,pdf,'o')
plt.plot(x,vec_gauss_pdf(x))#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x_i$')
plt.ylabel('$p_X(x_i)$')
plt.legend(["Numerical","Theory"])
```

```
#if using termux
#plt.savefig('../figs/uni_pdf.pdf')
#plt.savefig('../figs/uni_pdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/uni_pdf.pdf"))
#if using termux
plt.savefig('../figs/gauss_pdf.pdf')
plt.savefig('../figs/gauss_pdf.eps')
subprocess.run(shlex.split("termux-open ../figs/gauss_pdf.pdf"))
#else
#plt.show() #opening the plot window
```

9.2.4 Find the mean and variance of X by writing a C program.

9.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{9.7}$$

repeat the above exercise theoretically.

9.3. From Uniform to Other

9.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{9.8}$$

and plot its CDF.

9.3.2 Find a theoretical expression for $F_V(x)$.

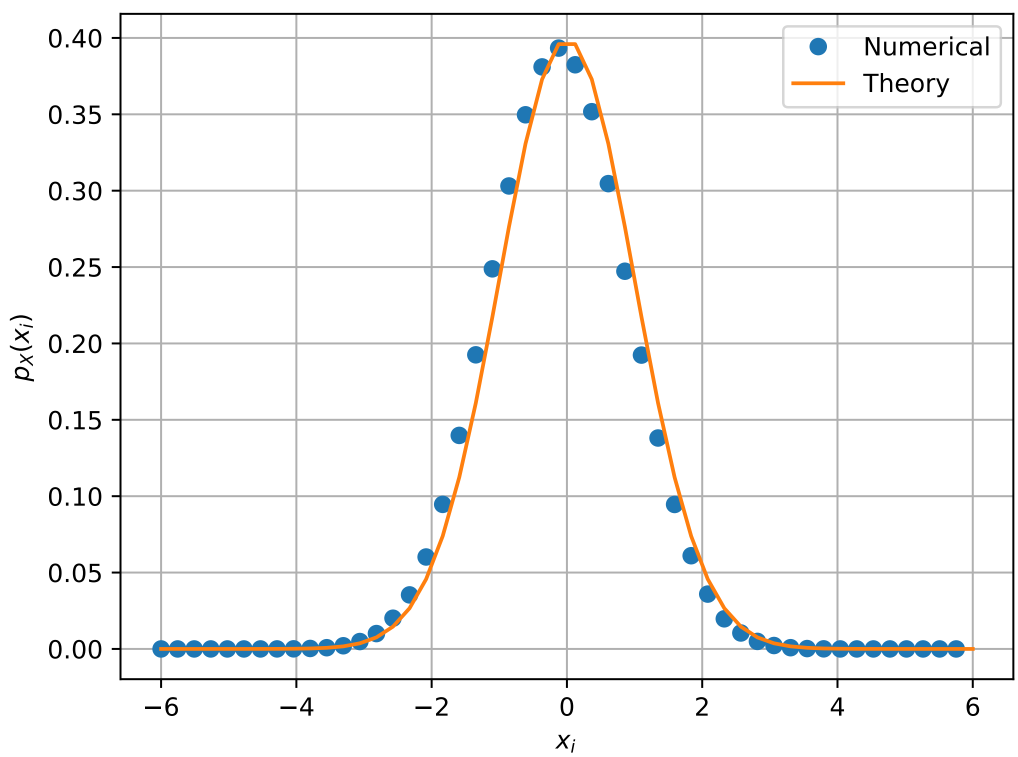

Figure 9.3: The PDF of X

9.4. Triangular Distribution

9.4.1 Generate

$$T = U_1 + U_2 \tag{9.9}$$

9.4.2 Find the CDF of T.

9.4.3 Find the PDF of T.

9.4.4 Find the theoretical expressions for the PDF and CDF of T.

9.4.5 Verify your results through a plot.

Chapter 10

Maximum Likelihood Detection: BPSK

10.1. Maximum Likelihood

10.1.1 Generate equiprobable $X \in \{1, -1\}$.

10.1.2 Generate

$$Y = AX + N, \tag{10.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

10.1.3 Plot Y using a scatter plot.

10.1.4 Guess how to estimate X from Y.

10.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{10.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{10.3}$$

10.1.6 Find P_e assuming that X has equiprobable symbols.

10.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

10.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

10.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{10.4}$$

10.1.10 Repeat the above exercise using the MAP criterion.

Chapter 11

Transformation of Random Variables

11.1. Gaussian to Other

11.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{11.1}$$

11.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{11.2}$$

find α.

11.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{11.3}$$

11.2. Conditional Probability

11.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{11.4}$$

for

$$Y = AX + N, \tag{11.5}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

11.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

11.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{11.6}$$

Find $P_e = E\left[P_e(N)\right]$.

11.2.4 Plot P_e in problems 11.2.1 and 11.2.3 on the same graph w.r.t γ. Comment.

Chapter 12

Bivariate Random Variables: FSK

12.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{12.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{12.3}$$

12.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{12.4}$$

on the same graph using a scatter plot.

12.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

12.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{12.5}$$

with respect to the SNR from 0 to 10 dB.

12.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 13

Exercises

13.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 13.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

   Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{13.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{13.2}$$

   where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 13.1 obtain a decision rule for BPSK

   Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{13.3}$$

3. Repeat the previous exercise using the MAP criterion.

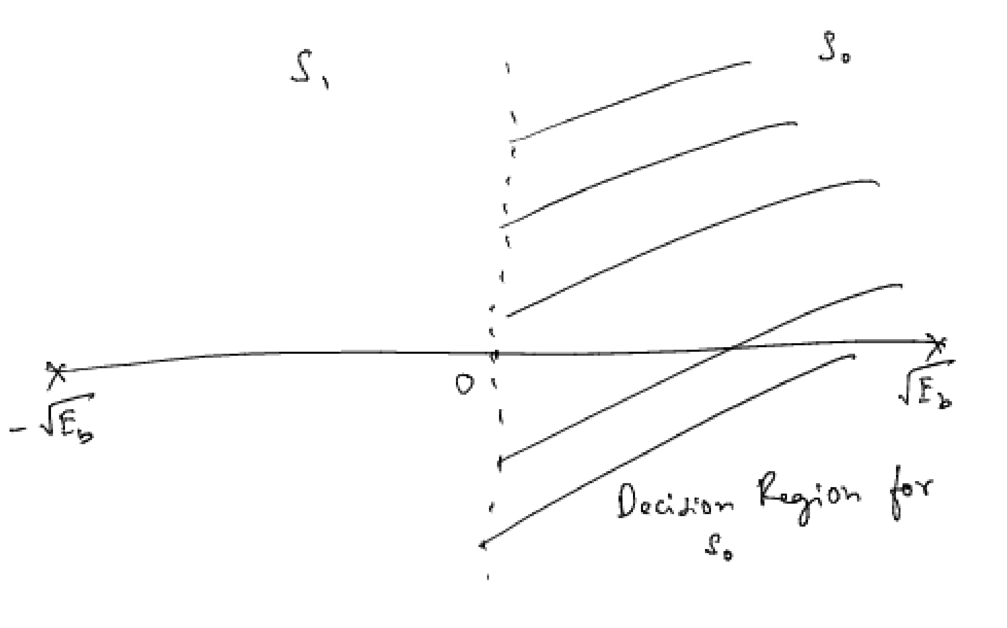

Figure 13.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0|s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{13.4}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{13.5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in

(13.5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{13.6}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{13.7}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{13.8}$$

and the complementary error function is defined as

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{13.9}$$

Show that

$$Q(x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{13.10}$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 13.2

7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{13.11}$$

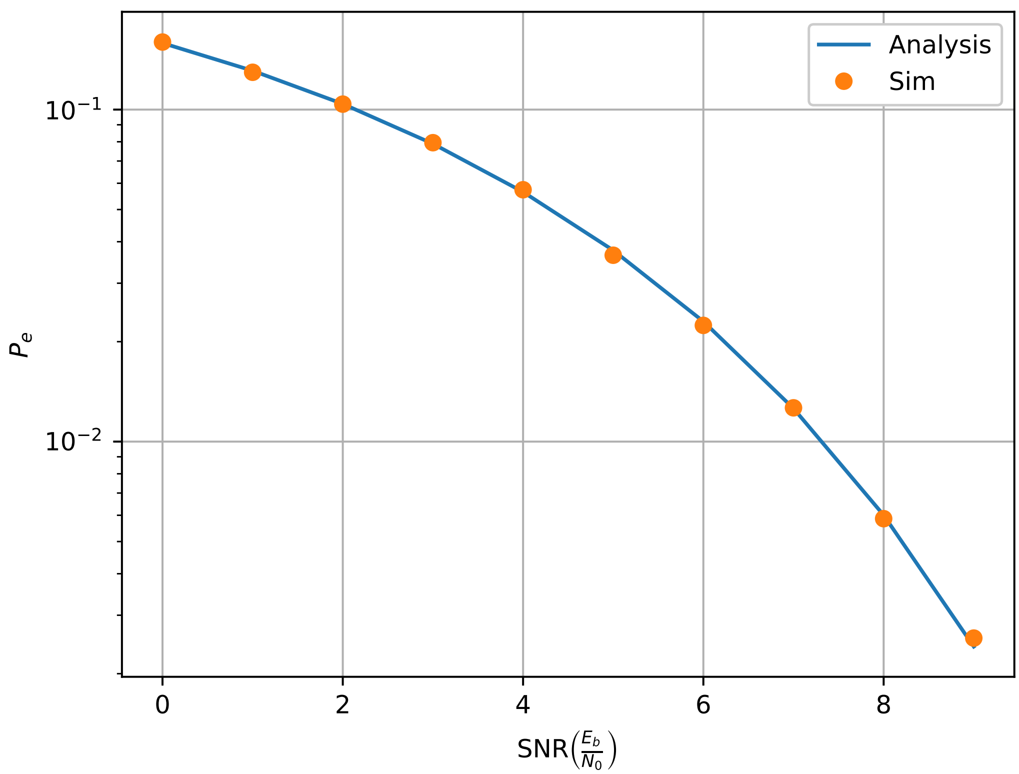

Figure 13.2:

13.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 13.3. Obtain the equations for the received symbols.

   Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{13.12}$$

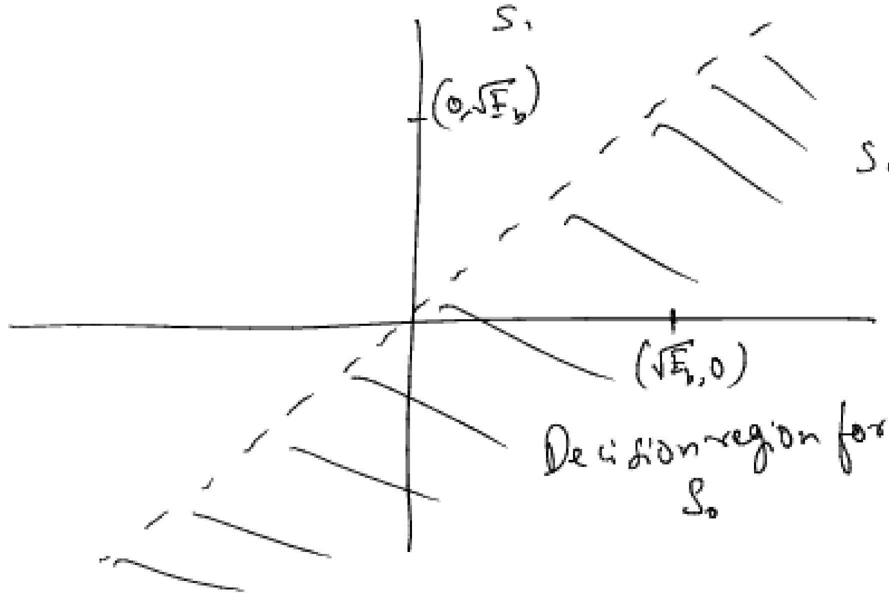

Figure 13.3:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{13.13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 13.3.

   Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{13.14}$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

13.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{13.15}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{13.16}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{13.17}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{13.18}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{13.19}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{13.20}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

13.4. M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{13.21}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{13.22}$$

$$r_2 = R\sin\theta \tag{13.23}$$

show that the joint pdf of R, θ is

$$p\left(R, \theta\right) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{13.24}$$

(b) Show that

$$\lim_{\alpha\to\infty} \int_0^{\infty} (V-\alpha)\, e^{-(V-\alpha)^2}\, dV = 0 \tag{13.25}$$

$$\lim_{\alpha\to\infty} \int_0^{\infty} e^{-(V-\alpha)^2}\, dV = \sqrt{\pi} \tag{13.26}$$

(c) Using the above, evaluate

$$\int_0^{\infty} V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}\, dV \tag{13.27}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2\theta} \cos\theta\, d\theta \tag{13.28}$$

(e) Find $P_{e|\mathbf{s}_0}$.

13.5. Noncoherent BFSK

13.5.1 Show that

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\theta}\, d\theta \tag{13.29}$$

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos(\theta-\phi)}\, d\theta \tag{13.30}$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\, d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \tag{13.31}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta \, d\theta \tag{13.32}$$

13.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{13.33}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p(\mathbf{r}|0, \phi_0)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{13.34}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{13.35}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{13.36}$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \tag{13.37}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{13.38}$$

13.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R} \cos \Theta \tag{13.39}$$

$$V = \sqrt{R} \sin \Theta \tag{13.40}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det \begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{13.41}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v) \left. J \right|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{13.42}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k}e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{13.43}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{13.44}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{13.45}$$

where ϕ is a constant.

(e) Find $\Pr(Y > Z)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{13.46}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{13.47}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{13.48}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{13.49}$$

find $\Pr(R < Z)$.

13.6. Craig's Formula and MGF

13.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{13.50}$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0,1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{13.51}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{13.52}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{13.53}$$

Show that

$$P_e = \int_0^{\infty} Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{13.54}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{13.55}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.

Appendix A

Axioms

A.1. Definitions

A.1.1

$$0 \leq \Pr(A) \leq 1 \tag{A.1.1.1}$$

A.1.2 If $AB = 0$,

$$\Pr(A + B) = \Pr(A) + \Pr(B). \tag{A.1.2.1}$$

A.1.3 If A, B are independent,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{A.1.3.1}$$

A.1.4

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{A.1.4.1}$$

A.2. Boolean Logic

A.2.1

$$A + A' = 1 \tag{A.2.1.1}$$

A.2.2

$$A'B' = (A + B)' \tag{A.2.2.1}$$

A.2.3

$$A + B = A\left(B + B'\right) + B \tag{A.2.3.1}$$

$$= B\left(A + 1\right) + AB' \tag{A.2.3.2}$$

$$= B + AB' \tag{A.2.3.3}$$

A.2.4

$$A = A\left(B + B'\right) = AB + AB' \tag{A.2.4.1}$$

and

$$\left(AB\right)\left(AB'\right) = 0, \because BB' = 0 \tag{A.2.4.2}$$

Hence, AB and AB' are mutually exclusive.

A.2.5 Let A, B and C be three events.

Let X be the event that exactly one of A, B and C occurs.

Let Y be the event that at least one of A, B or C occur.

Let Z be the event that at least two of A, B or C occur.

$$Y = A + B + C \tag{A.2.5.1}$$

Similarly,

$$Z = AB + BC + CA \tag{A.2.5.2}$$

And,

$$X = (AB'C' + A'BC' + A'B'C) \tag{A.2.5.3}$$

A.3. Properties

A.3.1

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right). \tag{A.3.1.1}$$

A.3.2

$$\Pr\left(A'B'\right) = \Pr\left(\left(A+B\right)'\right) \tag{A.3.2.1}$$

$$= 1 - \Pr\left(A+B\right) \tag{A.3.2.2}$$

A.3.3

$$\Pr\left(A+B\right) = \Pr\left(B + AB'\right) \tag{A.3.3.1}$$

$$= \Pr\left(B\right) + \Pr\left(AB'\right) \tag{A.3.3.2}$$

$$\because B\left(AB'\right) = 0 \tag{A.3.3.3}$$

A.3.4

$$\Pr\left(A\right) = \Pr\left(AB\right) + \Pr\left(AB'\right) \tag{A.3.4.1}$$

$$\implies \Pr\left(AB'\right) = \Pr\left(A\right) - \Pr\left(AB\right) \tag{A.3.4.2}$$

A.3.5 Substituting (A.3.4.2) in (A.3.3.2),

$$\Pr\left(A+B\right) = \Pr\left(A\right) + \Pr\left(B\right) - \Pr\left(AB\right) \tag{A.3.5.1}$$

Appendix B

Z-transform

B.1 The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{B.1.1}$$

B.2 If X_1 and X_2 are independent, the MGF of

$$X = X_1 + X_2 \tag{B.2.1}$$

is given by

$$M_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{B.2.2}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

B.3 Let X_i be independent. For

$$X = X_1 + X_2 + ... + X_n, \tag{B.3.1}$$

$$M_X(z) = \prod_{i=1}^{n} M_{X_i}(z) \tag{B.3.2}$$

B.4 The nth moment of X can be expressed as

$$E\left[X^n\right] = \frac{d^n M_X(z^{-1})}{dz^n}|_{z=1} \tag{B.4.1}$$

Appendix C

Distributions

C.1. Bernoulli

C.1.1. The pmf of a Bernoulli distribution is defined as

$$p_X(k) = \begin{cases} q = 1 - p & k = 0 \\ p & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{C.1.1.1}$$

C.1.2. For a Bernoulli random variable X with success probability p,

$$M_X(z) = q + pz^{-1} \tag{C.1.2.1}$$

C.1.3. The mean of the Bernoulli distribution is

$$E(X) = p \tag{C.1.3.1}$$

C.1.4. The following code simulates 100 coin tosses

```
#Code by GVV Sharma
```

```
#November 18, 2020
#Released under GNU/GPL
#Given a Bernoulli probability and
#number of samples, the code generates the event data

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli

#100 samples
simlen=int(1e2)

#Probability of the event
prob = 0.5

#Generating sample date using Bernoulli r.v.
data_bern = bernoulli.rvs(size=simlen,p=prob)
#Calculating the number of favourable outcomes
err_ind = np.nonzero(data_bern == 1)
#calculating the probability
err_n = np.size(err_ind)/simlen

#Theory vs simulation
print(err_n,prob)
print(data_bern)
```

C.2. Binomial Distribution

C.2.1. The Binomial distribution is defined as

$$X = X_1 + X_2 + ... + X_n, \tag{C.2.1.1}$$

Where X_i are i.i.d bernoulli.

C.2.2. For a Binomial random variable X with parameters n, p,

$$M_X(z) = \left(q + pz^{-1}\right)^n \tag{C.2.2.1}$$

C.2.3. The mean for the Binomial r.v. is

$$E[X] = np \tag{C.2.3.1}$$

Solution: From (B.4.1) and (C.2.2.1),

$$E[X] = \frac{d(q+pz)^n}{dz}|_{z=1} \tag{C.2.3.2}$$

$$= np(q+pz)^{n-1}|_{z=1} \tag{C.2.3.3}$$

$$= np(q+p)^{n-1} \tag{C.2.3.4}$$

yielding (C.2.3.1)

$$\because p + q = 1 \tag{C.2.3.5}$$

C.2.4. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution: See the following code

```
#Code by GVV Sharma
#November 20,2020
#Released under GNU/GPL
#To find the probability of an event using the binomial distribution

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli
from scipy.stats import norm
from scipy.stats import binom




#Simlen
simlen=1000

#Number of hurdles
n = 10

#Probability of clearing a hurdle
p = 1-5/6

#Mean
```

```
mu = p

#Variance
sigma = np.sqrt(p*(1-p))

#Theoretical probability of knocking down fewer than 2 hurdles
k = 1
print(binom.cdf(k, n, p),3*(5/6)**10)

#Using the Gaussian approximation for the binomial pdf
print(1/(sigma*np.sqrt(n))*(norm.pdf((k-n*mu)/(sigma*np.sqrt(n)))+norm.pdf((k
    -1-n*mu)/(sigma*np.sqrt(n)))))



#Simulating the probability using the binomial random variable
data_binom = binom.rvs(n,p,size=simlen) #Simulating the event of jumping 10
    hurdles
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
#print(data_binom)



#Simulating the probability using the bernoulli random variable
data_bern_mat = bernoulli.rvs(p,size=(n,simlen))
```

```
data_binom=np.sum(data_bern_mat, axis=0)
#print(data_bern_mat)
#print(data_binom)
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
```

C.3. Uniform Distribution

C.3.1. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the random variables representing the outcome for a die. Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{C.3.1.1}$$

C.3.2. The Z-transform of X is given by

$$P_X(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} = \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}, \quad |z| > 1 \tag{C.3.2.1}$$

upon summing up the geometric progression.

C.3.3. From (C.3.2.1), the CDF of X is given by

$$F_X(n) = \Pr(X \le n) = \begin{cases} 0 & n < 1 \\ \frac{n}{6} & 1 \le n \le 6 \\ 1 & \text{otherwise} \end{cases} \tag{C.3.3.1}$$

and plotted in Fig. C.3.3.1.

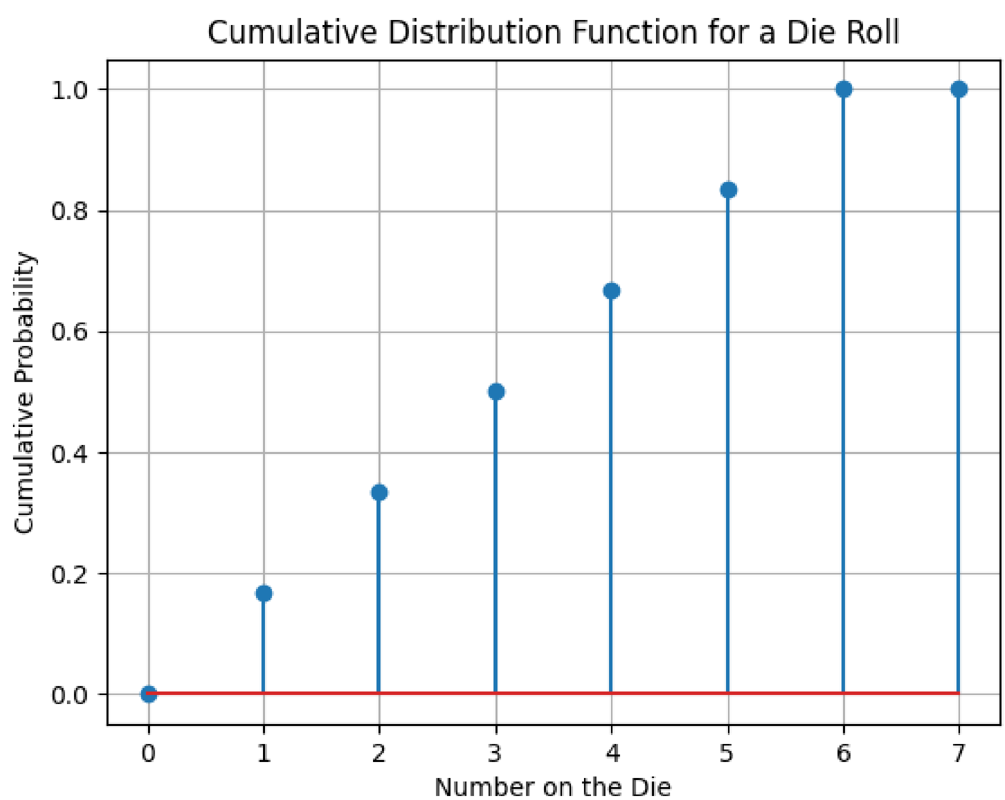

Figure C.3.3.1: CDF

C.4. Triangular Distribution

C.4.1. The desired outcome is

$$X = X_1 + X_2, \tag{C.4.1.1}$$

$$\implies X \in \{1, 2, \ldots, 12\} \tag{C.4.1.2}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{C.4.1.3}$$

C.4.2. Convolution: From (C.4.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{C.4.2.1}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{C.4.2.2}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k)$$

$$= \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{C.4.2.3}$$

From (C.4.2.2) and (C.4.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{C.4.2.4}$$

where $*$ denotes the convolution operation. Substituting from (C.3.1.1) in (C.4.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{C.4.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{C.4.2.6}$$

From (C.4.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{C.4.2.7}$$

Substituting from (C.3.1.1) in (C.4.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{C.4.2.8}$$

satisfying (C.4.1.3).

C.4.3. <u>The Z-transform:</u> From (C.3.2.1) and (B.2.2),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{C.4.3.1}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.2}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{C.4.3.3}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.4}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.5}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{C.4.3.6}$$

From (B.1.1), (C.4.3.2) and (C.4.3.5)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{C.4.3.7}$$

which is the same as (C.4.2.8). Note that (C.4.2.8) can be obtained from (C.4.3.5) using contour integration as well.

C.4.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure C.4.4.1. The theoretical pmf obtained in (C.4.2.8) is plotted for comparison.

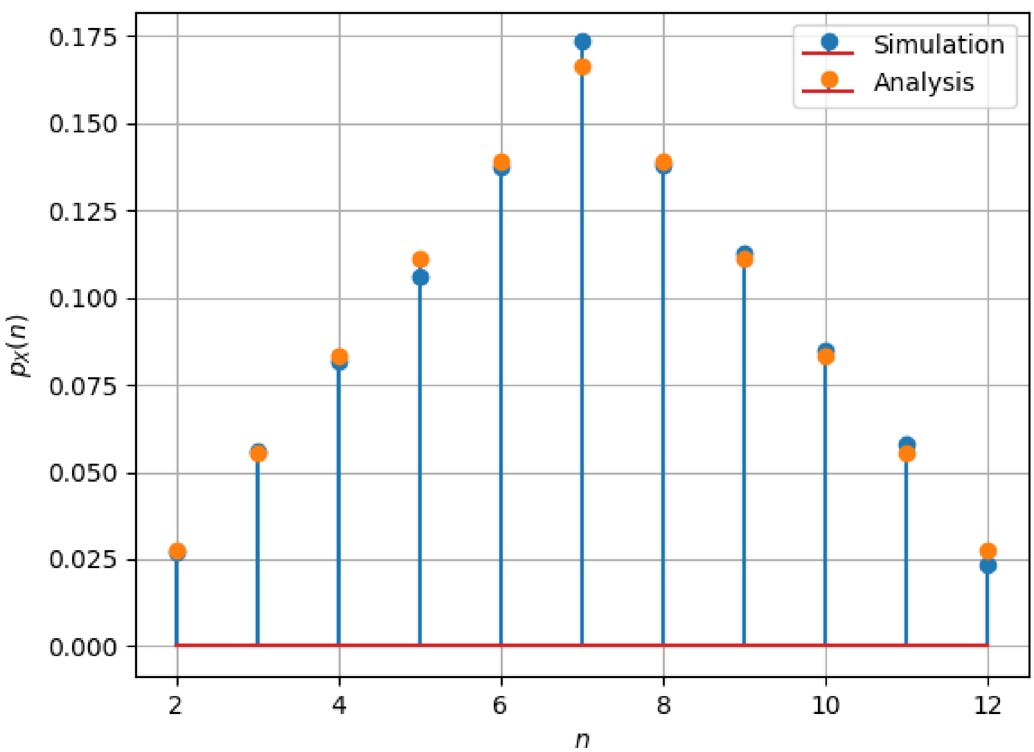

Figure C.4.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

C.4.5. The python code is available below

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

#Sample size
simlen = 10000
#Possible outcomes
n = range(2,13)
# Generate X1 and X2
y = np.random.randint(1,7, size=(2, simlen))

#Generate X
X = np.sum(y, axis = 0)
#Find the frequency of each outcome
unique, counts = np.unique(X, return_counts=True)
#Simulated probability
psim = counts/simlen
#Theoretical probability
n1 = range(2,8)
n2 = range(8,13)
panal1 = (n1 -np.ones((1,6)))
panal2 = (13*np.ones((1,5))-n2)
```

```
panal = np.concatenate((panal1,panal2),axis=None)/36


#Plotting
plt.stem(n,psim, markerfmt='o', use_line_collection=True, label='Simulation')
plt.stem(n,panal, markerfmt='o',use_line_collection=True, label='Analysis')
plt.xlabel('$n$')
plt.ylabel('$p_{X}(n)$')
plt.legend()
plt.grid()# minor


#If using termux
plt.savefig('figs/pmf.pdf')
plt.savefig('figs/pmf.png')
subprocess.run(shlex.split("termux-open figs/pmf.pdf"))
#else
#plt.show()
```

Appendix D

Central Limit Theorem

D.1. Binomial

D.1 Let

$$X = \mathrm{Bin}(n, p). \tag{D.1.1}$$

The mean and variance are then given by

$$\mu = np,\, \sigma^2 = npq. \tag{D.1.2}$$

D.2 For large values of $n, k, n-k$, by Stirling's Approximation.

$$\begin{aligned} n! &\approx n^n e^{-n}\sqrt{2\pi n} \\ k! &\approx k^k e^{-k}\sqrt{2\pi k} \\ (n-k)! &\approx (n-k)^{(n-k)}\, e^{-(n-k)}\sqrt{2\pi\,(n-k)} \end{aligned} \tag{D.2.1}$$

D.3 Then,

$$p_X(k) = \frac{n!}{k!(n-k)!}p^k q^{n-k} \approx \frac{n^n e^{-n}\sqrt{2\pi n}}{k^k e^{-k}\sqrt{2\pi k}(n-k)^{n-k}e^{-(n-k)}\sqrt{2\pi(n-k)}}p^k q^{n-k} \tag{D.3.1}$$

$$= \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \tag{D.3.2}$$

from (D.2.1)

D.4 For

$$\delta \ll np, nq, \tag{D.4.1}$$

and

$$k = np + \delta, \tag{D.4.2}$$

$$n - k = nq - \delta \tag{D.4.3}$$

$$\implies \begin{aligned} \frac{k}{np} &= 1 + \frac{\delta}{np} \\ \frac{n-k}{nq} &= 1 - \frac{\delta}{nq} \end{aligned} \tag{D.4.4}$$

D.5 Taking logarithms in (D.3.2),

$$\ln\left[p_X(k)\right] = -k\ln\left(\frac{np}{k}\right) - (n-k)\left(\frac{nq}{n-k}\right) + \frac{1}{2}\ln\left(\frac{n}{2\pi k(n-k)}\right) \tag{D.5.1}$$

$$\begin{aligned} &= -\left(np+\delta\right)\ln\left(1+\frac{\delta}{np}\right) - \left(nq-\delta\right)\left(1-\frac{\delta}{nq}\right) \\ &\quad + \frac{1}{2}\ln\left(\frac{n}{2\pi\left(np+\delta\right)\left(nq-\delta\right)}\right) \end{aligned} \tag{D.5.2}$$

upon substituting from (D.4.4). From (D.4.1),

$$\frac{1}{2}\ln\left(\frac{n}{2\pi\left(np+\delta\right)\left(nq-\delta\right)}\right)\approx\frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right) \tag{D.5.3}$$

From (D.4.1) and the approximation

$$\ln\left(1+x\right)\approx x-\frac{x^2}{2}, \tag{D.5.4}$$

the first sum in (D.5.2) can be expressed as,

$$\begin{aligned}
&-\left(np+\delta\right)\ln\left(1+\frac{\delta}{np}\right)-\left(nq-\delta\right)\left(1-\frac{\delta}{nq}\right)\\
&\approx-\left(np+\delta\right)\left(\frac{\delta}{np}-\frac{\delta^2}{2n^2p^2}\right)-\left(nq-\delta\right)\left(-\frac{\delta}{nq}-\frac{\delta^2}{2n^2q^2}\right)\\
&=-\delta\left[1+\frac{\delta}{2np}-1+\frac{\delta}{2nq}\right]=-\frac{\delta^2}{2npq}
\end{aligned} \tag{D.5.5}$$

Substituting from (D.5.5) and (D.5.5) in (D.5.2),

$$\ln\left[p_X(k)\right]\approx\frac{1}{2}\ln\left(\frac{1}{2\pi npq}\right)-\frac{\delta^2}{2npq} \tag{D.5.6}$$

$$\implies p_X(k)=\sqrt{\frac{1}{2\pi npq}}e^{-\frac{(k-np)^2}{2npq}}=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}} \tag{D.5.7}$$

D.6 A comparison of Binomial and Gaussian pmf/pdf is provided in Figs. D.6.1 and D.6.2.

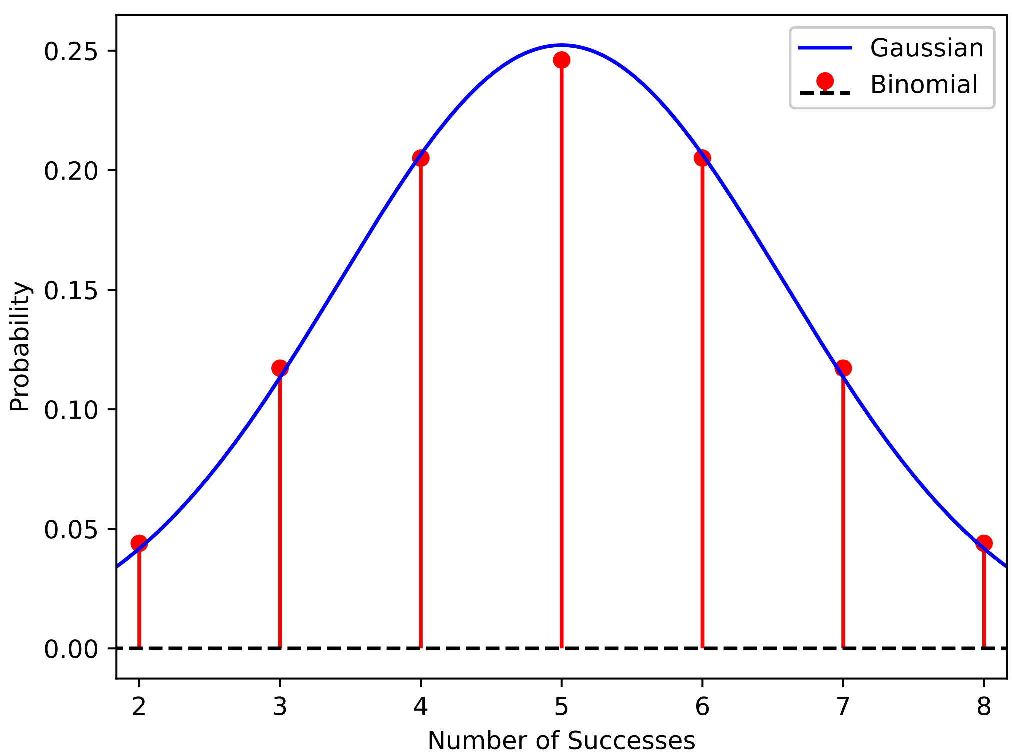

Figure D.6.1: 10 trials

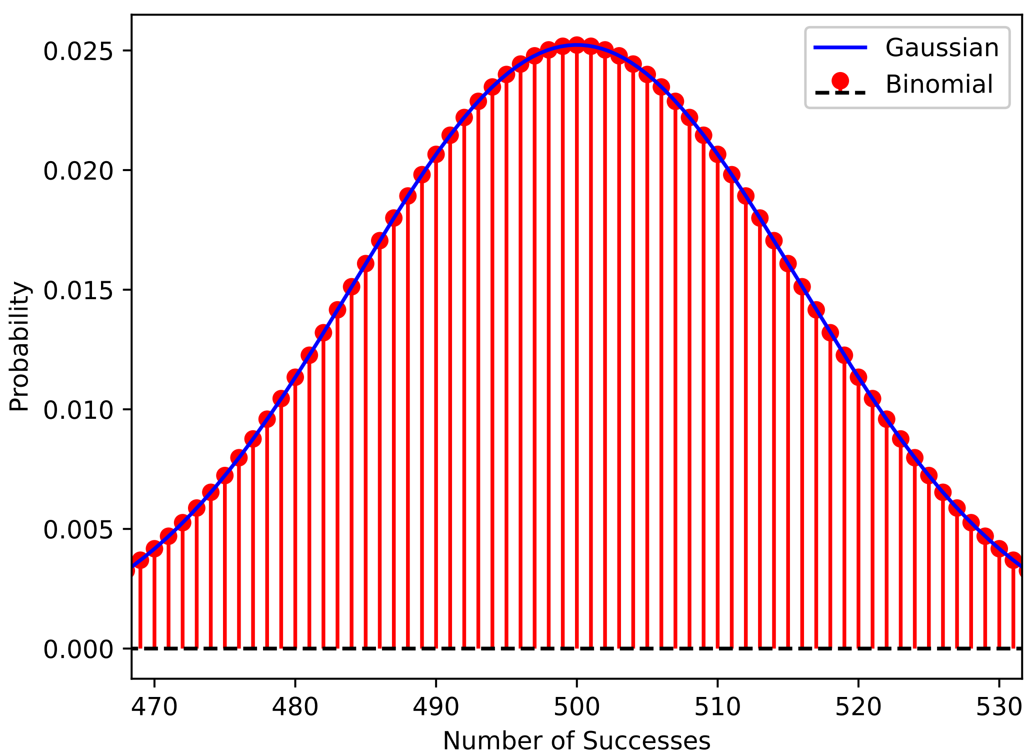

Figure D.6.2: 1000 trails

Appendix E

Identities

E.1 Let

$$N = R + B + G, n = r + b + g \tag{E.1.1}$$

where R, B, G and r, b, g represent the number of red, blue and green marbles respectively within N and n. Then

$$\Pr(r, b, g) = \frac{{}^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g}}{{}^{R+B+G}C_{r+b+g}} \tag{E.1.2}$$

Solution: The number of ways of choosing n marbles from N is

$${}^{N}C_{n} \tag{E.1.3}$$

The number of ways of choosing r,b,g marbles is

$${}^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g} \tag{E.1.4}$$

Using the definition of probability, we obtain (E.1.2).

E.2

$$ {}^{R+B}C_n = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \tag{E.2.1} $$

Solution: Since

$$ (x+1)^R = \sum_{k=0}^{R} {}^{R}C_k \, x^k, \tag{E.2.2} $$

$$ (x+1)^R \, (x+1)^B = \sum_{k=0}^{R} \sum_{m=0}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{E.2.3} $$

$$ \Longrightarrow \; (x+1)^{R+B} = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^n + \sum_{k=0}^{R} \sum_{m \neq n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{E.2.4} $$

(E.2.5)

yielding (E.2.1) upon comparing the coefficients of x^n on both sides.