GATE Problems in Probability

***Abstract*—These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

1) An urn contains 5 red balls and 5 black balls.In the first draw, one ball is picked at random and discarded without noticing its colour.The probability to get a red ball in the second draw is

a) $\frac{1}{2}$ b) $\frac{4}{9}$ c) $\frac{5}{9}$ d) $\frac{6}{9}$

Solution: Let $X_i \in \{0, 1\}$ represent the i^{th} draw where 1 denotes a red ball is drawn.

TABLE I

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	4/18	5/18
$X_2 = 1$	5/18	4/18

Table I represents the probabilities of all possible cases when two balls are drawn one by one from the urn.

$$\Pr(X_2 = 1) = \Pr(X_2 = 1|X_1 = 0) + \Pr(X_2 = 1|X_1 = 1) \tag{1}$$

$$= \frac{5}{18} + \frac{4}{18} \tag{2}$$

$$= \frac{1}{2} \tag{3}$$

The required option is (A).

2) There are 3 red socks, 4 green socks and 3 blue socks.You choose 2 socks.The probability that they are of the same colour is

a) $\frac{1}{5}$ b) $\frac{7}{30}$ c) $\frac{1}{4}$ d) $\frac{4}{15}$

Solution: Let $X_i \in \{1, 2, 3\}$ represent the i^{th} draw, where 1, 2, 3 correspond to the colour of socks drawn as Red, Blue and Green respectively

TABLE II

	$X_1 = 1$	$X_1 = 2$	$X_1 = 3$
$X_2 = 1$	6/90	12/90	9/90
$X_2 = 2$	12/90	12/90	12/90
$X_2 = 3$	9/90	12/90	6/90

TABLE II represents all the possibilities of choosing socks one by one.
The probability that the two socks drawn are of the same colour(by substituting values from table II)

$$= \Pr(X_1 = X_2) \tag{4}$$

$$= \sum_{i=1}^{3} \Pr(X_2 = i|X_1 = i) \Pr(X_1 = i) \tag{5}$$

$$= \frac{6}{90} + \frac{12}{90} + \frac{6}{90} \tag{6}$$

$$= \frac{4}{15} \tag{7}$$

So the correct option is (D)

3) The probability that a k-digit number does NOT contain the digits 0,5, or 9 is

a) 0.3^k b) 0.6^k c) 0.7^k d) 0.9^k

Solution: Let

$$X_i \in \{0, 1, 2, \ldots, 9\} \tag{8}$$

represent the digit at the i^{th} place.

$$\Pr(X_i \notin \{0, 5, 9\}) = \frac{7}{10} = 0.7 \tag{9}$$

If the k-digit number does not contain 0,5 or 9,

$$\Pr(X_1 \notin \{0, 5, 9\}, X_2 \notin \{0, 5, 9\}, \ldots, X_k \neq \{0, 5, 9\}) \tag{10}$$

Since the events are independent,

$$\Pr(X_1 \notin \{0,5,9\}, X_2 \notin \{0,5,9\}, \ldots, X_k \neq \{0,5,9\}) = \Pr(X_1 \notin \{0,5,9\}) \ldots \Pr(X_k \notin \{0,5,9\}) \quad (11)$$

$$= \prod_{i=1}^{k} 0.7 \quad (12)$$

$$= (0.7)^k \quad (13)$$

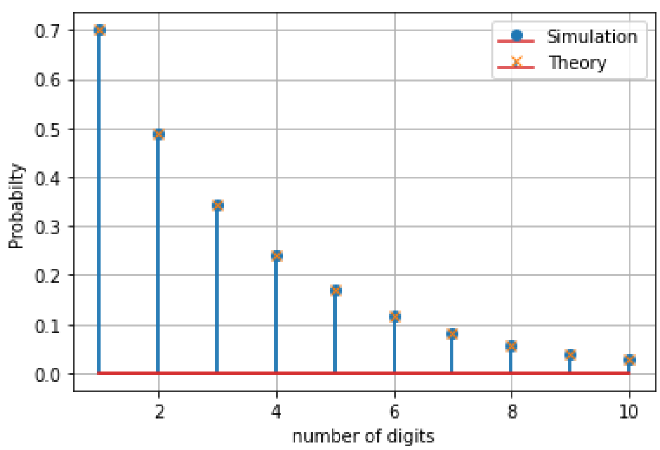


Fig. 1: Plot

4) Three fair cubical dice are thrown simultaneously. The probability that all three dice have the same number of dots on the faces showing up is (up to third decimal place)...........
Solution: Let

$$X_1, X_2, X_3 \in \{1,2,3,4,5,6\} \quad (14)$$

represent the three dice.
Since, all the three are fair dice, the probability of any dice showing a particular number is given by

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & \text{i=1,2,3,4,5,6} \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

If all the dice show a particular number i,

$$\implies \Pr(X_1 = X_2 = X_3 = i) \quad (16)$$

Since the events are independent,

$$\Pr(X_1 = X_2 = X_3 = i) = \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \quad (17)$$

where i=1,2,3,4,5,6.
There are 6 faces on a cubical dice. Hence, there are six cases in which all the dice show the same number

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = X_2 = X_3 = i) \quad (18)$$

From (17),we have

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \quad (19)$$

$$= \sum_{i=1}^{6} \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \quad (20)$$

$$= \frac{1}{36} \quad (21)$$

5) Candidates were asked to come to an interview with 3 pens each. Black,blue,green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is.......

6) The probability of getting a ”head” in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a ”head” is obtained. If the tosses are independent, then the probability of getting ”head” for the first time in the fifth toss is........
Solution: Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.
$X = k$ represents $k-1$ failures were obtained before getting 1 success. p represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \quad (22)$$

Using (22) we get

$$\Pr(X = 5) = (1-p)^{k-1} \times p = (0.7)^4 \times 0.3 = 0.07203 \quad (23)$$

7) Given Set A = [2,3,4,5] and Set B =

[11,12,13,14,15], two numbers are randomly selected,one from each set. What is probability that the sum of the two numbers equals 16?

a) 0.20 b) 0.25 c) 0.30 d) 0.33

Solution: Given,
Set A= [2,3,4,5]
Set B= [11,12,13,14,15]

Total number of element in the sample space is 20.

Let us define a random variable$X \in \{0, 1\}$

X=0	the event when A+B=16
X=1	the event when A+B $\neq$ 16

TABLE III: Random Variables

Now, probability of selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = 1 \tag{7.1}$$

So,the probability of selecting an element from set B after selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = \frac{1}{5} \tag{7.2}$$

Therefore,
Overall probability of randomly choosing elements from set A and set B such that $\Pr(X = 0)$is

$$\Pr(X = 0) = \Pr(A + B = 16) \tag{7.3}$$
$$\Pr(X = 0) = 1 \times \frac{1}{5} \tag{7.4}$$
$$\Pr(X = 0) = \frac{1}{5} = 0.2 \tag{7.5}$$

X	0	1
$\Pr(X)$	$\frac{1}{5}$	$\frac{4}{5}$

TABLE IV: Probability distribution table

Therefore, the correct option is (a).

8) Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is...... **Solution:** Let X be random variable.
$X \in \{1,2,3,4,5,6\}$
$p_X(n) \rightarrow$Probability of showing up n.
As $p_X(n)$ is proportional to n. We have,

$$p_X(n) = \begin{cases} kn & 1 \leq n \leq 6 \\ 0 & otherwise \end{cases} \tag{8.1}$$

Where k is some real constant.

TABLE V

n	1	2	3	4	5	6
$p_X(n)$	k	2k	3k	4k	5k	6k

We know that,

$$\sum_{n=1}^{6} p_X(n) = 1 \tag{8.2}$$

By substituting the values in 8.2, we have

$$k + 2k + 3k + 4k + 5k + 6k = 1 \tag{8.3}$$

$$\implies k = \frac{1}{21} \tag{8.4}$$

Probability of the face with three dots showing up

$$\implies p_X(3) = 3k \tag{8.5}$$

Substituting the value of k from 8.4

$$\implies p_X(3) = \frac{1}{7} \tag{8.6}$$

9) **Step 1.** Flip a coin twice.
Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.
The probability that the output of the experiment is Y is (upto two decimal places)......

10) Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from

X and Y. Let f be randomly chosen from F. The probability of f being one-to-one is........

11) The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is......
Solution: Let $X \in \{1, 2, ..., 100\}$ be the random variable representing the outcome for random selection of a number in $\{1, ..., 100\}$.
Since X has a uniform distribution, the probability mass function (pmf) is represented as

$$\Pr(X = n) = \begin{cases} \frac{1}{100} & 1 \leq n \leq 100 \\ 0 & otherwise \end{cases} \tag{24}$$

Let A represent the event that the number is divisible by 2. Let B represent the event that the number is divisible by 3. Let C represent the event that the number is divisible by 5.
We need to find the probability that the number is not divisible by 2, 3 or 5. Thus we need to find $1 - \Pr(A + B + C)$
We know

$$\Pr(A + B + C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(BC) - \Pr(AC) + \Pr(ABC) \tag{25}$$

Event	Interpretation	Probability
A	n is divisible by 2	$\frac{50}{100}$
B	n is divisible by 3	$\frac{33}{100}$
C	n is divisible by 5	$\frac{20}{100}$
AB	n is divisible by 6	$\frac{16}{100}$
BC	n is divisible by 15	$\frac{6}{100}$
AC	n is divisible by 10	$\frac{10}{100}$
ABC	n is divisible by 30	$\frac{3}{100}$

TABLE VI

Substituting in (25), we get

$$\Pr(A + B + C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} \tag{26}$$

Thus,

$$\Pr(A + B + C) = \frac{74}{100} \tag{27}$$

Thus required probability =

$$1 - \Pr(A + B + C) = \frac{26}{100} \tag{28}$$

12) P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for the job is

a) $\frac{4}{5}$ b) $\frac{5}{6}$ c) $\frac{7}{8}$ d) $\frac{11}{12}$

Solution: Let A represent the event that P applies for the job. Let B represent the event that Q applies for the job.
According to the given information in the question,

$$\Pr(A) = \frac{1}{4} \tag{29}$$

$$\Pr(A|B) = \frac{1}{2} \tag{30}$$

$$\Pr(B|A) = \frac{1}{3} \tag{31}$$

According to the definition of Conditional Probability,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{32}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{33}$$

On substituting the values of $\Pr(A)$, $\Pr(B|A)$ in

(33), we get

$$\frac{1}{3} = \frac{\Pr(AB)}{\frac{1}{4}} \tag{34}$$

$$\Pr(AB) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \tag{35}$$

$$\therefore \Pr(AB) = \frac{1}{12} \tag{36}$$

Now substituting the values of $\Pr(A|B)$, $\Pr(AB)$ in (32), we get

$$\frac{1}{2} = \frac{\frac{1}{12}}{\Pr(B)} \tag{37}$$

$$\Pr(B) = \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{2}\right)} \tag{38}$$

$$\therefore \Pr(B) = \frac{1}{6} \tag{39}$$

The probability that P does not apply for the job given that Q does not apply for the job is given by $\Pr\left(A'|B'\right)$.
Now,

$$A'B' = (A+B)' \tag{40}$$

$$\Rightarrow \Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{41}$$

$$\therefore \Pr\left(A'B'\right) = 1 - \Pr(A+B) \tag{42}$$

As we know that,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{43}$$

By substituting the values of $\Pr(A)$, $\Pr(B)$, $\Pr(AB)$ in (43), we get

$$\Pr(A+B) = \frac{1}{4} + \frac{1}{6} - \frac{1}{12} \tag{44}$$

$$= \frac{3+2-1}{12} \tag{45}$$

$$\therefore \Pr(A+B) = \frac{1}{3} \tag{46}$$

$\therefore$ Required Probability is

$$\Pr\left(A'|B'\right) = \frac{\Pr\left(A'B'\right)}{\Pr(B')} \tag{47}$$

By substituting the values of $\Pr(B)$, $\Pr\left(A'B'\right)$ [from (42)] in (47), we get

$$\Pr\left(A'|B'\right) = \frac{1-\Pr(A+B)}{1-\Pr(B)} \tag{48}$$

$$\Pr\left(A'|B'\right) = \frac{1-\left(\frac{1}{3}\right)}{1-\left(\frac{1}{6}\right)} \tag{49}$$

$$\Rightarrow \Pr\left(A'|B'\right) = \frac{\left(\frac{2}{3}\right)}{\left(\frac{5}{6}\right)} = \frac{4}{5} \tag{50}$$

(51)

$\therefore \Pr\left(A'|B'\right) = \frac{4}{5} = 0.8$
Hence, the probability that P does not apply for the job given that Q does not apply for the job is equal to $\frac{4}{5}$
$\therefore$ The correct option is (A) $\frac{4}{5}$.

13) Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

a) $\frac{5}{11}$ b) $\frac{1}{2}$ c) $\frac{7}{13}$ d) $\frac{6}{11}$

14) A continuous random variable X has a probability density function $f(x) = e^{-x}, 0 < x < \infty$. Then $P(X > 1)$ is

a) 0.368 b) 0.5 c) 0.632 d) 1.0

Solution:
Given,

$$f(x) = e^{-x}, 0 < x < \infty \tag{52}$$

We have to find $\Pr(X > 1)$,

$$\Pr(X > 1) = \int_1^\infty f(x)\,dx \tag{53}$$

Using (52) in (53)

$$\Pr(X > 1) = \int_1^\infty e^{-x}\,dx \tag{54}$$
$$= \left[-e^{-x}\right]_1^\infty \tag{55}$$
$$= (-e^{-\infty}) - (-e^{-1}) \tag{56}$$
$$= e^{-1} \tag{57}$$
$$= \frac{1}{e} \tag{58}$$
$$\implies \Pr(X > 1) = 0.368 \tag{59}$$

Finding the probability using uniform distribution,
Let $F_X(x)$ be the cumulative distribution function of random variable X.

$$F_X(x) = \int_0^x f(x)\,dx \tag{60}$$

$F_X(x)$ can be obtained from the uniform distribution of a random variable U on (0,1) and let U=e^{-x}.

$$0 < U < 1 \tag{61}$$

As for random variable X also,

$$0 < F_X(x) < 1 \tag{62}$$

This similarity between U and $F_X(x)$ is used to generate the random variable X from U.

$$F_X(x) = \Pr(X < x) \tag{63}$$
$$= \Pr(-\log_e U < x) \tag{64}$$
$$= \Pr(U < e^{-x}) \tag{65}$$
$$= F_U(e^{-x}) \tag{66}$$

From uniform distribution,

$$F_U(x) = x, 0 < x < 1 \tag{67}$$

In the figure 2,orange colour graph represents the pdf of the random variable X and blue colour graph represents the cdf of the random variable X. Using (67) in (66), Cumulative distribution function (CDF) of random variable X is,

$$F_X(x) = \Pr(X < x) \tag{68}$$
$$= 1 - e^{-x}, 0 < x < \infty \tag{69}$$

Now we have to find $\Pr(X > 1)$,

$$\Pr(X > 1) = 1 - \Pr(X < 1) \tag{70}$$

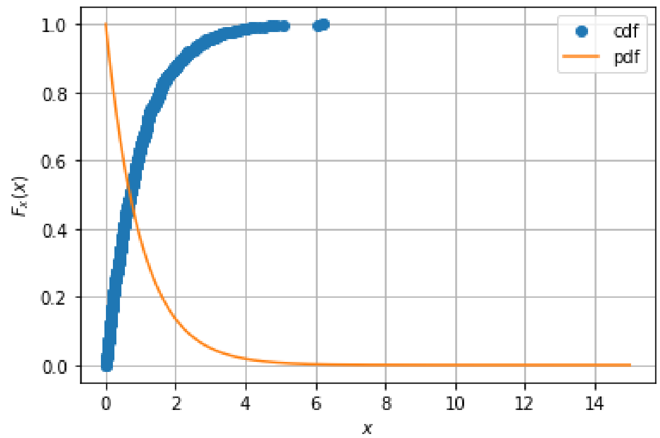


Fig. 2: CDF of random variable X

Using (69),

$$\Pr(X > 1) = 1 - (1 - e^{-1}) \tag{71}$$
$$\Pr(X > 1) = e^{-1} \tag{72}$$
$$\implies \Pr(X > 1) = 0.368 \tag{73}$$

15) A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

If the expected value $E[X] = \frac{2}{3}$, then $Pr[X < 0.5]$ is..........
Solution:
We know that the total probability is one,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{75}$$

Using (74) in (75),

$$\int_0^1 (a + bx)\,dx = 1 \tag{76}$$
$$\left[ax + \frac{bx^2}{2}\right]_0^1 = 1 \tag{77}$$
$$\left(a + \frac{b}{2}\right) - 0 = 1 \tag{78}$$
$$\implies a + \frac{b}{2} = 1 \tag{79}$$

We know that expectation value of X,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{80}$$

Using $E(X) = \frac{2}{3}$ and (74) in (80), we get

$$\frac{2}{3} = \int_0^1 x(a+bx)\,dx \tag{81}$$

$$= \int_0^1 ax + bx^2\,dx \tag{82}$$

$$= \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 \tag{83}$$

$$= \frac{a}{2} + \frac{b}{3} - 0 \tag{84}$$

$$\implies \frac{a}{2} + \frac{b}{3} = \frac{2}{3} \tag{85}$$

By solving (79) and (85), we get

$$a = 0 \; and \; b = 2. \tag{86}$$

Using values of a and b in (74), we get

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{87}$$

The graph of PDF of X is 3

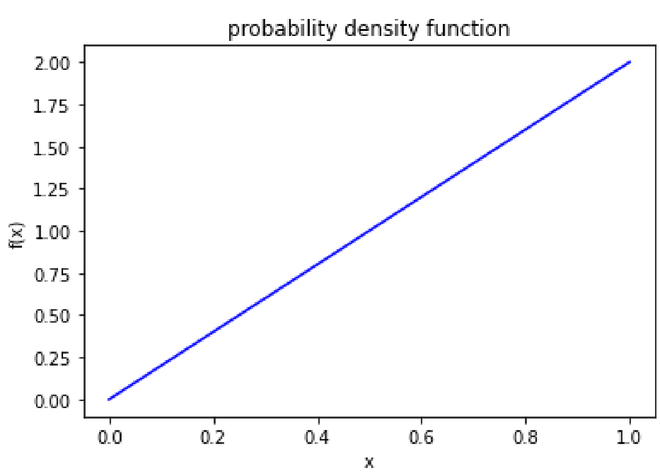


Fig. 3: Probability Density Function (PDF) of X

Let $F_X(x)$ be the cumulative distribution function of random variable X.

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx \tag{88}$$

$F_X(x)$ can be obtained from the uniform distribution of a random variable U on (0,1) and let U=X^2.

$$0 < U < 1 \tag{89}$$

As for random variable X also,

$$0 < F_X(x) < 1 \tag{90}$$

This similarity between U and $F_X(x)$ is used to generate the random variable X from U.

$$F_X(x) = \Pr(X < x) \tag{91}$$

$$= \Pr\left(\sqrt{U} < x\right) \tag{92}$$

$$= \Pr\left(U < x^2\right) \tag{93}$$

$$= F_U(x^2) \tag{94}$$

From uniform distribution,
The graph of Probability Density Function (PDF) of U is 4

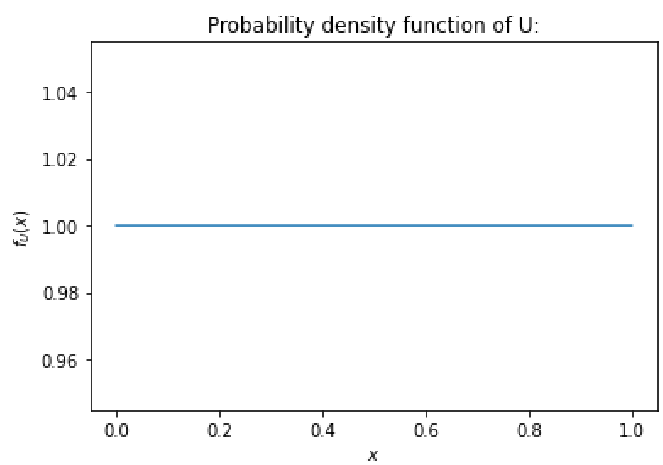


Fig. 4: Probability Density Function (PDF) of U

$$F_U(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{95}$$

Using (95) in (94),
Cumulative distribution function (CDF) of random variable X is,

$$F_X(x) = \Pr(X < x) = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{96}$$

The graph of Cumulative distribution function (CDF) of random variable X is 5

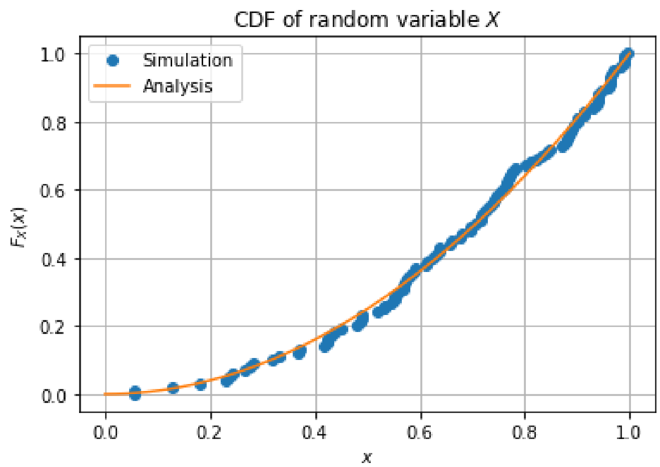


Fig. 5: Cumulative Density Function (CDF)

Now we have to find $\Pr(X < 0.5)$,Using (96),

$$\Pr(X < 0.5) = (0.5)^2 \tag{97}$$
$$\implies \Pr(X < 0.5) = 0.25 \tag{98}$$

16) Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$.The probability that $\max[X, Y]$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

17) The input X to the binary Symmetric Channel (BSC) shown in Fig. 6 is '1' with probability 0.8. The cross-over probability is $\frac{1}{7}$. If the received bit Y=0, the conditional probability that '1' was transmitted is........

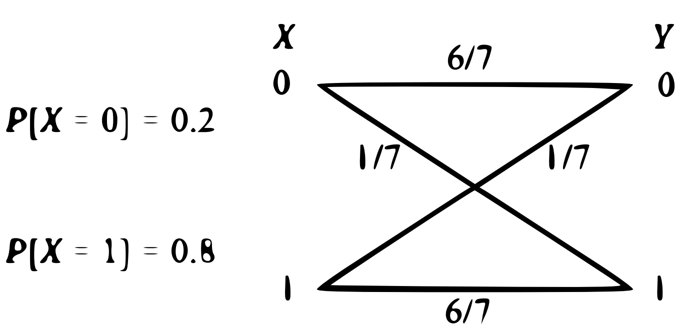


Fig. 6

Solution:

$$\Pr(X = 1|Y = 0) = \frac{\Pr(\{X = 1\}\{Y = 0\})}{\Pr(Y = 0)} \tag{99}$$
$$\Pr(Y = 0|X = 1) = \frac{\Pr(\{X = 1\}\{Y = 0\})}{\Pr(X = 1)} \tag{100}$$

From (100),

$$\Pr(\{X = 1\}\{Y = 0\}) = \Pr(Y = 0|X = 1)\Pr(X = 1) \tag{101}$$

Substituting (101) in (99),

$$\Pr(X = 1|Y = 0) = \frac{\Pr(Y = 0|X = 1)\Pr(X = 1)}{\Pr(Y = 0)} \tag{102}$$

Given data,

$$\Pr(Y = 0|X = 1) = \frac{1}{7}, \Pr(Y = 0|X = 0) = \frac{6}{7} \tag{103}$$

$$\Pr(Y = 0) = \Pr(Y = 0|X = 1)\Pr(X = 1) + \Pr(Y = 0|X = 0)\Pr(X = 0) \tag{104}$$

Substituting the values from (103) and the data given in the question in (104),

$$\Pr(Y = 0) = \frac{2}{7} \tag{105}$$

Substituting (103), (105) and the data given in the question in (102),

$$\Pr(X = 1|Y = 0) = 0.4 \tag{106}$$

18) A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

19) A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

a) $\frac{1}{3}$ b) $\frac{3}{7}$ c) $\frac{1}{2}$ d) $\frac{4}{7}$

Solution: Consider, Bernoulli random variables Say X_1 and X_2. Required probability is

$$Pr(X_2 = 0|X_1 = 1) = \frac{Pr(X_1 = 1, X_2 = 0)}{Pr(X_1 = 1)} \tag{107}$$

$$= \frac{\frac{2}{7}}{\frac{4}{7}} = \frac{1}{2} = \frac{1}{2} \tag{108}$$

Hence (C) is correct option.

20) Let X be a random variable with a probability density function

$$f(x) = \begin{cases} 0.2 & |x| \le 1 \\ 0.1 & 1 \le |x| \le 4 \\ 0 & otherwise \end{cases} \tag{109}$$

Find $\Pr(0.5 < X \le 5)$
Solution:
We know, if X is a continuous random variable, and its p.d.f is given by $f(x)$, then we define the c.d.f $F(x)$ as:

$$F(x) = \Pr(X \le x) \tag{110}$$

and is given by:

$$F(x) = \int_{-\infty}^{x} f(x)\, dx \tag{111}$$

$f(x)$ is a valid p.d.f because:

a) The area under the curve of the p.d.f is 1, i.e:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{112}$$

b) $f(x) \ge 0$ for all $x \in \mathbb{R}$

Since $f(x)$ is a valid p.d.f, from (111), we get the following c.d.f:

$$F(x) = \begin{cases} 0 & x \le -4 \\ 0.1(x+4) & -4 \le x \le -1 \\ 0.3 + 0.2(x+1) & -1 \le x \le 1 \\ 0.7 + 0.1(x-1) & 1 \le x \le 4 \\ 1 & 4 \le x \end{cases} \tag{113}$$

Thus,

$$\Pr(0.5 \le X \le 5) = F(5) - F(0.5) = 0.4 \tag{114}$$

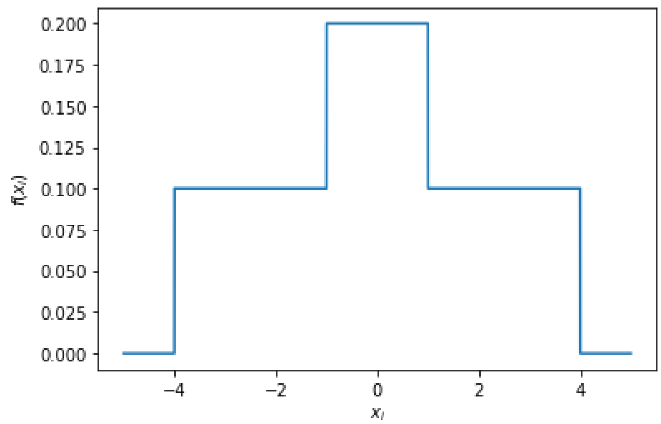


Fig. 7: plot of $f(x)$- p.d.f

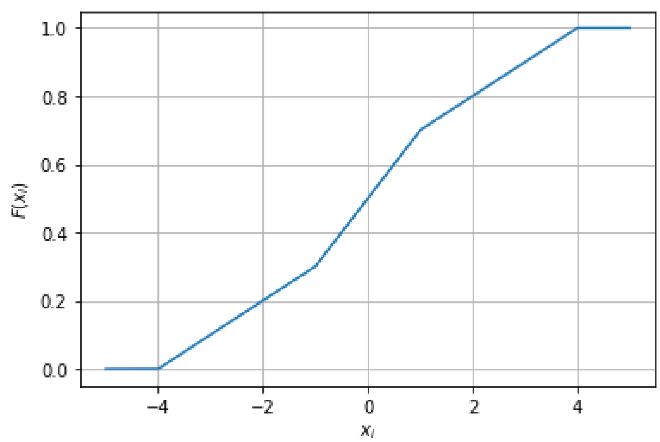


Fig. 8: plot of $F(x)$- c.d.f

21) Consider two identically distributed zero-mean random variables U and V. Let the cumulative distribution functions of U and 2V be F(x) and G(x) respectively. Then,for all values of x

a) $F(x) - G(x) \leqslant 0$

b) $F(x) - G(x) \geqslant 0$

c) $(F(x) - G(x))x \leqslant 0$

d) $(F(x) - G(x))x \geqslant 0$

Solution: If X is a random variable, the cumulative distribution functions of U and 2V can be written in terms of X as

$$F(x) = \Pr(X \le x) \tag{115}$$

$$G(x) = \Pr(2X \le x) \quad (116)$$

Or,

$$G(x) = \Pr(X \le x/2) \quad (117)$$

Using 115 in 117, we can see that

$$G(x) = F(x/2) \quad (118)$$

So,

$$F(x) - G(x) = F(x) - F(x/2) \quad (119)$$

As F is Cumulative Distribution Function, it is non-decreasing.
That means for $x \ge y$, $F(x) \ge F(y)$.

Using this, we can form the following table:

Case	$F(x) - F(x/2)$	$(F(x) - F(x/2))x$
$x \ge 0$	≥ 0	≥ 0
$x \le 0$	≤ 0	≥ 0

TABLE VII

From the table we can see that for any value of x,

$$(F(x) - F(x/2))x \ge 0 \quad (120)$$

Or, using 118,

$$(F(x) - G(x))x \ge x \quad (121)$$

22) Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $P(3V \geqslant 2U)$ is
a) $\frac{4}{9}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{5}{9}$

23) Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$. The probability that max $[X, Y]$ is less than $\frac{1}{2}$ is
a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

24) A binary symmetric channel (BSC) has a transition probability of $\frac{1}{8}$. If the binary transmit symbol X is such that $P(X = 0) = \frac{9}{10}$, then the probability of error for an optimum receiver will be

a) $\frac{7}{80}$ b) $\frac{63}{80}$ c) $\frac{9}{10}$ d) $\frac{1}{10}$

Solution:

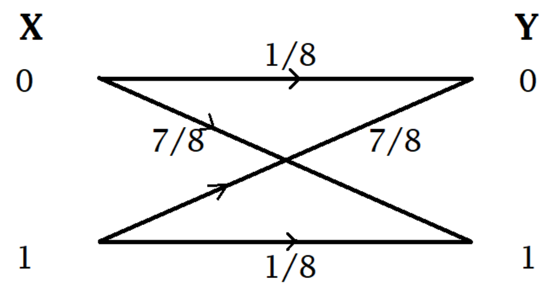


Fig. 9: Binary symmetric channel

Let random variables, $X \in \{0, 1\}$ denote the bit transmitted and $Y \in \{0, 1\}$ denote the output bit received.
From the given information,

$$\Pr(X = 0) = \frac{9}{10} \quad (122)$$

$$\Pr(X = 1) = 1 - \Pr(X = 0) = \frac{1}{10} \quad (123)$$

Also given, transition probability $= \frac{1}{8}$. Transition probability is the probability with which the bit is transmitted correctly. That gives,

$$\Pr(Y = 1|X = 1) = \Pr(Y = 0|X = 0) = \frac{1}{8} \quad (124)$$

$$\begin{aligned}\text{Probability that the bit is not transmitted correctly} \\ = 1 - \text{transition probability} \\ = 1 - \frac{1}{8} = \frac{7}{8}\end{aligned} \quad (125)$$

That gives,

$$\Pr(Y = 0|X = 1) = \Pr(Y = 1|X = 0) = \frac{7}{8} \quad (126)$$

Let E denote the event that bit is transmitted incorrectly. Probability of error, $\Pr(E)$

$$\Pr(E) = \Pr(X=0)\Pr(Y=1|X=0) + \Pr(X=1)\Pr(Y=0|X=1) \quad (127)$$

On substituting the values,

$$\Pr(E) = \frac{9}{10}\times\frac{7}{8} + \frac{1}{10}\times\frac{7}{8} \quad (128)$$
$$= \frac{63}{80} + \frac{7}{80} \quad (129)$$
$$= \frac{7}{8} \quad (130)$$

Answer: No option matches

25) A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

26) A fair dice is tossed two times. The probability that the second toss result in a value that is higher than the first toss is

a) $\frac{2}{36}$ b) $\frac{2}{6}$ c) $\frac{5}{12}$ d) $\frac{1}{2}$

Solution: Given, a fair die, which is tossed twice. Let the random variable $X_i \in \{1,2,3,4,5,6\}, i = 1,2$, represent the outcome of the number on the die in the first, second toss respectively. The probability mass function (PMF) for a fair die is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6}, & 1 \le n \le 6 \\ 0, & otherwise \end{cases} \quad (26.1)$$

Using (26.1), the cumulative distribution function (CDF) is obtained to be

$$F_{X_i}(r) = \Pr(X_i \le r) = \begin{cases} \frac{r}{6}, & 1 \le r \le 6 \\ 1, & r \ge 7 \\ 0, & otherwise \end{cases} \quad (26.2)$$

$$X_1 < X_2 \Rightarrow X_2 = k, X_1 \le k-1 \quad (26.3)$$

$\because X_1, X_2$ are independent,

$$Pr(X_1 < X_2) = E\left[F_{X_1}(X_2 - 1)\right] \quad (26.4)$$

After unconditioning (26.4), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} p_{X_2}(k) F_{X_1}(k-1) \quad (26.5)$$

Substituting (26.1) and (26.2), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} \frac{1}{6}\left(\frac{k-1}{6}\right) \quad (26.6)$$

On solving, we get

$$Pr(X_1 < X_2) = \frac{5}{12}\text{(option (C))} \quad (26.7)$$

TABLE VIII: Cases and their theoretical probabilities

Case	$X_1 < X_2$	$X_1 > X_2$	$X_1 = X_2$
Probability	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{6}$

27) A fair coin is tossed 10 times. What is the probability that ONLY the first two tosses will yield heads?

a) $\left(\frac{1}{2}\right)^2$
c) $\left(\frac{1}{2}\right)^{10}$
b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$
d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$

Solution: Let $M \sim B(n,h)$ be a random variable representing number of 'heads' in 10 tosses. So M has a binomial distribution :

$$\Pr(M = k) = {}^{n}C_k \times (h)^{n-k} \times (t)^k \quad (131)$$

Where

- n = Total number of tosses = 10
- h = Probability that 'head' appears in a toss $= \frac{1}{2}$
- t = Probability that 'tail' appears in a toss $= \frac{1}{2}$

So,

$$\Pr(M = k) = {}^{10}C_k \times \left(\frac{1}{2}\right)^{10-k} \times \left(\frac{1}{2}\right)^{k} \tag{132}$$

n	10
$\Pr(M = 2)$	${}^{10}C_2 \times \left(\frac{1}{2}\right)^{10-2} \times \left(\frac{1}{2}\right)^{2}$
Calculation	${}^{10}C_2 \times \left(\frac{1}{2}\right)^{10}$
Value	0.043945

- Number of ways of choosing 2 positions from 10 tosses = ${}^{10}C_2$

- Number of favourable outcome = 1 (Choosing FIRST and SECOND tosses as heads)

- Probability that chosen 2 'heads' are from FIRST and SECOND tosses = $\frac{1}{{}^{10}C_2}$

Probability that ONLY the first 2 tosses yield heads

$$= \Pr(M = 2) \times \frac{1}{{}^{10}C_2} \tag{133}$$

$$= {}^{10}C_2 \times \left(\frac{1}{2}\right)^{10} \times \frac{1}{{}^{10}C_2} \tag{134}$$

$$= \left(\frac{1}{2}\right)^{10} \tag{135}$$

28) Consider two independent random variables X and Y with identical distributions. The variables X and Y take value 0, 1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ rrespectively. What is the conditional probability $P(X+Y = 2|X-Y = 0)$?

a) 0 b) $\frac{1}{16}$ c) $\frac{1}{6}$ d) 1

Solution: The values that the random variable X can take along with its probabilities are given by

X	0	1	2
$\Pr(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Y	0	1	2
$\Pr(Y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The values that the random variable Y can take along with its probabilities are given by

$$\Pr(X - Y = 0) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{6}{16} \tag{136}$$

$$\Pr((X + Y = 2), (X - Y = 0)) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \tag{137}$$

$$\Pr(X + Y = 2 \mid X - Y = 0) = \frac{\Pr((X + Y = 2), (X - Y = 0))}{\Pr(X - Y = 0)} = \frac{\frac{1}{16}}{\frac{6}{16}} = \frac{1}{6} \tag{138}$$

29) A discrete random variable X takes values from 1 to 5 with probabilities as shown in the table. A student calculates the mean of X as 3.5 and her teacher calculates the variance of X as 1.5. Which of the following statements is true?

k	1	2	3	4	5
P(X=k)	0.1	0.2	0.4	0.2	0.1

a) Both the student and the teacher are right

b) Both the student and the teacher are wrong

c) The student is wrong but the teacher is right

d) The student is right but the teacher is wrong

30) If E denotes expectation, the variance of a random variable X is given by

a) $E[X^2] - E^2[X]$ c) $E[X^2]$

b) $E[X^2] + E^2[X]$ d) $E^2[X]$

Solution: Before we start the proof we need to know 3 properties of expectation

$$E[f(x) + g(x)] = E[f(x)] + E[g(x)] \quad (139)$$

If k is a constant value then

$$E[k \cdot g(x)] = k \cdot E[g(x)] \quad (140)$$
$$E[k] = k \quad (141)$$

Now variance of random X is given by

$Var(X) = E[(X-\mu)^2] \quad \text{where}\ \mu = E[X]$

$Var(X) = E[X^2 - 2\mu \cdot X + \mu^2]$

$$= E[X^2] - E[2\mu \cdot X] + E[\mu^2] \ \text{from (1)}$$

$$= E[X^2] - 2\mu \cdot E[X] + \mu^2 \ \text{from (2) and (3)}$$

$$= E[X^2] - 2\mu^2 + \mu^2 \quad (\because E[X] = \mu)$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - E^2[X] \quad (\because \ \mu = E[X])$$

31) An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is:

a) 0.5 b) 0.18 c) 0.12 d) 0.06

32) A probability density function is of the form
$p(x) = Ke^{-\alpha|x|}, x \in (-\infty, \infty)$
The value of K is

a) 0.5 b) 1 c) 0.5α d) α

33) Consider a binary digital communication system with equally likely 0's and 1's. When binary 0 is transmitted the voltage at the detector input can lie between the level -0.25V and +0.25V with equal probability: when binary 1 is transmitted, the voltage at the detector can have any value between 0 and 1V with equal probability. If the detector has a threshold of 2.0V (i.e., if the received signal is greater than 0.2V, the bit is taken as 1), the average bit error probability is

a) 0.15 b) 0.2 c) 0.05 d) 0.5

34) Let X and Y be two statistically independent random variables uniformly distributed in the range $(-1, 1)$ and $(-2, 1)$ respectively. Let $Z = X + Y$, then the probability that $[Z \leqslant -2]$ is

a) zero b) $\frac{1}{6}$ c) $\frac{1}{3}$ d) $\frac{1}{12}$

Solution:

1 Answer

Option (D) $\frac{1}{12}$

2 Solution

X and Y are two independent random variables. Let

$$p_X(x) = \Pr(X = x) \quad (142)$$
$$p_Y(y) = \Pr(Y = y) \quad (143)$$
$$p_Z(z) = \Pr(Z = z) \quad (144)$$

be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} p_X(x)\ dx = 1 \quad (145)$$
$$2 \times p_X(x) = 1 \quad (146)$$
$$p_X(x) = 1/2 \quad (147)$$

Similarly for Y we have,

$$\int_{-2}^{1} p_Y(y)\ dy = 1 \quad (148)$$
$$3 \times p_Y(y) = 1 \quad (149)$$
$$p_Y(y) = 1/3 \quad (150)$$

The density for X is

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \quad (151)$$

We have ,

$$Z = X + Y \iff z = x + y \iff x = z - y \tag{152}$$

The density of X can also be represented as,

$$p_X(z-y) = \begin{cases} \frac{1}{2} & -1 \le z-y \le 1 \\ 0 & otherwise \end{cases} \tag{153}$$

and the density of Y is,

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{154}$$

The density of Z i.e. $Z = X + Y$ is given by the convolution of the densities of X and Y

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z-y) p_Y(y)\, dy \tag{155}$$

From 153 and 154 we have,
The integrand is $\frac{1}{6}$ when,

$$2 \le y \le 1 \tag{156}$$
$$-1 \le z - y \le 1 \tag{157}$$
$$z - 1 \le y \le z + 1 \tag{158}$$

and zero, otherwise.
Now when $-3 \le z \le -2$ them we have,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{159}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{160}$$
$$= \frac{1}{6}(z+3) \tag{161}$$

For $-2 < z \le -1$,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{162}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{163}$$
$$= \frac{1}{6}(z+3) \tag{164}$$

For $-1 < z \le 0$,

$$p_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\, dy \tag{165}$$
$$= \frac{1}{6} \times (z + 1 - (z-1)) \tag{166}$$
$$= \frac{1}{3} \tag{167}$$

For $0 < z \le 2$,

$$p_Z(z) = \int_{z-1}^{1} \frac{1}{6}\, dy \tag{168}$$
$$= \frac{1}{6} \times (1 - (z-1)) \tag{169}$$
$$= \frac{1}{6}(2-z) \tag{170}$$

Therefore the density of Z is given by

$$p_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \le z \le -2 \\ \frac{1}{6}(z+3) & -2 < z \le -1 \\ \frac{1}{3} & -1 < z \le 0 \\ \frac{1}{6}(2-z) & 0 < z \le 2 \\ 0 & otherwise \end{cases} \tag{171}$$

The CDF of Z is defined as,

$$F_Z(z) = \Pr(Z \le z) \tag{172}$$

Now for $z \le -1$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\, dz \tag{173}$$
$$= \int_{-3}^{z} \frac{1}{6}(z+3)\, dz \tag{174}$$
$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Big|_{-3}^{z} \tag{175}$$
$$= \frac{1}{6} \times \left(\left(\frac{z^2}{2} + 3z\right) - \left(\frac{9}{2} - 9\right)\right) \tag{176}$$
$$= \frac{z^2 + 6z + 9}{12} \tag{177}$$

Similarly for $z \le 0$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\, dz \tag{178}$$
$$= \frac{1}{3} + \int_{-1}^{z} \frac{1}{3}\, dz \tag{179}$$
$$= \frac{z+2}{3} \tag{180}$$

finally for $z \leq 2$,

$$\Pr(Z \leq z) = \int_{-\infty}^{z} p_Z(z)\, dz \quad (181)$$

$$= \frac{2}{3} + \int_{0}^{z} \frac{1}{6}(2 - z)\, dz \quad (182)$$

$$= \frac{2}{3} + \frac{4z - z^2}{12} \quad (183)$$

$$= \frac{8 + 4z - z^2}{12} \quad (184)$$

The CDF is as below,

$$F_Z(z) = \begin{cases} 0 & z < 3 \\ \frac{z^2+6z+9}{12} & z \leq -1 \\ \frac{z+2}{3} & z \leq 0 \\ \frac{8+4z-z^2}{12} & z \leq 2 \\ 1 & z > 2 \end{cases} \quad (185)$$

So

$$\Pr(Z \leq -2) = F_Z(2) \quad (186)$$

$$= \frac{1}{12} \quad (187)$$

i.e. option (D).
The plot for PDF of Z can be observed at figure 10 and the plot for CDF of Z is at figure 11.

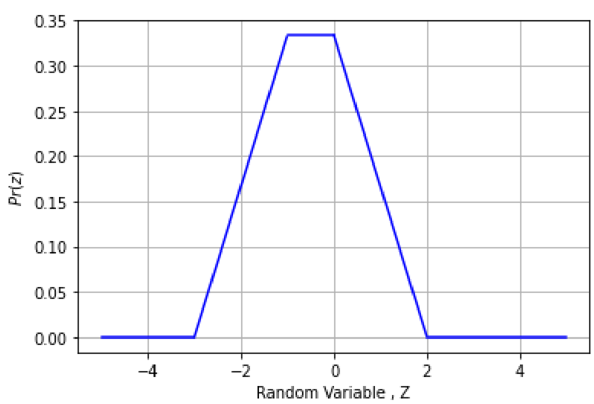


Fig. 10: The PDF of Z

35) Let X be the Gaussian random variable obtained by sampling the process at $t = t_i$ and let

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\, dy$$

The probability that $[X \leqslant 1]$ is ...

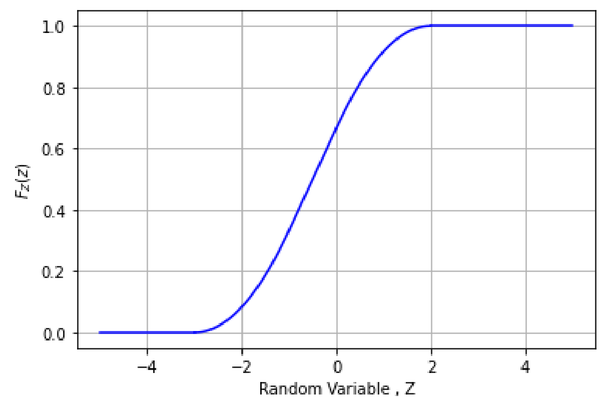


Fig. 11: The CDF of Z

36) Let Y and Z be the random variables obtained by sampling $X(t)$ at $t = 2$ and $t = 4$ respectively. Let W = Y-Z. The variance of W is

a) 13.36 b) 9.36 c) 2.64 d) 8.00

37) Let the random variable X represent the number of times a fair coin needs to be tossed till two consecutive heads appear for the first time. The expectation of X is......

38) Let $X \in [0, 1]$ and $Y \in [0, 1]$ be two independent binary random variables. If $P(X = 0) = p$ and $P(Y = 0) = q$, then $P(X + Y \geqslant 1)$ is equal to

a) $pq + (1 - p)(1 - q)$
b) pq
c) $p(1 - q)$
d) $1 - pq$

39) Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\widetilde{A}$ and $\widetilde{B}$ be their complements. Which one of the following statements is FALSE?

a) $P(A \cap B) = P(A)P(B)$
b) $P(A|B) = P(A)$
c) $P(A \cup B) = P(A) + P(B)$
d) $P(\widetilde{A} \cap \widetilde{B}) = P(\widetilde{A})P(\widetilde{B})$

Solution:

a) As A, B are independent events, By definition,

$$\Pr(A + B) = \Pr(A)\Pr(B)$$

Thus option 1 is true.

b)

$$\begin{aligned}\Pr(A|B) &= \frac{\Pr(A+B)}{\Pr(B)} \\ &= \frac{\Pr(A)\Pr(B)}{\Pr(B)} \\ &= \Pr(A)\end{aligned}$$

Thus option 2 is true.

c)

$$\begin{aligned}\Pr(AB) &= \Pr(A) + \Pr(B) - \Pr(A+B) \\ &= \Pr(A) + \Pr(B) - \Pr(A)\Pr(B)\end{aligned}$$

Thus option 3 is false.

d)

$$\begin{aligned}\Pr(A' + B') &= \Pr((AB)') \\ &= 1 - \Pr(AB) \\ &= 1 - \Pr(A) - \Pr(B) + \Pr(A+B) \\ &= (1 - \Pr(A))(1 - \Pr(B)) \\ &= \Pr(A')\Pr(B')\end{aligned}$$

Thus option 4 is true.

Hence, FALSE statement is option 3.

40) A digital communication system uses a repetition code for channel encoding/decoding. During transmission, each bit is repeated three times(0 is transmitted as 000, and 1 is transmitted as 111). It is assumed that the source puts out symbols independently and with equal probability. The decoder operates as follows: In a block of three received bits, if the number of zeros exceeds the number of ones, the decoder decides in favour of a 0, and if the number of ones exceeds the number of zeros, the decoder decides in favour of a 1. Assuming a binary symmetric channel with crossover probability p = 0.1, the average probability of error is ....

Solution: Let Y be the bit sent by the sender and X be the number of 1's received by the receiver and p = 0.1 is the crossover probability

Case 1: Y = 0

$$\Pr(X = i) = \binom{n}{i} \times p^i \times (1-p)^{n-i} \tag{188}$$

When $X \geq 2$ the receiver interprets it as 1, which is an error. And by Total Probability theorem we have

$$P_1 = \frac{P(X=2) + P(X=3)}{\sum_{i=0}^{3} P(X=i)} \tag{189}$$

where P_1 is the probability of error when Y = 0

Case 2: Y = 1

$$\Pr(X = i) = \binom{n}{i} \times p^{n-i} \times (1-p)^{i} \tag{190}$$

When $X \leq 1$ the receiver interprets it as 0, which is an error. And by Total Probability theorem we have

$$P_2 = \frac{\Pr(X=0) + \Pr(X=1)}{\sum_{i=0}^{3} \Pr(X=i)} \tag{191}$$

where P_2 is the probability of error when Y = 1

$$\sum_{i=0}^{3} \Pr(X=i) = 1 \times 0.9^3 + 3 \times 0.1 \times 0.9^2 + 3 \times 0.1^2 \times 0.9 + 1 \times 0.1^3 = 1 \tag{192}$$

$$P_1 = 0.028 \tag{193}$$

$$P_2 = 0.028 \tag{194}$$

The average probability is

$$P_{avg} = \Pr(Y=0) \times P_1 + \Pr(Y=1) \times P_2 = 0.028 \tag{195}$$

	X	0	1	2	3
Y=0	$\Pr(X)$	0.729	0.243	0.027	0.001
Y=1	$\Pr(X)$	0.001	0.027	0.243	0.729

TABLE IX: Probability of number of 1's recieved

41) Two random variables X and Y are distributed according to

$$f_{x,y}(x,y) = \begin{cases} (x+y) & 0 \leqslant x \leqslant 10 \leqslant y \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

The probability $P(X+Y \leqslant 1)$ is ....

42) Let X be a zero mean unit variance Gaussian random variable. $E[|X|]$ is equal to ....

43) If calls arrive at a telephone exchange such that the time of arrival of any call is independent of the time of arrival of earlier or future calls, the probability distribution function of the toatl number of calls in a fixed time interval will be

a) Poisson
c) Exponential
b) Gaussian
d) Gamma

Solution:

Symbol	Description	Property	Random
T	Total time period	$T = n\Delta t$	No
n	Total Number of intervals		No
Δt	One time interval	Δt=T/n	No
k	Number of calls arrived during the time interval (0,T)		Yes
t_i	Denotes the time of arrival of each call in interval $(0,T)$		Yes
p	Probability of receiving a call at time t_i		No
λ	Average number of calls $(0,T)$	$\lambda = np$	No
e	Euler's number		No

Lets denote the fixed time interval by [0,T]. To find the probability of k number of calls during this time interval, lets divide the interval into n parts of equal length Δt. Let us denote the probability of receiving a call at a particular time t_i by p. Suppose the telephone exchange receives an average of λ calls in time interval of length T.

Hence, we have

$$np = \lambda \tag{196}$$

$$\implies p = \frac{\lambda}{n} \tag{197}$$

In Fig. 12, the interval (0,T) has been divided

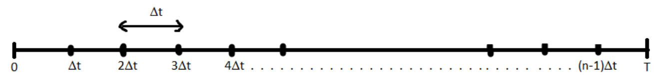


Fig. 12: Figure showing division of time intervals

into n equal parts, where length of each interval is Δt and the number of calls in each interval is a random variable. t_i where

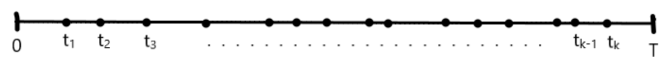


Fig. 13: Figure showing times of arrival of k calls

$i = \{1,2,3\cdots k\}$ are the time of arrival of k calls in the interval $(0,T)$.

A call has probability p for arriving at $t_i, \forall i = \{1,2,\cdots k\}$ and the probability of 1-p for not arriving at that instant.

In Binomial distribution we have certain number of intervals, i.e. n, with probability of arrival of each call as p and for a binomial random variable $X = \{0,1\cdots n\}$, the probability of call arriving in any k intervals is

$$\Pr(X=k) = {}^{n}C_k \cdot p^k \cdot (1-p)^k \tag{198}$$

But in Poisson distribution, we essentially have infinite intervals, so $n \to \infty$. Thus, the probability expression changes to:

$$\lim_{n\to\infty} \Pr(X=k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} \tag{199}$$

$$\lim_{n\to\infty} \Pr(X=k) = \left(\frac{\lambda^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k} \tag{200}$$

probability of k successes per period, with given parameter of λ.

$\therefore$The probability distribution function of the total number of calls in a fixed time interval will be **Poisson** distribution.
Answer: Option(A)

44) Consider a communication scheme where the binary valued signal X satisfies $P\{X = +1\} = 0.75$ and $P\{X = -1\} = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance σ^2. The received signal Y is fed to the threshold detector. The output of the threshold detector $\hat{X}$ is:

$$\hat{X} = \begin{cases} +1 & Y > \tau \\ -1 & Y \leqslant \tau \end{cases}$$

To achieve minimum probability of error $P\{\hat{X} \neq X\}$, the threshols τ should be

a) strictly positive
b) zero
c) strictly negative
d) strictly positive, zero or strictly negative depending on the nonzero value of σ^2

45) Consider a discrete-time channel $Y = X + Z$, where the additive noise Z is signal-dependent. In particular, given the transmitted symbol $X \in \{-a, +a\}$ at any instant, the noise sample Z is chosen independently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver.
When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^-8$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{\frac{-u^2}{2}} du\right.$$

, and for $v > 1$, use $\left. Q(v) \approx e^{\frac{-v^2}{2}}\right)$
When $\beta = -0.3$, the BER is closet to

a) 10^-7
b) 10^-6

Now lets take the limit of right-hand side one term at a time. We'll do this in three steps. The first step is to find the limit of

$$\begin{aligned}\lim_{n\to\infty}\frac{n!}{(n-k)!n^k} &= \lim_{n\to\infty}\frac{n(n-1)(n-2)..(n-k+1)}{n^k}\\ &= \lim_{n\to\infty}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)....\left(\frac{n-k+1}{n}\right)\\ &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{k-1}{n}\right)\\ &= 1\cdot 1\cdot 1........1\\ &= 1\end{aligned} \tag{201}$$

Now we have to find the limit of

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \tag{202}$$

We know that the definition e is given as

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \tag{203}$$

So, lets replace the value of $-\frac{n}{\lambda}$ by x in (202), we get

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda} \tag{204}$$

And the third part is to find the limit of

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} \tag{205}$$

As n approaches infinity, this term becomes 1^{-k} which is equal to one. So,

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = 1 \tag{206}$$

Now on substituting (201), (204) and (206) in equation (200), we get

$$\left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty}\frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} = \left(\frac{\lambda^k}{k!}\right)(1)\left(e^{-\lambda}\right)(1) \tag{207}$$

This just simplifies into

$$\Pr(X=k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{208}$$

(208) is equal to probability density function of Poisson distribution, which gives us

c) 10^-4 d) 10^-2

Solution:
Given that the threshold of the detector is zero. Define a detector function g such that

$$g(Y) = \begin{cases}+a & Y>0\\ -a & Y<0\end{cases} \tag{209}$$

It is given that $X \in \{-a, a\}$ is a random variable.

$$\therefore \Pr(X=a) = \Pr(X=-a) = \frac{1}{2} \tag{210}$$

Since the noise in the signal, Z is chosen independently from a Gaussian distribution with mean $\mu = \beta X$ and unit variance, it follows that

$$F_Z(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(z-\beta X)^2}{2}\right)dz \tag{211}$$

$$= \int_{-\infty}^{z-\beta X}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2}{2}\right)dz \tag{212}$$

$$= \int_{\beta X-z}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2}{2}\right)dz \tag{213}$$

$$= Q(\beta X - z) \tag{214}$$

Also, it is easy to see that

$$Q(-v) = 1 - Q(v)\ \forall\ v \in \mathbb{R} \tag{215}$$

The detector can record erroneous bits in the signal iff

$$X > 0\ ,\ g(Y) = -a \text{ (Call this BER}_{+a}\text{) or} \tag{216}$$

$$X < 0\ ,\ g(Y) = a \text{ (Call this BER}_{-a}\text{)} \tag{217}$$

$$\therefore \text{BER}_{+a} = \Pr(g(Y) = -a \mid X = a)\Pr(X=a) \tag{218}$$

$$= \Pr(Y<0 \mid X=a)\Pr(X=a) \tag{219}$$

$$= \frac{1}{2}\times\Pr(X+Z<0 \mid X=a) \tag{220}$$

$$= \frac{1}{2}\times F_Z(-a) \tag{221}$$

$$= \frac{1}{2}\times Q(\beta X + a) \quad \text{(From (214))} \tag{222}$$

$$= \frac{1}{2}\times Q(a(1+\beta)) \tag{223}$$

$$\text{BER}_{-a} = \Pr(g(Y) = a \mid X = -a)\Pr(X = -a) \tag{224}$$
$$= \Pr(Y > 0 \mid X = -a)\Pr(X = -a) \tag{225}$$
$$= \frac{1}{2} \times \Pr(X + Z > 0 \mid X = -a) \tag{226}$$
$$= \frac{1}{2} \times (1 - F_Z(a)) \tag{227}$$
$$= \frac{1}{2} \times (1 - Q(\beta X - a)) \quad \text{(From (214))} \tag{228}$$
$$= \frac{1}{2} \times Q(a(1+\beta)) \quad \text{(From (215))} \tag{229}$$

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \tag{230}$$
$$= Q(a(1+\beta)) \tag{231}$$

When $\beta = 0$, it is given that

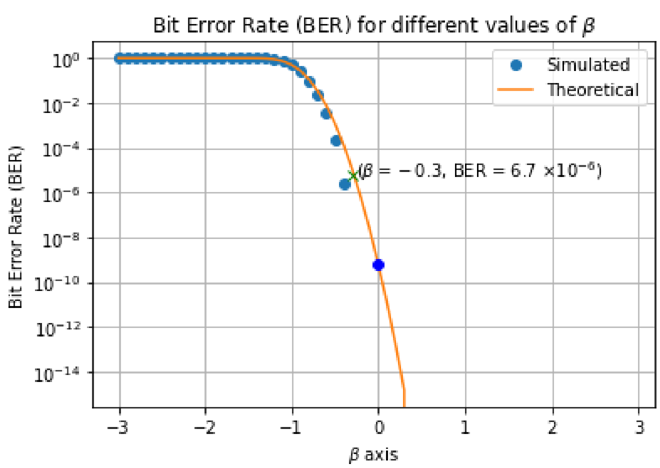


Fig. 14: Theory vs Simulated plot of BER

$$\text{BER} = Q(a) = 10^{-8} \tag{232}$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{233}$$
$$\Leftrightarrow a \approx 6.069 \tag{234}$$

When $\beta = -0.3$,

$$\text{BER} = Q(a(1+\beta)) = Q(6.069 \times (1 - 0.3)) \tag{235}$$
$$= Q(6.069 \times 0.7) \tag{236}$$
$$= Q(4.249) \tag{237}$$
$$\approx \exp(-\frac{4.249^2}{2}) \tag{238}$$
$$\approx 1.2 \times 10^{-4} \tag{239}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option 45c is correct.

46) Consider the random process
$X(t) = U + Vt$,
where U is a zero-mean Gaussian random variable and V is a random variable distributed between 0 and 2. Assume that U and V are statistically independent. The mean value of the random process at t=2 is........ **Solution:** Here U is a gaussian random variable of mean 0 and Let us consider V is uniformly distributed random variable in $(0, 2)$.

Random Variable	U	V	X(t)
Expected Value	0	1	t

TABLE X: Random Variables and Expected Values

From Table X we can deduce that,

$$E[X(t)] = E[U + Vt] \tag{240}$$
$$E[X(t)] = E[U] + tE[V] \tag{241}$$
$$E[X(t)] = 0 + t \tag{242}$$
$$E[X(t)] = t \tag{243}$$
$$E[X(2)] = 2 \tag{244}$$

$\therefore$ mean of random process X(t) at 2 is 2.

47) Consider the Z-channel given in Fig. 15. The input is 0 or 1 with equal probability.
If the output is 0, the probability that the input is also 0 equals......

48) If P and Q are two random events, then the following is TRUE:

a) Independence of P and Q implies that $\Pr P \cap Q = 0$

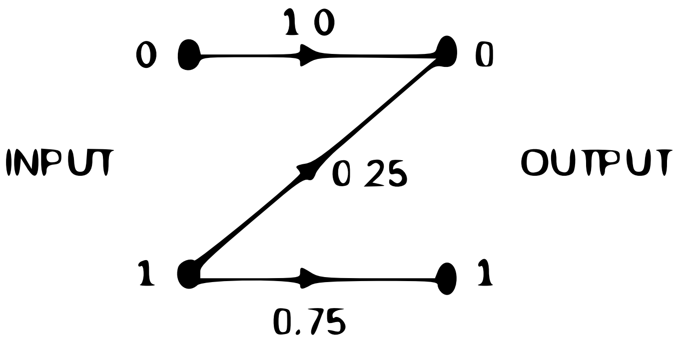


Fig. 15

b) $\Pr(P \cup Q) \geqslant \Pr(P) + \Pr(Q)$

c) If P and Q are mutually exclusive, then they must be independent

d) $\Pr(P \cap Q) \leqslant \Pr(P)$

Solution:

a) Independence of P and Q means if P happens, then outcome of Q won't be affected by that. so

$$\Pr(P/Q) = \Pr(P) \tag{245}$$
$$\frac{\Pr(PQ)}{\Pr(Q)} = \Pr(P) \tag{246}$$
$$\implies \Pr(PQ) = \Pr(P).\Pr(Q) \tag{247}$$

This is what we can say hence (A) is wrong

b) As

$$\Pr(P+Q) = \Pr(P) + \Pr(Q) - \Pr(PQ) \tag{248}$$
$$\Pr(P+Q) + \Pr(PQ) = \Pr(P) + \Pr(Q) \tag{249}$$
$$\Pr(PQ) \geq 0 \tag{250}$$
$$\implies \Pr(P) + \Pr(Q) \geq \Pr(P+Q) \tag{251}$$

Hence (B) is also wrong

c) When P and Q are mutually exclusive, then either P occurs or Q occurs but not both simultaneously.So if P happens, chance of Q happening gets ruled out and vice-versa.

Mutually exclusive refers

$$\Pr(PQ) = 0 \tag{252}$$
$$\Pr(PQ) \neq \Pr(P).\Pr(Q) \tag{253}$$

Hence, mutually exclusive events may not be independent.
Hence (C) is also wrong

d) As

$$\Pr(Q/P) = \frac{\Pr(PQ)}{\Pr(P)} \tag{254}$$

And

$$\Pr(Q/P) \leq 1 \tag{255}$$
$$\frac{\Pr(PQ)}{\Pr(P)} \leq 1 \tag{256}$$
$$\Pr(PQ) \leq \Pr(P) \tag{257}$$

Hence (D) is correct.

49) A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is:

a) $\frac{1}{8}$
b) $\frac{1}{2}$
c) $\frac{3}{8}$
d) $\frac{3}{4}$

50) The probability density function (PDF) of a random variable X is as shown in Fig. 16.
The corresponding cumulative distribution function (CDF) has the form

a) Fig. 17
b) Fig. 18
c) Fig. 19
d) Fig. 20

51) The distribution function $f_x(x)$ of a random variable X is shown in Fig. 21. The probability that X=1 is

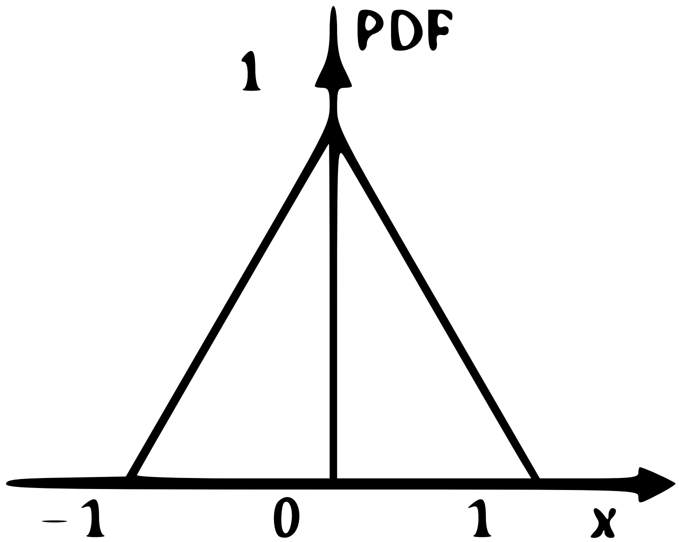


Fig. 16

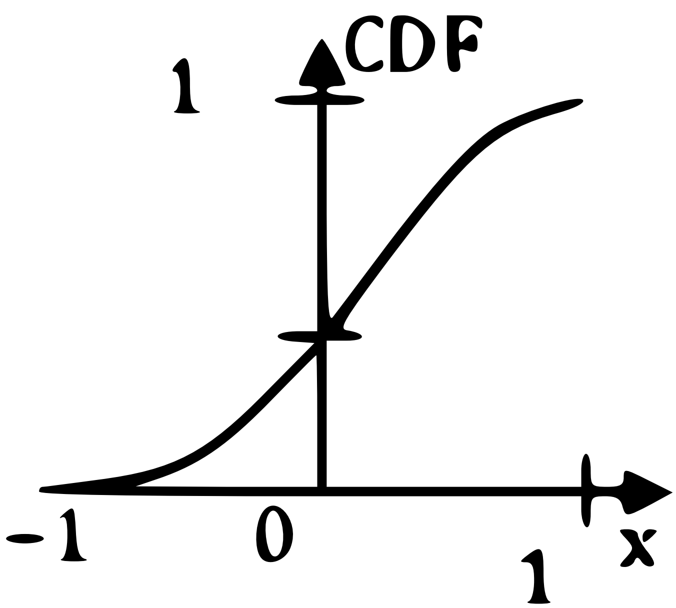


Fig. 17

a) Zero
b) 0.25
c) 0.55
d) 0.30

52) Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leq \ x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} < x \leq \ 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,the value of c is equal to ______________

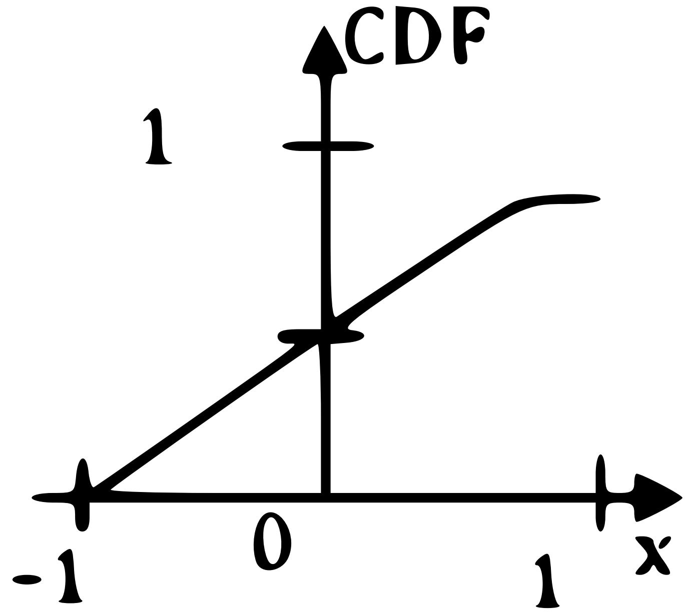


Fig. 18

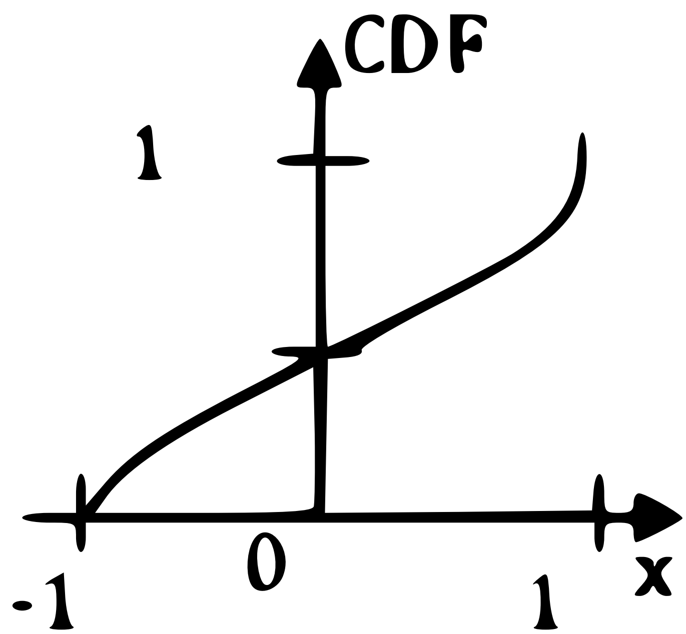


Fig. 19

53) Suppose X and Y are two random variables such that $aX + bY$ is a normal random variable for all $a, b \in \mathbb{R}$. Consider the following statements P,Q,R and S:
(P): X is a standard normal random variable.
(Q): The conditional distribution of X given Y is normal.
(R): The conditional distribution of X given $X + Y$ is normal.
(S): X - Y has mean 0.

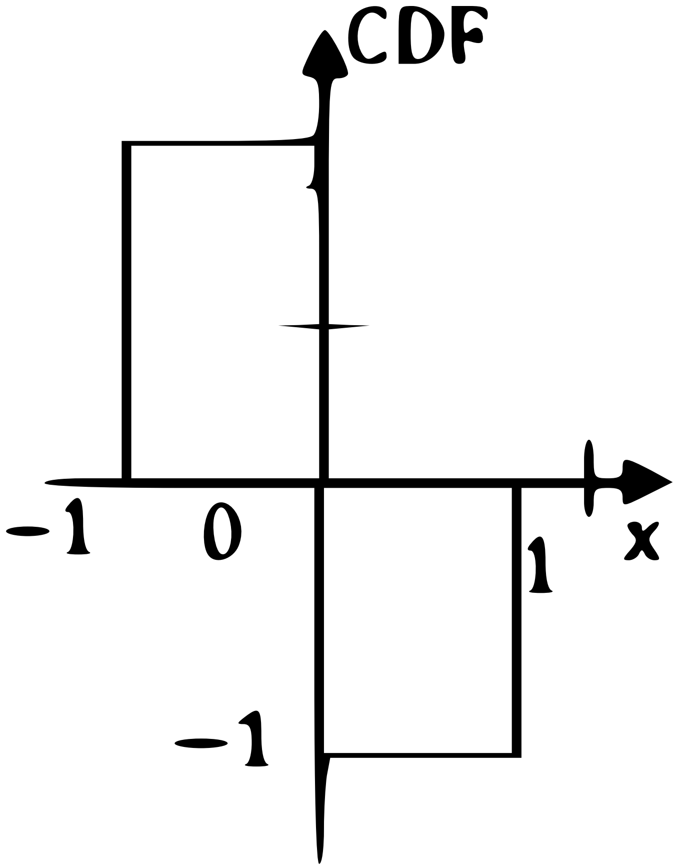


Fig. 20

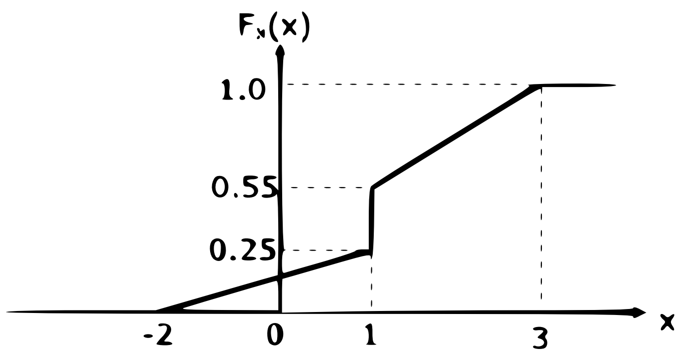


Fig. 21

Which of the above statements ALWAYS hold TRUE?

a) both P and Q
c) both Q and S
b) both Q and R
d) both P and S

54) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to ______________

55) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed,then $P(X_1 < X_2)$ is equal to ______________

Solution:

a) Given that X_1 is an exponential random variable. Let the P.D.F of X_1 be

$$p_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases} \tag{258}$$

C.D.F of x_1 is :

$$\begin{aligned} F_{X_1}(x_1) &= \int_{-\infty}^{x_1} p_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^{0} p_{X_1}(x_1) dx_1 + \int_{0}^{x_1} p_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^{0} 0 \times dx_1 + \int_{0}^{x_1} \lambda e^{-\lambda x_1} dx_1 \\ &= 1 - e^{-\lambda x_1} \end{aligned} \tag{259}$$

$$\text{As mean} = \lambda \tag{260}$$

$$\text{Given that mean} = 1 \tag{261}$$

$$\text{so } \lambda = 1 \tag{262}$$

b) Given that X_2 is an gamma random variable.Let the P.D.F of X_2 be:

$$p_{X_2}(x_2) = \begin{cases} \frac{a^b x_2^{b-1} e^{-a x_2}}{\Gamma(b)} & x_2 \geq 0 \\ 0 & x_2 < 0 \end{cases} \tag{263}$$

$$\text{Since mean} = \frac{b}{a} = 2 \tag{264}$$

$$\text{Also,variance} = \frac{b}{a^2} = 2 \tag{265}$$

From (264) and (265)

$$b = 2, a = 1 \tag{266}$$

Since the total probability of X_2 is 1 so,

$$\int_{-\infty}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{267}$$

$$\int_{-\infty}^{0} p_{X_2}(x_2)dx_2 + \int_{0}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{268}$$

$$\int_{-\infty}^{0} 0 \times dx_2 + \int_{0}^{\infty} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 = 1 \tag{269}$$

$$\frac{a^b}{\Gamma(b)} \int_{0}^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = 1 \tag{270}$$

$$\int_{0}^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = \frac{\Gamma(b)}{a^b} \tag{271}$$

now substituting $a + \lambda$ for a in (271) gives

$$\int_{0}^{\infty} x_2^{b-1} e^{-(a+\lambda)x_2} dx_2 = \frac{\Gamma(b)}{(a+\lambda)^b} \tag{272}$$

Now we have to find $P(X_1 < X_2)$

c) Given that X_1 and X_2 are independent random variables,so

$$P(X_1 < X_2|X_2) = F_{X_1}(X_2) = 1 - e^{-\lambda X_2} \tag{273}$$

Now,

$$P(X_1 < X_2) = \int_{0}^{\infty} F_{X_1}(X_2) \times p_{X_2}(x_2)dx_2 \tag{274}$$

from (263),(273)

$$P(X_1 < X_2) = \int_{0}^{\infty} (1 - e^{-\lambda X_2}) \times \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 \tag{275}$$

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \int_{0}^{\infty} x_2^{b-1} (e^{-ax_2} - e^{-(a+\lambda)x_2}) dx_2 \tag{276}$$

from (271) and (272)

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \left(\frac{\Gamma(b)}{a^b} - \frac{\Gamma(b)}{(a+\lambda)^b} \right) \tag{277}$$

$$P(X_1 < X_2) = 1 - \frac{a^b}{(a+\lambda)^b} \tag{278}$$

$$P(X_1 < X_2) = 1 - \left(\frac{a}{a+\lambda} \right)^b \tag{279}$$

from (262) and (266)

$$P(X_1 < X_2) = 1 - \left(\frac{1}{1+1} \right)^2 \tag{280}$$

$$P(X_1 < X_2) = 1 - \frac{1}{4} = \frac{3}{4} \tag{281}$$

Common Data for the next two Questions :

Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given $X = x$, is uniform on the interval $(x-1, x+1)$. Suppose $E(X) = 1$ and $Var(X) = \frac{5}{3}$.

56) The mean of the random variable Y is

a) $\frac{1}{2}$

b) 1

c) $\frac{3}{2}$

d) 2

Solution: We know that,

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \tag{282}$$

Given that $f_{Y|X=x}(y)$ is uniform over the interval (x-1,x+1).

$$\Rightarrow f_{Y|X=x}(y) = \begin{cases} \frac{1}{2} & y \in (x-1, x+1) \\ 0 & \text{otherwise} \end{cases} \tag{283}$$

Given $E(X) = 1$

$$\Rightarrow \int_{-\infty}^{\infty} x f_X(x) dx = 1 \tag{284}$$

Now consider $E(Y|X=x)$,

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy \qquad (285)$$

From (283) it simplifies to,

$$\Rightarrow E(Y|X=x) = \int_{-\infty}^{x-1} y f_{Y|X=x}(y)dy +$$

$$\int_{x-1}^{x+1} y f_{Y|X=x}(y)dy + \int_{x+1}^{\infty} y f_{Y|X=x}(y)dy \qquad (286)$$

$$\Rightarrow E(Y|X=x) = \int_{x-1}^{x+1} y\left(\frac{1}{2}\right)dy \qquad (287)$$

$$= x \qquad (288)$$

Now we can write ,

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X=x) f_X(x)dx \qquad (289)$$

$$= \int_{-\infty}^{\infty} x f_X(x)dx \qquad (290)$$

$$= E(X) \qquad (291)$$

From (284) we get

$$E(Y) = 1. \qquad (292)$$

57) The variance of the random variable Y is

a) $\frac{1}{2}$

b) $\frac{2}{3}$

c) 1

d) 2

58) Let the random variable X have the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

Then $P(2 \le X < 4)$ is equal to ______________

59) Let X be a random variable having the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{1}{3} & 1 \le x < 2 \\ \frac{1}{2} & 2 \le x < \frac{11}{3} \\ 1 & x \ge \frac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to ______________

60) Let X and Y be two random variables having the joint probability density function

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \le \frac{2}{3}|Y = \frac{3}{4}\right)$ is equal to ______________

a) $\frac{5}{9}$ b) $\frac{2}{3}$ c) $\frac{7}{9}$ d) $\frac{8}{9}$

61) Let $\Omega = (0,1]$ be the sample space and let $P(\cdot)$ be a probability function defined by

$$P((0,x]) = \begin{cases} \frac{x}{2} & 0 \le x < \frac{1}{2} \\ x & \frac{1}{2} \le x \le 1. \end{cases}$$

Then $P\left(\{\frac{1}{2}\}\right)$ is equal to ______________

62) Suppose the random variable U has uniform distribution on $[0,1]$ and $X = -2\log U$. The density of X is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$

63) Suppose X is a real-valued random variable.Which of the following values **CANNOT** be attained by $E[X]$ and $E[X^2]$, respectively?

a) 0 *and* 1
b) 2 *and* 3
c) $\frac{1}{2}$ *and* $\frac{1}{3}$
d) 2 *and* 5

Solution: We know that

$$var(X) = E[(X - E[X])^2] \tag{293}$$
$$var(X) = E[X^2] - (E[X])^2 \tag{294}$$

For uniform distribution in the interval $[a, b]$

$$var(X) = \frac{(b-a)^2}{12} \tag{295}$$

For uniform distribution,$(b - a)^2 \geq 0$
By definition of variance,it is average value of $(X - E[X])^2$.
Since $(X - E[X])^2 \geq 0$,average $E[(X - E[X])^2] \geq 0$.

$$\therefore var(X) \geq 0 \tag{296}$$
$$\therefore E[X^2] - (E[X])^2 \geq 0 \tag{297}$$

a) $E[X] = 0$ and $E[X^2] = 1$

$$E[X^2] - (E[X])^2 = 1 - 0 \tag{298}$$
$$= 1 \tag{299}$$
$$\therefore E[X^2] - (E[X])^2 \geq 0 \tag{300}$$

$\therefore E[X] = 0$ and $E[X^2] = 1$ can be attained

b) $E[X] = \frac{1}{2}$ and $E[X^2] = \frac{1}{3}$

$$E[X^2] - (E[X])^2 = \frac{1}{3} - \frac{1}{4} \tag{301}$$
$$= \frac{1}{12} \tag{302}$$
$$\therefore E[X^2] - (E[X])^2 \geq 0 \tag{303}$$

$\therefore E[X] = \frac{1}{2}$ and $E[X^2] = \frac{1}{3}$ can be attained

c) $E[X] = 2$ and $E[X^2] = 3$

$$E[X^2] - (E[X])^2 = 3 - 4 \tag{304}$$
$$= -1 \tag{305}$$
$$\therefore E[X^2] - (E[X])^2 \leq 0 \tag{306}$$

$\therefore E[X] = 2$ and $E[X^2] = 3$ cannot be attained

d) $E[X] = 2$ and $E[X^2] = 5$

$$E[X^2] - (E[X])^2 = 5 - 4 \tag{307}$$
$$= 1 \tag{308}$$
$$\therefore E[X^2] - (E[X])^2 \geq 0 \tag{309}$$

$\therefore E[X] = 2$ and $E[X^2] = 5$ can be attained

$\therefore E[X] = 2$ and $E[X^2] = 3$ cannot be attained

64) Let X_n denote the sum of points obtained when n fair dice are rolled together. The expectation and variance of X_n are

a) $\frac{7}{2}n$ and $\frac{35}{12}n^2$ respectively.
b) $\frac{7}{2}n$ and $\frac{35}{12}n$ respectively.
c) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively.
d) None of the above

65) Let X and Y be jointly distributed random variables having the joint probability density function
$f(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$
Then $P(Y > max(X, -X)) =$

a) $\dfrac{1}{2}$
b) $\dfrac{1}{3}$
c) $\dfrac{1}{4}$
d) $\dfrac{1}{6}$

66) Consider two identical boxes B_1 and B_2, where the box $B(i = 1, 2)$ contains $i+2$ red and $5-i-1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are drawn with

replacement from the box B_1, otherwise, two balls are drawn with replacement from the box B_2. The probability that the two drawn balls are of different colours is

a) $\frac{7}{25}$ c) $\frac{12}{25}$

b) $\frac{9}{25}$ d) $\frac{16}{25}$

Solution: Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the random variables of a die,

$$\Pr(X = N) = \begin{cases} \frac{1}{6} & 1 \leq N \leq 6 \\ 0 & otherwise \end{cases} \tag{310}$$

$$\Pr(X = m) . \Pr(X = n) = 0 \tag{311}$$

$\forall\ m, n \in \{1, 2, 3, 4, 5, 6\}$ as a single die cannot show more than one outcome at a roll.
Let $Y \in \{0, 1\}$ represent the die where,
$1 \implies$ the die with outcome $N = \{2, 4, 5, 6\}$,
$0 \implies N = \{1, 3\}$.

$$\Pr(Y = 1) = \\ \Pr((X = 2) + (X = 4) + (X = 5) + (X = 6)) \tag{312}$$

by using Boolean logic and (311),

$$\Pr(Y = 1) = \frac{2}{3} \tag{313}$$

$$\Pr(Y = 0) = 1 - \Pr(Y = 1) = \frac{1}{3} \tag{314}$$

$$\implies \Pr(B_1) = \Pr(Y = 1) = \frac{2}{3} \tag{315}$$

$$\implies \Pr(B_2) = \Pr(Y = 0) = \frac{1}{3} \tag{316}$$

Let $C \in \{0, 1\}$ where,
$0 \implies$ red balls,
$1 \implies$ white balls.

TABLE XI: Table of number of balls

Box	No. of red balls $(i + 2)$	No. of white balls $(5 - i - 1)$	Total balls
B_1	$n(C = 0\|B_1) = 3$	$n(C = 1\|B_1) = 3$	$n(C\|B_1) = 6$
B_2	$n(C = 0\|B_2) = 4$	$n(C = 1\|B_2) = 2$	$n(C\|B_2) = 6$

TABLE XII: Table of probability of taking balls from each box

Box	Probability of taking red ball	Probability of taking white ball
B_1	$\Pr(C = 0\|B_1) = 1/2$	$\Pr(C = 1\|B_1) = 1/2$
B_2	$\Pr(C = 0\|B_2) = 2/3$	$\Pr(C = 1\|B_2) = 1/3$

The probability of picking 2^{nd} ball is not effected by picking 1^{st} ball because the 2^{nd} ball is chose after replacement.
Selecting two balls with replacement is a Bernoulli distribution of 2 trails,

TABLE XIII: Table of no. of ways of selecting two different coloured balls

Cases	Trail 1	Trail 2
$(B_1, C = 0, C = 1)$	$\Pr(C = 0\|B_1)$	$\Pr(C = 1\|B_1)$
$(B_1, C = 1, C = 0)$	$\Pr(C = 1\|B_1)$	$\Pr(C = 0\|B_1)$
$(B_2, C = 0, C = 1)$	$\Pr(C = 0\|B_2)$	$\Pr(C = 1\|B_2)$
$(B_2, C = 1, C = 0)$	$\Pr(C = 1\|B_2)$	$\Pr(C = 0\|B_2)$

$$\implies \Pr((C = 0, C = 1)|B_1) = \\ \Pr(C = 0|B_1) . \Pr(C = 1|B_1) \\ + \Pr(C = 1|B_1) . \Pr(C = 0|B_1) \tag{317}$$

$$\Pr((C = 0, C = 1)|B_1) = \frac{1}{2} \tag{318}$$

$$\implies \Pr((C = 0, C = 1)|B_2) = \\ \Pr(C = 0|B_2) . \Pr(C = 1|B_2) \\ + \Pr(C = 1|B_2) . \Pr(C = 0|B_2) \tag{319}$$

$$\Pr((C = 0, C = 1)|B_1) = \frac{4}{9} \tag{320}$$

TABLE XIV: Table of variables description

Variables	Description
$\Pr((C = 0, C = 1)\|B_1)$	Probability of selecting two different coloured balls from box B_1
$\Pr((C = 0, C = 1)\|B_2)$	Probability of selecting two different coloured balls from box B_2
$\Pr(T)$	Total probability of selecting two different coloured balls

by using Bayes theorem,

$$\Pr(T) = \Pr((C=0, C=1)|B_1) \cdot \Pr(B_1) + \Pr((C=0, C=1)|B_2) \cdot \Pr(B_2) \quad (321)$$

$$\Pr(T) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{4}{9}\right)\left(\frac{1}{3}\right) \quad (322)$$

Hence, the probability of selecting two different coloured balls from the boxes is

$$\Pr(T) = \frac{13}{27} \quad (323)$$

Common Data for the next two Questions :

Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, \\ & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

67) The variance of the random variable X is

a) $\frac{1}{12}$
b) $\frac{1}{4}$
c) $\frac{7}{12}$
d) $\frac{5}{12}$

68) The covariance between the random variables X and Y

a) $\frac{1}{3}$
b) $\frac{1}{4}$
c) $\frac{1}{6}$
d) $\frac{1}{12}$

Common Data for the next two Questions :

Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

69) The value of a is

a) 4
b) 2
c) 1
d) 0.5

70) The value of $E(X|Y=2)$ is

a) 4
b) 3
c) 2
d) 1

71) Let X and Y be two random variables having the joint probability density function

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the conditional probability $P(X \leqslant \frac{2}{3}|Y = \frac{3}{4})$ is equal to

a) $\frac{5}{9}$
b) $\frac{2}{3}$
c) $\frac{7}{9}$
d) $\frac{8}{9}$

72) Let $\Omega = (0, 1]$ be the sample space and let $P(.)$ be a probability function defined by

$$P((0,x]) = \begin{cases} \frac{x}{2} & 0 \leqslant x < \frac{1}{2} \\ x & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

Then $P\left(\{\frac{1}{2}\}\right)$ is equal to.......

73) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leqslant x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leqslant x < 1 \\ 1 & x \geqslant 1 \end{cases}$$

Then $P(\frac{1}{4} < x < 1)$ is equal to........

Solution:

We know that,

$$P(p < X < q) = F(q^-) - F(p) \tag{324}$$
$$P\left(\frac{1}{4} < X < 1\right) = F(1^-) - F\left(\frac{1}{4}\right) \tag{325}$$
$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \tag{326}$$
$$= \frac{11}{16} \tag{327}$$
$$= 0.6875 \tag{328}$$

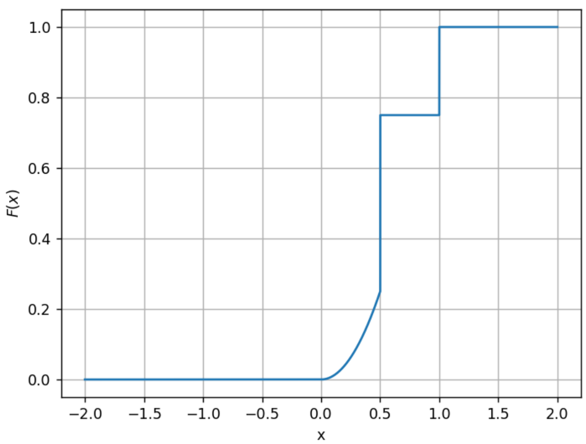


Fig. 22: The figure depicts the CDF of X

74) Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $P(X_1 < X_2)$ is equal to.....

Common Data for the next two Questions :

Let X and Y be two continuous random variables with the joint probability density function
$f(x, y) = \begin{cases} 2 & 0 < x + y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases}$

75) $P(X + Y < \frac{1}{2})$ is

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) $\frac{3}{4}$

d) 1

76) $E(X|Y = \frac{1}{2})$

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) 1

d) 2

Solution: Let X and Y be two continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} 2 & 0 < x + y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{329}$$

Then $E\left(X|Y = \frac{1}{2}\right)$ is

3 Solution

Given X and Y are two continuous random variables with joint probability density function,

$$f(x, y) = \begin{cases} 2 & 0 < x + y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{330}$$

We know that,
$0 < x+y < 1 \implies 0 < y < 1-x$ for $0 < x < 1$.
Then,

$$f_X(x) = \int f_{XY}(x, y)\, dy \tag{331}$$

$$= \int_0^{1-x} (2) dy \tag{332}$$

$$= 2(1 - x) \tag{333}$$

$$\implies f_X(x) = \begin{cases} 2(1 - x) & 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{334}$$

Similarly,
$0 < x+y < 1 \implies 0 < x < 1-y$ for $0 < y < 1$

Then,

$$f_y(y) = \int f_{XY}(x, y)\, dx \tag{335}$$

$$= \int_0^{1-y} (2) dx \tag{336}$$

$$= 2(1 - y) \tag{337}$$

$$\implies f_Y(y) = \begin{cases} 2(1 - y) & 0 \le y < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{338}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \tag{339}$$

$$= \begin{cases} \frac{2}{2(1-y)} & \text{if } 0 \le x + y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{340}$$

Then,

$$E(X|Y = y) = \int_{-\infty}^{\infty} (x) \left(\frac{1}{1 - y}\right) dx \tag{341}$$

$$= \frac{1}{1 - y} \int_0^{1-y} (x) dx \tag{342}$$

$$= \frac{1}{1 - y} \left[\frac{x^2}{2}\right]_0^{1-y} \tag{343}$$

$$\therefore E(X|Y = y) = \frac{1 - y}{2} \tag{344}$$

$$\implies E\left(X|Y = \frac{1}{2}\right) = \frac{1 - \frac{1}{2}}{2} \tag{345}$$

$$\therefore E\left(X|Y = \frac{1}{2}\right) = \frac{1}{4} \tag{346}$$

77) If a random variable X assumes only positive integral values, with the probability
$P(X = x) = \frac{2}{3}(\frac{1}{3})^{x-1}$, $x = 1, 2, 3, ...,$

then $E(X)$ is

a) $\frac{2}{9}$

b) $\frac{2}{3}$

c) 1

d) $\frac{3}{2}$

Solution: Let Y={0,1} be a set of random variables of a Bernoulli's distribution with 0 representing a loss and 1 a win and let $Y_i \in Y$ for i=1,2,3...,Y_i is the outcome of i^{th} try of choosing 0 or 1 from Y.

So the Random variable X is generated by assigning value of i to X where $Y_i = 1$ for the first time.

$$X = \{x : Y_{i=x} = 1, Y_{i<x} = 0\}$$
$$\implies X = \{Y_1 = 0, Y_2 = 0, Y_3 = 0, ..., Y_x = 1\}$$

For given bernouli's trail $p = \frac{2}{3}$ and $q = 1-p = \frac{1}{3}$. The given probability distribution is

$$P(X = x) = P(Y_{i=x} = 1)P(Y_{i<x} = 0)$$
$$\implies P(X = x) = p(1 - p)^{x-1}$$
$$\implies P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1}$$

The expectation value of X represented by E(X) is given by

$$E(X) = \sum_{i=1}^{\infty} Pr(x = i) \times i$$

Let S=E(X),

$$\implies E(X) = S = \sum_{i=1}^{\infty} Pr(x = i) \times i \quad (347)$$

$$\implies S = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times i \quad (348)$$

$$\implies S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times i \quad (349)$$

Multiplying (348) with $\frac{1}{3}$ on both sides gives

$$\frac{1}{3}S = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \times i \quad (350)$$

In (349) $\sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \times i$ can be written as $\sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i-1)$

$$\implies \frac{1}{3}S = \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i-1) \quad (351)$$

(349)-(351) gives :$\frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i - (i-1))$ (352)

$$\implies \frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \quad (353)$$

$$\implies S = 1 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i} \quad (354)$$

$$\implies S = 1 + \frac{1/3}{1 - \frac{1}{3}} \quad (355)$$

$$\implies S = \frac{3}{2} \quad (356)$$

The Variance Var(X) is given by $\sum x^2P(x) - E(X)$ for the given distribution,

$$Var(X) = \sum_{i=1}^{\infty} i^2P(x = i) - E(X) \quad (357)$$

$$\text{let } S = \sum_{i=1}^{\infty} i^2P(x = i) = \sum_{i=1}^{\infty} i^2\frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \quad (358)$$

$$S/3 = \sum_{i=1}^{\infty} i^2\frac{2}{3}\left(\frac{1}{3}\right)^{i} = \sum_{i=0}^{\infty} i^2\frac{2}{3}\left(\frac{1}{3}\right)^{i} \quad (359)$$

$$= \sum_{i=1}^{\infty} (i-1)^2\frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \quad (360)$$

(358)-(360) gives us

$$\frac{2S}{3} = \sum_{i=1}^{\infty}(i^2 - (i-1)^2)\frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{361}$$

$$S = \sum_{i=1}^{\infty}(2i-1)\left(\frac{1}{3}\right)^{i-1} \tag{362}$$

$$\implies S = 3\sum_{i=1}^{\infty}\frac{2}{3}\left(\frac{1}{3}\right)^{i-1} i - \sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^{i-1} \tag{363}$$

$$\implies S = 3E(X) - \frac{1}{1-1/3} \tag{364}$$

$$\implies S = \frac{9}{2} - \frac{3}{2} = 3 \tag{365}$$

From (365) and (357) we can write

$$Var(X) = 3 - \frac{3}{2} = \frac{3}{2}$$

From (356) we can say that the expectation value of X given by E(X)=S=$\frac{3}{2}$ and $Var(X) = \frac{3}{2}$

78) The joint probability density function of two random variables X and Y is given as

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leqslant x \leqslant 10 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere} \end{cases}$$

$E(X)$ and $E(Y)$ are, respectively,

a) $\frac{2}{5}$ and $\frac{3}{5}$

b) $\frac{3}{5}$ and $\frac{3}{5}$

c) $\frac{3}{5}$ and $\frac{6}{5}$

d) $\frac{4}{5}$ and $\frac{6}{5}$

Solution: For a continuous joint probability distribution $E(X)$
and $E(Y)$ are obtained using the following equations
(366) and (367)

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \cdot f(x,y)\,dx\,dy \tag{366}$$

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \cdot f(x,y)\,dx\,dy \tag{367}$$

Using equation (366) $E(X)$ is calculated as

$$\begin{aligned}
E(X) &= \int_0^1\int_0^1 x\frac{6}{5}\left(x+y^2\right)dx\,dy + 0 \\
&= \int_0^1 \frac{6}{5}\left(\int_0^1 x^2\,dx\right) + \frac{6}{5}y^2\left(\int_0^1 x\,dx\right)dy \\
&= \int_0^1 \frac{6}{5}\left(\frac{1}{3}\right) + \frac{6}{5}y^2\left(\frac{1}{2}\right)dy \\
&= \frac{2}{5}\int_0^1 dy + \frac{3}{5}\int_0^1 y^2\,dy \\
&= \frac{2}{5} + \frac{3}{5}\left(\frac{1}{3}\right) \\
E(X) &= \frac{3}{5}
\end{aligned}$$

Using equation (367) $E(Y)$ is calculated as

$$\begin{aligned}
E(Y) &= \int_0^1\int_0^1 y\frac{6}{5}\left(x+y^2\right)dx\,dy + 0 \\
&= \int_0^1 \frac{6}{5}x\left(\int_0^1 y\,dy\right) + \frac{6}{5}\left(\int_0^1 y^3\,dy\right)dx \\
&= \int_0^1 \frac{6}{5}x\left(\frac{1}{2}\right) + \frac{6}{5}\left(\frac{1}{4}\right)dx \\
&= \frac{3}{5}\int_0^1 x\,dx + \frac{3}{10}\int_0^1 dx \\
&= \frac{3}{5}\left(\frac{1}{2}\right) + \frac{3}{10} \\
E(Y) &= \frac{3}{5}
\end{aligned}$$

$$\therefore E(X) = \frac{3}{5} \text{ and } E(Y) = \frac{3}{5}$$

Hence the answer is **option b**

79) Suppose the random variable U has uniform distribution on $[0,1]$ and $X = -2\log U$. The density of X is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$

Solution: U - uniformly distributed random variable on $\in$ [0,1]. Probability density function of U is:

$$f_U(u) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{368}$$

X is given by :

$$X = -2\ln(U) \tag{369}$$
$$\implies 0 \le X \le \infty \tag{370}$$

CDF of X is defined as

$$\begin{aligned} F_X(x) &= \Pr(X \le x) && (371) \\ &= \Pr(-2\ln(U) \le x) && (372) \\ &= \Pr(\ln(U) \ge (-x)/2) && (373) \\ &= \Pr(U \ge \exp(-x/2)) && (374) \\ &= 1 - \Pr(U \le exp(-x/2)) && (375) \\ &= 1 - exp(-x/2) && (376) \end{aligned}$$

where x $\in$ [0,∞]
PDF of X :

$$\begin{aligned} f_X(x) &= \frac{d(F_X(x))}{dx} && (377) \\ &= \frac{1}{2}exp((-x)/2) && (378) \end{aligned}$$

we have

$$0 \le X \le \infty \tag{379}$$

$$f_X(x) = \begin{cases} \frac{1}{2}exp(\frac{-x}{2}) & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{380}$$

$\therefore$ answer will be option (3)

80) Suppose X is a real-valued random variable. Which of the following values CANNOT be attained by $E[X]$ and $E[X^2]$, respectively?

a) 0 and 1

b) 2 and 3

c) $\frac{1}{2}$ and $\frac{1}{3}$

d) 2 and 5

Solution: The variance of a distribution is given by

$$\sigma^2 = E[X^2] - E[X]^2 \tag{381}$$

As variance is always positive,

$$E[X^2] - E[X]^2 \ge 0 \tag{382}$$

is a necessary condition for any real valued random variable. Computing the value of $E[X^2] - E[X]^2$ for the options, we have

(A) 0 and 1

$$\implies E[X^2] - E[X]^2 = 1 - 0^2 = 1 \ge 0 \tag{383}$$

(B) 2 and 3

$$\implies E[X^2] - E[X]^2 = 3 - 2^2 = -1 \le 0 \tag{384}$$

(C) $\frac{1}{2}$ and $\frac{1}{3}$

$$\implies E[X^2] - E[X]^2 = \frac{1}{3} - \frac{1}{2}^2 = \frac{1}{12} \ge 0 \tag{385}$$

(D) 2 and 5

$$\implies E[X^2] - E[X]^2 = 5 - 2^2 = 1 \ge 0 \tag{386}$$

81) Probability density function $p(x)$ of a random variable x is as shown below. The value of α is

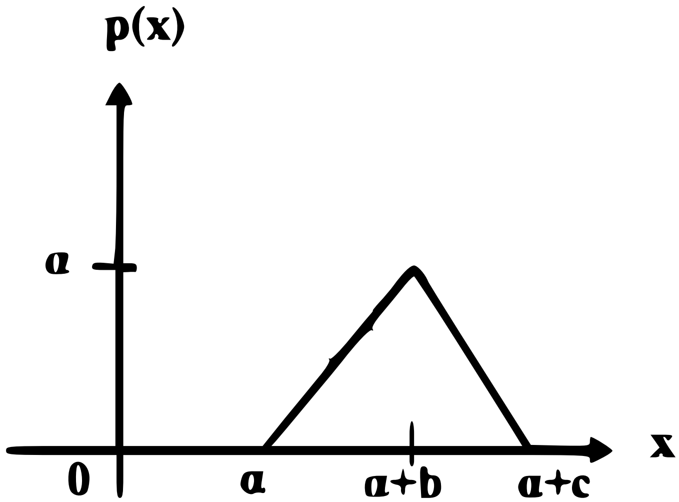


Fig. 23

a) $\frac{2}{c}$

b) $\frac{1}{c}$

c) $\frac{2}{(b+c)}$

d) $\frac{1}{(b+c)}$

82) A player throws a ball at a basket kept at a distance. The probability that the ball falls into the basket in a single attempt is 0.1. The player attempts to throw the ball twice. Considering each attempt to be independent, the probability that this player puts the ball into the basket only in the second attempt is.........
Solution: Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.
$X = k$ represents $k-1$ failures were obtained before getting 1 success. p represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \tag{387}$$

Using (387) we get

$$\Pr(X=2) = (1-p)^{k-1} \times p$$
$$= (0.9) \times 0.1 = 0.09 \tag{388}$$

83) A screening test is carried out to detect a certain disease. It is found that 12% of the positive reports and 15% of the negative reports are incorrect. Assuming that the probability of a person getting positive report is 0.01, the probability that a person tested gets an incorrect report is ...
Solution: Let $X \in \{0,1\}$ represent the random variable, where 0 represents the case where a person gets a positive report while 1 represents the case where a person gets a negative report. From the question,

$$\Pr(X=0) = 0.01 \tag{389}$$
$$\Pr(X=0) + \Pr(X=1) = 1 \tag{390}$$
$$\Pr(X=1) = 1 - 0.01 = 0.99 \tag{391}$$

Let $Y \in \{0,1\}$ represent the random variable, where 0 represents a correct report whereas 1 represents an incorrect report.

$$\Pr(Y=1|X=0) = 12\% = 0.12 \tag{392}$$
$$\Pr(Y=1|X=1) = 15\% = 0.15 \tag{393}$$

Then, from total probability theorem,

$$\Pr(Y=1) = \Pr(Y=1, X=0) + \Pr(Y=1, X=1) \tag{394}$$

Using Bayes theorem,

$$\Pr(Y=1) = \Pr(Y=1|X=0) \times \Pr(X=0) + \Pr(Y=1|X=1) \times \Pr(X=1) \tag{395}$$

$$\Pr(Y=1) = 0.12 \times 0.01 + 0.15 \times 0.99 \tag{396}$$
$$= 0.0012 + 0.1485 \tag{397}$$
$$= 0.1497 \tag{398}$$

84) Shaquille O' Neal is a 60% career free throw shooter, meaning that he successfully makes 60 free throws out of 100 attempts on average. What is the probability that he will successfully make exactly 6 free throws in 10 attempts?

A) 0.2508

B) 0.2816

C) 0.2934

D) 0.6000

Solution: Let

$$X_i \in \{0, 1\} \tag{399}$$

represent the i^{th} free throw, where 1 represents a successful free throw attempt and 0 represents an unsuccessful attempt. Let

$$X = \sum_{i=1}^{n} X_i \tag{400}$$

where n is the total number of free throws. Then, X has a binomial distribution with

$$\Pr(X = k) = {}^{n}C_k p^k q^{n-k} \tag{401}$$

Where,

$$p = \frac{6}{10} \tag{402}$$

$$q = 1 - p = \frac{4}{10} \tag{403}$$

$$n = 10 \tag{404}$$

from the given information. Then,

$$\Pr(X = 6) = {}^{10}C_6 \left(\frac{6}{10}\right)^6 \left(\frac{4}{10}\right)^4 \tag{405}$$

On simplifying we get,

$$\Pr(X = 6) = 0.2508 \tag{406}$$

Therefore, the probability that he will successfully make exactly 6 free throws in 10 attempts is 0.2508 and hence option (A) is correct.

85) Consider a sequence of tossing a fair coin where outcomes of tosses are independent.The probability of getting the head for the third time in the fifth toss is

(A) $\frac{5}{16}$

(B) $\frac{3}{16}$

(C) $\frac{3}{5}$

(D) $\frac{9}{16}$

Solution: Let the random variable $X \in \{0, 1\}$ denotes head and tail in a toss.As both are equally probable.

$$\Pr(X = 0) = \frac{1}{2} \tag{407}$$

$$\Pr(X = 1) = \frac{1}{2} \tag{408}$$

Event	Description
A	nth toss is a head
B	Exactly k-1 heads in first four tosses
C	nth toss is the third head

TABLE XV: Description of events used in problem

$$\Pr(A) = \Pr(X = 1) = \frac{1}{2} \tag{409}$$

$$\Pr(B) = \frac{{}^{n-1}C_{k-1}}{2^{n-1}} \tag{410}$$

$$C = AB \tag{411}$$

$$\Pr(C) = \Pr(AB) \tag{412}$$

As A and B are independent events.

$$\Pr(C) = \Pr(A)\Pr(B) \tag{413}$$

$$= \frac{1}{2} \times \frac{{}^{n-1}C_{k-1}}{2^{n-1}} \tag{414}$$

$$= \frac{{}^{n-1}C_{k-1}}{2^{n}} \tag{415}$$

Here n=5,k=3

$$\Pr(C | n = 5, k = 2) = \frac{{}^{4}C_2}{2^5} \tag{416}$$

$$= \frac{6}{32} \tag{417}$$

Therefore probability of getting the head for the third time in the fifth toss is $\frac{3}{16}$.

86) A random variable X takes values -1 and +1 with probabilities 0.2 and 0.8, respectively. It is transmitted across a channel which adds noise N, so that the random variable at the channel output is $Y = X+N$. The noise N is independent of X, and is uniformly distributed over the

interval [-2 , 2]. The receiver makes a decision

$$\hat{X} = \begin{cases} -1, & \text{if} \quad Y \le \theta \\ +1, & \text{if} \quad Y \ge \theta \end{cases}$$

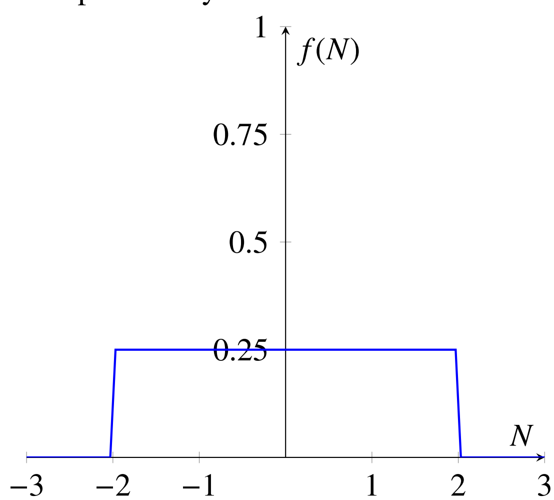
where the threshold $\theta \in [-1, 1]$ is chosen so as to minimize the probability of error $\Pr\left(\hat{X} \neq X\right)$. The minimum probability of error, rounded off to 1 decimal place, is ...

Solution:

We know that

$$X \in \{-1, +1\} \tag{418}$$
$$N \in [-2, 2] \tag{419}$$
$$Y = X + N \tag{420}$$
$$\Pr(X = -1) = 0.2 \tag{421}$$
$$\Pr(X = +1) = 0.8 \tag{422}$$

Since N is uniformly distributed
$\therefore$ the probability distribution function of N is:



The cdf of this uniform probability distribution function is

$$F_X(x) = \int_{-\infty}^{x} f(N)dN \tag{423}$$
$$= \int_{-2}^{x} \frac{1}{4} dN \tag{424}$$
$$= \frac{x+2}{4} \tag{425}$$

For $X \neq \hat{X}$ we need to check for each case. Using equation (425)

$$\therefore \Pr(N > \theta + 1) = 1 - \Pr(N < \theta + 1) \tag{426}$$
$$= 1 - F_X(\theta + 1) \tag{427}$$
$$= 1 - \frac{\theta + 3}{4} \tag{428}$$
$$= \frac{1}{4}(1 - \theta) \tag{429}$$

$$\therefore \Pr(N < \theta - 1) = F_X(\theta - 1) \tag{430}$$
$$= \frac{1}{4}(1 + \theta) \tag{431}$$

The probability of error:

$$\Pr\left(\hat{X} \neq X\right) = P(-1) \cdot P(\theta < -1 + N) + P(1) \cdot P(\theta > N + 1) \tag{432}$$

Substituting (429) and (431) in (432). We get:

$$\Pr\left(\hat{X} \neq X\right) = 0.2 \cdot \frac{1}{4}(1 - \theta) + 0.8 \cdot \frac{1}{4}(1 + \theta) \tag{433}$$

On simplifying the equation we get

$$\Pr\left(\hat{X} \neq X\right) = \frac{1}{4} + \frac{3}{20}\theta \tag{434}$$

Since this is a linear equation in θ, the minimum will occur at boundary points. Putting $\theta = +1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.4 \tag{435}$$

but on putting $\theta = -1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.1 \tag{436}$$

Hence the value of probability of error is:

$$\therefore \Pr\left(\hat{X} \neq X\right) = 0.1 \tag{437}$$

87) You have gone to a cyber-cafe with a friend. You found that the cyber-café has only three terminals. All terminals are unoccupied. You and your friend have to make a random choice of selecting a terminal. What is the probability that both of you will NOT select the same terminal?
Solution: There are three terminals, each with an equal
probability of $\frac{1}{3}$ to be picked.

Defining random variables $X_1, X_2 \in \{0,1,2\}$
Where,
$X_i = 0$ when ith man picks first terminal.
$X_i = 1$ when ith man picks second terminal.
$X_i = 2$ when ith man picks third terminal.

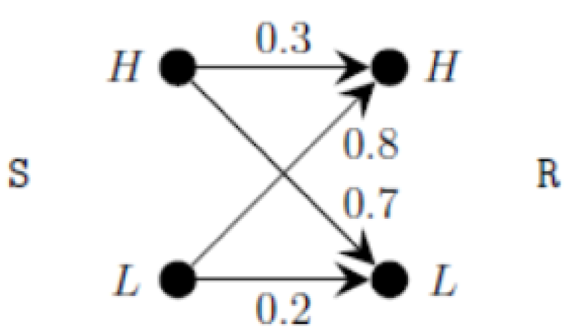
$$\Pr(X_1 \neq X_2) = 1 - \Pr(X_1 = X_2). \quad (438)$$

$$\implies \Pr(X_1 = X_2) = \sum_{j=1}^{3} \Pr(X_1 = X_2 = j) \quad (439)$$

$$\implies \Pr(X_1 = X_2) = \sum_{j=1}^{3} \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{1}{3} \quad (440)$$

$$\therefore \Pr(X_1 \neq X_2) = \frac{2}{3}. \quad (441)$$

88) A sender(S) transmits a signal,which can be one of two kinds: H and L with probabilities 0.1 and 0.9 respectively, to a receiver (R).
In the graph below,the weight of an edge(u,v) is the probability of receiving v when u is transmitted,where u,v $\in$ {H,L}.For example the probability that the received signal is L given the transmitted signal is H is 0.7. If the received



signal is H, the probability that the transmitted signal was H is ______ ?
Solution:
In our problem we have a binary channel which is not symmetric as crossover probabilities differ
Let $A \in \{0,1\}$ represent the random variable, where 0 represents H being transmitted, 1 represents L being transmitted.
Let $B \in \{0,1\}$ represent the random variable, where 0 represents H being received, 1 represents L being received.

TABLE XVI: Probability for random variables

$\Pr(A = 0)$	0.1	$\Pr(A = 1)$	0.9
$\Pr(B = 0 \mid A = 0)$	0.3	$\Pr(B = 0 \mid A = 1)$	0.8
$\Pr(B = 1 \mid A = 0)$	0.7	$\Pr(B = 1 \mid A = 1)$	0.2

Now we need to find $\Pr(A = 0|B = 0)$
Using Bayes theorem

$$\Pr(A = 0|B = 0) = \frac{\Pr(A = 0) \times \Pr(B = 0|A = 0)}{\sum_{i=0}^{1} \Pr(A = i) \times \Pr(B = 0|A = i)} \quad (442)$$

Putting in values given in question

$$\Pr(A = 0|B = 0) = \frac{1}{25} = 0.04 \quad (443)$$

The probability that transmitted signal was H is 0.04

89) A box has ten light bulbs out of which two are defective, Two light bulbs are drawn from this box one after the other without replacement. The probability that both light bulbs drawn are not defective is

A) $\frac{8}{45}$ B) $\frac{28}{45}$ C) $\frac{16}{25}$ D) $\frac{4}{5}$

Solution: Let $X_i \in \{0,1\}$ represent the i^{th} draw, where 0 denotes a defective bulb and 1 denotes a non-defective bulb.

TABLE XVII

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	2/90	16/90
$X_2 = 1$	16/90	56/90

Table II represents the probabilities of all possible cases when two bulbs are drawn one by one without replacement. Probability that both of the bulbs are non-defective (by substituting values from table II)

$$= \Pr(X_2 = 1|X_1 = 1)\Pr(X_1 = 1) \quad (444)$$

$$= \frac{56}{90} \quad (445)$$

$$= \frac{28}{45} \quad (446)$$

So the correct option is (B)

90) Three fair dies are rolled simultaneously. The probability of getting a sum of 5 is

a) $\frac{1}{108}$

b) $\frac{1}{72}$

c) $\frac{1}{54}$

d) $\frac{1}{36}$

Solution:

Let $X_i \in \{1,2,3,4,5,6\}$, i = 1,2,3, be the random variables representing the outcome for each die. As the dies are fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \quad (447)$$

Let X be a random variable denotes the desired outcome,

$$X = X_1 + X_2 + X_3 \quad (448)$$
$$\implies X \in \{3,4,\cdots,18\} \quad (449)$$

We have to find $P_X(n) = \Pr(X_1 + X_2 + X_3 = n)$

$$\begin{aligned} p_X(n) &= \Pr(X_1 + X_2 + X_3 = n) \\ &= \Pr(X_1 + X_2 = n - X_3) \\ &= \Sigma_k \Pr(X_1 + X_2 = n - k | X_3 = k)\, p_{X_3}(k) \quad (450) \end{aligned}$$

As X_1, X_2, X_3 are independent, After unconditioning

$$\Pr(X_1 + X_2 = n - k | X_3 = k) = \Pr(X_1 + X_2 = n - k) \quad (451)$$

Using (451) in (450)

$$\begin{aligned} p_X(n) &= \Sigma_k \Pr(X_1 + X_2 = n - k | X_3 = k)\, p_{X_3}(k) \\ &= \Sigma_k \Pr(X_1 + X_2 = n - k)\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a \Pr(X_1 = n - k - a | X_2 = a) \Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a \Pr(X_1 = n - k - a) \Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k (\Sigma_a p_{X_1}(n - k - a)\, p_{X_2}(a))\, p_{X_3}(k) \quad (452) \end{aligned}$$

Equation (452) can be written as follows using convolution operation,

$$\begin{aligned} p_X(n) &= \Sigma_k (\Sigma_a p_{X_1}(n - k - a)\, p_{X_2}(a))\, p_{X_3}(k) \\ &= p_{X_1}(n) * p_{X_2}(n) * p_{X_3}(n) \quad (453) \end{aligned}$$

The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \quad (454)$$

From (447) and (454),

$$\begin{aligned} P_{X_1}(z) = P_{X_2}(z) &= P_{X_3}(z) \\ &= \frac{1}{6} \sum_{n=1}^{6} z^{-n} \quad (455) \\ &= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \quad (456) \end{aligned}$$

upon summing up the geometric progression. From (453),

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n) * p_{X-3}(n), \quad (457)$$
$$P_X(z) = P_{X_1}(z) P_{X_2}(z) P_{X_3}(z) \quad (458)$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (456) and (458),

$$\begin{aligned} P_X(z) &= \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^3 \quad (459) \\ &= \frac{1}{216} \frac{z^{-3}\left(1 - 3z^{-6} + 3z^{-12} - z^{-18}\right)}{\left(1 - z^{-1}\right)^3} \quad (460) \end{aligned}$$

Using the fact that,

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \quad (461)$$
$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \quad (462)$$
$$n^2 u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \quad (463)$$
$$(n^2 + n) u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{2z^{-1}}{\left(1 - z^{-1}\right)^2} \quad (464)$$

after some algebra, it can be shown that,

$$\frac{1}{216\times 2}\Big[\Big((n-2)^2+n-2\Big)u(n-2) \\ -3\Big((n-8)^2+n-8\Big)u(n-8) \\ +3\Big((n-14)^2+n-14\Big)u(n-14) \\ -\Big((n-20)^2+n-20\Big)u(n-20)\Big] \\ \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{216}\frac{z^{-3}\left(1-3z^{-6}+3z^{-12}-z^{-18}\right)}{(1-z^{-1})^3} \tag{465}$$

where

$$u(n)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases} \tag{466}$$

From (454),(460) and (465),

$$p_X(n)=\frac{1}{216\times 2}\Big[\Big((n-2)^2+n-2\Big)u(n-2) \\ -3\Big((n-8)^2+n-8\Big)u(n-8) \\ +3\Big((n-14)^2+n-14\Big)u(n-14) \\ -\Big((n-20)^2+n-20\Big)u(n-20)\Big] \tag{467}$$

From (466) and (467),

$$p_X(n)=\begin{cases}0 & n<3\\ \frac{n^2-3n+2}{432} & 3\leq n\leq 8\\ \frac{42n-2n^2-166}{432} & 8<n\leq 14\\ \frac{n^2-39n+380}{432} & 14<n\leq 18\\ 0 & n>18\end{cases} \tag{468}$$

We need probability of getting sum of 5,

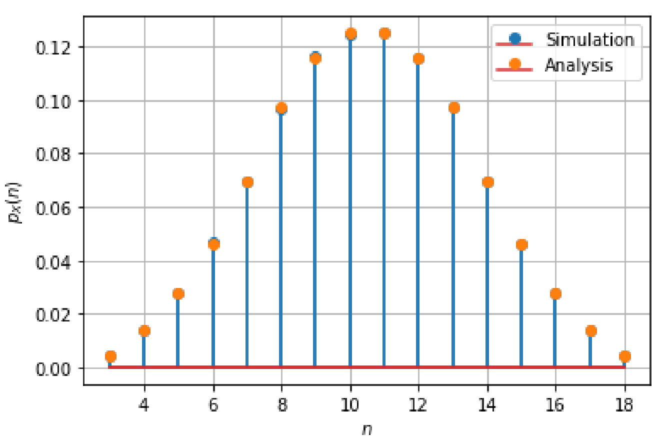


Fig. 24: Probability mass function of X (simulations are close to analysis)

$\implies$ n=5

from (468) and using n=5,

$$p_X(5)=\frac{5^2-3(5)+2}{432} \tag{469}$$

$$p_X(5)=\frac{12}{432} \tag{470}$$

$$p_X(5)=\frac{1}{36} \tag{471}$$

Therefore the probability of getting a sum of 5 when three fair dies are rolled is $\frac{1}{36}$.

Ans: Option (D)

91) What is the chance that a leap year,selected at random,will contain 53 Saturdays?

a) $\frac{2}{7}$

b) $\frac{3}{7}$

c) $\frac{1}{7}$

d) $\frac{5}{7}$

Solution:

92) Let $\mathcal{R}$ be the set of all binary relations on the set $\{1,2,3\}$. Suppose a relation is chosen from $\mathcal{R}$ at random. The probability that the chosen relation is reflexive is?

Solution: Let A be a set of n numbers. No. of pairs formed from elements of A:

$$^nC_1\times {}^nC_1=n^2 \tag{472}$$

For each pair we have 2 choices, whether to include it in the relation or not.

$\therefore$ Number of binary relations on A:

$$2\times 2\times \ldots\ n^2 \text{ times } = 2^{n^2} \tag{473}$$

Definition 1. *A reflexive relation is one in which every element maps to itself, i.e., a relation R on set A is reflexive if $(a,a)\in R\ \forall\ a\in A$.*

For example, consider the set $A=\{1,2,3\}$. A possible reflexive relation on A is R_1

$= \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$ as every element in A is related to itself in R_1 while relation $R_2 = \{(1,1),(2,2),(1,2)\}$ is not a reflexive relation on A as $3 \in A$ but $(3,3) \notin R_2$. In a reflexive relation, out of the n^2 pairs (17), n have to be included (n pairs of the form (a,a)) which means there is only 1 way to include them. For the remaining $n^2 - n$ pairs we have 2 choices, whether to include it in the relation or not.
$\therefore$ Number of reflexive relations are:

$$1 \times 2^{n^2-n} = 2^{n^2-n} \tag{474}$$

Let $X \in \{0,1\}$ be a random variable where 0 represents reflexive relation chosen from $\mathcal{R}$ and 1 represents non-reflexive relation chosen from $\mathcal{R}$. In this case, n=3.

$$\Pr(X=0) = \frac{2^{n^2-n}}{2^{n^2}} = \frac{2^6}{2^9} \tag{475}$$

$$\therefore \text{ Answer } = \frac{1}{8} \tag{476}$$

93) If P and Q are two random events, then the following is true:
   (a) Independence of P and Q implies that probability $(P \cap Q) = 0$

   (b) Probability $(P \cup Q) \leq$ Probability (P) + Probability(Q)

   (c) If P and Q are mutually exclusive, then they must be independent

   (d) Probability $(P \cap Q) \leq$ Probability (P)

   Solution: For two random events A and B that are independent, we know that,

$$Pr(AB) = Pr(A)Pr(B) \tag{477}$$

and for two mutually exclusive events C and D,

$$Pr(CD) = 0 \tag{478}$$

(a) Independence of P and Q implies that the occurrence of one is unaffected by the other.

$$\Rightarrow Pr(PQ) = Pr(P)Pr(Q) \tag{479}$$

The given option will be true only when either Pr(P) or Pr(Q) will be zero, therefore, (a) is incorrect.

(b) From set theory,

$$A \cup B = A + B - A \cap B \tag{480}$$

$$\Rightarrow Pr(P+Q) = Pr(P) + Pr(Q) - Pr(PQ) \tag{481}$$

$$\Rightarrow Pr(P+Q) \leq Pr(P) + Pr(Q) \tag{482}$$

thus, (b) is incorrect.

(c) Two events can be both mutually exclusive and independent only when one of them have a zero probability. Since it isn't necessary that $Pr(P) = 0$ or $Pr(Q) = 0$, (c) is incorrect.

(d) The set P will have either have the same or more elements than the set $P \cap Q$

$$Pr(PQ) \leq Pr(P) \tag{483}$$

(d) is correct.

Thus, the only correct option is (d).

94) A diagnostic test for a certain disease is 90% accurate. That is, the probability of a person having (respectively, not having) the disease tested positive (respectively, negative) is 0.9. Fifty percent of the population has the disease. What is the probability that a randomly chosen person has the disease given that the person

tested negative?
Solution: Let X and Y be two Bernoulli random variables such that X,Y$\in\{0,1\}$ and as given fifty percent of the population has the disease, the probability mass function of X is

$$p_X(n) = \Pr(X = n) = \begin{cases} 0.5 & n = 1 \\ 0.5 & n = 0 \\ 0 & otherwise \end{cases} \tag{484}$$

where X denotes the health status of a person(X=1 if person is healthy and X=0 if person is diseased) and Y denotes the diagnostic test result (Y=1 if it is positive and Y=0 if it is negative).
Given the probabilities of,

$$\Pr(Y = 1|X = 0) = 0.9 \tag{485}$$
$$\Pr(Y = 0|X = 1) = 0.9 \tag{486}$$

we need to find $\Pr(X = 0|Y = 0)$,

$$\Pr(X = 0|Y = 0) = \frac{\Pr(X = 0 \cap Y = 0)}{\Pr(Y = 0)} \tag{487}$$
$$\Pr(X = 0|Y = 0) = \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\Pr(Y = 0)} \tag{488}$$

$$\Pr(Y = 0) = \Pr(Y = 0|X = 1)\Pr(X = 1) + \Pr(Y = 0|X = 0)\Pr(X = 0) \tag{489}$$

Using (484),(485) and (486) in (489),

$$\Pr(Y = 0) = 0.9(0.5) + (1 - 0.9)0.5$$
$$\Pr(Y = 0) = 0.5 \tag{490}$$

Using (484),(485) and (490) in (488)

$$\Pr(X = 0|Y = 0) = \frac{(1 - 0.9)0.5}{0.5} \tag{491}$$
$$\Pr(X = 0|Y = 0) = 0.1 \tag{492}$$

95) Box-S has 2 white and 4 black balls and box-T has 5 white and 3 black balls.A ball is drawn at
random from box-S and put in box-T.Subsequently,the probability of drawing a white ball from box-T is? (rounding off to 2 decimal places)
Solution: Box-0 has 2 white and 4 black balls.
Box-1 has 5 white and 3 black balls.

Event	definition
W	Event of transfering white balls from box-0 to box-1
B	Event of transfering black balls from box-0 to box-1
C	Event of drawing white balls from box-1
$\Pr(W = 1)$	Probability of transfering one whiteball from box-0 to box-1
$\Pr(B = 1)$	Probability of transfering one blackball from box-0 to box-1
$\Pr(C = 1\|W = 1)$	Probability of drawing a whiteball from box-1 after transfering white ball to box-1.
$\Pr(C = 1\|B = 1)$	Probability of drawing a whiteball from box-1 after transfering black ball to box-1.

TABLE XVIII: Table 1

Probability	value
$\Pr(W = 1)$	$\frac{1}{3}$
$\Pr(B = 1)$	$\frac{2}{3}$
$\Pr(C = 1\|W = 1)$	$\frac{6}{9}$
$\Pr(C = 1\|B = 1)$	$\frac{5}{9}$

TABLE XIX: Table 2

$$\Pr(\text{drawn ball is white}) = \Pr(C = 1) \tag{493}$$
$$\tag{494}$$

From Baye's theorem

$$\Pr(C = 1) = \Pr(C = 1|W = 1) \times \Pr(W = 1) + \Pr(C = 1|B = 1) \times \Pr(B = 1) \tag{495}$$

Substiting values from table (XIX) in (495)

$$\Pr(C = 1) = \frac{6}{9} \times \frac{1}{3} + \frac{5}{9} \times \frac{2}{3} \tag{496}$$
$$= \frac{16}{27} \tag{497}$$

96) A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected

and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

Solution: Let $X \in \{0, 1\}$ be the random variable where X=0 represents that the first removed ball is white. Let $Y \in \{0, 1\}$ be the random variable, where Y=1 represents that the second removed ball is red.

After the first ball is removed (given to be white which means X=0), number of white balls reduces to 3 and total number of balls reduces to 6.

Probability that the second removed ball is red when the first removed ball is white is

$$\Pr(Y = 1|X = 0) = \frac{3}{6} = \frac{1}{2} \tag{498}$$

So,

$$\Pr(Y = 1|X = 0) = 0.5 \tag{499}$$

$\therefore$ The answer is option (C) $\frac{1}{2}$.

97) Two dice are thrown simultaneously. The probability that the product of the numbers appearing on the top faces of the dice is a perfect square is

(A)$\frac{1}{9}$ (B) $\frac{2}{9}$ (C)$\frac{1}{3}$ (D)$\frac{4}{9}$

Solution: Let X be a random variable which is equal to 1, when the product of the numbers appearing on the top faces of the dice is a perfect square and 0 when it is not a perfect square.

The total no. of possible outcomes is 36.

Outcomes corresponding to $X = 1$ are listed in table XX The total no. of favourable outcomes are 8. Therefore we have,

$$\Pr(X = 1) = \frac{8}{36} \tag{500}$$
$$= \frac{2}{9} \tag{501}$$

Similarly we have that the probability of not getting a perfect square as a product i.e. $X = 0$

$$\Pr(X = 0) = 1 - \Pr(X = 1) \tag{502}$$
$$= 1 - \frac{2}{9} \tag{503}$$
$$= \frac{7}{9} \tag{504}$$

Squares	Favourable outcomes
1	(1,1)
4	(1,4) , (2,2) , (4,1)
9	(3,3)
16	(4,4)
25	(5,5)
36	(6,6)

TABLE XX: Outcomes for X=1

98) Consider that X and Y are independent continuous valued random variables with uniform PDF given by $X \sim U(2, 3)$ and $Y \sim U(1, 4)$. Then $\Pr(Y \leq X)$ is equal to .....

Solution:

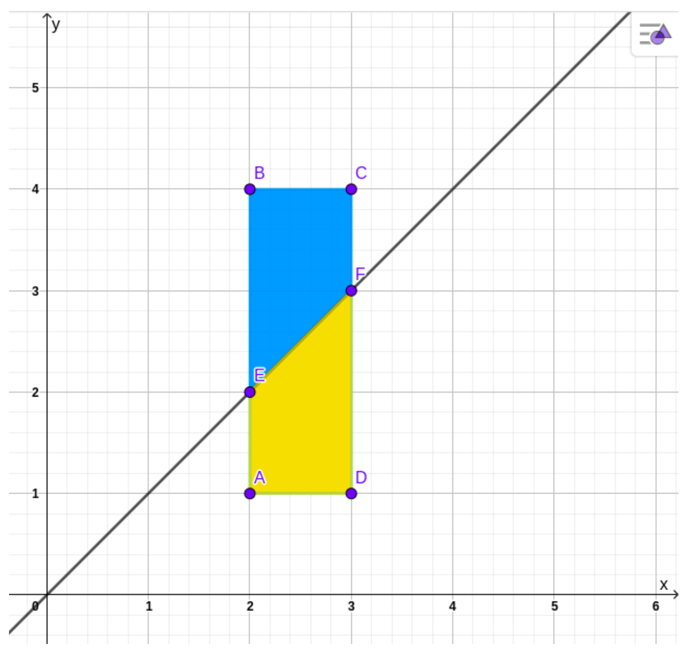


Fig. 25: Probability Distribution of (X, Y)

In figure 25, rectangle ABCD represents sample space of (X, Y). $Y \leq X$ for any point (X, Y) if and only if the point lies on or below line EF. Therefore

$$\Pr(Y \leq X) = \frac{Area\ of\ AEFD}{Area\ of\ ABCD} \tag{505}$$
$$= \frac{1}{2} \tag{506}$$

Alternately, we have PDF and CDF of X and Y given by

$$f_X(x) = \begin{cases} 1 & 2 \le x \le 3 \\ 0 & otherwise \end{cases} \quad (507)$$

$$F_X(x) = \begin{cases} 0 & x < 2 \\ x-2 & 2 \le x \le 3 \\ 1 & x > 3 \end{cases} \quad (508)$$

$$f_Y(x) = \begin{cases} 1 & 1 \le x \le 4 \\ 0 & otherwise \end{cases} \quad (509)$$

$$F_Y(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x-1}{3} & 1 \le x \le 4 \\ 1 & x > 4 \end{cases} \quad (510)$$

Thus

$$\Pr(Y \le X) = \int_{-\infty}^{\infty} F_Y(x) f_X(x) dx \quad (511)$$
$$= \int_2^3 \frac{x-1}{3} dx \quad (512)$$
$$= \frac{1}{2} \quad (513)$$

99) What is the chance that a leap year,selected at random,will contain 53 Saturdays?

a) $\frac{2}{7}$

b) $\frac{3}{7}$

c) $\frac{1}{7}$

d) $\frac{5}{7}$

Solution:
Let X be a random variable
We Define, X ∈ 0,1

P(X = 0)	denotes for 52 Saturday
P(X = 1)	denotes for 53 Saturdays

TABLE XXI: $\Pr(X = x)$

$$\implies \text{Remaining Days} = 366 - 364 = 2 \quad (514)$$
$$\implies \Pr(X = 1) = \frac{2}{7} \quad (515)$$

∴ The correct answer is **Option A**

100) Let X be the Poisson random variable with parameter $\lambda = 1$. Then, the probability $\Pr(2 \le X \le 4)$ equals ....
Solution:
Let

$$X \in \{0, 1, 2, 3, 4, 5...\} \quad (516)$$

We know that, for a poisson random variable X with a given parameter λ, probability of $X = k$ is:

$$\Pr(X = k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \quad (517)$$

CDF is:

$$F(X = k) = \sum_{x=0}^{k} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \quad (518)$$

And also,

$$\Pr(x < X \le y) = F(y) - F(x) \quad (519)$$

Now by using (519),

$$\Pr(2 \le X \le 4) = \Pr(1 < X \le 4) \quad (520)$$
$$= F(4) - F(1) \quad (521)$$
$$= \frac{65}{24e} - \frac{2}{e} \quad (522)$$
$$= \frac{17}{24e} \quad (523)$$

101) Let the probability density function of random variable,X,be given as:

$f_x(x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}u(-x)$

where u(x) is the unit step function.Then the value of a and Prob$\{X \le 0\}$, respectively,are:

(A) $2,\frac{1}{2}$

(B) $4,\frac{1}{2}$

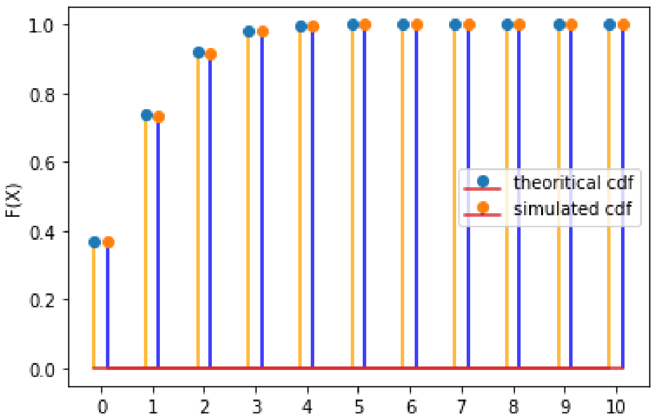


Fig. 26: Theoretical CDF vs Simulated CDF

(C) $2,\frac{1}{4}$

(D) $4,\frac{1}{4}$

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1. \tag{524}$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\infty} f_x(x)\,dx = 1 \tag{525}$$

$$\int_{-\infty}^{0} ae^{4x}\,dx + \int_{0}^{\infty} \frac{3}{2}e^{-3x}\,dx = 1 \tag{526}$$

The expression (526) was written from (525) since,

$$u(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Simplifying (526) we have:

$$\int_{-\infty}^{0} ae^{4x}\,dx + \int_{0}^{\infty} \frac{3}{2}e^{-3x}\,dx = 1$$

$$\implies a\left[\frac{e^{4x}}{4}\right]_{-\infty}^{0} + \frac{3}{2}\left[\frac{e^{-3x}}{-3}\right]_{0}^{\infty} = 1 \tag{527}$$

$$\implies a\left[\frac{1}{4} - 0\right] - \frac{1}{2}\left[0 - 1\right] = 1 \tag{528}$$

$$\implies \frac{a}{4} + \frac{1}{2} = 1 \implies a = 2 \tag{529}$$

Therefore,

$$f_x(x) = \begin{cases} \frac{3}{2}e^{-3x}, & \text{for } x \geq 0 \\ 2e^{4x}, & \text{for } x < 0 \end{cases} \tag{530}$$

The plot for PDF of X can be observed at figure 27

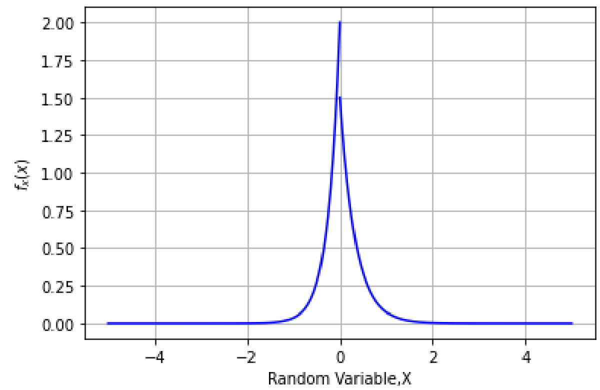


Fig. 27: The PDF of X

The CDF of X is defined as follows:

$$F_X(x) = \Pr\left(X \leq x\right) \tag{531}$$

Now for $x < 0$,

$$\Pr\left(X \leq x\right) = \int_{-\infty}^{x} f_x(x)\,dx \tag{532}$$

$$= \int_{-\infty}^{x} 2e^{4x}\,dx \tag{533}$$

$$= 2\left[\frac{e^{4x}}{4}\right]_{-\infty}^{x} \tag{534}$$

$$= 2\left[\frac{e^{4x}}{4} - 0\right] \tag{535}$$

$$= \frac{e^{4x}}{2} \tag{536}$$

Similarly for $x \geq 0$,

$$\Pr\left(X \leq x\right) = \int_{-\infty}^{x} f_x(x)\,dx \tag{537}$$

$$= \int_{-\infty}^{0} 2e^{4x}\,dx + \int_{0}^{x} \frac{3}{2}e^{-3x}\,dx \tag{538}$$

$$= 2\left[\frac{e^{4x}}{4}\right]_{-\infty}^{0} + \left[\frac{-e^{-3x}}{2}\right]_{0}^{x} \tag{539}$$

$$= 2\left[\frac{1}{4} - 0\right] - \frac{1}{2}\left[e^{-3x} - 1\right] \tag{540}$$

$$= 1 - \frac{e^{-3x}}{2} \tag{541}$$

The CDF of X is as below:

$$F_X(x) = \begin{cases} 1 - \frac{e^{-3x}}{2}, & \text{for } x \geq 0 \\ \frac{e^{4x}}{2}, & \text{for } x < 0 \end{cases} \tag{542}$$

The plot for CDF of X can be observed at figure 27.

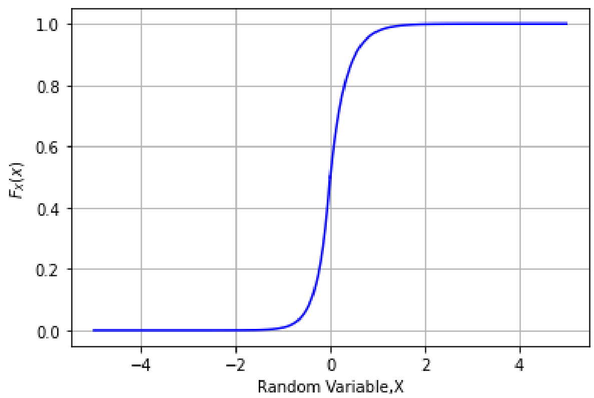


Fig. 28: The CDF of X

$$\therefore \Pr(X \leq 0) = F_X(0) = \frac{1}{2} \tag{543}$$

102) Suppose p is the number of cars per minute passing through a certain road junction between 5 PM and 6 PM, and p has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

a) $8/(2e^3)$

b) $9/(2e^3)$

c) $17/(2e^3)$

d) $26/(2e^3)$

Solution:
Probability of Poison Distribution is,

$$\Pr(X = p) = \frac{e^{-\mu}\mu^p}{p!} \tag{544}$$

Here, p refers to no. of cars per minute, $p \in \{0, 1, 2, \ldots, \infty\}$ Mean of poison distribution,

$$\mu = 3 \tag{545}$$

$$\Pr(X = p) = \frac{e^{-3}3^p}{p!} \tag{546}$$

by Boolean logic,

TABLE XXII: Table of probability of no. of cars passing per minute

p	0	1	2	3	...
$\Pr(X = p)$	$1/e^3$	$3/e^3$	$9/(2e^3)$	$9/(2e^3)$	...

$$\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \tag{547}$$

$$\Pr(X < 3) = \frac{17}{2e^3} \tag{548}$$

Option (C) is correct

103) An automobile plant contracted to buy shock absorbers from two suppliers X and $Y.X$ supplies 60% and Y supplies 40% of the shock absorbers. All shock absorbers are subjected to a quality test. The ones that pass the quality test are considered reliable. Of X's shock absorbers, 96% are reliable. Of Y's shock absorbers, 72% are reliable. The probability that a randomly chosen shock absorber, which is found to be reliable, is made by Y is

a) 0.288

b) 0.334

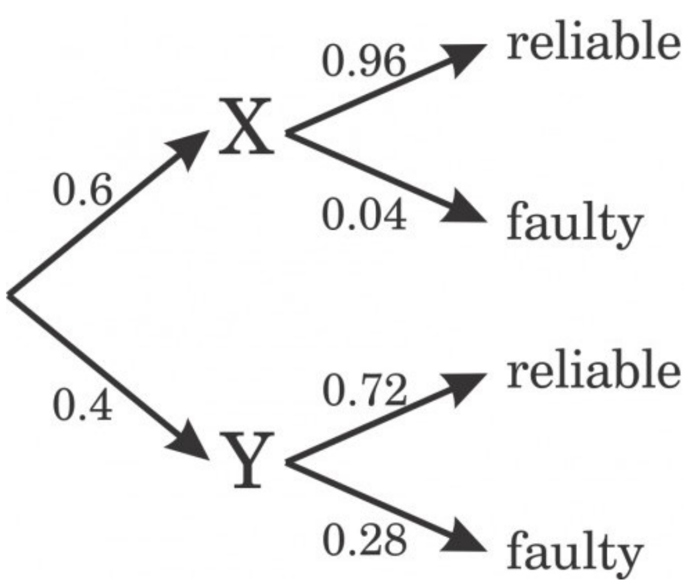
c) 0.667

d) 0.720

Solution: Let Consider, Bernoulli random variables say X, Y and R. Required probability is

	Refer to probability that product	Result
$Pr(X = 1)$	from supplier X	0.6
$Pr(Y = 1)$	from supplier X	0.4
$Pr(R = 1)$	is reliable	
$Pr(R = 0)$	is faulty	
$Pr(R = 1/X = 1)$	from supplier X is reliable	0.96
$Pr(R = 1/Y = 1)$	from supplier Y is reliable	0.72

TABLE XXIII: probability of random variables.



$Pr(Y=1|R=1)$.So,

$$Pr(Y=1|R=1)=\frac{Pr(Y=1,R=1)}{Pr(R=1)} \tag{549}$$

$$=\frac{Pr(Y=1)Pr(R=1/Y=1)}{Pr(X=1)Pr(R=1/X=1)+Pr(Y=1)P(R=1/Y=1)} \tag{550}$$

$$=\frac{(0.4)(0.72)}{(0.6)(0.96)+(0.4)(0.72)}=0.334 \tag{551}$$

104) The probability of a resistor being defective is 0.02. There are 50 such resistors in a circuit. The probability of two or more defective resistors in the circuit (round off to two decimal places) is — **Solution:** Consider, Probability of a defective resistor $= P = 0.02 = \frac{1}{50}$.
Total number of resistors = n = 50.
Let X be number of defective resistors.
By Binomial distribution,

$$Pr(X=k)=\binom{n}{k}(P)^k(1-P)^{n-k} \tag{552}$$

$$Pr(X=0)=\binom{50}{0}\left(\frac{1}{50}\right)^0\left(1-\frac{1}{50}\right)^{50-0} \tag{553}$$

$$\implies Pr(X=0)=\left(\frac{49}{50}\right)^{50} \tag{554}$$

$$Pr(X=1)=\binom{50}{1}\left(\frac{1}{50}\right)^1\left(1-\frac{1}{50}\right)^{50-1} \tag{555}$$

$$\implies Pr(X=1)=\left(\frac{49}{50}\right)^{49} \tag{556}$$

$$\Pr(X\geq 2)=1-\Pr(X<2)$$
$$\Pr(X\geq 2)=1-(\Pr(X=0)+\Pr(X=1))$$
$$\Pr(X\geq 2)=0.2642$$

105) A person who speaks truth 3 out of 4 times,throws a fair dice with six faces and informs the outcome is 5.The probability that the outcome is really 5 is
Solution: Let $X\in\{0,1\}$ represent the random variable,where 0 represents person speaking false,1 represents person speaking truth.
Let $Y\in\{1,2,3,4,5,6\}$ represent random variable,where $1,2,3,4,5,6$ represents person informs outcome of dice is $1,2,3,4,5,6$,respectively.
From Baye's theorem

Event	definition	value
$\Pr(X=1)$	Probability of person speaking truth	$\frac{3}{4}$
$\Pr(X=0)$	Probability of person speaking false	$\frac{1}{4}$
$\Pr(Y=5\|X=1)$	Probability of person informing outcome is 5 if person speaks truth	$\frac{1}{6}$
$\Pr(Y=5\|X=0)$	Probability of person informing outcome is 5 if person speaks false	$\frac{5}{6}$

TABLE XXIV: Table 1

$$\Pr(Y=5)=\Pr(Y=5|X=1)\times\Pr(X=1)$$
$$+\Pr(Y=5|X=0)\times\Pr(X=0) \tag{557}$$

Substiting values from table (XXIV) in (557)

$$\Pr(Y=5)=\frac{8}{24} \tag{558}$$

$$\Pr((X=1)(Y=5))=\Pr(Y=5|X=1)$$
$$\times\Pr(X=1) \tag{559}$$
$$=\frac{3}{24} \tag{560}$$

We need to find $\Pr(X = 1|Y = 5)$

$$\Pr(X = 1|Y = 5) = \frac{\Pr((X = 1)(Y = 5))}{\Pr(Y = 5)} \quad (561)$$
$$= \frac{3}{8} \quad (562)$$

106) The probabilities that a student passes in Mathematics, Physics and Chemistry are m,p, and c respectively. Of these subjects, the student has 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Following relations are drawn in m, p, c:

(I) p + m + c = 27/20

(II) p + m + c = 13/20

(III) (p)× (m) × (c) =1/10

(A) Only relation I is true

(B) Only relation II is true

(C) Relations II and III are true

(D) Relations I and III are true

Solution:
Let M,P,C be the events representing student passes in Mathematics,Physics,Chemistry respectively.

$$\Pr(M) = m \quad (563)$$
$$\Pr(P) = p \quad (564)$$
$$\Pr(C) = c \quad (565)$$

The given information can be represented as

$$\Pr(M + P + C) = 75\% = \frac{3}{4} \quad (566)$$
$$\Pr(MP + PC + CA) = 50\% = \frac{1}{2} \quad (567)$$
$$\Pr(MP + PC + CA - 3MPC) = 40\% = \frac{2}{5} \quad (568)$$

(567) and (568) can also be written as

$$\Pr(MP) + \Pr(PC) + \Pr(CM) - 2\Pr(MPC) = \frac{1}{2} \quad (569)$$
$$\Pr(MP) + \Pr(PC) + \Pr(CM) - 3\Pr(MPC) = \frac{2}{5} \quad (570)$$

Subtracting and solving the above two equations we get,

$$\Pr(MPC) = \frac{1}{10} \quad (571)$$
$$\Pr(MP) + \Pr(PC) + \Pr(CM) = \frac{7}{10} \quad (572)$$

Using inclusion-exclusion principle, We can express (566) as

$$\Pr(M) + \Pr(P) + \Pr(C) - [\Pr(MP) + \Pr(PC) + \Pr(CM)] + \Pr(MPC) = \frac{3}{4} \quad (573)$$
$$p + m + c - \frac{7}{10} + \frac{1}{10} = \frac{3}{4} \quad (574)$$
$$p + m + c = \frac{27}{10} \quad (575)$$

There is no constant answer for the product of p,m,c which is shown in simulation.

$\therefore$ Only relation I is true.

107) **Solution:**
Let $F_X(t)$ denote the Cumulative Distribution Function for random variable X.

$$F_X(t) = \int_{-\infty}^{t} f(t)\,dt \tag{576}$$
$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{t} e^{-t}\,dt \tag{577}$$
$$= -e^{-t}\Big|_0^t \tag{578}$$
$$= 1 - e^{-t} \tag{579}$$

$$P(X \le b|X \ge a) = \frac{P((X \le b),(X \ge a))}{P(X \ge a)} \tag{580}$$
$$= \frac{P(a \le X \le b)}{P(X \ge a)} \tag{581}$$
$$= \frac{F_X(b) - F_X(a)}{\lim_{k\to\infty} F_X(k) - F_X(a)} \tag{582}$$
$$= \frac{e^{-a} - e^{-b}}{e^{-a}} \tag{583}$$
$$= 1 - e^{-(b-a)} \tag{584}$$

Therefore the required probability depends on $b - a$

108) Let X and Y denote the sets consisting 2 and 20 distinct elements respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one to one is :

Solution: We know, every $x \in X$ can be mapped to one of 20 elements in Y.

$$n(F) = 20 \times 20 = 400 \tag{585}$$

For one to one functions, the first element in X has 20 elements it can be mapped to, and second element in X has only 19 elements.(to avoid repetition).

$$n(f) = 20 \times 19 = 380 \tag{586}$$

Required probability:

$$\frac{n(f)}{n(F)} = \frac{380}{400} = \frac{19}{20} \tag{587}$$

109) Let S be a sample space and two mutually exclusive events A and B be such that $A + B = S$. If $P(\cdot)$ denotes the probability of the event, the maximum value of $P(A)P(B)$ is

Solution:

$$\Pr(A + B) = 1 \tag{588}$$
$$\Pr(A) + \Pr(B) = 1 \tag{589}$$
$$\Pr(A)\Pr(B) = \Pr(A)(1 - \Pr(A)) \tag{590}$$
$$= \Pr(A) - (\Pr(A))^2 \tag{591}$$
$$= \frac{1}{4} - \left(\Pr(A) - \frac{1}{2}\right)^2 \tag{592}$$
$$\le \frac{1}{4} \tag{593}$$
$$\Pr(A) = \Pr(B) = \frac{1}{2} \Rightarrow \Pr(A)\Pr(B) = \frac{1}{4} \tag{594}$$
$$\therefore \max(\Pr(A)\Pr(B)) = \frac{1}{4} \tag{595}$$

110) Given Set A = {2,3,4,5} and Set B = {11,12,13,14,15}, two numbers are randomly selected, one from each set. What is probability that the sum of the two numbers equals 16?

Solution:

Let $X_1 \in \{2, 3, 4, 5\}$ and $X_2 \in \{11, 12, 13, 14, 15\}$ be the random variables such that X_1 represents the number chosen from set A and X_2 the number chosen from set B.

Then, the probability mass functions are

$$p_{X_1}(n) = \Pr(X_1 = n) = \begin{cases} \frac{1}{4} & 2 \le n \le 5 \\ 0 & otherwise \end{cases} \tag{596}$$

$$p_{X_2}(n) = \Pr(X_2 = n) = \begin{cases} \frac{1}{5} & 11 \le n \le 15 \\ 0 & otherwise \end{cases} \tag{597}$$

Let X be the random variable denoting the sum (X=X_1+X_2). Then, X can take the values $\{13, 14, 15, 16, 17, 18, 19, 20\}$.

$$p_X(n) = \Pr(X_1 + X_2 = n) \tag{598}$$
$$= \Pr(X_1 = n - X_2) \tag{599}$$
$$= \Sigma_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{600}$$

As X_1, X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k) = \Pr(X_1 = n - k) \tag{601}$$

from (600) and (601)

$$p_X(n) = \Sigma_k p_{X_1}(n-k)p_{X_2}(n) = p_{X_1}(n) * p_{X_2}(n) \tag{602}$$

where * denotes the convolution operator.
As,

$$p_X(n) = \Sigma_k p_{X_1}(n-k)p_{X_2}(k) \tag{603}$$

$$= \frac{1}{5}\sum_{k=11}^{15} p_{X_1}(n-k) \tag{604}$$

$$= \frac{1}{5}\sum_{k=n-15}^{n-11} p_{X_1}(k) \tag{605}$$

Since $p_{X_1}(k) = 0$ for $k < 2, k > 5$
Therefore, we get

$$p_x(n) = \begin{cases} 0 & n \le 12 \\ \frac{1}{5}\sum_{k=2}^{n-11} p_{X_1}(k) & 2 \le n-11 \le 5 \\ \frac{1}{5}\sum_{k=n-15}^{5} p_{X_1}(k) & 2 \le n-15 \le 5 \\ 0 & n > 20 \end{cases} \tag{606}$$

Therefore, from (596) we get

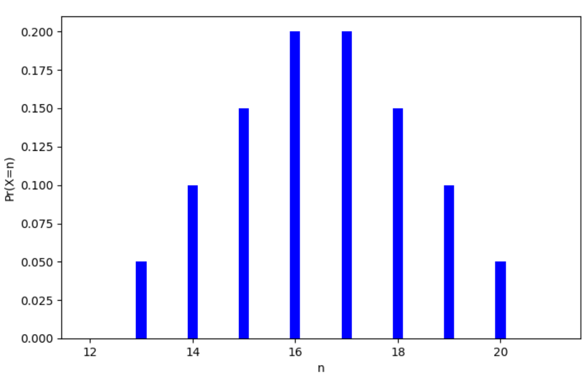


Fig. 29: Probability mass function of X

$$p_x(n) = \begin{cases} 0 & n \le 12 \\ \frac{n-12}{20} & 13 \le n \le 16 \\ \frac{21-n}{20} & 17 \le n \le 20 \\ 0 & n > 20 \end{cases} \tag{607}$$

Required probability is the probability of the sum of numbers selected from the sets, one from each set to be 16.

Therefore from (607),

$$p_X(16) = \left(\frac{16-12}{20}\right) \tag{608}$$

$$\implies p_X(16) = \frac{4}{20} \tag{609}$$

$$\implies \Pr(X_1 + X_2 = 16) = \frac{1}{5} \tag{610}$$

$$\therefore \Pr(X_1 + X_2 = 16) = 0.2 \tag{611}$$

111) For each element in a set of size 2n, an unbiased coin is tossed. The 2n coin tosses are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is:

a) $\frac{^{2n}C_n}{4^n}$

b) $\frac{^{2n}C_n}{2^n}$

c) $\frac{1}{^{2n}C_n}$

d) $\frac{1}{2}$

Solution: The number of elements chosen is equal to the number of heads obtained by 2n coin tosses. Let X be a random variable with value of X equal to the number of heads obtained.
Probability of getting a head, $p = \frac{1}{2}$
Probability of getting a tail, $q = \frac{1}{2}$
Probability that n elements are chosen out of 2n elements is $\Pr(X = n)$
From binomial distribution we know that,

$$\Pr(X = r) = {}^{2n}C_r p^r q^{2n-r} \tag{612}$$

$$\Pr(X = n) = {}^{2n}C_n \times \left(\frac{1}{2}\right)^n \times \left(\frac{1}{2}\right)^n \tag{613}$$

$$= \frac{^{2n}C_n}{4^n} \tag{614}$$

Hence option (A) is correct.

112) Players A and B take turns to throw a fair dice with six faces. If A is the first player to throw, then the probability of B being the first one to get a six is — (round of to two

decimal places).
Solution:
Let the random variable X represent which player gets six first. That is $X = 0$ when A gets a six first and $X = 1$ when B gets six first.
Let another random variable Y represent getting a six on the dice. $Y = 1$ for six and $Y = 0$ for any other number.
Let N be the number of turns until we get a six.

$$\Pr(Y = 0) = \frac{5}{6} \tag{615}$$
$$\Pr(Y = 1) = \frac{1}{6} \tag{616}$$

The event success is when B gets a six for first time and failure is when neither A nor B gets six. Let p denote probability of success

$$p = \Pr(Y = 1) \tag{617}$$
$$\Pr(Y = 0) = 1 - p \tag{618}$$
$$p = \frac{1}{6} \tag{619}$$

To get $X = 1$ in N turns we have to get $N - 1$ failures for B and N failures for A and finally one success for B. Therefore the geometric distribution is,

$$f(N) = (1-p)^{n-1} \times p \times (1-p)^n \tag{620}$$
$$= (1-p)^{2n-1} \times p \tag{621}$$
$$= \left(\frac{5}{6}\right)^{2n-1} \times \frac{1}{6} \tag{622}$$

The result has been summarized in table XXV.

No. of turns	Probability
1	$5^1/6^2$
2	$5^3/6^4$
$\vdots$ n	$\vdots$ $5^{2n-1}/6^{2n}$
$\vdots$	$\vdots$

TABLE XXV: Summary of turns

Thus the total probability is sum of these individual probabilities i.e.

$$\Pr(X = 1) = \sum_{N=1}^{\infty} f(N) \tag{623}$$
$$= \frac{5}{6^2} + \frac{5^3}{6^4} + \ldots + \frac{5^{2n-1}}{6^{2n}} + \ldots \tag{624}$$
$$= \frac{5}{6^2} \times \left(1 + \frac{5^2}{6^2} + \frac{5^4}{6^4} + \ldots\right) \tag{625}$$

By Using sum of infinite GP we have,

$$\Pr(X = 1) = \frac{5}{6^2} \times \left(\frac{1}{1 - \frac{25}{36}}\right) \tag{626}$$
$$= \frac{5}{36} \times \frac{36}{11} \tag{627}$$
$$= \frac{5}{11} = 0.45 \tag{628}$$

113) The probability that a number selected at random between 100 and 999 (both inclusive) will not contain digit 7 is. **Solution:**
Let's assume a random 3-digit number be xyz. Where x, y, z are 3 random single-digit integers such that

$$x \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \tag{629}$$
$$y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \tag{630}$$
$$z \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \tag{631}$$

a) Probability of selecting x without including 7

$$\Pr(x \neq 7) = \frac{8}{9} \tag{632}$$

b) Probability of selecting y without including 7

$$\Pr(y \neq 7) = \frac{9}{10} \tag{633}$$

c) Probability of selecting z without including 7

$$\Pr(z \neq 7) = \frac{9}{10} \tag{634}$$

So,the total probability of a random 3-digit number xyz will not contain 7

$$= \Pr(x \neq 7) \times \Pr(y \neq 7) \times \Pr(z \neq 7) \quad (635)$$
$$= \frac{8}{9} \times \frac{9}{10} \times \frac{9}{10} \quad (636)$$
$$= \frac{18}{25} \quad (637)$$

The probability of a number selected at random between 100 and 999 (both inclusive) will not contain digit 7 is $\frac{18}{25}$

114) A class of twelve children has two more boys than girls. A group of three children are randomly picked from this class to accompany the teacher on the field trip. What is the probability that the group accompanying the teacher contains more girls than boys.
   (a) 0
   (b) $\frac{325}{864}$
   (c) $\frac{525}{864}$
   (d) $\frac{5}{12}$

Solution:
Let $X \in \{0, 3\}$ be a discrete random variable which denotes the number of girls in the group of 3.
Since, there are two more boys than girls:

$$\text{Total number of boys}(B) = 7$$
$$\text{Total number of girls}(G) = 5$$
$$\Pr(X = c) = \frac{{}^5C_c \times {}^7C_{3-k}}{{}^{12}C_3} \quad (638)$$

For number of girls more than boys required probability $= \Pr(2 \leq X \leq 3)$ and hence, following cases are possible.

a) X=3

$$\Pr(X = 3) = \frac{{}^5C_3}{{}^{12}C_3}, \text{using (638)}$$

b) X=2

$$\Pr(X = 2) = \frac{{}^5C_2 \times {}^7C_1}{{}^{12}C_3}, \text{using (638)}$$

$$\text{So, required probability } = \frac{{}^5C_2 \times {}^7C_1}{{}^{12}C_3} + \frac{{}^5C_3}{{}^{12}C_3} = \frac{4}{11}$$

115) Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is

Solution:
Let the random variable $X \in \{1, 2\}$ represent the type of the chosen bulb. $X = 1$ denotes a Type 1 bulb, while $X = 2$ denotes a Type 2 bulb. Given,

$$n(X = 1) = n(X = 2) \quad (5.1)$$
$$\implies p_X(1) = p_X(2) = \frac{1}{2} \quad (5.2)$$

Let the random variable $Y \in \{0, 1\}$ represent if a bulb lasts more than 100 hours. $Y = 1$ denotes that it lasts, while $Y = 0$ denotes that it doesn't. Given,

$$p_{Y|X}(1|1) = 0.7 \quad (5.3)$$
$$p_{Y|X}(1|2) = 0.4 \quad (5.4)$$

To find : $p_Y(1)$

$$p_Y(1) = p_{Y|X}(1|1)p_X(1) + p_{Y|X}(1|2)p_X(2) \quad (5.5)$$
$$p_Y(1) = (0.7)(0.5) + (0.4)(0.5) \quad (5.6)$$
$$\therefore p_Y(1) = 0.55 \quad (5.7)$$

116) Suppose X_i for $i = 1, 2, 3$ are independent and identically distributed random variables whose probability mass functions are $\Pr(X_i = 0) = \Pr(X_i = 1) = \frac{1}{2}$ for $i = 1, 2, 3$. Define another random variable $Y = X_1X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $\Pr(Y = 0|X_3 = 0) =$

Solution:

For

$$\because Y = (X_1X_2) \oplus X_3 = 0 \tag{639}$$
$$\implies X_1X_2 = X_3 \tag{640}$$

$$\Pr(Y = 0|X_3 = 0) = \frac{\Pr(Y = 0, X_3 = 0)}{\Pr(X_3 = 0)} \tag{641}$$
$$= \frac{\Pr(X_1X_2 = X_3, X_3 = 0)}{\Pr(X_3 = 0)} \tag{642}$$

$$\Pr(X_3 = 0) = \frac{1}{2} \tag{643}$$

if $X_3 = 0$, from (640)

$$X_1X_2 = 0 \tag{644}$$

The random variables are independent of each other:

$\Pr(X_1 = 0, X_2 = 0)$	$\Pr(X_1 = 0) \cdot \Pr(X_2 = 0)$	0.25
$\Pr(X_1 = 1, X_2 = 0)$	$\Pr(X_1 = 1) \cdot \Pr(X_2 = 0)$	0.25
$\Pr(X_1 = 0, X_2 = 1)$	$\Pr(X_1 = 0) \cdot \Pr(X_2 = 1)$	0.25

TABLE XXVI: Probabilities

$$\Pr(X_1X_2 = 0) = \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1) + \Pr(X_1 = 1, X_2 = 0) \tag{645}$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \tag{646}$$

$$\Pr(Y = 0, X_3 = 0) = \Pr(X_1X_2 = X_3 = 0) \tag{647}$$
$$= \Pr(X_1X_2 = 0) \cdot \Pr(X_3 = 0) \tag{648}$$
$$= \frac{3}{4} \cdot \frac{1}{2} \tag{649}$$
$$= \frac{3}{8} \tag{650}$$

Upon substituting (650) and (643) in (641)

$$\Pr(Y = 0|X_3 = 0) = \frac{3}{4} = 0.75 \tag{651}$$

117) The probability that a given positive integer lying between 1 and 100 (both inclusive) and is NOT divisible by 2 or 3 or 5 is ...
Solution: Let A, B, C are events where a positive integer between 1 and 100 (both inclusive) is divisible by 2, 3, 5 respectively.

$$\Pr(A) = \frac{1}{2} \tag{652}$$
$$\Pr(B) = \frac{33}{100} \tag{653}$$
$$\Pr(C) = \frac{1}{5} \tag{654}$$
$$\Pr(AB) = \frac{16}{100} \tag{655}$$
$$\Pr(BC) = \frac{6}{100} \tag{656}$$
$$\Pr(AC) = \frac{1}{10} \tag{657}$$
$$\Pr(ABC) = \frac{3}{100} \tag{658}$$

Required probability : $\Pr(A + B + C)'$

$$\Pr(A + B + C)' = 1 - \Pr(A + B + C) = 1 - \Pr(A) - \Pr(B) - \Pr(C) + \Pr(AB) + \Pr(BC) + \Pr(AC) - \Pr(ABC) = 0.26 \tag{659}$$