# GATE Problems in Probability

***Abstract*—These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

1) An urn contains 5 red balls and 5 black balls.In the first draw, one ball is picked at random and discarded without noticing its colour.The probability to get a red ball in the second draw is

   a) $\frac{1}{2}$ b) $\frac{4}{9}$ c) $\frac{5}{9}$ d) $\frac{6}{9}$

   **Solution:** Let $X_i \in \{0, 1\}$ represent the $i^{th}$ draw where 1 denotes a red ball is drawn.

   TABLE I

   | | $X_1 = 0$ | $X_1 = 1$ |
   |---|---|---|
   | $X_2 = 0$ | 4/18 | 5/18 |
   | $X_2 = 1$ | 5/18 | 4/18 |

   Table I represents the probabilities of all possible cases when two balls are drawn one by one from the urn.

$$\Pr(X_2 = 1) = \Pr(X_2 = 1|X_1 = 0) + \Pr(X_2 = 1|X_1 = 1) \tag{1}$$

$$= \frac{5}{18} + \frac{4}{18} \tag{2}$$

$$= \frac{1}{2} \tag{3}$$

   The required option is (A).

2) There are 3 red socks, 4 green socks and 3 blue socks.You choose 2 socks.The probability that they are of the same colour is

   a) $\frac{1}{5}$ b) $\frac{7}{30}$ c) $\frac{1}{4}$ d) $\frac{4}{15}$

3) The probability that a $k$-digit number does NOT contain the digits 0,5, or 9 is

   a) $0.3^k$ b) $0.6^k$ c) $0.7^k$ d) $0.9^k$

**Solution:** Let

$$X_i \in \{0, 1, 2, \ldots, 9\} \tag{4}$$

represent the digit at the $i^{th}$ place.

$$\Pr(X_i \notin \{0, 5, 9\}) = \frac{7}{10} = 0.7 \tag{5}$$

If the k-digit number does not contain 0,5 or 9,

$$\Pr(X_1 \notin \{0, 5, 9\}, X_2 \notin \{0, 5, 9\}, \ldots, X_k \neq \{0, 5, 9\}) \tag{6}$$

Since the events are independent,

$$\Pr(X_1 \notin \{0, 5, 9\}, X_2 \notin \{0, 5, 9\}, \ldots, X_k \neq \{0, 5, 9\}) = \Pr(X_1 \notin \{0, 5, 9\}) \ldots \Pr(X_k \notin \{0, 5, 9\}) \tag{7}$$

$$= \prod_{i=1}^{k} 0.7 \tag{8}$$

$$= (0.7)^k \tag{9}$$

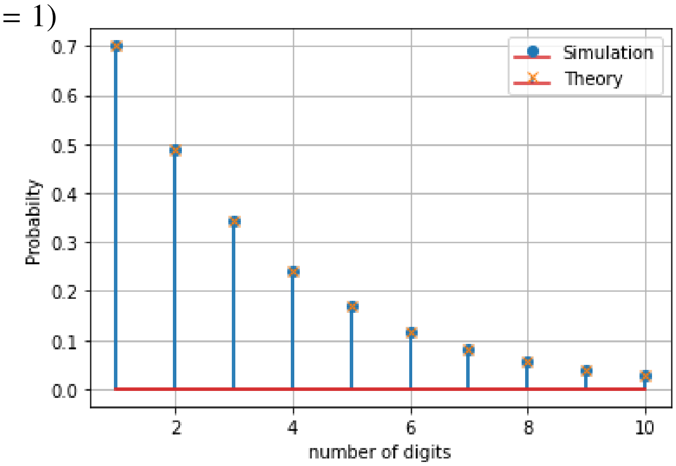


Fig. 1: Plot

4) Three fair cubical dice are thrown simultaneously. The probability that all three dice

have the same number of dots on the faces showing up is (up to third decimal place)..........
**Solution:** Let

$$X_1, X_2, X_3 \in \{1, 2, 3, 4, 5, 6\} \tag{10}$$

represent the three dice.
Since, all the three are fair dice, the probability of any dice showing a particular number is given by

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & \text{i=1,2,3,4,5,6} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

If all the dice show a particular number i,

$$\implies \Pr(X_1 = X_2 = X_3 = i) \tag{12}$$

Since the events are independent,

$$\Pr(X_1 = X_2 = X_3 = i) \\ = \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \tag{13}$$

where i=1,2,3,4,5,6.
There are 6 faces on a cubical dice. Hence, there are six cases in which all the dice show the same number

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = X_2 = X_3 = i) \tag{14}$$

From (13),we have

$$\Pr(X_1 = X_2 = X_3) \\ = \sum_{i=1}^{6} \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \tag{15}$$

$$= \sum_{i=1}^{6} \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \tag{16}$$

$$= \frac{1}{36} \tag{17}$$

5) Candidates were asked to come to an interview with 3 pens each. Black,blue,green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is.......

6) The probability of getting a ”head” in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a ”head” is obtained. If the tosses are independent, then the probability of getting ”head” for the first time in the fifth toss is........
**Solution:** Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.
$X = k$ represents $k - 1$ failures were obtained before getting 1 success. $p$ represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \tag{18}$$

Using (18) we get

$$\Pr(X = 5) = (1-p)^{k-1} \times p \\ = (0.7)^4 \times 0.3 = 0.07203 \tag{19}$$

7) Given Set A = [2,3,4,5] and Set B = [11,12,13,14,15], two numbers are randomly selected,one from each set. What is probability that the sum of the two numbers equals 16?

a) 0.20 b) 0.25 c) 0.30 d) 0.33

8) Consider a dice with the property that the probability of a face with $n$ dots showing up is proportional to $n$. The probability of the face with three dots showing up is......

9) **Step 1.** Flip a coin twice.
**Step 2.** If the outcomes are (TAILS, HEADS) then output Y and stop.
**Step 3.** If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
**Step 4.** If the outcomes are (TAILS, TAILS), then go to Step 1.
The probability that the output of the experiment is Y is (upto two decimal places)......

10) Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one-to-one is........

11) The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is......
**Solution:** Let $X \in \{1, 2, ..., 100\}$ be the random variable representing the outcome for random selection of a number in $\{1, ..., 100\}$.
Since X has a uniform distribution, the probability mass function (pmf) is represented as

$$\Pr(X = n) = \begin{cases} \frac{1}{100} & 1 \le n \le 100 \\ 0 & otherwise \end{cases} \tag{20}$$

Let A represent the event that the number is divisible by 2. Let B represent the event that the number is divisible by 3. Let C represent the event that the number is divisible by 5.
We need to find the probability that the number is not divisible by 2, 3 or 5. Thus we need to find $1 - \Pr(A + B + C)$
We know

$$\begin{aligned}\Pr(A + B + C) = \Pr(A) + \Pr(B) + \Pr(C) \\ - \Pr(AB) - \Pr(BC) \\ - \Pr(AC) + \Pr(ABC)\end{aligned} \tag{21}$$

| Event | Interpretation | Probability |
|---|---|---|
| A | n is divisible by 2 | $\frac{50}{100}$ |
| B | n is divisible by 3 | $\frac{33}{100}$ |
| C | n is divisible by 5 | $\frac{20}{100}$ |
| AB | n is divisible by 6 | $\frac{16}{100}$ |
| BC | n is divisible by 15 | $\frac{6}{100}$ |
| AC | n is divisible by 10 | $\frac{10}{100}$ |
| ABC | n is divisible by 30 | $\frac{3}{100}$ |

TABLE II

Substituting in (21), we get

$$\begin{aligned}\Pr(A + B + C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} \\ - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100}\end{aligned} \tag{22}$$

Thus,

$$\Pr(A + B + C) = \frac{74}{100} \tag{23}$$

Thus required probability =

$$1 - \Pr(A + B + C) = \frac{26}{100} \tag{24}$$

12) P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for the job is

a) $\frac{4}{5}$ b) $\frac{5}{6}$ c) $\frac{7}{8}$ d) $\frac{11}{12}$

13) Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

a) $\frac{5}{11}$ b) $\frac{1}{2}$ c) $\frac{7}{13}$ d) $\frac{6}{11}$

14) A continuous random variable X has a probability density function $f(x) = e^{-x}, 0 < x < \infty$. Then $P(X > 1)$ is

a) 0.368 b) 0.5 c) 0.632 d) 1.0

15) A random variable X has probability density function $f(x)$ as given below:
$$f(x) = \begin{cases} a + bx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
If the expected value $E[X] = \frac{2}{3}$, then $Pr[X < 0.5]$ is..........

16) Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$.The probability that $\max[X, Y]$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

17) The input X to the binary Symmetric Channel (BSC) shown in Fig. 2 is '1' with probability 0.8. The cross-over probability is $\frac{1}{7}$. If the received bit Y=0, the conditional probability that '1' was transmitted is........

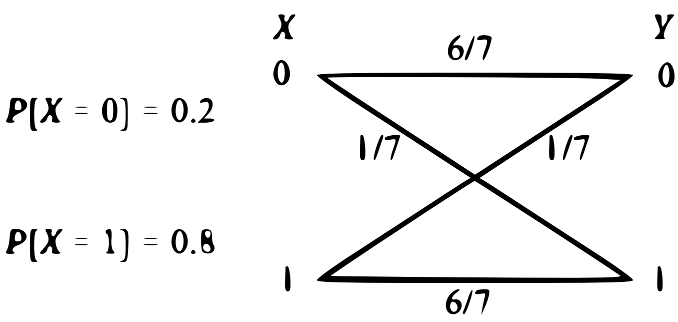


Fig. 2

18) A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

19) A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

a) $\frac{1}{3}$ b) $\frac{3}{7}$ c) $\frac{1}{2}$ d) $\frac{4}{7}$

20) Let X be a random variable with probability density function $f(x) = \begin{cases} 0.2 & |x| \leqslant 1 \\ 0.1 & 1 \leqslant |x| \leqslant 4 \\ 0 & \text{otherwise.} \end{cases}$ The probability $P(0.5 < X, 5)$ is..........

21) Consider two identically distributed zero-mean random variables U and V. Let the cumulative distribution functions of U and 2V be F($x$) and G($x$) respectively. Then,for all values of $x$

a) $F(x) - G(x) \leqslant 0$ c) $(F(x) - G(x))x \leqslant 0$

b) $F(x) - G(x) \geqslant 0$ d) $(F(x) - G(x))x \geqslant 0$

22) Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $P(3V \geqslant 2U)$ is

a) $\frac{4}{9}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{5}{9}$

23) Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$. The probability that max $[X, Y]$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

24) A binary symmetric channel (BSC) has a transition probability of $\frac{1}{8}$. If the binary transmit symbol X is such that $P(X = 0) = \frac{9}{10}$, then the probability of error for an optimum receiver will be

a) $\frac{7}{80}$ b) $\frac{63}{80}$ c) $\frac{9}{10}$ d) $\frac{1}{10}$

**Solution:**

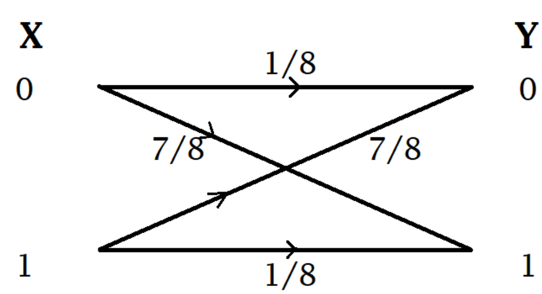


Fig. 3: Binary symmetric channel

Let random variables, $X \in \{0, 1\}$ denote the bit transmitted and $Y \in \{0, 1\}$ denote the output bit

received.
From the given information,

$$\Pr(X=0) = \frac{9}{10} \tag{25}$$

$$\Pr(X=1) = 1 - \Pr(X=0) = \frac{1}{10} \tag{26}$$

Also given, transition probability $= \frac{1}{8}$. Transition probability is the probability with which the bit is transmitted correctly. That gives,

$$\Pr(Y=1|X=1) = \Pr(Y=0|X=0) = \frac{1}{8} \tag{27}$$

Probability that the bit is not transmitted correctly

$$= 1 - \text{transition probability}$$

$$= 1 - \frac{1}{8} = \frac{7}{8} \tag{28}$$

That gives,

$$\Pr(Y=0|X=1) = \Pr(Y=1|X=0) = \frac{7}{8} \tag{29}$$

Let E denote the event that bit is transmitted incorrectly. Probability of error, $\Pr(E)$

$$\Pr(E) = \Pr(X=0)\Pr(Y=1|X=0) + \Pr(X=1)\Pr(Y=0|X=1) \tag{30}$$

On substituting the values,

$$\Pr(E) = \frac{9}{10}\times\frac{7}{8} + \frac{1}{10}\times\frac{7}{8} \tag{31}$$

$$= \frac{63}{80} + \frac{7}{80} \tag{32}$$

$$= \frac{7}{8} \tag{33}$$

Answer: No option matches

25) A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

26) A fair dice is tossed two times. The probability that the second toss result in a value that is higher than the first toss is

a) $\frac{2}{36}$ b) $\frac{2}{6}$ c) $\frac{5}{12}$ d) $\frac{1}{2}$

27) A fair coin is tossed 10 times. What is the probability that ONLY the first two tosses will yield heads?

a) $\left(\frac{1}{2}\right)^2$
b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$
c) $\left(\frac{1}{2}\right)^{10}$
d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$

28) Consider two independent random variables X and Y with identical distributions. The variables X and Y take value 0, 1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ rrespectively. What is the conditional probability $P(X+Y=2|X-Y=0)$?

a) 0 b) $\frac{1}{16}$ c) $\frac{1}{6}$ d) 1

29) A discrete random variable X takes values from 1 to 5 with probabilities as shown in the table. A student calculates the mean of X as 3.5 and her teacher calculates the variance of X as 1.5. Which of the following statements is true?

| k | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(X=k) | 0.1 | 0.2 | 0.4 | 0.2 | 0.1 |

a) Both the student and the teacher are right
b) Both the student and the teacher are wrong
c) The student is wrong but the teacher is right
d) The student is right but the teacher is wrong

30) If E denotes expectation, the variance of a random variable X is given by

a) $E[X^2] - E^2[X]$
b) $E[X^2] + E^2[X]$
c) $E[X^2]$
d) $E^2[X]$

31) An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper

1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is:

a) 0.5 b) 0.18 c) 0.12 d) 0.06

32) A probability density function is of the form
$$p(x) = Ke^{-\alpha|x|}, x \in (-\infty, \infty)$$
The value of K is

a) 0.5 b) 1 c) $0.5\alpha$ d) $\alpha$

33) Consider a binary digital communication system with equally likely 0's and 1's. When binary 0 is transmitted the voltage at the detector input can lie between the level -0.25V and +0.25V with equal probability: when binary 1 is transmitted, the voltage at the detector can have any value between 0 and 1V with equal probability. If the detector has a threshold of 2.0V (i.e., if the received signal is greater than 0.2V, the bit is taken as 1), the average bit error probability is

a) 0.15 b) 0.2 c) 0.05 d) 0.5

34) Let X and Y be two statistically independent random variables uniformly distributed in the range $(-1, 1)$ and $(-2, 1)$ respectively. Let $Z = X + Y$, then the probability that $[Z \leqslant -2]$ is

a) zero b) $\frac{1}{6}$ c) $\frac{1}{3}$ d) $\frac{1}{12}$

35) Let X be the Gaussian random variable obtained by sampling the process at $t = t_i$ and let
$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$
The probability that $[X \leqslant 1]$ is ...

36) Let Y and Z be the random variables obtained by sampling $X(t)$ at $t = 2$ and $t = 4$ respectively. Let W = Y-Z. The variance of W is

a) 13.36 b) 9.36 c) 2.64 d) 8.00

37) Let the random variable X represent the number of times a fair coin needs to be tossed till two consecutive heads appear for the first time. The expectation of X is......

38) Let $X \in [0, 1]$ and $Y \in [0, 1]$ be two independent binary random variables. If $P(X = 0) = p$ and $P(Y = 0) = q$, then $P(X + Y \geqslant 1)$ is equal to

a) $pq + (1 - p)(1 - q)$
b) pq
c) $p(1 - q)$
d) $1 - pq$

39) Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\widetilde{A}$ and $\widetilde{B}$ be their complements. Which one of the following statements is FALSE?

a) $P(A \cap B) = P(A)P(B)$
b) $P(A|B) = P(A)$
c) $P(A \cup B) = P(A) + P(B)$
d) $P(\widetilde{A} \cap \widetilde{B}) = P(\widetilde{A})P(\widetilde{B})$

40) A digital communication system uses a repetition code for channel encoding/decoding. During transmission, each bit is repeated three times(0 is transmitted as 000, and 1 is transmitted as 111). It is assumed that the source puts out symbols independently and with equal probability. The decoder operates as follows: In a block of three received bits, if the number of zeros exceeds the number of ones, the decoder decides in favour of a 0, and if the number of ones exceeds the number of zeros, the decoder decides in favour of a 1. Assuming a binary symmetric channel with crossover probability p = 0.1, the average probability of error is ........

41) Two random variables X and Y are distributed according to
$$f_{x,y}(x, y) = \begin{cases} (x + y) & 0 \leqslant x \leqslant 10 \leqslant y \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$
The probability $P(X + Y \leqslant 1)$ is .......

42) Let X be a zero mean unit variance Gaussian random variable. $E[|X|]$ is equal to .........

43) If calls arrive at a telephone exchange such that the time of arrival of any call is independent of the time of arrival of earlier or future calls, the probability distribution function of the toatl number of calls in a fixed time interval will be

a) Poisson
b) Gaussian
c) Exponential
d) Gamma

44) Consider a communication scheme where the binary valued signal X satisfies $P\{X = +1\} = 0.75$ and $P\{X = -1\} = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance $\sigma^2$. The received signal Y is fed to the threshold detector. The output of the threshold detector $\hat{X}$ is:

$$\hat{X} = \begin{cases} +1 & Y > \tau \\ -1 & Y \leqslant \tau \end{cases}$$

To achieve minimum probability of error $P\{\hat{X} \neq X\}$, the threshols $\tau$ should be

a) strictly positive
b) zero
c) strictly negative
d) strictly positive, zero or strictly negative depending on the nonzero value of $\sigma^2$

45) Consider a discrete-time channel $Y = X + Z$, where the additive noise Z is signal-dependent. In particular, given the transmitted symbol $X \in \{-a, +a\}$ at any instant, the noise sample Z is chosen independently from a Gaussian distribution with mean $\beta X$ and unit variance. Assume a threshold detector with zero threshold at the receiver.
When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^{-}8$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{\frac{-u^2}{2}} du\right.$$

, and for $v > 1$, use $\left. Q(v) \approx e^{\frac{-v^2}{2}}\right)$
When $\beta = -0.3$, the BER is closet to

a) $10^{-}7$
b) $10^{-}6$
c) $10^{-}4$
d) $10^{-}2$

46) Consider the random process
$X(t) = U + Vt$,
where U is a zero-mean Gaussian random variable and V is a random variable distributed between 0 and 2. Assume that U and V are statistically independent. The mean value of the random process at t=2 is........ **Solution:** Here U is a gaussian random variable of mean 0 and Let us consider V is uniformly distributed random variable in $(0, 2)$.

| Random Variable | U | V | X(t) |
|---|---|---|---|
| Expected Value | 0 | 1 | t |

TABLE III: Random Variables and Expected Values

From Table III we can deduce that,

$$E[X(t)] = E[U + Vt] \tag{34}$$
$$E[X(t)] = E[U] + tE[V] \tag{35}$$
$$E[X(t)] = 0 + t \tag{36}$$
$$E[X(t)] = t \tag{37}$$
$$E[X(2)] = 2 \tag{38}$$

$\therefore$ mean of random process X(t) at 2 is 2.

47) Consider the Z-channel given in Fig. 4. The input is 0 or 1 with equal probability.

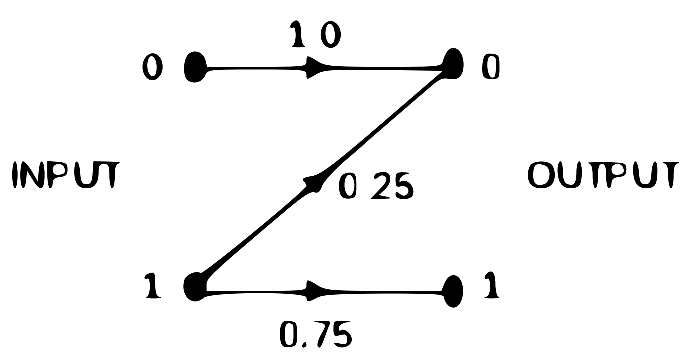

Fig. 4

If the output is 0, the probability that the input is also 0 equals......

48) If P and Q are two random events, then the following is TRUE:

a) Independence of P and Q implies that $\Pr P \cap Q = 0$

b) $\Pr(P \cup Q) \geqslant \Pr(P) + \Pr(Q)$

c) If P and Q are mutually exclusive, then they must be independent

d) $\Pr(P \cap Q) \leqslant \Pr(P)$

49) A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is:

a) $\frac{1}{8}$  c) $\frac{3}{8}$

b) $\frac{1}{2}$  d) $\frac{3}{4}$

50) The probability density function (PDF) of a random variable X is as shown in Fig. 5.

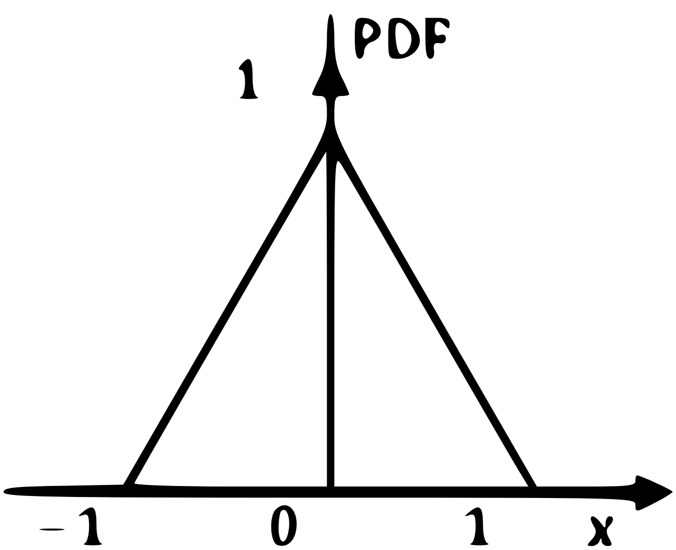


Fig. 5

The corresponding cumulative distribution function (CDF) has the form

a) Fig. 6  c) Fig. 8

b) Fig. 7  d) Fig. 9

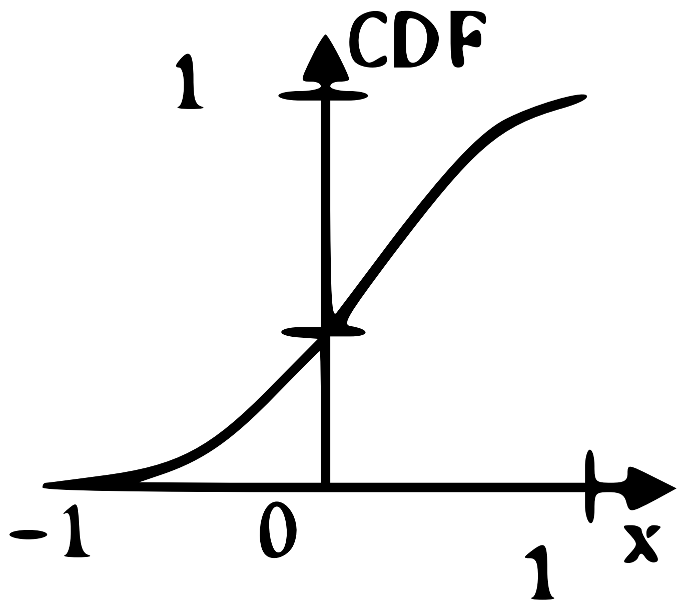


Fig. 6

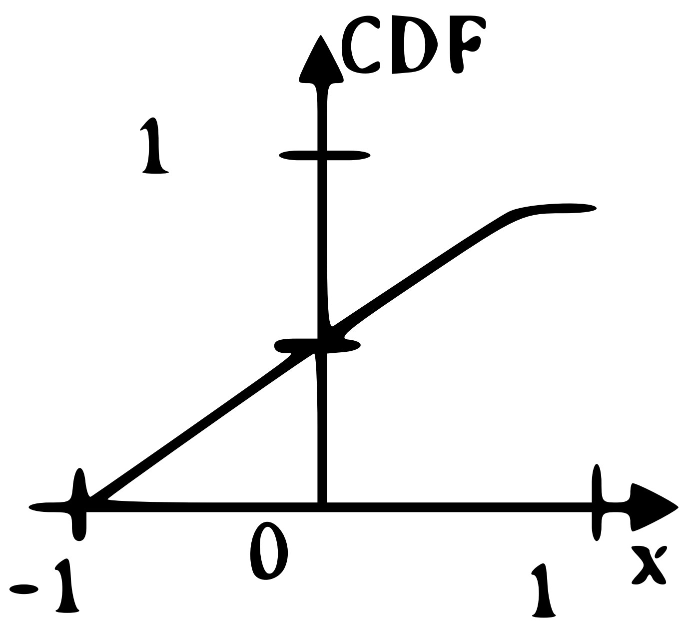


Fig. 7

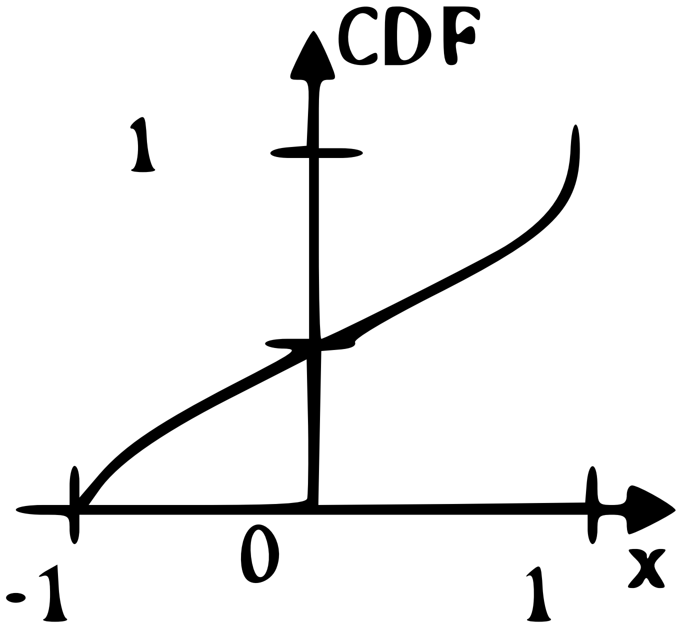


Fig. 8

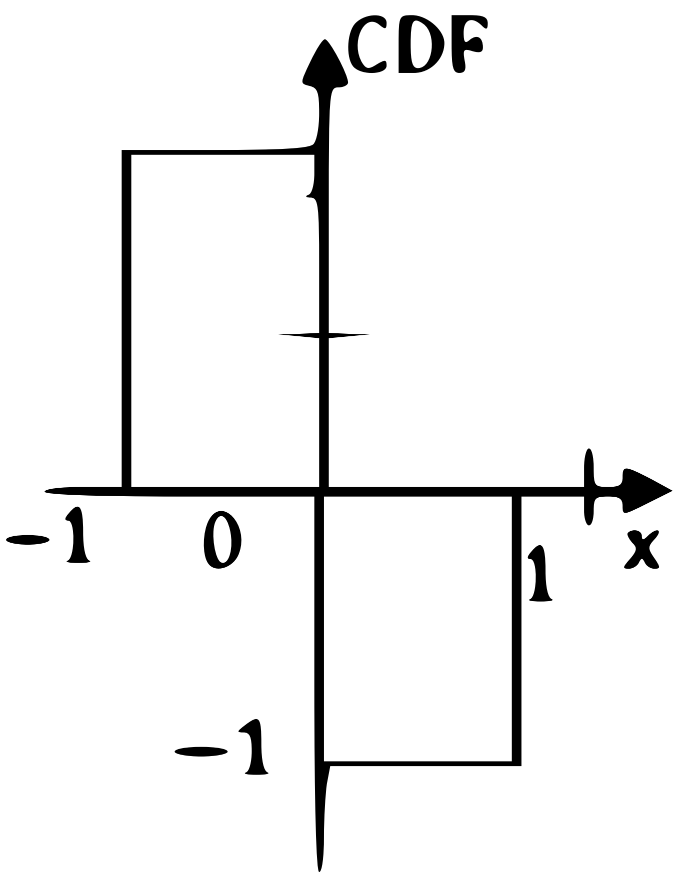


Fig. 9

51) The distribution function $f_x(x)$ of a random variable X is shown in Fig. 10. The probability that X=1 is

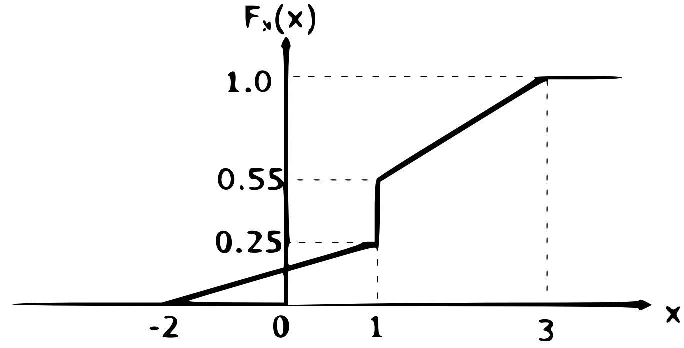


Fig. 10

a) Zero
b) 0.25
c) 0.55
d) 0.30

52) Let the probability density function of a random variable $X$ be

$$f(x) = \begin{cases} x & 0 \leq \ x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} < x \leq \ 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,the value of $c$ is equal to ______________

53) Suppose $X$ and $Y$ are two random variables such that $aX + bY$ is a normal random variable for all $a, b \in \mathbb{R}$. Consider the following statements P,Q,R and S:
(P): $X$ is a standard normal random variable.
(Q): The conditional distribution of $X$ given $Y$ is normal.
(R): The conditional distribution of $X$ given $X + Y$ is normal.
(S): $X$ - $Y$ has mean 0.

Which of the above statements ALWAYS hold TRUE?

a) both P and Q
b) both Q and R
c) both Q and S
d) both P and S

54) Let $X$ be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to _______________

55) Let $X_1$ be an exponential random variable with mean 1 and $X_2$ a gamma random variable with mean 2 and variance 2. If $X_1$ and $X_2$ are independently distributed,then $P(X_1 < X_2)$ is equal to _______________

**Common Data for the next two Questions :**

Let $X$ and $Y$ be jointly distributed random variables such that the conditional distribution of $Y$, given $X = x$, is uniform on the interval $(x-1, x+1)$. Suppose $E(X) = 1$ and $Var(X) = \frac{5}{3}$.

56) The mean of the random variable $Y$ is

a) $\frac{1}{2}$

b) 1

c) $\frac{3}{2}$

d) 2

57) The variance of the random variable $Y$ is

a) $\frac{1}{2}$

b) $\frac{2}{3}$

c) 1

d) 2

58) Let the random variable $X$ have the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 1 \\ \frac{3}{5} & 1 \leq x < 2 \\ \frac{1}{2} + \frac{x}{8} & 2 \leq x < 3 \\ 1 & x \geq 3. \end{cases}$$

Then $P(2 \leq X < 4)$ is equal to _______________

59) Let $X$ be a random variable having the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{1}{3} & 1 \leq x < 2 \\ \frac{1}{2} & 2 \leq x < \frac{11}{3} \\ 1 & x \geq \frac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to _______________

60) Let $X$ and $Y$ be two random variables having the joint probability density function

$$f(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3} | Y = \frac{3}{4}\right)$ is equal to _______________

a) $\frac{5}{9}$  b) $\frac{2}{3}$  c) $\frac{7}{9}$  d) $\frac{8}{9}$

61) Let $\Omega = (0, 1]$ be the sample space and let $P(\cdot)$ be a probability function defined by

$$P((0, x]) = \begin{cases} \frac{x}{2} & 0 \leq x < \frac{1}{2} \\ x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $P\left(\{\frac{1}{2}\}\right)$ is equal to _______________

62) Suppose the random variable $U$ has uniform distribution on $[0, 1]$ and $X = -2\log U$. The density of $X$ is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

$B_2$. The probability that the two drawn balls are of different colours is

a) $\frac{7}{25}$

b) $\frac{9}{25}$

c) $\frac{12}{25}$

d) $\frac{16}{25}$

**Common Data for the next two Questions :**

Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, \\ & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

67) The variance of the random variable X is

a) $\frac{1}{12}$

b) $\frac{1}{4}$

c) $\frac{7}{12}$

d) $\frac{5}{12}$

68) The covariance between the random variables X and Y

a) $\frac{1}{3}$

b) $\frac{1}{4}$

c) $\frac{1}{6}$

d) $\frac{1}{12}$

**Common Data for the next two Questions :**

Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

69) The value of a is

a) 4

b) 2

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$

63) Suppose $X$ is a real-valued random variable.Which of the following values **CANNOT** be attained by $E[X]$ and $E[X^2]$, respectively?

a) $0$ *and* $1$

c) $\frac{1}{2}$ *and* $\frac{1}{3}$

b) $2$ *and* $3$

d) $2$ *and* $5$

64) Let $X_n$ denote the sum of points obtained when n fair dice are rolled together. The expectation and variance of $X_n$ are

a) $\frac{7}{2}n$ and $\frac{35}{12}n^2$ respectively.

c) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively.

b) $\frac{7}{2}n$ and $\frac{35}{12}n$ respectively.

d) None of the above

65) Let X and Y be jointly distributed random variables having the joint probability density function
$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$
Then $P(Y > max(X, -X)) =$

a) $\frac{1}{2}$

c) $\frac{1}{4}$

b) $\frac{1}{3}$

d) $\frac{1}{6}$

66) Consider two identical boxes $B_1$ and $B_2$, where the box $B(i = 1, 2)$ contains $i+2$ red and $5-i-1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are drawn with replacement from the box $B_1$, otherwise, two balls are drawn with replacement from the box

c) 1

d) 0.5

70) The value of $E(X|Y = 2)$ is

a) 4

c) 2

b) 3

d) 1

71) Let X and Y be two random variables having the joint probability density function

$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$
Then the conditional probability $P(X \leqslant \frac{2}{3}|Y = \frac{3}{4})$ is equal to

a) $\frac{5}{9}$

c) $\frac{7}{9}$

b) $\frac{2}{3}$

d) $\frac{8}{9}$

72) Let $\Omega = (0, 1]$ be the sample space and let $P(.)$ be a probability function defined by
$P((0,x]) = \begin{cases} \frac{x}{2} & 0 \leqslant x < \frac{1}{2} \\ x & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$
Then $P\left(\{\frac{1}{2}\}\right)$ is equal to.......

73) Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leqslant x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leqslant x < 1 \\ 1 & x \geqslant 1 \end{cases}$$

Then $P(\frac{1}{4} < x < 1)$ is equal to........

74) Let $X_1$ be an exponential random variable with mean 1 and $X_2$ a gamma random variable with mean 2 and variance 2. If $X_1$ and $X_2$ are independently distributed, then $P(X_1 < X_2)$ is equal to.....

**Common Data for the next two Questions :**

Let X and Y be two continuous random variables with the joint probability density function
$$f(x,y) = \begin{cases} 2 & 0 < x+y < 1, x > 0, y > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

75) $P(X+Y<\frac{1}{2})$ is

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) $\frac{3}{4}$

d) 1

76) $E(X|Y=\frac{1}{2})$

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) 1

d) 2

77) If a random variable X assumes only positive integral values, with the probability
$P(X=x) = \frac{2}{3}(\frac{1}{3})^{x-1}$, $x = 1, 2, 3, ...,$

then $E(X)$ is

a) $\frac{2}{9}$

b) $\frac{2}{3}$

c) 1

d) $\frac{3}{2}$

78) The joint probability density function of two random variables X and Y is given as
$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leqslant x \leqslant 1 0 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere} \end{cases}$$
$E(X)$ and $E(Y)$ are, respectively,

a) $\frac{2}{5}$ and $\frac{3}{5}$

b) $\frac{3}{5}$ and $\frac{3}{5}$

c) $\frac{3}{5}$ and $\frac{6}{5}$

d) $\frac{4}{5}$ and $\frac{6}{5}$

79) Suppose the random variable U has uniform distribution on $[0,1]$ and $X = -2\log U$. The density of X is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$

80) Suppose X is a real-valued random variable. Which of the following values CANNOT be attained by $E[X]$ and $E[X^2]$, respectively?

a) 0 and 1

b) 2 and 3

c) $\frac{1}{2}$ and $\frac{1}{3}$

d) 2 and 5

81) Probability density function $p(x)$ of a random variable x is as shown below. The value of $\alpha$ is

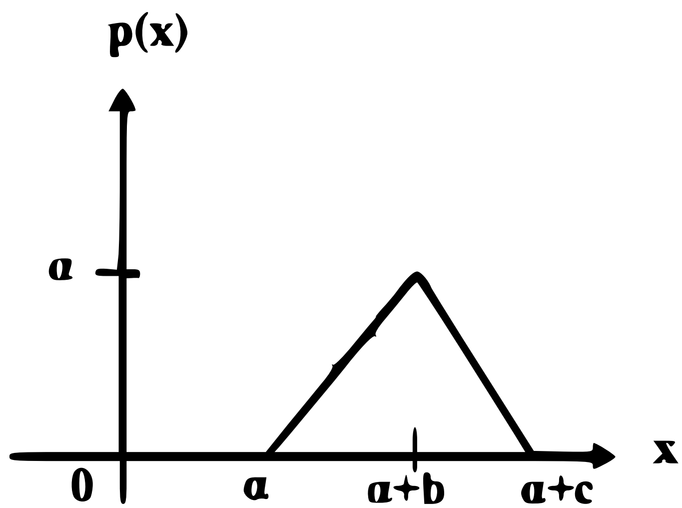


Fig. 11

a) $\frac{2}{c}$

b) $\frac{1}{c}$

c) $\frac{2}{(b+c)}$

d) $\frac{1}{(b+c)}$

82) A player throws a ball at a basket kept at a distance. The probability that the ball falls into the basket in a single attempt is 0.1. The player attempts to throw the ball twice. Considering each attempt to be independent, the probability that this player puts the ball into the basket only in the second attempt is.........
**Solution:** Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.
$X = k$ represents $k-1$ failures were obtained before getting 1 success. $p$ represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \tag{39}$$

Using (39) we get

$$\begin{aligned} \Pr(X=2) &= (1-p)^{k-1} \times p \\ &= (0.9) \times 0.1 = 0.09 \end{aligned} \tag{40}$$

83) A screening test is carried out to detect a certain disease. It is found that 12% of the positive reports and 15% of the negative reports are incorrect. Assuming that the probability of a person getting positive report is 0.01, the probability that a person tested gets an incorrect report is . . .
**Solution:** Let $X \in \{0,1\}$ represent the random variable, where 0 represents the case where a person gets a positive report while 1 represents the case where a person gets a negative report. From the question,

$$\Pr(X=0) = 0.01 \tag{41}$$
$$\Pr(X=0) + \Pr(X=1) = 1 \tag{42}$$
$$\Pr(X=1) = 1 - 0.01 = 0.99 \tag{43}$$

Let $Y \in \{0,1\}$ represent the random variable, where 0 represents a correct report whereas 1 represents an incorrect report.

$$\Pr(Y=1|X=0) = 12\% = 0.12 \tag{44}$$
$$\Pr(Y=1|X=1) = 15\% = 0.15 \tag{45}$$

Then, from total probability theorem,

$$\begin{aligned} \Pr(Y=1) = \Pr(Y=1, X=0) \\ + \Pr(Y=1, X=1) \end{aligned} \tag{46}$$

Using Bayes theorem,

$$\begin{aligned} \Pr(Y=1) = \Pr(Y=1|X=0) \times \Pr(X=0) \\ + \Pr(Y=1|X=1) \times \Pr(X=1) \end{aligned} \tag{47}$$

$$\Pr(Y=1) = 0.12 \times 0.01 + 0.15 \times 0.99 \tag{48}$$
$$= 0.0012 + 0.1485 \tag{49}$$
$$= 0.1497 \tag{50}$$

84) Shaquille O' Neal is a 60% career free throw shooter, meaning that he successfully makes 60 free throws out of 100 attempts on average. What is the probability that he will successfully make exactly 6 free throws in 10 attempts?

A) 0.2508

B) 0.2816

C) 0.2934

D) 0.6000

**Solution:** Let

$$X_i \in \{0,1\} \tag{51}$$

represent the $i^{th}$ free throw, where 1 represents a successful free throw attempt and 0 represents an unsuccessful attempt. Let

$$X = \sum_{i=1}^{n} X_i \tag{52}$$

where n is the total number of free throws. Then, X has a binomial distribution with

$$\Pr(X=k) = {}^nC_k p^k q^{n-k} \tag{53}$$

Where,

$$p = \frac{6}{10} \tag{54}$$

$$q = 1 - p = \frac{4}{10} \tag{55}$$

$$n = 10 \tag{56}$$

from the given information. Then,

$$\Pr(X = 6) = {}^{10}C_6 \left(\frac{6}{10}\right)^6 \left(\frac{4}{10}\right)^4 \tag{57}$$

On simplifying we get,

$$\Pr(X = 6) = 0.2508 \tag{58}$$

Therefore, the probability that he will successfully make exactly 6 free throws in 10 attempts is 0.2508 and hence option (A) is correct.