Q: Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, each having probability density function $f(\cdot)$ and median θ. We want to test
$H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$
Consider a test that rejects H_0 if $S > c$ for some c depending on the size of the test, where S is the cardinality of the set $\{i : X_i > \theta_0, 1 \le i \le n\}$. Then which one of the following statements is true?
(A) Under H_0, the distribution of S depends on $f(\cdot)$.
(B) Under H_1, the distribution of S does not depend on $f(\cdot)$.
(C) The power function depends on θ.
(D) The power function does not depend on θ.
Solution:

Definition 1. *Median θ is defined as*
$\Pr(X_i \le \theta) = 0.5$ *for all i from 1 to n.*

Definition 2. *S is defined as*
$S = \sum_{i=1}^{n} I(X_i > \theta_0)$
where $I(X_i > \theta_0$ represents an indicator function.

$$I(X_i > \theta_0) = \begin{cases} 1, & \text{if } X_i > \theta_0 \\ 0, & \text{if } X_i \le \theta_0 \end{cases} \tag{1}$$

$$E(S) = E\left(\sum_{i=1}^{n} I(X_i > \theta_0)\right) \tag{2}$$
$$= \sum_{i=1}^{n} E(I(X_i > \theta_0)) \tag{3}$$

Since,

$$E(I(X_i > \theta_0)) = P(X_i > \theta_0) = \int_{\theta_0}^{\infty} f(x)\,dx \tag{4}$$

Therefore,

$$E(S) = \sum_{i=1}^{n} \int_{\theta_0}^{\infty} f(x)\,dx \tag{5}$$

1) From (5), under H_0, the distribution of S depends on $f(\cdot)$.

2) The power function can be expressed as:

$$\pi(\theta) = \Pr(\text{Reject } H_0 \,|\, H_1 \text{ is true}) \tag{6}$$
$$= \Pr(S > c|\theta) \tag{7}$$

Therefore, power function depends on value of θ.