EE23010 NCERT Exemplar

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Let X_1, X_2, X_3....,X_n be a random sample of size $n\,(\geq 2)$ from a population having probability density function

$$f\,(x;\theta) = \begin{cases} \frac{2}{\theta x}\,(\log_e x)\,e^{-\frac{(\log_e x)^2}{\theta}} & ,0 < x < 1 \\ 0 & ,otherwise \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which of the following statements is true,

(A) $\frac{1}{n}\sum_{i=1}^{n}(\ln X_i)^2$ is the maximum likelihood estimator of θ
(B) $\frac{1}{n-1}\sum_{i=1}^{n}(\ln X_i)^2$ is the maximum likelihood estimator of θ
(C) $\frac{1}{n}\sum_{i=1}^{n}\ln X_i$ is the maximum likelihood estimator of θ
(D) $\frac{1}{n-1}\sum_{i=1}^{n}\ln X_i$ is the maximum likelihood estimator of θ

Solution:

$$L\,(\theta) = f\,(x_1, x_2, .., x_n; \theta) \tag{1}$$

The product of pdfs can be used to approximate the likelihood function even if the variables are dependent. This is a general approach that is often used in practice to estimate MLE of θ. Therefore,

$$L\,(\theta) = \prod_{i=1}^{n} f\,(x_i;\theta) \tag{2}$$

Maximizing $L\,(\theta)$ is equivalent to maximizing the the $\ln L\,(\theta)$ as ln is a monotonically increasing function.

$$l\,(\theta) = \ln L\,(\theta) \tag{3}$$

$$= \ln\left(\prod_{i=1}^{n} f\,(x_i;\theta)\right) \tag{4}$$

$$= \sum_{i=1}^{n} \ln f\,(x_i;\theta) \tag{5}$$

$$= -n\ln 2 - n\ln\theta + \sum_{i=1}^{n}\ln\left(-\ln x_i\right) - \sum_{i=1}^{n}\left(\ln x_i\right) - \sum_{i=1}^{n}\frac{(\ln x_i)^2}{\theta} \tag{6}$$

Maximizing $l\,(\theta)$ with respect to θ gives the MLE estimation, therefore

$$\frac{\partial l\,(\theta)}{\partial\theta} = 0 \tag{7}$$

$$\frac{-n}{\theta} + \frac{1}{(\theta)^2}\sum_{i=1}^{n}(\ln X_i)^2 = 0 \tag{8}$$

$$\theta = \frac{1}{n}\sum_{i=1}^{n}(\ln X_i)^2 \tag{9}$$

Hence (A) is the true statement.