

# Assignment

Karthikeya hanu prakash kanithi (EE22BTECH11026)

Question : Suppose that  $(X, Y)$  has joint probability mass function

$$P(X = 0, Y = 0) = P(X = 1, Y = 1) = \theta, \quad (1)$$

$$P(X = 1, Y = 0) = P(X = 0, Y = 1) = \frac{1}{2} - \theta. \quad (2)$$

where  $0 \leq \theta \leq \frac{1}{2}$  is an unknown parameter. Consider testing  $H_0 : \theta = \frac{1}{4}$  against  $H_1 : \theta = \frac{1}{3}$ ; based on a random sample  $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$  from the above probability mass function. Let  $M$  be the cardinality of the set  $\{i : X_i = Y_i, 1 \leq i \leq n\}$ . If  $m$  is the observed value of  $M$ , then which one of the following statements is true?

- A) The likelihood ratio test rejects  $H_0$  if  $m > c$  for some  $c$ .
- B) The likelihood ratio test rejects  $H_0$  if  $m < c$  for some  $c$ .
- C) The likelihood ratio test rejects  $H_0$  if  $c_1 < m < c_2$  for some  $c_1$  and  $c_2$ .
- D) The likelihood ratio test rejects  $H_0$  if  $m < c_1$  or  $m > c_2$  for some  $c_1$  and  $c_2$ .

**Solution:** Given that,

$$H_0 : \theta = \theta_0 = \frac{1}{4}, \quad (3)$$

$$H_1 : \theta = \theta_1 = \frac{1}{3}. \quad (4)$$

and the pmf is given by

$$p_{XY}(0, 0) = p_{XY}(1, 1) = \theta \quad (5)$$

$$p_{XY}(0, 1) = p_{XY}(1, 0) = \frac{1}{2} - \theta \quad (6)$$

Then for the given random sample of data,

$$p_{X_i, Y_i}(x, y) = \begin{cases} 2\theta & x = y \\ 1 - 2\theta & x \neq y \end{cases} \quad (7)$$

(8)

Then the likelihood of the data under  $H_0$  is given by:

$$L(\theta_0 | data) = \prod_{i=1}^n p_{X_i, Y_i}(x, y) \quad (9)$$

$$= (2\theta_0)^m (1 - 2\theta_0)^{n-m} \quad (10)$$

$$= \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m} \quad (11)$$

Then the likelihood of the data under  $H_1$  is given by:

$$L(\theta_1 | data) = \prod_{i=1}^n p_{X_i, Y_i}(x, y) \quad (12)$$

$$= (2\theta_1)^m (1 - 2\theta_1)^{n-m} \quad (13)$$

$$= \left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m} \quad (14)$$

The likelihood ratio will be

$$\lambda(data) = \frac{L(\theta_1 | x)}{L(\theta_0 | x)} \quad (15)$$

$$= \frac{\left(\frac{2}{3}\right)^m \left(\frac{1}{3}\right)^{n-m}}{\left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m}} = (2)^m \left(\frac{2}{3}\right)^n \quad (16)$$

Let the critical value be denoted by  $c_1$ , then the likelihood ratio test rejects  $H_0$  if

$$\Rightarrow \lambda(data) \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \quad (17)$$

(18)

From (16),

$$\Rightarrow (2)^m \left(\frac{2}{3}\right)^n \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \quad (19)$$

$$\Rightarrow (2)^m \underset{H_0}{\overset{H_1}{\gtrless}} c_1 \left(\frac{2}{3}\right)^n \quad (20)$$

$$\Rightarrow m \underset{H_0}{\overset{H_1}{\gtrless}} \log_2 \left( c_1 \left(\frac{2}{3}\right)^n \right) \quad (21)$$

$$\Rightarrow m \underset{H_0}{\overset{H_1}{\gtrless}} c \quad \exists c \in \mathbb{R} \quad (22)$$

where,

$$c = \log_2 \left( c_1 \left(\frac{2}{3}\right)^n \right) \quad (23)$$

$\therefore$  From (22), Option A is correct and Options B,C,D are incorrect