Assignment 4

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Download the python code from

https://github.com/ArunSiddardha/Assignment4/assignment4.py

and latex-tikz code from

https://github.com/ArunSiddardha/Assignment4/Assignment4.tex

1 Problem GATE 2014 CS Q48

Four fair six-sided dice are rolled.The probability that sum of results being 22 is $\frac{X}{1296}$.the value of X is

2 Solution

Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, i = 1,2,3,4 be the random variables representing the outcome for each die. As the dies are fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{2.0.1}$$

Let X be a random variable denotes the desired outcome,

$$X = X_1 + X_2 + X_3 + X_4 \tag{2.0.2}$$
$$\implies X \in \{4, 5, \cdots, 24\} \tag{2.0.3}$$

We have to find $P_X(n) = \Pr(X_1 + X_2 + X_3 + X_4 = n)$ The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.0.4}$$

From (2.0.1) and (2.0.4),

$$P_{X_1}(z) = P_{X_2}(z) = P_{X_3}(z) = P_{X_3}(z)$$
$$= \frac{1}{6} \sum_{n=1}^{6} z^{-n} \tag{2.0.5}$$
$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{2.0.6}$$

upon summing up the geometric progression.From convolution

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n) * p_{X_3}(n) * p_{X_4}(n), \tag{2.0.7}$$
$$P_X(z) = P_{X_1}(z) P_{X_2}(z) P_{X_3}(z) p_{X_4}(z) \tag{2.0.8}$$

The above property follows from Fourier analysis and is fundamental to signal processing.
From (2.0.6) and (2.0.8),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^4 \tag{2.0.9}$$
$$= \frac{1}{1296} \frac{z^{-4}\left(1 - 4z^{-6} + 6z^{-12} - 4z^{-24} + z^{-24}\right)}{\left(1 - z^{-1}\right)^4} \tag{2.0.10}$$

Using the fact that,

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{2.0.11}$$
$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.0.12}$$
$$n^2 u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \tag{2.0.13}$$
$$(n^2 + n) u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{2z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.0.14}$$
$$(n^3 + 3n^2 + 2n) u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{6z^{-1}}{\left(1 - z^{-1}\right)^4} \tag{2.0.15}$$

after some algebra, it can be shown that,

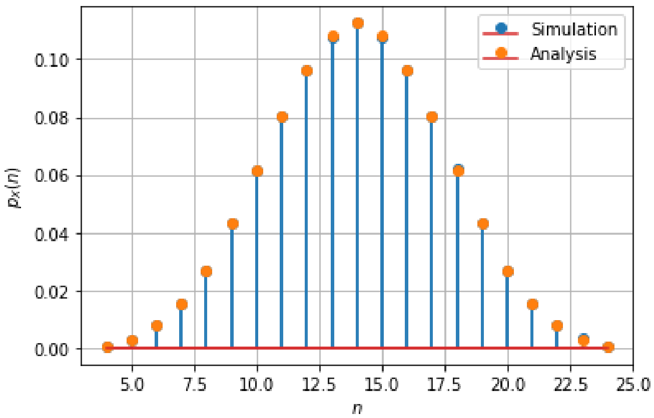

Fig. 1: Probability of getting sum of 22

$$\frac{1}{1296 \times 6}\Big[\Big((n-3)^3 + 3(n-3)^2 + 2(n-3)\Big)u(n-3)$$
$$-4\Big((n-9)^3 + 3(n-9)^2 + 2(n-9)\Big)u(n-9)$$
$$+6\Big((n-15)^3 + 3(n-15)^2 + 2(n-15)\Big)u(n-15)$$
$$-4\Big((n-21)^3 + 3(n-21)^2 + 2(n-21)\Big)u(n-21)$$
$$+\Big((n-27)^3 + 3(n-27)^2 + 2(n-27)\Big)u(n-27)\Big]$$
$$\overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{1296}\frac{z^{-4}\left(1-4z^{-6}+6z^{-12}-4z^{-24}+z^{-24}\right)}{(1-z^{-1})^4} \tag{2.0.16}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{2.0.17}$$

From (2.0.4),(2.0.10) and (2.0.16),

$$p_X(n) = \frac{1}{1296 \times 6}\Big[$$
$$\Big((n-3)^3 + 3(n-3)^2 + 2(n-3)\Big)u(n-3)$$
$$-4\Big((n-9)^3 + 3(n-9)^2 + 2(n-9)\Big)u(n-9)$$
$$+6\Big((n-15)^3 + 3(n-15)^2 + 2(n-15)\Big)u(n-15)$$
$$-4\Big((n-21)^3 + 3(n-21)^2 + 2(n-21)\Big)u(n-21)$$
$$+\Big((n-27)^3 + 3(n-27)^2 + 2(n-27)\Big)u(n-27)\Big] \tag{2.0.18}$$

From (2.0.17) and (2.0.18),

$$p_X(n) = \begin{cases} 0 & n < 4 \\ \frac{n^3-6n^2+11n-6}{7776} & 4 \leq n \leq 9 \\ \frac{90n^2-3n^3-753n+2010}{7776} & 9 < n \leq 15 \\ \frac{3n^3-162n^2+2769n-14370}{7776} & 15 < n \leq 21 \\ \frac{-n^3+78n^2-2027n+17550}{7776} & 21 < n \leq 24 \\ 0 & 24 < n \end{cases} \tag{2.0.19}$$

We need probability of getting sum of 22,

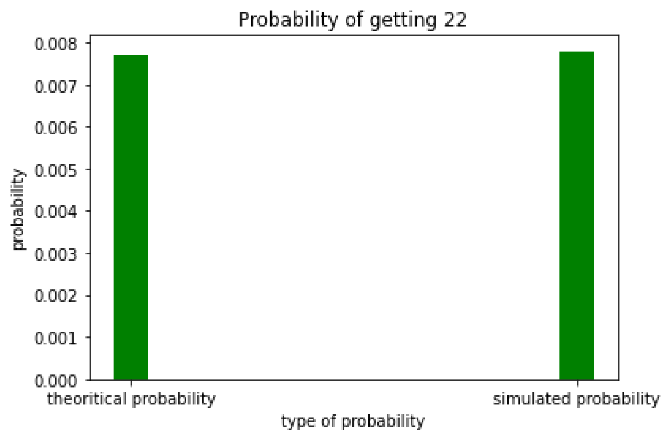

Fig. 2: Probability mass function of X (simulations are close to analysis)

$\implies$ n=22
from (2.0.19) and using n=22,

$$p_X(22) = \frac{-(22)^3 + 78(22)^2 - 2027(22) + 17550}{7776} \tag{2.0.20}$$

$$p_X(22) = \frac{60}{7776} \tag{2.0.21}$$

$$p_X(22) = \frac{10}{1296} \tag{2.0.22}$$

Therefore the probability of getting a sum of 22 when four fair dies are rolled is $\frac{10}{1296}$.
Ans X = 10