GATE PROBABILITY

Through Simulations

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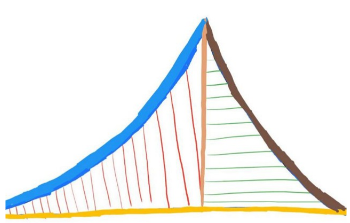

Contents

Introduction

This book solves probability problems in GATE question papers.

Chapter 1

Axioms

Chapter 2

Distributions

1. Let $\phi(.)$ denote the cumulative distribution function of a standard normal random variable. If the random variable X has the cumulative distribution function

$$F(x) = \begin{cases} \phi(x), & x < -1 \\ \phi(x+1), & x \geq -1 \end{cases} \tag{2.1}$$

then which one of the following statements is true?

(a) $P(X \leq -1) = \frac{1}{2}$

(b) $P(X = -1) = \frac{1}{2}$

(c) $P(X < -1) = \frac{1}{2}$

(d) $P(X \leq 0) = \frac{1}{2}$

(GATE ST 2023)

Solution: Gaussian

Q function is defined

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-u^2}{2}} du \tag{2.2}$$

From question and (2.2);

$$F_X(x) = \begin{cases} Q\left(-x\right), & x < -1 \\ 1 - Q\left(x+1\right), & x \geq -1 \end{cases} \tag{2.3}$$

From (2.3);

(a)

$$\Pr\left(X \leq -1\right) = F_X(-1) = 1 - Q\left(0\right) \tag{2.4}$$

$$= 0.5 \tag{2.5}$$

So Option A i.e., $P(X < -1) = \frac{1}{2}$ is correct

(b) The pdf of X can be defined in terms of cdf as

$$\Pr\left(X = b\right) = F_X(b) - \lim_{x \to b^-} F_X(x) \tag{2.6}$$

From (2.6);

$$\Pr\left(X = -1\right) = F_X(-1) - \lim_{x \to -1^-} F_X(x) \tag{2.7}$$

$$= 1 - Q\left(0\right) - Q\left(-(-1)\right) \tag{2.8}$$

$$= 0.341 \tag{2.9}$$

So Option B i.e., $P(X = -1) = \frac{1}{2}$ is incorrect

(c)

$$\Pr\left(X < -1\right) = \lim_{x \to -1^-} F_X(x) = F_X(-1) \tag{2.10}$$

$$= Q\left(-(-1)\right) \tag{2.11}$$

$$= 0.159 \tag{2.12}$$

So Option C i.e., $P(X < -1) = \frac{1}{2}$ is incorrect

(d)

$$\Pr\left(X \le 0\right) = F_X(0) = 1 - Q\left(1\right) \tag{2.13}$$

$$= 0.8413 \tag{2.14}$$

So Option D i.e., $P(X \le 0) = \frac{1}{2}$ is incorrect

Guassian CDF plot of X is given in fig2.1

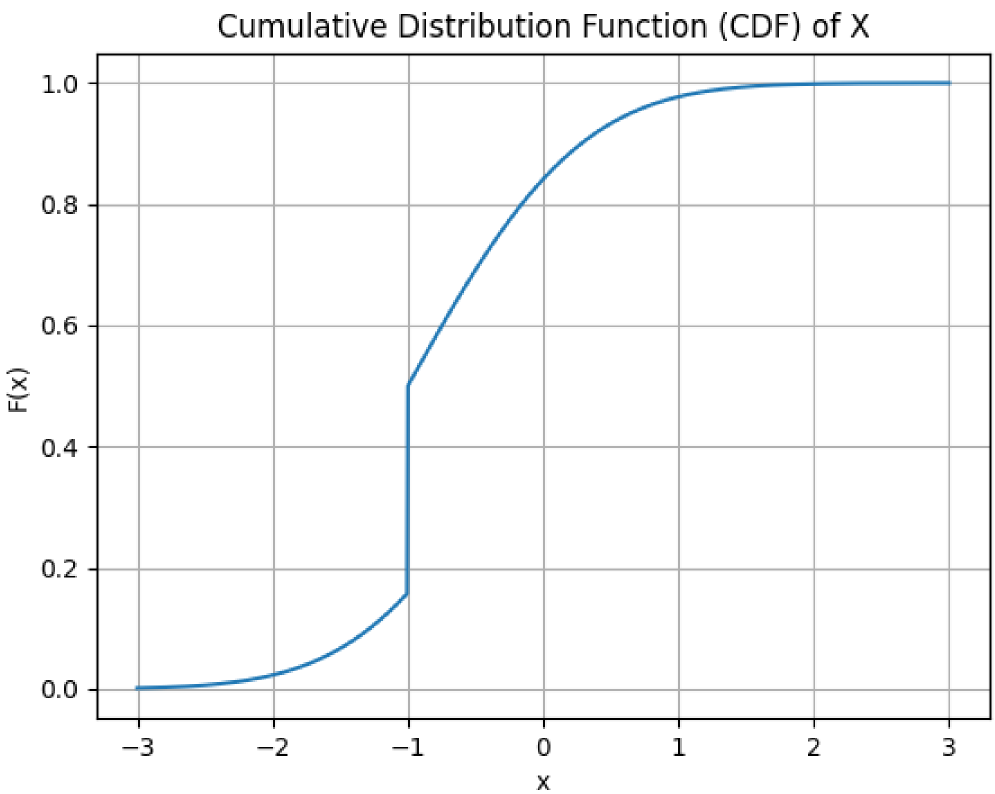

Figure 2.1:

2. Let X be a random variable with the probability density function $f(x)$ such that

$$f(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & -\sqrt{3} \leq x \leq \sqrt{3} \\ 0, & \text{otherwise} \end{cases} \tag{2.15}$$

Then the variance of X is? (GATE XH-C1 2023)

Solution:

The mean of X

$$\mu_X = \int_{-\infty}^{\infty} x f(x) dx \tag{2.16}$$

As the integrand is odd

$$\implies \mu_X = 0 \tag{2.17}$$

The variance of X is:

$$\sigma_X^2 = \mathbb{E}\left(X - \mu_X\right)^2 \tag{2.18}$$

From (2.17)

$$\implies \sigma_X^2 = \mathbb{E}\left(X^2\right) \tag{2.19}$$

$$= \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x^2 dx \tag{2.20}$$

$$= 1 \tag{2.21}$$

Chapter 3

Conditional Probability

Chapter 4

Moments

Chapter 5

Random Algebra

1. Let (X, Y) have joint probability density function

$$p_{XY}(x, y) = \begin{cases} 8xy & if 0 < x < y < 1 \\ 0 & otherwise \end{cases} \tag{5.1}$$

if $E(X|Y = y_0) = \frac{1}{2}$,then y_0 equals

(a) $\frac{3}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{2}{3}$

(GATE ST 2023)

Solution:

$$E(X|Y) = \int_{-\infty}^{\infty} x p_{X|Y} dx \tag{5.2}$$

where

$$p_{X|Y} = \frac{p_{XY}(x,y)}{p_Y(y)} \tag{5.3}$$

$$p_Y(y) = \int_0^y p_{X|Y}(x,y)\,dx \tag{5.4}$$

for $0 < y < 1$

$$= \int_0^y 8xydx \tag{5.5}$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y \tag{5.6}$$

$$= 4y^3 \tag{5.7}$$

For $0 < x < y < 1$,on substituting $p_Y(y)$ we get

$$p_{X|Y} = \frac{8xy}{4y^3} \tag{5.8}$$

$$= \frac{2x}{y^2} \tag{5.9}$$

and

$$
\begin{aligned}
E\left(X|Y=y_0\right) &= \int_0^{y_0} x.\frac{2x}{y_0^2}dx && (5.10)\\
&= \frac{2}{y_0^2}\left[\frac{x^3}{3}\right]_0^{y_0} && (5.11)\\
&= \frac{2y_0}{3} && (5.12)\\
\implies \frac{2y_0}{3} &= \frac{1}{2} && (5.13)\\
y_0 &= \frac{3}{4} && (5.14)
\end{aligned}
$$

Chapter 6

Hypothesis Testing

Chapter 7

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Chapter 8

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Convergence

9.1 Let $\{X_n\}_{n\geq 1}$ and Let $\{Y_n\}_{n\geq 1}$ be two sequences of random variables and X and Y be two random variables, all of them defined on the same probability space. Which one of the following statements is true?

(A) If $\{X_n\}_{n\geq 1}$ converges in distribution to a real constant c, then $\{X_n\}_{n\geq 1}$ converges in probability to c.

(B) If $\{X_n\}_{n\geq 1}$ converges in probability to X, then $\{X_n\}_{n\geq 1}$ converges in 3^{rd} mean to X.

(C) If $\{X_n\}_{n\geq 1}$ converges in distribution to X and $\{Y_n\}_{n\geq 1}$ converges in distribution to Y, then $\{X_n + Y_n\}_{n\geq 1}$ converges in distribution to $X + Y$.

(D) If $\{E(X_n)\}_{n\geq 1}$ converges to $E(X)$, then $\{X_n\}_{n\geq 1}$ converges in 1^{st} mean to X.

(GATE ST 2023) **Solution:**

(a) X_n converges in distribution to X, $X_n \xrightarrow{d} X$, then for all x,

$$lim_{n\to\infty} F_{X_n}(x) = F_X(x) \tag{9.1}$$

(b) X_n converges in probability to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,

$$lim_{n \to \infty} \Pr\left(|X_n - X| > \epsilon\right) = 0 \tag{9.2}$$

(c) X_n converges in p^{th} mean to X, then we have

$$lim_{n \to \infty} E(|X_n - X|^p) = 0 \tag{9.3}$$

(A) For $\epsilon > 0$, B be defined as

$$B = \{x : |x - c| \geq \epsilon\} \tag{9.4}$$

Now,

$$\Pr\left(|X_n - c| \geq \epsilon\right) = \Pr\left(X_n \in B\right) \tag{9.5}$$

Using Portmanteau Lemma, if $X_n \xrightarrow{d} c$, we have

$$\begin{aligned}
\limsup_{n \to \infty} \Pr\left(X_n \in B\right) &\leq \Pr\left(c \in B\right) && (9.6)\\
&\leq \Pr\left(|0 - 0| \geq \epsilon\right) && (9.7)\\
&\leq \Pr\left(0 \geq \epsilon\right) && (9.8)\\
&\leq 0 && (9.9)\\
&= 0 && (9.10)\\
lim_{n \to \infty} \Pr\left(|X_n - c| > \epsilon\right) &= 0 && (9.11)
\end{aligned}$$

From (9.2), $X_n \xrightarrow{p} c$. So, we have

$$X_n \xrightarrow{d} c \implies X_n \xrightarrow{p} c \tag{9.12}$$

Option (A) is correct.

(B) Statement (B) is may or may not correct. Counter Example: Consider distribution

X_n	0	n
$\Pr(X_n)$	$1-\frac{1}{n}$	$\frac{1}{n}$

For $\epsilon > 0$, X_n converges in probability to $X = 0$

$$lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = lim_{n\to\infty} \Pr(X_n > \epsilon) \tag{9.13}$$

$X_n > \epsilon$vis subset of $X_n = n$ since every time X_n equals n, it's also true that X_n is greater than ϵ. But there may be times when X_n is greater than ϵ without X_n being equal to n. So,

$$\Pr(X_n > \epsilon) \leq \Pr(X_n = n) \tag{9.14}$$

$$lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) \leq lim_{n\to\infty} \Pr(X_n = n) \tag{9.15}$$

$$\leq lim_{n\to\infty} \frac{1}{n} \tag{9.16}$$

$$\leq 0 \tag{9.17}$$

$$= 0 \tag{9.18}$$

But X_n does not converges in 3^{rd} mean to $X = 0$.

$$lim_{n\to\infty}E(|X_n - X|^3) = lim_{n\to\infty}E(X_n^3) \tag{9.19}$$

$$= lim_{n\to\infty}0^3\left(1 - \frac{1}{n}\right) + n^3\left(\frac{1}{n}\right) \tag{9.20}$$

$$= lim_{n\to\infty}n^2 \neq 0 \tag{9.21}$$

(C) Statement (C) is may or may not correct. Counter Example: Consider distribution

$$Z \sim \mathcal{N}(0,1) \tag{9.22}$$

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random variables such that they both converge in distribution as Z and $(-1)^n Z$. Proof that Y_n converges in distribution.

For n even

$$lim_{n\to\infty}F_{Y_n}(x) = \Pr(Z \leq x) \tag{9.23}$$

For n odd

$$lim_{n\to\infty}F_{Y_n}(x) = \Pr(-Z \leq x) \tag{9.24}$$

$$= \Pr(Z \leq x) \tag{9.25}$$

Proved. So,we have

$$F_{X_n+Y_n}(x) = \Pr\left(X_n + Y_n \leq x\right) \tag{9.26}$$

$$= \Pr\left(Z + (-1)^n Z \leq x\right) \tag{9.27}$$

For n is even

$$F_{X_n+Y_n}(x) = \Pr\left(2Z \leq x\right) \tag{9.28}$$

$$= \Pr\left(Z \leq \frac{x}{2}\right) \tag{9.29}$$

$$= 1 - \Pr\left(Z > \frac{x}{2}\right) \tag{9.30}$$

$$\approx 1 - Q\left(\frac{x}{2}\right) \tag{9.31}$$

For n is odd

$$F_{X_n+Y_n}(x) = \Pr\left(0 \leq x\right) \tag{9.32}$$

$$= \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = H(x) \tag{9.33}$$

So, on generalizing

$$F_{X_n+Y_n}(x) = \begin{cases} 1 - Q\left(\frac{x}{2}\right) & \text{if } n \text{ is even} \\ H(x) & \text{if } n \text{ is odd} \end{cases} \tag{9.34}$$

$lim_{n\to\infty} F_{X_n+Y_n}(x)$ oscillate between $1 - Q\left(\frac{x}{2}\right)$ and $H(x)$. This doesnot imply convergence.

(D) Statement (D) is may or may not correct. Counter Example: Consider

X_n	0	n
$\Pr(X_n)$	$1-\frac{1}{n}$	$\frac{1}{n}$

$$lim_{n\to\infty}E(X_n) = 0\left(1-\frac{1}{n}\right)+n\left(\frac{1}{n}\right) \tag{9.35}$$

$$= 1 \tag{9.36}$$

As $n \to \infty$, $E(X_n)$ converges to $E(X) = 1$.

$$lim_{n\to\infty}X_n = 0 = X \tag{9.37}$$

To find 1^{st} mean convergennce of X_n. From (9.36)

$$lim_{n\to\infty}E(|X_n - X|) = lim_{n\to\infty}E(X_n) \tag{9.38}$$

$$= 1 \neq 0 \tag{9.39}$$

So, X_n does not converges in 1^{st} mean to X.

Chapter 10

Information Theory

1. The frequency of occurrence of 8 symbols (a-h) is shown in the table below. A symbol is chosen and it is determined by asking a series of "yes/no" questions which are assumed to be truthfully answered. The average number of questions when asked in the most efficient sequence, to determine the chosen symbol, is

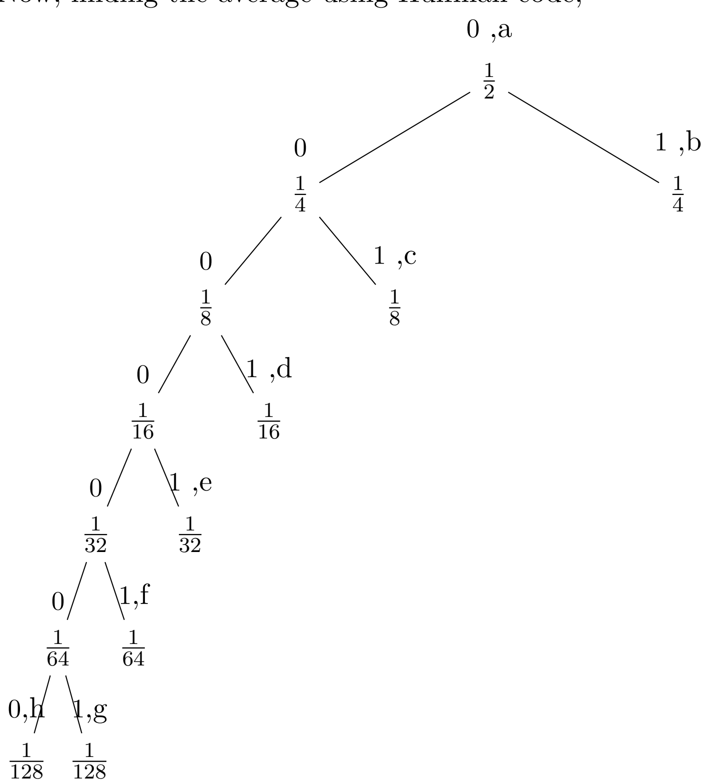
Symbols	**Frequency of occurance**
a	$\frac{1}{2}$
b	$\frac{1}{4}$
c	$\frac{1}{8}$
d	$\frac{1}{16}$
e	$\frac{1}{32}$
f	$\frac{1}{64}$
g	$\frac{1}{128}$
h	$\frac{1}{128}$

Solution:

Parameter	**Value**	**Description**
X	$1 \leq X \leq 8$	number of symbols
l	2	base of algorithm
$H(X)$	$\sum_i p_X(i) \log_l \left(\frac{1}{p_X(i)}\right)$	average number of question

$$
\begin{aligned}
H(X) &= \sum_i p_X(i) \log_b \left(\frac{1}{p_X(i)} \right) && (10.1)\\
&= \frac{1}{2} \log_2(2) + \frac{1}{4} \log_2(4) + ... + \frac{1}{128} \log_2(128) && (10.2)\\
&= 0.5 + 0.5 + 0.375 + ... + 0.0078125 && (10.3)\\
&= 1.984375 && (10.4)
\end{aligned}
$$

Now, finding the average using Huffman code,



Using the above binary table following code is generated;

Symbols	**Frequency**	**Code**	**Size**
a	$\frac{1}{2}$	1	0.5
b	$\frac{1}{4}$	01	0.25
c	$\frac{1}{8}$	001	0.125
d	$\frac{1}{16}$	0001	0.0625
e	$\frac{1}{32}$	00001	0.03125
f	$\frac{1}{64}$	000001	0.015625
g	$\frac{1}{128}$	0000001	0.0078125
h	$\frac{1}{128}$	0000000	0.0078125

Table 10.1: Huffman table
The average number of question = Weighted path length = 1.9844