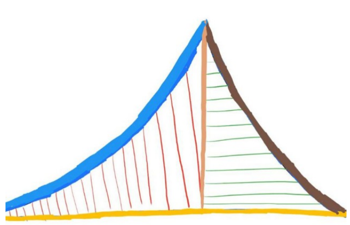
# GATE PROBABILITY

## Through Simulations

**G. V. V. Sharma**



# Contents

# Introduction

This book solves probability problems in GATE question papers.

# Chapter 1

# Axioms

# Chapter 2

# Distributions

1. Let $\phi(.)$ denote the cumulative distribution function of a standard normal random variable. If the random variable $X$ has the cumulative distribution function

$$F(x) = \begin{cases} \phi(x), & x < -1 \\ \\ \phi(x+1), & x \geq -1 \end{cases} \tag{2.1}$$

then which one of the following statements is true?

(a) $P(X \leq -1) = \frac{1}{2}$

(b) $P(X = -1) = \frac{1}{2}$

(c) $P(X < -1) = \frac{1}{2}$

(d) $P(X \leq 0) = \frac{1}{2}$

(GATE ST 2023)

**Solution: Gaussian**

Q function is defined

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-u^2}{2}} du \tag{2.2}$$

From question and (2.2);

$$F_X(x) = \begin{cases} Q\left(-x\right), & x < -1 \\ 1 - Q\left(x+1\right), & x \geq -1 \end{cases} \tag{2.3}$$

From (2.3);

(a)

$$\Pr\left(X \leq -1\right) = F_X(-1) = 1 - Q\left(0\right) \tag{2.4}$$

$$= 0.5 \tag{2.5}$$

So Option A i.e., $P(X < -1) = \frac{1}{2}$ is correct

(b) The pdf of X can be defined in terms of cdf as

$$\Pr\left(X = b\right) = F_X(b) - \lim_{x \to b^-} F_X(x) \tag{2.6}$$

From (2.6);

$$\Pr\left(X = -1\right) = F_X(-1) - \lim_{x \to -1^-} F_X(x) \tag{2.7}$$

$$= 1 - Q\left(0\right) - Q\left(-(-1)\right) \tag{2.8}$$

$$= 0.341 \tag{2.9}$$

So Option B i.e., $P(X = -1) = \frac{1}{2}$ is incorrect

(c)

$$\Pr\left(X < -1\right) = \lim_{x \to -1^-} F_X(x) = F_X(-1) \tag{2.10}$$

$$= Q\left(-(-1)\right) \tag{2.11}$$

$$= 0.159 \tag{2.12}$$

So Option C i.e., $P(X < -1) = \frac{1}{2}$ is incorrect

(d)

$$\Pr\left(X \leq 0\right) = F_X(0) = 1 - Q\left(1\right) \tag{2.13}$$

$$= 0.8413 \tag{2.14}$$

So Option D i.e., $P(X \leq 0) = \frac{1}{2}$ is incorrect

Guassian CDF plot of X is given in fig2.1

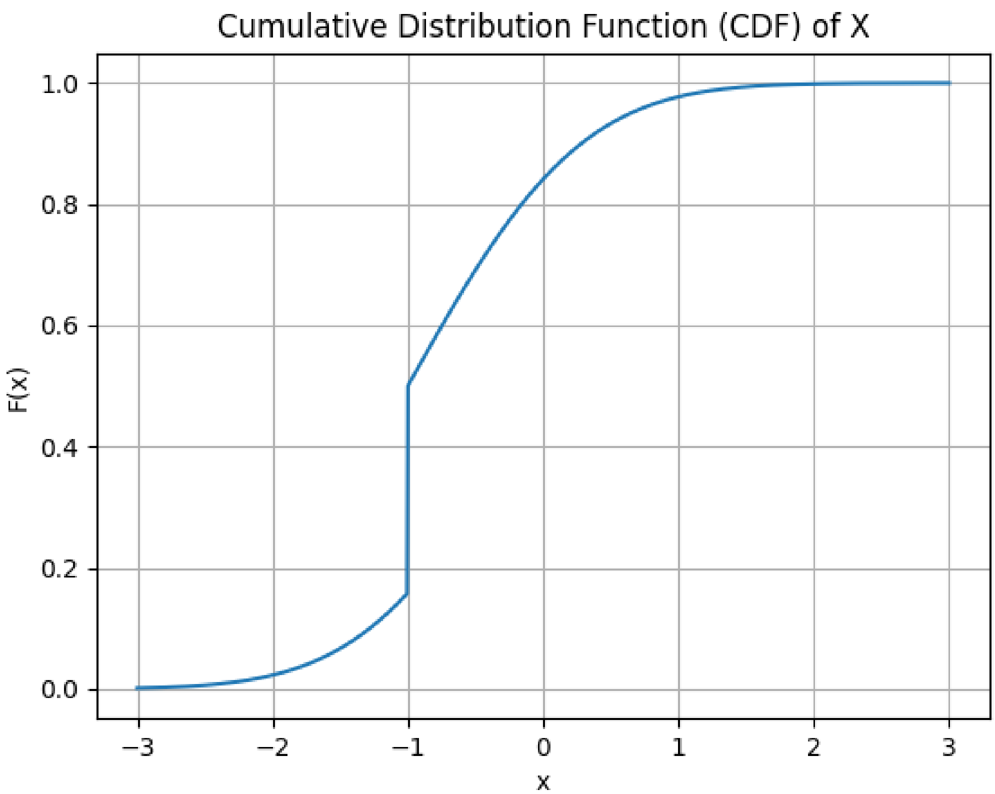



Figure 2.1:

2. Let $X$ be a random variable with the probability density function $f(x)$ such that

$$f(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & -\sqrt{3} \leq x \leq \sqrt{3} \\ 0, & \text{otherwise} \end{cases} \tag{2.15}$$

Then the variance of $X$ is? (GATE XH-C1 2023)

**Solution:**

The mean of $X$

$$\mu_X = \int_{-\infty}^{\infty} x f(x) dx \tag{2.16}$$

As the integrand is odd

$$\implies \mu_X = 0 \tag{2.17}$$

The variance of $X$ is:

$$\sigma_X^2 = \mathbb{E}\left(X - \mu_X\right)^2 \tag{2.18}$$

From (2.17)

$$\begin{align}
\implies \sigma_X^2 &= \mathbb{E}\left(X^2\right) \tag{2.19}\\
&= \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x^2 dx \tag{2.20}\\
&= 1 \tag{2.21}
\end{align}$$

# Chapter 3

# Conditional Probability

# Chapter 4

# Moments

# Chapter 5

# Random Algebra

1. Let $(X, Y)$ have joint probability density function

$$p_{XY}(x, y) = \begin{cases} 8xy & if 0 < x < y < 1 \\ 0 & otherwise \end{cases} \tag{5.1}$$

if $E(X|Y = y_0) = \frac{1}{2}$,then $y_0$ equals

(a) $\frac{3}{4}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{2}{3}$

(GATE ST 2023)

**Solution:**

$$E(X|Y) = \int_{-\infty}^{\infty} x p_{X|Y} dx \tag{5.2}$$

where

$$p_{X|Y} = \frac{p_{XY}(x,y)}{p_Y(y)} \tag{5.3}$$

$$p_Y(y) = \int_0^y p_{X|Y}(x,y)\,dx \tag{5.4}$$

for $0 < y < 1$

$$= \int_0^y 8xydx \tag{5.5}$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y \tag{5.6}$$

$$= 4y^3 \tag{5.7}$$

For $0 < x < y < 1$,on substituting $p_Y(y)$ we get

$$p_{X|Y} = \frac{8xy}{4y^3} \tag{5.8}$$

$$= \frac{2x}{y^2} \tag{5.9}$$

and

$$E\left(X|Y=y_0\right) = \int_0^{y_0} x.\frac{2x}{y_0^2}dx \tag{5.10}$$

$$= \frac{2}{y_0^2}\left[\frac{x^3}{3}\right]_0^{y_0} \tag{5.11}$$

$$= \frac{2y_0}{3} \tag{5.12}$$

$$\implies \frac{2y_0}{3} = \frac{1}{2} \tag{5.13}$$

$$y_0 = \frac{3}{4} \tag{5.14}$$

# Chapter 6

# Hypothesis Testing

## Chapter 7

# Bivariate Random Variables

# Chapter 8

# Random Processes

# Chapter 9

# Information Theory

1. The frequency of occurrence of 8 symbols (a-h) is shown in the table below. A symbol is chosen and it is determined by asking a series of "yes/no" questions which are assumed to be truthfully answered. The average number of questions when asked in the most efficient sequence, to determine the chosen symbol, is

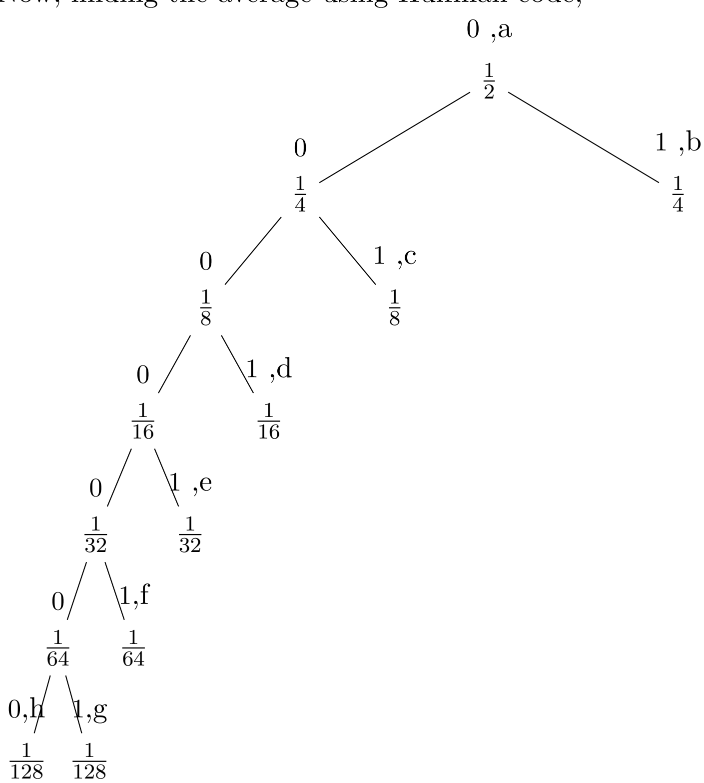
| **Symbols** | **Frequency of occurance** |
|---|---|
| a | $\frac{1}{2}$ |
| b | $\frac{1}{4}$ |
| c | $\frac{1}{8}$ |
| d | $\frac{1}{16}$ |
| e | $\frac{1}{32}$ |
| f | $\frac{1}{64}$ |
| g | $\frac{1}{128}$ |
| h | $\frac{1}{128}$ |

**Solution:**

| **Parameter** | **Value** | **Description** |
|---|---|---|
| $X$ | $1 \leq X \leq 8$ | number of symbols |
| $l$ | 2 | base of algorithm |
| $H(X)$ | $\sum_i p_X(i) \log_l \left(\frac{1}{p_X(i)}\right)$ | average number of question |

$$
\begin{aligned}
H(X) &= \sum_i p_X(i) \log_b \left( \frac{1}{p_X(i)} \right) && (9.1) \\
&= \frac{1}{2} \log_2 (2) + \frac{1}{4} \log_2 (4) + ... + \frac{1}{128} \log_2 (128) && (9.2) \\
&= 0.5 + 0.5 + 0.375 + ... + 0.0078125 && (9.3) \\
&= 1.984375 && (9.4)
\end{aligned}
$$

Now, finding the average using Huffman code,




Using the above binary table following code is generated;

| Symbols | Frequency | Code | Size |
|---|---|---|---|
| $a$ | $\frac{1}{2}$ | 1 | 0.5 |
| $b$ | $\frac{1}{4}$ | 01 | 0.25 |
| $c$ | $\frac{1}{8}$ | 001 | 0.125 |
| $d$ | $\frac{1}{16}$ | 0001 | 0.0625 |
| $e$ | $\frac{1}{32}$ | 00001 | 0.03125 |
| $f$ | $\frac{1}{64}$ | 000001 | 0.015625 |
| $g$ | $\frac{1}{128}$ | 0000001 | 0.0078125 |
| $h$ | $\frac{1}{128}$ | 0000000 | 0.0078125 |

Table 9.1: Huffman table

The average number of question = Weighted path length = 1.9844