Probability and Random Processes

Sarvesh K
EE22BTECH11046*

Question:
Let (X, Y) have joint probability density function

$$p_{XY}(x, y) = \begin{cases} 8xy & if 0 < x < y < 1 \\ 0 & otherwise \end{cases} \tag{1}$$

if $E(X|Y = y_0) = \frac{1}{2}$,then y_0 equals

1) $\frac{3}{4}$
2) $\frac{1}{2}$
3) $\frac{1}{3}$
4) $\frac{2}{3}$

(GATE ST 2023)

Solution:

$$E(X|Y) = \int_{-\infty}^{\infty} x p_{X|Y} dx \tag{2}$$

where

$$p_{X|Y} = \frac{p_{XY}(x, y)}{p_Y(y)} \tag{3}$$

$$p_Y(y) = \int_0^y p_{X|Y}(x, y)\, dx \tag{4}$$

for $0 < y < 1$

$$= \int_0^y 8xy dx \tag{5}$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y \tag{6}$$

$$= 4y^3 \tag{7}$$

For $0 < x < y < 1$,on substituting $p_Y(y)$ we get

$$p_{X|Y} = \frac{8xy}{4y^3} \tag{8}$$

$$= \frac{2x}{y^2} \tag{9}$$

and

$$E(X|Y = y_0) = \int_0^{y_0} x.\frac{2x}{y_0^2} dx \tag{10}$$

$$= \frac{2}{y_0^2}\left[\frac{x^3}{3}\right]_0^{y_0} \tag{11}$$

$$= \frac{2y_0}{3} \tag{12}$$

$$\implies \frac{2y_0}{3} = \frac{1}{2} \tag{13}$$

$$y_0 = \frac{3}{4} \tag{14}$$