Probability and Random Processes

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Q)Suppose that x is an observed sample of size 1 from a population with probability density function $f(\cdot)$. Based on x, consider testing

$$H_0 : f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}; \quad y \in \mathbb{R}$$

against

$$H_1 : f(y) = \frac{1}{2} e^{-|y|}; \quad y \in \mathbb{R}.$$

Then which one of the following statements is true?

1) The most powerful test rejects H_0 if $|x| > c$ for some $c > 0$
2) The most powerful test rejects H_0 if $|x| < c$ for some $c > 0$
3) The most powerful test rejects H_0 if $||x| - 1| > c$ for some $c > 0$
4) The most powerful test rejects H_0 if $||x| - 1| < c$ for some $c > 0$

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Solution:

$$L = \prod_{i=1}^{1} f(x) = f(x) \tag{1}$$

To determine the most powerful test, we need to consider the likelihood ratio test

$$\frac{L(H_1)}{L(H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} k \tag{2}$$

$$\implies \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{2} e^{-2|x|}} \underset{H_0}{\overset{H_1}{\gtrless}} k \tag{3}$$

$$\implies e^{\frac{x^2 - 2|x|}{2}} \underset{H_0}{\overset{H_1}{\gtrless}} k \frac{\sqrt{\pi}}{\sqrt{2}} \tag{4}$$

$$(|x| - 1)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2 \log \left(\frac{k \sqrt{\pi}}{\sqrt{2}} \right) + 1 \tag{5}$$

Taking square root on both sides,

$$||x| - 1| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{2 \log \left(\frac{k \sqrt{\pi}}{\sqrt{2}} \right) + 1} \tag{6}$$

$$\implies |x| \underset{H_0}{\overset{H_1}{\gtrless}} 1 + \sqrt{2 \log \left(\frac{k \sqrt{\pi}}{\sqrt{2}} \right) + 1} \tag{7}$$

Hence, the correct answer is (3)