Probability and Random Processes

Sarvesh K
EE22BTECH11046*

Question:
Let X be a random variable with probability density function

$$p_X(x) = \begin{cases} e^{-x} & if x \geq 0 \\ 0 & otherwise \end{cases} \tag{1}$$

For $a < b$, if $U(a,b)$ denotes the uniform distribution over the interval (a,b), then which of the following statements is/are true?

(A) e^{-X} follows $U(-1,0)$ distribution
(B) $1 - e^{-X}$follows $U(0,2)$ distribution
(C) $2e^{-X} - 1$follows $U(-1,1)$ distribution
(D) The probability mass function of $Y = [X]$ is $\Pr(Y = k) = e^{-k}\left(1 - e^{-1}\right)$ for k= 0, 1, 2, $\dots$, where $[X]$denotes the largest integer not exceeding x

(GATE ST 2023)

Solution: Let $Y \sim U(a,b)$, then

$$p_Y(y) = \begin{cases} \frac{1}{b-a} & a < y < b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and for $a < y < b$

$$F_Y(y) = \Pr(Y \leq y) \tag{3}$$
$$= \int_a^y \frac{1}{b-a} dy \tag{4}$$
$$= \frac{y-a}{b-a} \tag{5}$$

Similarly,for $x \geq 0$

$$F_X(x) = \Pr(X \leq x) \tag{6}$$
$$= \int_0^x e^{-x} dx \tag{7}$$
$$= 1 - e^{-x} \tag{8}$$

(A) $Y = e^{-X} = U(a,b)$
for $a < y < b$

$$F_Y(y) = \Pr\left(e^{-X} \leq y\right) \tag{9}$$
$$= \Pr(X \geq -\ln y) \tag{10}$$
$$= 1 - F_X(-\ln y) \tag{11}$$
$$= 1 - (1 - y) \tag{12}$$
$$= y \tag{13}$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = 0, b = 1 \tag{14}$$
$$\therefore Y \sim U(0,1) \tag{15}$$

(B) $Y = 1 - e^{-X} = U(a,b)$
for $a < y < b$

$$F_Y(y) = \Pr\left(1 - e^{-X} \leq y\right) \tag{16}$$
$$= \Pr\left(e^{-X} \geq 1 - y\right) \tag{17}$$
$$= \Pr(X \leq -\ln(1 - y)) \tag{18}$$
$$= F_X(-\ln(1 - y)) \tag{19}$$
$$= 1 - (1 - y) \tag{20}$$
$$= y \tag{21}$$
$$\implies Y \sim U(0,1) \tag{22}$$

(C) $Y = 2e^{-X} - 1 = U(a,b)$
for $a < y < b$

$$F_Y(y) = \Pr\left(2e^{-X} - 1 \leq y\right) \tag{23}$$
$$= \Pr\left(X \geq -\ln\left(\frac{y+1}{2}\right)\right) \tag{24}$$
$$= 1 - F_X\left(-\ln\left(\frac{y+1}{2}\right)\right) = 1 - \left(1 - \frac{y+1}{2}\right) \tag{25}$$
$$= \frac{y+1}{2} \tag{26}$$

Comparing this with CDF of Uniform distribution, we obtain

$$a = -1, b = 1 \tag{27}$$
$$\therefore Y \sim U(-1,1) \tag{28}$$

(D) $Y = [X]$

$$\Pr(Y = k) = \Pr([X] = k) \tag{29}$$
$$= \Pr(k \leq X < k+1) \tag{30}$$
$$= \int_k^{k+1} e^{-x} dx \tag{31}$$
$$= e^{-k}\left(1 - e^{-1}\right) \text{for k=0,1,2..} \tag{32}$$

(E) Generation of Random Variable X in C language
(i) rand() / (double)RAND_MAX:

This generates a random variable between 0 and RAND_MAX and divides it by RAND_MAX to obtain a uniform distribution between 0 and 1.

(ii) -log(rand() / (double)RAND_MAX) :
This transforms the uniform distribution between 0 and 1 into an exponential distribution by making the values vary from 0 to ∞.

(iii) Alternatively the Uniform distribution can be converted into Gaussian distribution using the Central Limit Theorem.

(iv) Gaussian is then converted into chi-square distribution with degree of freedom 2 which is similar to an exponential distribution.

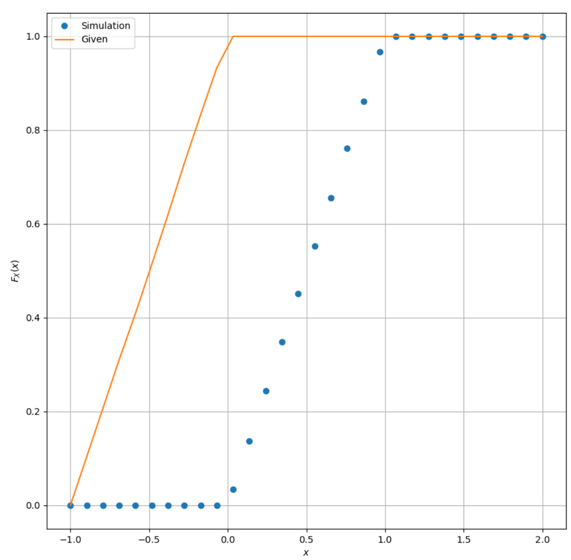

Fig. 5: e^{-X} vs. $U(-1, 0)$
Graphs don't match, $\therefore$ wrong option

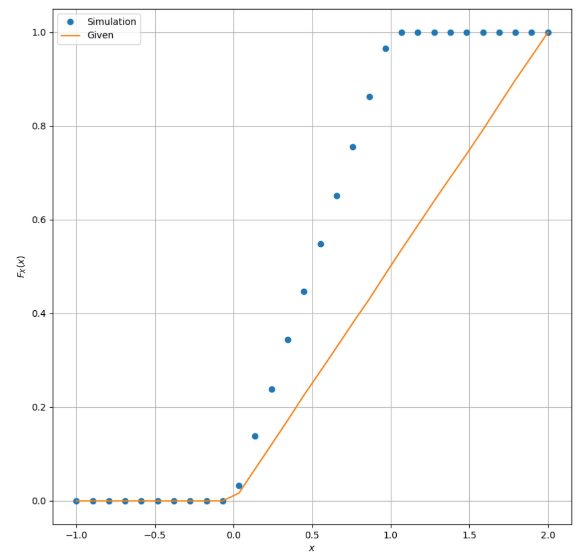

Fig. 5: $1 - e^{-X}$ vs. $U(0, 2)$
Graphs don't match, $\therefore$ wrong option

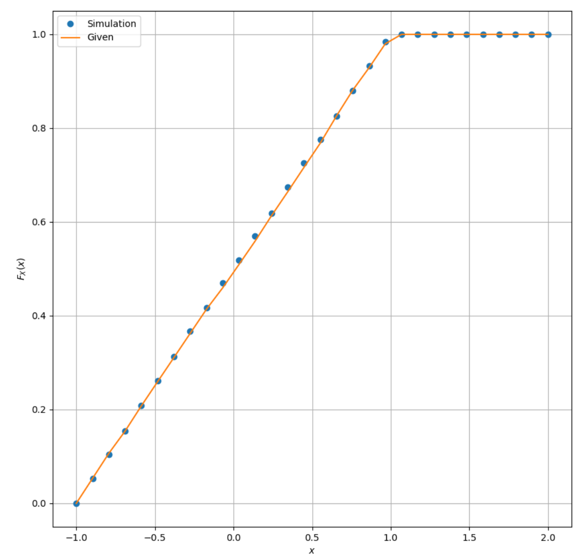

Fig. 5: $2e^{-X} - 1$ vs. $U(-1, 1)$
Graphs match, $\therefore$ correct option