GATE Problems in Probability

Abstract—**These problems have been selected from GATE question papers and can be used for conducting tutorials in courses related to a first course in probability.**

Contents

1 Axioms

1.1. The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2,3 or 5 is......

Solution: Let $X \in \{1, 2, ..., 100\}$ be the random variable representing the outcome for random selection of a number in $\{1, ..., 100\}$.

Since X has a uniform distribution, the probability mass function (pmf) is represented as

$$\Pr(X = n) = \begin{cases} \frac{1}{100} & 1 \le n \le 100 \\ 0 & otherwise \end{cases} \tag{1.1.1}$$

Let A represent the event that the number is divisible by 2. Let B represent the event that the number is divisible by 3. Let C represent the event that the number is divisible by 5.

We need to find the probability that the number is not divisible by 2, 3 or 5. Thus we need to find $1 - \Pr(A + B + C)$

We know

$$\Pr(A + B + C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(AB) - \Pr(BC) - \Pr(AC) + \Pr(ABC) \tag{1.1.2}$$

Substituting in (1.1.2), we get

$$\Pr(A + B + C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{6}{100} - \frac{10}{100} + \frac{3}{100} \tag{1.1.3}$$

Thus,

$$\Pr(A + B + C) = \frac{74}{100} \tag{1.1.4}$$

Thus required probability =

$$1 - \Pr(A + B + C) = \frac{26}{100} \tag{1.1.5}$$

Event	Interpretation	Probability
A	n is divisible by 2	$\frac{50}{100}$
B	n is divisible by 3	$\frac{33}{100}$
C	n is divisible by 5	$\frac{20}{100}$
AB	n is divisible by 6	$\frac{16}{100}$
BC	n is divisible by 15	$\frac{6}{100}$
AC	n is divisible by 10	$\frac{10}{100}$
ABC	n is divisible by 30	$\frac{3}{100}$

TABLE 1.1.1

1.2. P and Q are considering to apply for a job. The probability that P applies for the job is $\frac{1}{4}$, the probability that P applies for the job given that Q applies for the job is $\frac{1}{2}$, and the probability that Q applies for the job given that P applies for the job is $\frac{1}{3}$. Then the probability that P does not apply for the job given that Q does not apply for the job is

a) $\frac{4}{5}$ b) $\frac{5}{6}$ c) $\frac{7}{8}$ d) $\frac{11}{12}$

Solution: Let A represent the event that P applies for the job. Let B represent the event that Q applies for the job.
According to the given information in the question,

$$\Pr(A) = \frac{1}{4} \tag{1.2.1}$$

$$\Pr(A|B) = \frac{1}{2} \tag{1.2.2}$$

$$\Pr(B|A) = \frac{1}{3} \tag{1.2.3}$$

According to the definition of Conditional Probability,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.2.4}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.2.5}$$

On substituting the values of $\Pr(A)$, $\Pr(B|A)$ in (1.2.5), we get

$$\frac{1}{3} = \frac{\Pr(AB)}{\frac{1}{4}} \tag{1.2.6}$$

$$\Pr(AB) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \tag{1.2.7}$$

$$\therefore \Pr(AB) = \frac{1}{12} \tag{1.2.8}$$

Now substituting the values of $\Pr(A|B)$, $\Pr(AB)$ in (1.2.4), we get

$$\frac{1}{2} = \frac{\frac{1}{12}}{\Pr(B)} \tag{1.2.9}$$

$$\Pr(B) = \frac{\left(\frac{1}{12}\right)}{\left(\frac{1}{2}\right)} \tag{1.2.10}$$

$$\therefore \Pr(B) = \frac{1}{6} \tag{1.2.11}$$

The probability that P does not apply for the job given that Q does not apply for the job is given by $\Pr\left(A'|B'\right)$.
Now,

$$A'B' = (A+B)' \tag{1.2.12}$$

$$\Rightarrow \Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{1.2.13}$$

$$\therefore \Pr\left(A'B'\right) = 1 - \Pr(A+B) \tag{1.2.14}$$

As we know that,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.2.15}$$

By substituting the values of $\Pr(A)$, $\Pr(B)$, $\Pr(AB)$ in (1.2.15), we get

$$\Pr(A+B) = \frac{1}{4} + \frac{1}{6} - \frac{1}{12} \tag{1.2.16}$$

$$= \frac{3+2-1}{12} \tag{1.2.17}$$

$$\therefore \Pr(A+B) = \frac{1}{3} \tag{1.2.18}$$

$\therefore$ Required Probability is

$$\Pr\left(A'|B'\right) = \frac{\Pr\left(A'B'\right)}{\Pr(B')} \tag{1.2.19}$$

By substituting the values of $\Pr(B)$, $\Pr\left(A'B'\right)$

[from (1.2.14)] in (1.2.19), we get

$$\Pr\left(A'|B'\right) = \frac{1 - \Pr(A+B)}{1 - \Pr(B)} \quad (1.2.20)$$

$$\Pr\left(A'|B'\right) = \frac{1 - \left(\frac{1}{3}\right)}{1 - \left(\frac{1}{6}\right)} \quad (1.2.21)$$

$$\Rightarrow \Pr\left(A'|B'\right) = \frac{\left(\frac{2}{3}\right)}{\left(\frac{5}{6}\right)} = \frac{4}{5} \quad (1.2.22)$$

$$(1.2.23)$$

$\therefore \Pr\left(A'|B'\right) = \frac{4}{5} = 0.8$

Hence, the probability that P does not apply for the job given that Q does not apply for the job is equal to $\frac{4}{5}$

$\therefore$ The correct option is (A) $\frac{4}{5}$.

1.3. Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\widetilde{A}$ and $\widetilde{B}$ be their complements. Which one of the following statements is FALSE?

a) $P(A \cap B) = P(A)P(B)$

b) $P(A|B) = P(A)$

c) $P(A \cup B) = P(A) + P(B)$

d) $P(\widetilde{A} \cap \widetilde{B}) = P(\widetilde{A})P(\widetilde{B})$

Solution:

a) As A, B are independent events, By definition,

$$\Pr(A+B) = \Pr(A)\Pr(B)$$

Thus option 1 is true.

b)

$$\Pr(A|B) = \frac{\Pr(A+B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A)$$

Thus option 2 is true.

c)

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B)$$

Thus option 3 is false.

d)

$$\begin{aligned}\Pr(A' + B') &= \Pr((AB)') \\ &= 1 - \Pr(AB) \\ &= 1 - \Pr(A) - \Pr(B) + \Pr(A+B) \\ &= (1 - \Pr(A))(1 - \Pr(B)) \\ &= \Pr(A')\Pr(B')\end{aligned}$$

Thus option 4 is true.

Hence, FALSE statement is option 3.

1.4. If P and Q are two random events, then the following is TRUE:

a) Independence of P and Q implies that $\Pr P \cap Q = 0$

b) $\Pr(P \cup Q) \geqslant \Pr(P) + \Pr(Q)$

c) If P and Q are mutually exclusive, then they must be independent

d) $\Pr(P \cap Q) \leqslant \Pr(P)$

Solution:

a) Independence of P and Q means if P happens, then outcome of Q won't be affected by that. so

$$\Pr(P/Q) = \Pr(P) \quad (1.4.1)$$

$$\frac{\Pr(PQ)}{\Pr(Q)} = \Pr(P) \quad (1.4.2)$$

$$\implies \Pr(PQ) = \Pr(P) . \Pr(Q) \quad (1.4.3)$$

This is what we can say hence (A) is wrong

b) As

$$\Pr(P+Q) = \Pr(P) + \Pr(Q) - \Pr(PQ) \quad (1.4.4)$$

$$\Pr(P+Q) + \Pr(PQ) = \Pr(P) + \Pr(Q) \quad (1.4.5)$$

$$\Pr(PQ) \geq 0 \quad (1.4.6)$$

$$\implies \Pr(P) + \Pr(Q) \geq \Pr(P+Q) \quad (1.4.7)$$

Hence (B) is also wrong

c) When P and Q are mutually exclusive, then either P occurs or Q occurs but not both simultaneously.So if P happens, chance of Q happening gets ruled out and vice-versa. Mutually exclusive refers

$$\Pr(PQ) = 0 \quad (1.4.8)$$

$$\Pr(PQ) \neq \Pr(P) . \Pr(Q) \quad (1.4.9)$$

Hence, mutually exclusive events may not be independent.

Hence (C) is also wrong

d) As

$$\Pr(Q/P) = \frac{\Pr(PQ)}{\Pr(P)} \tag{1.4.10}$$

And

$$\Pr(Q/P) \le 1 \tag{1.4.11}$$
$$\frac{\Pr(PQ)}{\Pr(P)} \le 1 \tag{1.4.12}$$
$$\Pr(PQ) \le \Pr(P) \tag{1.4.13}$$

Hence (D) is correct.

1.5. If P and Q are two random events, then the following is true:

(a) Independence of P and Q implies that probability $(P \cap Q) = 0$

(b) Probability $(P \cup Q)$ $\le$ Probability (P) + Probability(Q)

(c) If P and Q are mutually exclusive, then they must be independent

(d) Probability $(P \cap Q) \le$ Probability (P)

Solution: For two random events A and B that are independent, we know that,

$$Pr(AB) = Pr(A)Pr(B) \tag{1.5.1}$$

and for two mutually exclusive events C and D,

$$Pr(CD) = 0 \tag{1.5.2}$$

(a) Independence of P and Q implies that the occurrence of one is unaffected by the other.

$$\implies Pr(PQ) = Pr(P)Pr(Q) \tag{1.5.3}$$

The given option will be true only when either Pr(P) or Pr(Q) will be zero, therefore, (a) is incorrect.

(b) From set theory,

$$A \cup B = A + B - A \cap B \tag{1.5.4}$$

$$\implies Pr(P+Q) = Pr(P) + Pr(Q) - Pr(PQ) \tag{1.5.5}$$

$$\implies Pr(P+Q) \le Pr(P) + Pr(Q) \tag{1.5.6}$$

thus, (b) is incorrect.

(c) Two events can be both mutually exclusive and independent only when one of them have a zero probability. Since it isn't necessary that $Pr(P) = 0$ or $Pr(Q) = 0$, (c) is incorrect.

(d) The set P will have either have the same or more elements than the set $P \cap Q$

$$Pr(PQ) \le Pr(P) \tag{1.5.7}$$

(d) is correct.

Thus, the only correct option is (d).

1.6. Let S be a sample space and two mutually exclusive events A and B be such that $A + B = S$. If $P(\cdot)$ denotes the probability of the event, the maximum value of $P(A)P(B)$ is

Solution:

$$\Pr(A+B) = 1 \tag{1.6.1}$$
$$\Pr(A) + \Pr(B) = 1 \tag{1.6.2}$$
$$\Pr(A)\Pr(B) = \Pr(A)(1 - \Pr(A)) \tag{1.6.3}$$
$$= \Pr(A) - (\Pr(A))^2 \tag{1.6.4}$$
$$= \frac{1}{4} - \left(\Pr(A) - \frac{1}{2}\right)^2 \tag{1.6.5}$$
$$\le \frac{1}{4} \tag{1.6.6}$$
$$\Pr(A) = \Pr(B) = \frac{1}{2} \implies \Pr(A)\Pr(B) = \frac{1}{4} \tag{1.6.7}$$
$$\therefore \max(\Pr(A)\Pr(B)) = \frac{1}{4} \tag{1.6.8}$$

1.7. P and Q are considering to apply for a job.The probability that P applies for the job is 1/4 ,the probability that P applies for the job given that Q applies for the job is 1/2, and the probability that Q applies for the job given that P applies for the job is 1/3.Then the probability that P does not apply for the job given that Q does not apply for the job is

(A) 4/5 (B)5/6 (C)7/8 (D)11/12

Solution:

Let A be the event that P is applying for the job.

Let B be the event that Q is applying for the job.
Using values given in question

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} \tag{1.7.1}$$

$$\implies \Pr(AB) = \frac{1}{12} \tag{1.7.2}$$

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{1.7.3}$$

$$\implies \Pr(B) = \frac{1}{6} \tag{1.7.4}$$

TABLE 1.7.1: Probability for random variables

$\Pr(A)$	1/4	$\Pr(B)$	1/6
$\Pr(A\|B)$	1/2	$\Pr(B\|A)$	1/3
$\Pr(AB)$	1/12		

Now using above values and De Morgan's Laws

$$\Pr(A'|B') = \frac{\Pr(A'B')}{\Pr(B')} \tag{1.7.5}$$

$$\implies \frac{1-\Pr(A+B)}{1-\Pr(B)} \tag{1.7.6}$$

$$\implies \frac{1-\Pr(A)-\Pr(B)+\Pr(AB)}{1-\Pr(B)} \tag{1.7.7}$$

$$\implies \Pr(A'|B') = \frac{4}{5} \tag{1.7.8}$$

The probability that P doesn't apply given Q doesn't apply is 0.8

1.8. E_1, E_2 are independent events such that,

$\Pr(E_1) = \frac{1}{4}$, $\Pr(E_2|E_1) = \frac{1}{2}$ and $\Pr(E_1|E_2) = \frac{1}{4}$

Define random variables X and Y by

$$X = \begin{cases} 1, & \text{if } E_1 \text{ occurs} \\ 0, & \text{if } E_1 \text{ does not occur} \end{cases} \tag{1.8.1}$$

$$Y = \begin{cases} 1, & \text{if } E_2 \text{ occurs} \\ 0, & \text{if } E_2 \text{ does not occur} \end{cases} \tag{1.8.2}$$

Consider the following statements

α : X is uniformly distributed on the set $\{0, 1\}$
β : X and Y are identically distributed
γ : $\Pr\left(X^2 + Y^2 = 1\right) = \frac{1}{2}$
δ : $\Pr\left(XY = X^2Y^2\right) = 1$

Choose the correct combination

(a) (α, β)
(b) (α, γ)
(c) (β, γ)
(d) (γ, δ)

Solution: Since events E_1 and E_2 are independent,

$$\Pr(E_1E_2) = \Pr(E_1) \times \Pr(E_2)$$

$$\Pr(E_2|E_1) = \frac{\Pr(E_1E_2)}{\Pr(E_1)} = \Pr(E_2)$$

$$\therefore \Pr(E_2) = \frac{1}{2} \tag{1.8.3}$$

From the given information we get,

$$F_X(x) = \begin{cases} 1, & x \geq 1 \\ \frac{3}{4}, & 0 \leq x \leq 1 \\ 0, & x < 0 \end{cases} \quad F_Y(y) = \begin{cases} 1, & y \geq 1 \\ \frac{1}{2}, & 0 \leq y \leq 1 \\ 0, & y < 0 \end{cases}$$

	0	1
$\Pr(X)$	$\frac{3}{4}$	$\frac{1}{4}$
$\Pr(Y)$	$\frac{1}{2}$	$\frac{1}{2}$

TABLE 1.8.1: Probability of $X \in \{0,1\}$ and $Y \in \{0,1\}$

(1) X is not uniformly distributed on the set $\{0,1\}$ as it is not continuous in $\{0,1\}$ (both X and Y are Bernoulli Distributions).
$\therefore$ Statement α is incorrect.

(2) Since $F_X(x) \neq F_Y(y)$, X and Y are not identically distributed.
$\therefore$ Statement β is incorrect.

(3) $\Pr\left(X^2 + Y^2 = 1\right)$

$$= \Pr(X = 0, Y = 1) + \Pr(X = 1, Y = 0)$$
$$= \frac{1}{2} \tag{1.8.4}$$

$\therefore$ Statement γ is correct.

(4) $\Pr\left(XY = X^2Y^2\right)$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} \Pr(X = i, Y = j)$$
$$= 1 \tag{1.8.5}$$

$\therefore$ Statement δ is correct.

(a) This option is incorrect as statement α is incorrect (1) and statement β is incorrect (2).

(b) This option is incorrect as statement γ is correct (3) but statement α is incorrect (1).

(c) This option is incorrect as statement γ is correct (3) but statement β is incorrect (2).

(d) This option is correct as statement γ is correct (3) and statement δ is correct (4).

$\therefore$ Option (d), (γ, δ), is the answer.

1.9. Two dice are thrown simultaneously. The probability that at least one of them will have 6 facing up is

A) 1/36
B) 1/3
C) 25/36
D) 11/36

Solution: Probability of at least one six facing up

Number of dices	$n = 2$
The total no. of outcomes	36
Probability of 6 facing-up	$p = 1/6$
Probability of 6 'NOT' facing-up	$q = 5/6$
Number of sixes in the outcome	X

$$= Pr(X = 1) + Pr(X = 2) \tag{1.9.1}$$
$$= {}^2C_1 pq + {}^2C_2 p^2 q^0 \tag{1.9.2}$$
$$= {}^2C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + {}^2C_2\left(\frac{1}{6}\right)^2 \tag{1.9.3}$$
$$= 2\left(\frac{5}{36}\right) + \frac{1}{36} \tag{1.9.4}$$
$$= \frac{11}{36} \tag{1.9.5}$$

1.10. A box contains two coins,one of which is fair and the other is two headed.One coin is choosen at random and tossed twice.If two heads appear,then the probability that the choosen coin is two headed is? **Solution:**

1.11. The probability that it will rain today is 0.5. The probability that it will rain tomorrow is 0.6. The probability that it will rain either today or tomorrow is 0.7 What is the probability that it will rain today and tomorrow? **Solution:** let X_0 be an event of raining today, X_1 be an event of raining tomorrow. Given that, Probability that it will rain today

$$\Pr(X_0) = 0.5 \tag{1.11.1}$$

Probability that it will rain tomorrow

$$\Pr(X_1) = 0.6 \tag{1.11.2}$$

Probability that it will either today or tomorrow is

$$\Pr(X_0 + X_1) = 0.7 \tag{1.11.3}$$

We have to find the probability that it will rain today and tomorrow which is

$$\Pr(X_0X_1) \tag{1.11.4}$$

we know that

$$\Pr(X_0X_1) = \Pr(X_0) + \Pr(X_1) - \Pr(X_0 + X_1) \tag{1.11.5}$$

On Substituting the values in (1.11.5)

$$\Pr(X_0X_1) = 0.5 + 0.6 - 0.7 = 0.4 \tag{1.11.6}$$

So, therefore the probability that it will rain today and tomorrow is 0.4.

1.12. Let P(E) denote the probability of the event E. Given P(A)=1, P(B)=$\frac{1}{2}$, the values of P(A|B) and P(B|A) respectively are
   a) $\frac{1}{4}, \frac{1}{2}$
   b) $\frac{1}{2}, \frac{1}{4}$
   c) $\frac{1}{2}, 1$
   d) $1, \frac{1}{2}$

   Solution: Applying Boolean Logic,

$$P(A) = 1 \implies A = 1 \tag{1.12.1}$$

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{1.12.2}$$

   Using (1.12.1),

$$P(A|B) = \frac{P(1 \times B)}{P(B)} \tag{1.12.3}$$

$$= \frac{P(B)}{P(B)} = 1 \tag{1.12.4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} \tag{1.12.5}$$

$$= \frac{P(B)}{P(A)} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \tag{1.12.6}$$

   Hence the correct answer is option 4).

1.13. Let E and F be any two events with $P(E \cup F) = 0.8$, $P(E) = 0.4$ and $P(E|F) = 0.3$ then P(F) is
   a) $\frac{3}{7}$
   b) $\frac{4}{7}$
   c) $\frac{3}{5}$
   d) $\frac{2}{5}$

   Solution: Given,

$$\Pr(E) = 0.4 \tag{1.13.1}$$

$$\Pr(E + F) = 0.8 \tag{1.13.2}$$

$$\Pr(E|F) = 0.3 \tag{1.13.3}$$

   By definition,

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} \tag{1.13.4}$$

$$\implies \Pr(EF) = \Pr(E|F) \times \Pr(F) \tag{1.13.5}$$

$$\implies \Pr(EF) = 0.3 \times \Pr(F) \tag{1.13.6}$$

   Now using the identity,

$$\Pr(E + F) = \Pr(E) + \Pr(F) - \Pr(EF) \tag{1.13.7}$$

   From (1.13.1),(1.13.2) and (1.13.6)

$$\implies 0.8 = 0.4 + \Pr(F) - (0.3 \times \Pr(F)) \tag{1.13.8}$$

$$\implies 0.4 = (1 - 0.3) \times \Pr(F) \tag{1.13.9}$$

$$\implies \Pr(F) = \frac{0.4}{0.7} \tag{1.13.10}$$

$$\boxed{\Pr(F) = \frac{4}{7}} \tag{1.13.11}$$

1.14. Let E and F be any two events with $\Pr(E) = 0.4, \Pr(F) = 0.3$ and $\Pr(F|E) = 3\Pr(F|E')$. Then $\Pr(E|F)$ equals ......

   Solution: Given
   a) $\Pr(E) = 0.4$
   b) $\Pr(F) = 0.3$
   c) $\Pr(F|E) = 3\Pr(F|E')$

   From given data

$$\Pr(F|E) = 3\Pr(F|E') \tag{1.14.1}$$

$$\frac{\Pr(FE)}{\Pr(E)} = 3 \times \frac{\Pr(FE')}{\Pr(E')} \tag{1.14.2}$$

$$\Pr(EF) = 2 \times \Pr(E'F) \tag{1.14.3}$$

   We know that

$$\Pr(F) = \Pr(EF) + \Pr(E'F) \tag{1.14.4}$$

Using (1.14.3) and (1.14.4), we get

$$\Pr(F) = \frac{3}{2} \times \Pr(EF) \tag{1.14.5}$$

$$\frac{\Pr(EF)}{\Pr(F)} = \frac{2}{3} \tag{1.14.6}$$

$$\Pr(E|F) = \frac{2}{3} \approx 0.66 \tag{1.14.7}$$

1.15. If A and B are two events and the probability $\Pr(B) \neq 1$,then

$$\frac{\Pr(A) - \Pr(A \cap B)}{1 - \Pr(B)} \tag{1.15.1}$$

equals

a) $\Pr(A|\bar{B})$
c) $\Pr(\bar{A}|B)$

b) $\Pr(A|B)$
d) $\Pr(\bar{A}|\bar{B})$

Solution:
Given A and B are two events,
We know that,

$$A = A(B + \bar{B}) \tag{1.15.2}$$

$$= AB + A\bar{B} \tag{1.15.3}$$

Since AB and $A\bar{B}$ are disjoint events,

$$\Pr(A) = \Pr(AB) + \Pr(A\bar{B}) \tag{1.15.4}$$

Hence,

$$\Pr(A\bar{B}) = \Pr(A) - \Pr(AB) \tag{1.15.5}$$

Since B and $\bar{B}$ are disjoint events,

$$\Pr(B) + \Pr(\bar{B}) = 1 \tag{1.15.6}$$

$$\Pr(\bar{B}) = 1 - \Pr(B) \tag{1.15.7}$$

We know that,

$$\Pr(A|\bar{B}) = \frac{\Pr(A\bar{B})}{\Pr(\bar{B})} \tag{1.15.8}$$

From (1.15.7) and (1.15.5)

$$\frac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)} = \frac{\Pr(A\bar{B})}{\Pr(\bar{B})} \tag{1.15.9}$$

From (1.15.8)

$$\frac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)} = \Pr(A|\bar{B}) \tag{1.15.10}$$

Hence **option A is correct**

1.16. Let A and b be two events such that $\Pr(B) = \frac{3}{4}$ and $\Pr(A + B') = \frac{1}{2}$.If A and B are independent, then $\Pr(A)$ equals
Solution:
Given,

$$\Pr(B) = \frac{3}{4} \tag{1.16.1}$$

$$\Pr(A + B') = \frac{1}{2} \tag{1.16.2}$$

we know that,

$$\Pr(B') = 1 - \Pr(B) \tag{1.16.3}$$

using (1.16.1) in (1.16.3),

$$\Pr(B') = \frac{1}{4} \tag{1.16.4}$$

we know that,

$$\Pr(A + B') = \Pr(A) + \Pr(B') - \Pr(A, B') \tag{1.16.5}$$

A and B are independent $\iff$ A and B' are independent

$$\Pr(A + B') = \Pr(A) + \Pr(B') - \Pr(A)\Pr(B') \tag{1.16.6}$$

using (1.16.2) and (1.16.4) in (1.16.6),

$$\frac{1}{2} = \Pr(A) + \frac{1}{4} - \frac{\Pr(A)}{4} \tag{1.16.7}$$

$$\frac{1}{4} = \frac{3\Pr(A)}{4} \tag{1.16.8}$$

$$\therefore \Pr(A) = \frac{1}{3} \tag{1.16.9}$$

1.17. Two independent events E and F are such that $P(E \cap F) = \frac{1}{6}$,$P(E^c \cap F^c) = \frac{1}{3}$ and $P(E) > P(F)$. Then $P(E)$ is

(A) $\frac{1}{2}$
(B) $\frac{2}{3}$
(C) $\frac{1}{3}$
(D) $\frac{1}{4}$

Solution:
If E and F are independent, E’ and F’ are also independent.

So,

$$\Pr(EF) = \Pr(E)\Pr(F) = \frac{1}{6} \tag{1.17.1}$$

$$\Pr(E'F') = \Pr(E')\Pr(F') = (1-\Pr(E))(1-\Pr(F)) = \frac{1}{3} \tag{1.17.2}$$

From (1.17.1) and (1.17.2)

$$\Pr(E) + \Pr(F) = \frac{5}{6} \tag{1.17.3}$$

From (1.17.1) and (1.17.3),

$$\Pr(E)\left(\frac{5}{6} - \Pr(E)\right) = \frac{1}{6}$$

$$\equiv \Pr(E) = \frac{1}{3}\text{or}\frac{1}{2}$$

$\Pr(E) = \frac{1}{2}$ satisfies $\Pr(E) > \Pr(F)$ while $\Pr(E) = \frac{1}{3}$ does not.
$\therefore \Pr(E) = \frac{1}{2}$
Solution: Option A

1.18. If A and B are two events and the probability $\Pr(B) \neq 1$,then $\dfrac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)}$ equals

a) $\Pr\left(A|\bar{B}\right)$

b) $\Pr(A|B)$

c) $\Pr\left(\bar{A}|B\right)$

d) $\Pr\left(\bar{A}|\bar{B}\right)$

Solution: From Laws of complimentary of Boolean algebra

$$B + \bar{B} = 1 \tag{1.18.1}$$

$$\Pr(B) + \Pr\left(\bar{B}\right) = 1 \tag{1.18.2}$$

$$1 - \Pr(B) = \Pr\left(\bar{B}\right) \tag{1.18.3}$$

And also as

$$A - AB = A(1 - B) \tag{1.18.4}$$

$$A - AB = A(\bar{B}) \tag{1.18.5}$$

$$\Pr(A) - \Pr(AB) = \Pr\left(A\bar{B}\right) \tag{1.18.6}$$

Truth table

A	B	AB	$\bar{B}$	A-AB	A$\bar{B}$
1	1	1	0	0	0
1	0	0	1	1	1
0	1	0	0	0	0
0	0	0	1	0	0

Using the above equations (1.18.3)and (1.18.6)

$$\frac{\Pr(A) - \Pr(AB)}{1 - \Pr(B)} = \frac{\Pr\left(A\bar{B}\right)}{\Pr\left(\bar{B}\right)} \tag{1.18.7}$$

$$= \Pr\left(A|\bar{B}\right) \tag{1.18.8}$$

Hence, option (1) is correct.

1.19. Let E,F and G be mutually independent events with $P(E) = \frac{1}{2}$,$P(F) = \frac{1}{3}$ and $P(G) = \frac{1}{4}$.Let p be the probability that at least two of the events among E,F and G occur.Then $12 \times p =$
Solution:

$$p = P(EFG) + \sum P(EFG') \tag{1.19.1}$$

since the events are mutually independent

$$P(EFG) = P(E)P(F)P(G) \tag{1.19.2}$$

$$\implies p = P(E)P(F)P(G) + P(E')P(F)P(G) + P(E)P(F')P(G) + P(E)P(F)P(G') \tag{1.19.3}$$

$$\implies 12 \times p = \frac{7}{2} \tag{1.19.4}$$

1.20. Let F, G and H be pair wise independent events such that $\Pr(F) = \Pr(G) = \Pr(H) = \dfrac{1}{3}$ and $\Pr(F \cap G \cap H) = \dfrac{1}{4}$ Then the probability that at least one event among F, G and H occurs is

a) $\dfrac{11}{12}$

b) $\dfrac{7}{12}$

c) $\dfrac{5}{12}$

d) $\frac{3}{4}$

Solution:

2 Elementary Probability

2.1. A box contains two coins,one of which is fair and the other is two headed.One coin is choosen at random and tossed twice.If two heads appear,then the probability that the choosen coin is two headed is?

2.2. Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one-to-one......

2.3. In the following table, X is a discrete random variable and $p(X = x)$ is the probability density. The standard deviation of X is

X	1	2	3
$p_X(k)$	0.3	0.6	0.1

TABLE 2.3.1: Probability Distribution

Solution: From the given information,

$$\mu = \sum_{k=1}^{3} k p_X(k) \tag{2.3.1}$$
$$= 1.8 \tag{2.3.2}$$

and

$$\sigma^2 = E(X^2) - \mu^2 \tag{2.3.3}$$
$$= \sum_{k=1}^{3} k^2 p_X(k) - \mu^2 \tag{2.3.4}$$
$$= 0.36 \tag{2.3.5}$$
$$\implies \sigma = 0.6 \tag{2.3.6}$$

2.4. An urn contains 5 red balls and 5 black balls.In the first draw, one ball is picked at random and discarded without noticing its colour.The probability to get a red ball in the second draw is

a) $\frac{1}{2}$ b) $\frac{4}{9}$ c) $\frac{5}{9}$ d) $\frac{6}{9}$

Solution: Let $X_i \in \{0, 1\}$ represent the i^{th} draw where 1 denotes a red ball is drawn.
Table 2.4.1 represents the probabilities of all possible cases when two balls are drawn one by one from the urn.

TABLE 2.4.1

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	4/18	5/18
$X_2 = 1$	5/18	4/18

$$\Pr(X_2 = 1) = \Pr(X_2 = 1|X_1 = 0) + \Pr(X_2 = 1|X_1 = 1) \tag{2.4.1}$$
$$= \frac{5}{18} + \frac{4}{18} \tag{2.4.2}$$
$$= \frac{1}{2} \tag{2.4.3}$$

The required option is (A).

2.5. Out of all the 2-digit integers between 1 and 100, a 2-digit number has to be selected at random. What is the probability that the selected number is not divisible by 7?

(A) $\frac{13}{90}$

(B) $\frac{12}{90}$

(C) $\frac{78}{90}$

(D) $\frac{77}{90}$

Solution: Let $X = \{10, 11, \ldots, 99\}$ be a random variable. Here, $\lfloor x \rfloor$ rounds off x to the greatest integer less than x.

$$\Pr(X \pmod 7 \neq 0) = 1 - \frac{n(X \pmod 7 = 0)}{n(X)} \tag{2.5.1}$$
$$= 1 - \frac{\left\lfloor \frac{100}{7} \right\rfloor - \left\lfloor \frac{10}{7} \right\rfloor}{90} \tag{2.5.2}$$
$$= 1 - \frac{13}{90} \tag{2.5.3}$$
$$= \frac{77}{90} \tag{2.5.4}$$

So, the correct option is (D).

2.6. There are 3 red socks, 4 green socks and 3 blue socks.You choose 2 socks.The probability that they are of the same colour is

a) $\frac{1}{5}$ b) $\frac{7}{30}$ c) $\frac{1}{4}$ d) $\frac{4}{15}$

Solution: Let $X_i \in \{1, 2, 3\}$ represent the i^{th} draw, where 1, 2, 3 correspond to the colour of socks drawn as Red, Blue and Green respectively

TABLE 2.6.1

	$X_1 = 1$	$X_1 = 2$	$X_1 = 3$
$X_2 = 1$	6/90	12/90	9/90
$X_2 = 2$	12/90	12/90	12/90
$X_2 = 3$	9/90	12/90	6/90

TABLE 2.6.1 represents all the possibilities of choosing socks one by one.
The probability that the two socks drawn are of the same colour(by substituting values from table 2.6.1)

$$= \Pr(X_1 = X_2) \tag{2.6.1}$$

$$= \sum_{i=1}^{3} \Pr(X_2 = i | X_1 = i) \Pr(X_1 = i) \tag{2.6.2}$$

$$= \frac{6}{90} + \frac{12}{90} + \frac{6}{90} \tag{2.6.3}$$

$$= \frac{4}{15} \tag{2.6.4}$$

So the correct option is (D)

2.7. The probability that a k-digit number does NOT contain the digits 0,5, or 9 is

a) 0.3^k b) 0.6^k c) 0.7^k d) 0.9^k

Solution: Let

$$X_i \in \{0, 1, 2, \dots, 9\} \tag{2.7.1}$$

represent the digit at the i^{th} place.

$$\Pr(X_i \notin \{0, 5, 9\}) = \frac{7}{10} = 0.7 \tag{2.7.2}$$

If the k-digit number does not contain 0,5 or 9,

$$\Pr(X_1 \notin \{0, 5, 9\}, X_2 \notin \{0, 5, 9\}, \dots, X_k \neq \{0, 5, 9\}) \tag{2.7.3}$$

Since the events are independent,

$$\Pr(X_1 \notin \{0, 5, 9\}, X_2 \notin \{0, 5, 9\}, \dots, X_k \neq \{0, 5, 9\}) = \Pr(X_1 \notin \{0, 5, 9\}) \dots \Pr(X_k \notin \{0, 5, 9\}) \tag{2.7.4}$$

$$= \prod_{i=1}^{k} 0.7 \tag{2.7.5}$$

$$= (0.7)^k \tag{2.7.6}$$

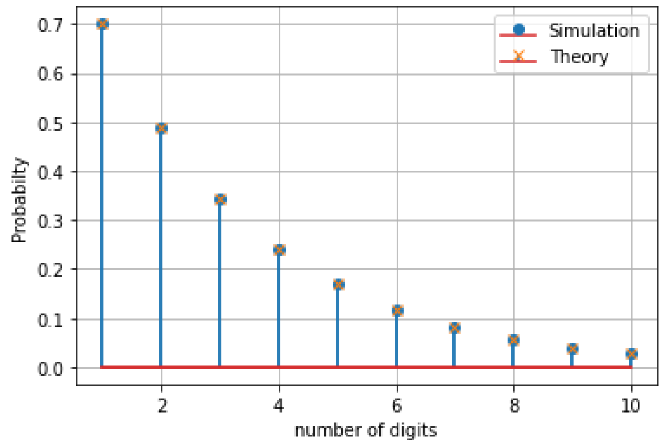


Fig. 2.7.1: Plot

2.8. Candidates were asked to come to an interview with 3 pens each. Black,blue,green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is.......

2.9. Let X and Y denote the sets containing 2 and 20 distinct objects respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one-to-one is........

2.10. Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is...... **Solution:** Let X be random variable.
X ∈ {1,2,3,4,5,6}
$p_X(n)$ →Probability of showing up n.
As $p_X(n)$ is proportional to n. We have,

$$p_X(n) = \begin{cases} kn & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{8.1}$$

Where k is some real constant.

TABLE 2.10.1

n	1	2	3	4	5	6
$p_X(n)$	k	2k	3k	4k	5k	6k

We know that,

$$\sum_{n=1}^{6} p_X(n) = 1 \tag{8.2}$$

By substituting the values in 8.2, we have

$$k + 2k + 3k + 4k + 5k + 6k = 1 \tag{8.3}$$

$$\implies k = \frac{1}{21} \tag{8.4}$$

Probability of the face with three dots showing up

$$\implies p_X(3) = 3k \tag{8.5}$$

Substituting the value of k from 8.4

$$\implies p_X(3) = \frac{1}{7} \tag{8.6}$$

2.11. A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

a) $\frac{1}{3}$ b) $\frac{3}{7}$ c) $\frac{1}{2}$ d) $\frac{4}{7}$

Solution: Consider, Bernoulli random variables Say X_1 and X_2. Required probability is

$$Pr(X_2 = 0|X_1 = 1) = \frac{Pr(X_1 = 1, X_2 = 0)}{Pr(X_1 = 1)} \tag{2.11.1}$$

$$= \frac{\frac{2}{7}}{\frac{4}{7}} = \frac{1}{2} = \frac{1}{2} \tag{2.11.2}$$

Hence (C) is correct option.

2.12. A discrete random variable X takes values from 1 to 5 with probabilities as shown in the table. A student calculates the mean of X as 3.5 and her teacher calculates the variance of X as 1.5. Which of the following statements is true?

k	1	2	3	4	5
P(X=k)	0.1	0.2	0.4	0.2	0.1

a) Both the student and the teacher are right
b) Both the student and the teacher are wrong
c) The student is wrong but the teacher is right
d) The student is right but the teacher is wrong

2.13. If E denotes expectation, the variance of a random variable X is given by

a) $E[X^2] - E^2[X]$ c) $E[X^2]$

b) $E[X^2] + E^2[X]$ d) $E^2[X]$

Solution: Before we start the proof we need to know 3 properties of expectation

$$E[f(x) + g(x)] = E[f(x)] + E[g(x)] \tag{2.13.1}$$

If k is a constant value then

$$E[k \cdot g(x)] = k \cdot E[g(x)] \tag{2.13.2}$$
$$E[k] = k \tag{2.13.3}$$

Now variance of random X is given by

$$Var(X) = E[(X - \mu)^2] \quad \text{where } \mu = E[X]$$

$$Var(X) = E[X^2 - 2\mu \cdot X + \mu^2]$$

$$= E[X^2] - E[2\mu \cdot X] + E[\mu^2] \text{ from (1)}$$

$$= E[X^2] - 2\mu \cdot E[X] + \mu^2 \text{ from (2) and (3)}$$

$$= E[X^2] - 2\mu^2 + \mu^2 \quad (\because E[X] = \mu)$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - E^2[X] \quad (\because \mu = E[X])$$

2.14. An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is:

a) 0.5 b) 0.18 c) 0.12 d) 0.06

Solution: Let A be the event that a student fails in Paper 1
Let B be the event that a student fails in Paper 2
Given

$$\Pr(\mathbf{A}) = 0.3, \Pr(\mathbf{B}) = 0.2, \Pr(\mathbf{A}|\mathbf{B}) = 0.6$$

By definition

$$\Pr(\mathbf{A}|\mathbf{B}) = \frac{\Pr(\mathbf{AB})}{\Pr(\mathbf{B})} \tag{1}$$

$$\Pr(\mathbf{AB}) = \Pr(\mathbf{A}|\mathbf{B}) \times \Pr(\mathbf{B}) \tag{2}$$

$$\Pr(\mathbf{AB}) = 0.6 \times 0.2 \tag{3}$$

$$\Pr(\mathbf{AB}) = 0.12 \tag{4}$$

2.15. Let U and V be two independent and identically distributed random variables such that $P(U = +1) = P(U = -1) = \frac{1}{2}$. The entropy H(U+V) in bits is

a) $\frac{3}{4}$ b) 1 c) $\frac{3}{2}$ d) $\log_2 3$

2.16. Let (X_1, X_2) be independent random variables. X_1 has mean 0 and variance 1, while X_2 has mean 1 and variance 4. The mutual information I $(X_1; X_2)$ between X_1 and X_2 in bits is.......

2.17. A binary communication system makes use of the symbols ”zero” and ”one”. There are channel errors. Consider the following events:

- x_0:a ”zero” is transmitted
- x_1:a ”one” is transmitted
- y_0:a ”zero” is received
- y_1:a ”one” is received

The following probabilities are given: $P(x_0) = \frac{1}{2}$, $P(y_0|x_0) = \frac{3}{4}$, and $P(y_0|x_1) = \frac{1}{2}$. The information in bits that you obtain when you learn which symbol has been received (while you know that a ”zero” has been transmitted) is ....

2.18. Consider two identical boxes B_1 and B_2, where the box $B(i = 1, 2)$ contains $i+2$ red and $5-i-1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are drawn with replacement from the box B_1, otherwise, two balls are drawn with replacement from the box B_2. The probability that the two drawn balls are of different colours is

a) $\frac{7}{25}$ c) $\frac{12}{25}$

b) $\frac{9}{25}$ d) $\frac{16}{25}$

Solution: Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the random variables of a die,

$$\Pr(X = N) = \begin{cases} \frac{1}{6} & 1 \le N \le 6 \\ 0 & otherwise \end{cases} \tag{2.18.1}$$

$$\Pr(X = m) . \Pr(X = n) = 0 \tag{2.18.2}$$

$\forall\ m, n \in \{1, 2, 3, 4, 5, 6\}$ as a single die cannot show more than one outcome at a roll.
Let $Y \in \{0, 1\}$ represent the die where,
$1 \implies$ the die with outcome $N = \{2, 4, 5, 6\}$,
$0 \implies N = \{1, 3\}$.

$$\Pr(Y = 1) = \Pr((X = 2) + (X = 4) + (X = 5) + (X = 6)) \tag{2.18.3}$$

by using Boolean logic and (2.18.2),

$$\Pr(Y = 1) = \frac{2}{3} \tag{2.18.4}$$

$$\Pr(Y = 0) = 1 - \Pr(Y = 1) = \frac{1}{3} \tag{2.18.5}$$

$$\implies \Pr(B_1) = \Pr(Y = 1) = \frac{2}{3} \tag{2.18.6}$$

$$\implies \Pr(B_2) = \Pr(Y = 0) = \frac{1}{3} \tag{2.18.7}$$

Let $C \in \{0, 1\}$ where,
$0 \implies$ red balls,
$1 \implies$ white balls.

TABLE 2.18.1: Table of number of balls

Box	No. of red balls $(i+2)$	No. of white balls $(5-i-1)$	Total balls
B_1	$n(C = 0|B_1) = 3$	$n(C = 1|B_1) = 3$	$n(C|B_1) = 6$
B_2	$n(C = 0|B_2) = 4$	$n(C = 1|B_2) = 2$	$n(C|B_2) = 6$

TABLE 2.18.2: Table of probability of taking balls from each box

Box	Probability of taking red ball	Probability of taking white ball
B_1	$\Pr(C = 0|B_1) = 1/2$	$\Pr(C = 1|B_1) = 1/2$
B_2	$\Pr(C = 0|B_2) = 2/3$	$\Pr(C = 1|B_2) = 1/3$

The probability of picking 2^{nd} ball is not effected by picking 1^{st} ball because the 2^{nd} ball is chose after replacement.

Selecting two balls with replacement is a Bernoulli distribution of 2 trails,

TABLE 2.18.3: Table of no. of ways of selecting two different coloured balls

Cases	Trail 1	Trail 2
$(B_1, C = 0, C = 1)$	$\Pr(C = 0 \mid B_1)$	$\Pr(C = 1 \mid B_1)$
$(B_1, C = 1, C = 0)$	$\Pr(C = 1 \mid B_1)$	$\Pr(C = 0 \mid B_1)$
$(B_2, C = 0, C = 1)$	$\Pr(C = 0 \mid B_2)$	$\Pr(C = 1 \mid B_2)$
$(B_2, C = 1, C = 0)$	$\Pr(C = 1 \mid B_2)$	$\Pr(C = 0 \mid B_2)$

$$\implies \Pr((C = 0, C = 1) \mid B_1) = \Pr(C = 0 \mid B_1) . \Pr(C = 1 \mid B_1) + \Pr(C = 1 \mid B_1) . \Pr(C = 0 \mid B_1) \quad (2.18.8)$$

$$\Pr((C = 0, C = 1) \mid B_1) = \frac{1}{2} \quad (2.18.9)$$

$$\implies \Pr((C = 0, C = 1) \mid B_2) = \Pr(C = 0 \mid B_2) . \Pr(C = 1 \mid B_2) + \Pr(C = 1 \mid B_2) . \Pr(C = 0 \mid B_2) \quad (2.18.10)$$

$$\Pr((C = 0, C = 1) \mid B_1) = \frac{4}{9} \quad (2.18.11)$$

by using Bayes theorem,

$$\Pr(T) = \Pr((C = 0, C = 1) \mid B_1) . \Pr(B_1) + \Pr((C = 0, C = 1) \mid B_2) . \Pr(B_2) \quad (2.18.12)$$

$$\Pr(T) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{4}{9}\right)\left(\frac{1}{3}\right) \quad (2.18.13)$$

TABLE 2.18.4: Table of variables description

Variables	Description
$\Pr((C = 0, C = 1) \mid B_1)$	Probability of selecting two different coloured balls from box B_1
$\Pr((C = 0, C = 1) \mid B_2)$	Probability of selecting two different coloured balls from box B_2
$\Pr(T)$	Total probability of selecting two different coloured balls

Hence, the probability of selecting two different coloured balls from the boxes is

$$\Pr(T) = \frac{13}{27} \quad (2.18.14)$$

2.19. You have gone to a cyber-cafe with a friend. You found that the cyber-café has only three terminals. All terminals are unoccupied. You and your friend have to make a random choice of selecting a terminal. What is the probability that both of you will NOT select the same terminal?

Solution: There are three terminals, each with an equal
probability of $\frac{1}{3}$ to be picked.
Defining random variables $X_1, X_2 \in \{0, 1, 2\}$
Where,
$X_i = 0$ when ith man picks first terminal.
$X_i = 1$ when ith man picks second terminal.
$X_i = 2$ when ith man picks third terminal.

$$\Pr(X_1 \neq X_2) = 1 - \Pr(X_1 = X_2) . \quad (2.19.1)$$

$$\implies \Pr(X_1 = X_2) = \sum_{j=1}^{3} \Pr(X_1 = X_2 = j) \quad (2.19.2)$$

$$\implies \Pr(X_1 = X_2) = \sum_{j=1}^{3} \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{1}{3} \quad (2.19.3)$$

$$\therefore \Pr(X_1 \neq X_2) = \frac{2}{3}. \quad (2.19.4)$$

2.20. What is the chance that a leap year,selected at random,will contain 53 Saturdays?

a) $\frac{2}{7}$
b) $\frac{3}{7}$
c) $\frac{1}{7}$
d) $\frac{5}{7}$

Solution:

2.21. Let $\mathcal{R}$ be the set of all binary relations on the set $\{1, 2, 3\}$. Suppose a relation is chosen from $\mathcal{R}$ at random. The probability that the chosen relation is reflexive is?

Solution: Let A be a set of n numbers. No. of pairs formed from elements of A:

$$ {}^{n}C_1 \times {}^{n}C_1 = n^2 \tag{2.21.1} $$

For each pair we have 2 choices, whether to include it in the relation or not.
$\therefore$ Number of binary relations on A:

$$ 2 \times 2 \times \ldots\ n^2 \text{ times } = 2^{n^2} \tag{2.21.2} $$

Definition 1. *A reflexive relation is one in which every element maps to itself, i.e., a relation R on set A is reflexive if* $(a, a) \in R\ \forall\ a \in A$.

For example, consider the set $A = \{1, 2, 3\}$. A possible reflexive relation on A is $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ as every element in A is related to itself in R_1 while relation $R_2 = \{(1, 1), (2, 2), (1, 2)\}$ is not a reflexive relation on A as $3 \in A$ but $(3, 3) \notin R_2$. In a reflexive relation, out of the n^2 pairs (11.2.4), n have to be included (n pairs of the form (a,a)) which means there is only 1 way to include them. For the remaining $n^2 - n$ pairs we have 2 choices, whether to include it in the relation or not.
$\therefore$ Number of reflexive relations are:

$$ 1 \times 2^{n^2-n} = 2^{n^2-n} \tag{2.21.3} $$

Let $X \in \{0, 1\}$ be a random variable where 0 represents reflexive relation chosen from $\mathcal{R}$ and 1 represents non-reflexive relation chosen from $\mathcal{R}$. In this case, n=3.

$$ \Pr(X = 0) = \frac{2^{n^2-n}}{2^{n^2}} = \frac{2^6}{2^9} \tag{2.21.4} $$

$$ \therefore \text{ Answer } = \frac{1}{8} \tag{2.21.5} $$

2.22. Box-S has 2 white and 4 black balls and box-T has 5 white and 3 black balls.A ball is drawn at
random from box-S and put in box-T.Subsequently,the probability of drawing a white ball from box-T is? (rounding off to 2 decimal places)
Solution: Box-0 has 2 white and 4 black balls.
Box-1 has 5 white and 3 black balls.

Event	definition
W	Event of transfering white balls from box-0 to box-1
B	Event of transfering black balls from box-0 to box-1
C	Event of drawing white balls from box-1
$\Pr(W = 1)$	Probability of transfering one whiteball from box-0 to box-1
$\Pr(B = 1)$	Probability of transfering one blackball from box-0 to box-1
$\Pr(C = 1\|W = 1)$	Probability of drawing a whiteball from box-1 after transfering white ball to box-1.
$\Pr(C = 1\|B = 1)$	Probability of drawing a whiteball from box-1 after transfering black ball to box-1.

TABLE 2.22.1: Table 1

Probability	value
$\Pr(W = 1)$	$\frac{1}{3}$
$\Pr(B = 1)$	$\frac{2}{3}$
$\Pr(C = 1\|W = 1)$	$\frac{6}{9}$
$\Pr(C = 1\|B = 1)$	$\frac{5}{9}$

TABLE 2.22.2: Table 2

$$ \Pr(\text{drawn ball is white}) = \Pr(C = 1) \tag{2.22.1} $$

$$ \tag{2.22.2} $$

From Baye's theorem

$$ \Pr(C = 1) = \Pr(C = 1|W = 1) \times \Pr(W = 1) + \Pr(C = 1|B = 1) \times \Pr(B = 1) \tag{2.22.3} $$

Substiting values from table (2.22.2) in (2.22.3)

$$\Pr(C = 1) = \frac{6}{9} \times \frac{1}{3} + \frac{5}{9} \times \frac{2}{3} \quad (2.22.4)$$
$$= \frac{16}{27} \quad (2.22.5)$$

2.23. A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is
Solution: Let $X \in \{0, 1\}$ be the random variable where X=0 represents that the first removed ball is white. Let $Y \in \{0, 1\}$ be the random variable, where Y=1 represents that the second removed ball is red.
After the first ball is removed (given to be white which means X=0), number of white balls reduces to 3 and total number of balls reduces to 6.
Probability that the second removed ball is red when the first removed ball is white is

$$\Pr(Y = 1|X = 0) = \frac{3}{6} = \frac{1}{2} \quad (2.23.1)$$

So,

$$\Pr(Y = 1|X = 0) = 0.5 \quad (2.23.2)$$

$\therefore$ The answer is option (C) $\frac{1}{2}$.

2.24. Two dice are thrown simultaneously. The probability that the product of the numbers appearing on the top faces of the dice is a perfect square is

(A)$\frac{1}{9}$ (B) $\frac{2}{9}$ (C)$\frac{1}{3}$ (D)$\frac{4}{9}$

Solution: Let X be a random variable which is equal to 1, when the product of the numbers appearing on the top faces of the dice is a perfect square and 0 when it is not a perfect square.
The total no. of possible outcomes is 36.
Outcomes corresponding to $X = 1$ are listed in table 2.24.1 The total no. of favourable outcomes are 8. Therefore we have,

$$\Pr(X = 1) = \frac{8}{36} \quad (2.24.1)$$
$$= \frac{2}{9} \quad (2.24.2)$$

Similarly we have that the probability of not getting a perfect square as a product i.e. $X = 0$

Squares	**Favourable outcomes**
1	(1,1)
4	(1,4) , (2,2) , (4,1)
9	(3,3)
16	(4,4)
25	(5,5)
36	(6,6)

TABLE 2.24.1: Outcomes for X=1

$$\Pr(X = 0) = 1 - \Pr(X = 1) \quad (2.24.3)$$
$$= 1 - \frac{2}{9} \quad (2.24.4)$$
$$= \frac{7}{9} \quad (2.24.5)$$

2.25. What is the chance that a leap year,selected at random,will contain 53 Saturdays?

a) $\frac{2}{7}$
b) $\frac{3}{7}$
c) $\frac{1}{7}$
d) $\frac{5}{7}$

Solution:
Let X be a random variable
We Define, X $\in$ 0,1

P(X = 0)	denotes for 52 Saturday
P(X = 1)	denotes for 53 Saturdays

TABLE 2.25.1: $\Pr(X = x)$

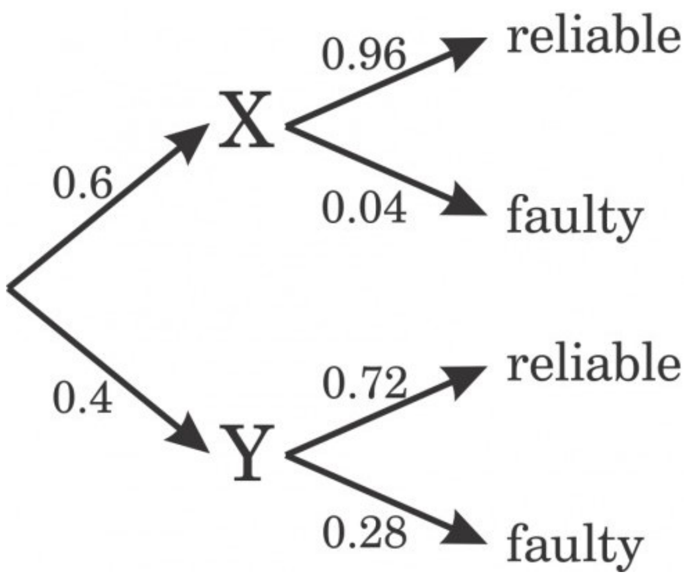
$$\implies \text{Remaining Days} = 366 - 364 = 2 \quad (2.25.1)$$
$$\implies \Pr(X = 1) = \frac{2}{7} \quad (2.25.2)$$

$\therefore$ The correct answer is **Option A**

2.26. An automobile plant contracted to buy shock absorbers from two suppliers X and Y.X supplies 60% and Y supplies 40% of the shock absorbers. All shock absorbers are subjected to a quality test. The ones that pass the quality test are considered reliable. Of X's shock absorbers, 96% are reliable. Of Y's shock absorbers, 72% are reliable. The probability that a randomly chosen shock absorber, which is found to be reliable, is made by Y is

a) 0.288
b) 0.334
c) 0.667
d) 0.720

Solution: Let Consider, Bernoulli random vari-



ables say X, Y and R. Required probability is

	Refer to probability that product	Result
$Pr(X = 1)$	from supplier X	0.6
$Pr(Y = 1)$	from supplier X	0.4
$Pr(R = 1)$	is reliable	
$Pr(R = 0)$	is faulty	
$Pr(R = 1/X = 1)$	from supplier X is reliable	0.96
$Pr(R = 1/Y = 1)$	from supplier Y is reliable	0.72

TABLE 2.26.1: probability of random variables.

$Pr(Y = 1|R = 1)$.So,

$$Pr(Y = 1|R = 1) = \frac{Pr(Y = 1, R = 1)}{Pr(R = 1)} \tag{2.26.1}$$

$$= \frac{Pr(Y = 1)Pr(R = 1/Y = 1)}{Pr(X = 1)Pr(R = 1/X = 1) + Pr(Y = 1)P(R = 1/Y = 1)} \tag{2.26.2}$$

$$= \frac{(0.4)(0.72)}{(0.6)(0.96) + (0.4)(0.72)} = 0.334 \tag{2.26.3}$$

2.27. A person who speaks truth 3 out of 4 times,throws a fair dice with six faces and informs the outcome is 5.The probability that the outcome is really 5 is

Solution: Let $X \in \{0, 1\}$ represent the random variable,where 0 represents person speaking false,1 represents person speaking truth.
Let $Y \in \{1, 2, 3, 4, 5, 6\}$ represent random variable,where $1, 2, 3, 4, 5, 6$ represents person informs outcome of dice is $1, 2, 3, 4, 5, 6$,respectively.
From Baye's theorem

Event	definition	value
$\Pr(X = 1)$	Probability of person speaking truth	$\frac{3}{4}$
$\Pr(X = 0)$	Probability of person speaking false	$\frac{1}{4}$
$\Pr(Y = 5\|X = 1)$	Probability of person informing outcome is 5 if person speaks truth	$\frac{1}{6}$
$\Pr(Y = 5\|X = 0)$	Probability of person informing outcome is 5 if person speaks false	$\frac{5}{6}$

TABLE 2.27.1: Table 1

$$\Pr(Y = 5) = \Pr(Y = 5|X = 1) \times \Pr(X = 1) + \Pr(Y = 5|X = 0) \times \Pr(X = 0) \tag{2.27.1}$$

Substiting values from table (2.27.1) in (2.27.1)

$$\Pr(Y = 5) = \frac{8}{24} \tag{2.27.2}$$

$$\Pr((X = 1)(Y = 5)) = \Pr(Y = 5|X = 1) \times \Pr(X = 1) \tag{2.27.3}$$

$$= \frac{3}{24} \tag{2.27.4}$$

We need to find $\Pr(X = 1|Y = 5)$

$$\Pr(X = 1|Y = 5) = \frac{\Pr((X = 1)(Y = 5))}{\Pr(Y = 5)} \tag{2.27.5}$$

$$= \frac{3}{8} \tag{2.27.6}$$

2.28. The probabilities that a student passes in Mathematics, Physics and Chemistry are m,p, and c respectively. Of these subjects, the student has 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Following relations are drawn in m, p, c:

(I) p + m + c = 27/20
(II) p + m + c = 13/20
(III) (p)× (m) × (c) =1/10
(A) Only relation I is true

(B) Only relation II is true
(C) Relations II and III are true
(D) Relations I and III are true

Solution:
Let M,P,C be the events representing student passes in Mathematics,Physics,Chemistry respectively.

$$\Pr(M) = m \quad (2.28.1)$$
$$\Pr(P) = p \quad (2.28.2)$$
$$\Pr(C) = c \quad (2.28.3)$$

The given information can be represented as

$$\Pr(M + P + C) = 75\% = \frac{3}{4} \quad (2.28.4)$$
$$\Pr(MP + PC + CA) = 50\% = \frac{1}{2} \quad (2.28.5)$$
$$\Pr(MP + PC + CA - 3MPC) = 40\% = \frac{2}{5} \quad (2.28.6)$$

(2.28.5) and (2.28.6) can also be written as

$$\Pr(MP) + \Pr(PC) + \Pr(CM) - 2\Pr(MPC) = \frac{1}{2} \quad (2.28.7)$$
$$\Pr(MP) + \Pr(PC) + \Pr(CM) - 3\Pr(MPC) = \frac{2}{5} \quad (2.28.8)$$

Subtracting and solving the above two equations we get,

$$\Pr(MPC) = \frac{1}{10} \quad (2.28.9)$$
$$\Pr(MP) + \Pr(PC) + \Pr(CM) = \frac{7}{10} \quad (2.28.10)$$

Using inclusion-exclusion principle, We can express (2.28.4) as

$$\Pr(M) + \Pr(P) + \Pr(C) - [\Pr(MP) + \Pr(PC) + \Pr(CM)] + \Pr(MPC) = \frac{3}{4} \quad (2.28.11)$$
$$p + m + c - \frac{7}{10} + \frac{1}{10} = \frac{3}{4} \quad (2.28.12)$$
$$p + m + c = \frac{27}{10} \quad (2.28.13)$$

There is no constant answer for the product of p,m,c which is shown in simulation.

$\therefore$ Only relation I is true.

2.29. Let X and Y denote the sets consisting 2 and 20 distinct elements respectively and F denote the set of all possible functions defined from X and Y. Let f be randomly chosen from F. The probability of f being one to one is :
Solution: We know, every $x \in X$ can be mapped to one of 20 elements in Y.

$$n(F) = 20 \times 20 = 400 \quad (2.29.1)$$

For one to one functions, the first element in X has 20 elements it can be mapped to, and second element in X has only 19 elements.(to avoid repetition).

$$n(f) = 20 \times 19 = 380 \quad (2.29.2)$$

Required probability:

$$\frac{n(f)}{n(F)} = \frac{380}{400} = \frac{19}{20} \quad (2.29.3)$$

2.30. The probability that a number selected at random between 100 and 999 (both inclusive) will not contain digit 7 is. **Solution:**
Let's assume a random 3-digit number be xyz. Where x, y, z are 3 random single-digit integers such that

$$x \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad (2.30.1)$$
$$y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad (2.30.2)$$
$$z \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \quad (2.30.3)$$

a) Probability of selecting x without including

7

$$\Pr(x \neq 7) = \frac{8}{9} \tag{2.30.4}$$

b) Probability of selecting y without including 7

$$\Pr(y \neq 7) = \frac{9}{10} \tag{2.30.5}$$

c) Probability of selecting z without including 7

$$\Pr(z \neq 7) = \frac{9}{10} \tag{2.30.6}$$

So,the total probability of a random 3-digit number xyz will not contain 7

$$= \Pr(x \neq 7) \times \Pr(y \neq 7) \times \Pr(z \neq 7) \tag{2.30.7}$$

$$= \frac{8}{9} \times \frac{9}{10} \times \frac{9}{10} \tag{2.30.8}$$

$$= \frac{18}{25} \tag{2.30.9}$$

The probability of a number selected at random between 100 and 999 (both inclusive) will not contain digit 7 is $\frac{18}{25}$

2.31. A class of twelve children has two more boys than girls. A group of three children are randomly picked from this class to accompany the teacher on the field trip. What is the probability that the group accompanying the teacher contains more girls than boys.

(a) 0
(b) $\frac{325}{864}$
(c) $\frac{525}{864}$
(d) $\frac{5}{12}$

Solution:

Let $X \in \{0, 3\}$ be a discrete random variable which denotes the number of girls in the group of 3.

Since, there are two more boys than girls:

$$\text{Total number of boys}(B) = 7$$

$$\text{Total number of girls}(G) = 5$$

$$\Pr(X = c) = \frac{{}^{5}C_{c} \times {}^{7}C_{3-k}}{{}^{12}C_{3}} \tag{2.31.1}$$

For number of girls more than boys required probability $= \Pr(2 \leq X \leq 3)$ and hence, following cases are possible.

a) X=3

$$\Pr(X = 3) = \frac{{}^{5}C_{3}}{{}^{12}C_{3}}, \text{using (2.31.1)}$$

b) X=2

$$\Pr(X = 2) = \frac{{}^{5}C_{2} \times {}^{7}C_{1}}{{}^{12}C_{3}}, \text{using (2.31.1)}$$

$$\text{So, required probability } = \frac{{}^{5}C_{2} \times {}^{7}C_{1}}{{}^{12}C_{3}} + \frac{{}^{5}C_{3}}{{}^{12}C_{3}} = \frac{4}{11}$$

2.32. Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is

Solution:

Let the random variable $X \in \{1, 2\}$ represent the type of the chosen bulb. $X = 1$ denotes a Type 1 bulb, while $X = 2$ denotes a Type 2 bulb. Given,

$$n(X = 1) = n(X = 2) \tag{5.1}$$

$$\Rightarrow p_X(1) = p_X(2) = \frac{1}{2} \tag{5.2}$$

Let the random variable $Y \in \{0, 1\}$ represent if a bulb lasts more than 100 hours. $Y = 1$ denotes that it lasts, while $Y = 0$ denotes that it doesn't. Given,

$$p_{Y|X}(1|1) = 0.7 \tag{5.3}$$

$$p_{Y|X}(1|2) = 0.4 \tag{5.4}$$

To find : $p_Y(1)$

$$p_Y(1) = p_{Y|X}(1|1)p_X(1) + p_{Y|X}(1|2)p_X(2) \tag{5.5}$$

$$p_Y(1) = (0.7)(0.5) + (0.4)(0.5) \tag{5.6}$$

$$\therefore p_Y(1) = 0.55 \tag{5.7}$$

2.33. The probability that a given positive integer lying between 1 and 100 (both inclusive) and is NOT divisible by 2 or 3 or 5 is …

Solution: Let A, B, C are events where a positive integer between 1 and 100 (both inclusive) is divisible by 2, 3, 5 respectively.

$$\Pr(A) = \frac{1}{2} \quad (2.33.1)$$
$$\Pr(B) = \frac{33}{100} \quad (2.33.2)$$
$$\Pr(C) = \frac{1}{5} \quad (2.33.3)$$
$$\Pr(AB) = \frac{16}{100} \quad (2.33.4)$$
$$\Pr(BC) = \frac{6}{100} \quad (2.33.5)$$
$$\Pr(AC) = \frac{1}{10} \quad (2.33.6)$$
$$\Pr(ABC) = \frac{3}{100} \quad (2.33.7)$$

Required probability : $\Pr(A + B + C)'$

$$\begin{aligned}\Pr(A + B + C)' &= 1 - \Pr(A + B + C)\\ &= 1 - \Pr(A) - \Pr(B) - \Pr(C) +\\ &\quad \Pr(AB) + \Pr(BC) + \Pr(AC)\\ &\quad - \Pr(ABC) = 0.26 \quad (2.33.8)\end{aligned}$$

2.34. What is the probability that a divisor of 10^{99} is a multiple of 10^{96} ? (A) $\frac{1}{625}$ (B) $\frac{4}{625}$ (C) $\frac{12}{625}$ (D) $\frac{16}{625}$

Solution: Let

$$X = \{(x, y) : 0 \le x \le 99, 0 \le y \le 99\}$$

be a set of random variables, $N = 2^x 5^y$,

$$\implies \forall (x, y) \in X,\ \text{N is a divisor of } 10^{99} \quad (2.34.1)$$
$$\implies n(X) = 100 \times 100 = 10^4 \quad (2.34.2)$$

Let

$$Y = \{(x, y) : (x, y) \in X, x \ge 96, y \ge 96\}$$
$$N_1 = 2^x 5^y$$

$$\implies \forall (x, y) \in Y, N_1 | 10^{99} \text{ and is multiple of } 10^{96} \quad (2.34.3)$$
$$\implies n(Y) = 4 \times 4 = 16 \quad (2.34.4)$$

Let P denotes the probability that a divisor of 10^{99} is a multiple of 10^{96} then

$$P = \frac{n(Y)}{n(X)}$$

From 2.34.2 and 2.34.4 we can write

$$P = \frac{16}{10^4} = \frac{1}{625}$$

So the probability is $\frac{1}{625}$, option (**A**).

2.35. There are five bags each containing identical sets of ten distinct chocolates. One chocolate is picked from each bag.
The probability that atleast two chocolates are identical is?

Solution:

Let random variable $X \in \{0, 2, 3, 4, 5\}$ denote the maximum number of identical chocolates picked in an experiment.

$$P(X \ge 2) = 1 - P(X = 0) \quad (2.35.1)$$
$$= 1 - \frac{10 \times 9 \times 8 \times 7 \times 6}{10^5} \quad (2.35.2)$$
$$= 1 - 0.3024 \quad (2.35.3)$$
$$= 0.6976 \quad (2.35.4)$$

2.36. Raju has four fair coins and one fair dice. At first Raju tosses a coin. If the coin shows head then he rolls the dice and the number that dice shows is taken as his score. If the coin shows tail then he tosses three more coins and the total number of tails shown (including the first one) is taken as his score. If Raju tells that his score is 2 then the probability that he rolled the dice is (up to two decimal places):

Solution: Let X_i denote the random variable function for the ith coin $i \in \{1, 2, 3, 4\}$.
$X_i \in (0, 1)$ where 0 represents head and 1 represents tail $i \in \{1, 2, 3, 4\}$.

	Head	Tail
$X_i = k$	0	1

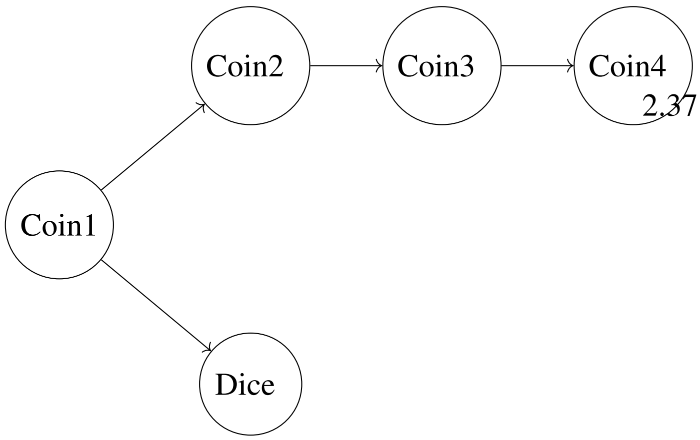
$$\Pr(X_i = k) = \frac{1}{2} \quad (2.36.1)$$

$k \in \{0, 1\}$ and $i \in \{1, 2, 3, 4\}$.
Let Y denote the random variable function for the dice.
$Y \in (1, 2, 3, 4, 5, 6)$ where 1 represents dice showing 1 and so on.

Dice number	1	2	3	4	5	6
$Y = k$	1	2	3	4	5	6

$$\Pr(Y = k) = \frac{1}{6} \tag{2.36.2}$$

$k \in \{1, 2, 3, 4, 5, 6\}$.



Let B denote the event that out of last three coins, only one shows tail.
By Binomial Distribution,

$$\Pr(B) = {}^3C_1 \left(\frac{1}{2}\right)^3 \tag{2.36.3}$$
$$= \frac{3}{8} \tag{2.36.4}$$

Since tossing a coin and rolling a dice are independent events,

$$\Pr((X_i = k), (Y = l)) = \Pr(X_i = k) \cdot \Pr(Y = l) \tag{2.36.5}$$

$k \in \{0, 1\}$ and $l \in \{1, 2, 3, 4, 5, 6\}$.
Let A denote the event that the score is 2.
Clearly,

$$\Pr(A) = \Pr(Y = 2|X_1 = 0) \cdot \Pr(X_1 = 0) + \Pr(B|X_1 = 1) \cdot \Pr(X_1 = 1) \tag{2.36.6}$$
$$= \Pr((X_1 = 0), (Y = 2)) + \Pr((X_1 = 1), B) \tag{2.36.7}$$
$$= \Pr(X_1 = 0) \cdot \Pr(Y = 2) + \Pr(X_1 = 1) \cdot \Pr(B) \tag{2.36.8}$$
$$= \frac{13}{48} \tag{2.36.9}$$

We have to find $\Pr(X_1 = 0|A)$

$$\Pr(X_1 = 0|A) = \frac{\Pr(A, (X_1 = 0))}{\Pr(A)} \tag{2.36.10}$$
$$= \frac{\Pr(X_1 = 0) \cdot \Pr(Y = 2)}{\Pr(A)} \tag{2.36.11}$$
$$= 0.31 \tag{2.36.12}$$

Therefore, required probability = 0.31

2.37. A bag contains 10 white balls and 15 black balls . Two balls are drawn in succession. The probability that one of them is white and the other is black is.
Solution:
Let X denote the number of white balls in the first draw and Y be the number of white balls in second draw and let E be the event mentioned in question.

$$\Pr(E) = \Pr(X = 1) \times \Pr(Y = 0/X = 1) + \Pr(X = 0) \times \Pr(Y = 1/X = 0) \tag{2.37.1}$$

Let m and n be the number of black and white balls in the box.

$$\Pr(X = 0) = \frac{m}{m+n} \tag{2.37.2}$$
$$\Pr(X = 1) = \frac{n}{m+n} \tag{2.37.3}$$
$$\Pr(Y = 0/X = 0) = \frac{m-1}{m+n-1} \tag{2.37.4}$$
$$\Pr(Y = 1/X = 0) = \frac{n}{m+n-1} \tag{2.37.5}$$
$$\Pr(Y = 0/X = 1) = \frac{m}{m+n-1} \tag{2.37.6}$$
$$\Pr(Y = 1/X = 1) = \frac{n-1}{m+n-1} \tag{2.37.7}$$

$$\Pr(E) = \frac{n}{m+n} \times \frac{m}{m+n-1} + \frac{m}{m+n} \times \frac{n}{m+n-1} = \frac{1}{2} \tag{2.37.8}$$

2.38. The box 1 contains chips numbered 3, 6, 9, 12 and 15. The box 2 contains chips numbered 6, 11, 16, 21 and 26. Two chips, one from each box are drawn at random. The numbers written on these chips are multiplied. The probability for the product to be an even number is

a) $\frac{6}{25}$

b) $\frac{2}{5}$
c) $\frac{3}{5}$
d) $\frac{19}{25}$

Solution: Consider two independent random variables X and Y which denotes the number on the chip drawn from box 1 and box 2 respectively.
X can take the values 3, 6, 9, 12, 15
X can take the values 6, 11, 16, 21, 26

$$\Pr(X \times Y = even) = \Pr(X = even, Y = odd) + \Pr(X = odd, Y = even) + \Pr(X = even, Y = even) \tag{2.38.1}$$

$$= \frac{2}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \tag{2.38.2}$$

$$= \frac{19}{25} \tag{2.38.3}$$

2.39. A box contains 25 parts of which 10 are defective. Two parts are being drawn simultaenously in a random manner from the box. The probability of both parts being good is
$(A)\frac{7}{20}$ $(B)\frac{42}{125}$ $(C)\frac{25}{29}$ $(D)\frac{5}{9}$ **Solution:** Let $X_1, X_2 \in \{0, 1\}$ represent the parts, where 0 represents good part, 1 represent defective part. From the given information

$$\Pr(X_1 = 0) = \frac{15}{25} = \frac{3}{5} \tag{2.39.1}$$

$$\Pr(X_2 = 0|X_1 = 0) = \frac{14}{24} = \frac{7}{12} \tag{2.39.2}$$

Then,

$$\Pr(X_1 = 0, X_2 = 0) = \Pr(X_2 = 0|X_1 = 0) \times \Pr(X_1 = 0) = \frac{7}{20} \tag{2.39.3}$$

2.40. From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be Kings, if the first card is NOT replaced?
a) $\frac{1}{26}$
b) $\frac{1}{52}$
c) $\frac{1}{169}$
d) $\frac{1}{221}$

Solution:
Let $A, B \in \{0, 1\}$, where 1 denotes that card is a King, and 0 denotes that card is not a King. A denotes the first card is picked, B denotes second card is picked.

$$\Pr(A = 1) = \frac{4}{52} \tag{2.40.1}$$

$$\Pr(B = 1|A = 1) = \frac{3}{51} \tag{2.40.2}$$

Applying Bayes Theorem, we need to find the value of $\Pr(A = 1, B = 1)$:

$$= \Pr(B = 1|A = 1) \cdot \Pr(A = 1) \tag{2.40.3}$$

$$= \frac{4}{52} \cdot \frac{3}{51} \tag{2.40.4}$$

$$= \frac{1}{221} \tag{2.40.5}$$

The Probability that both cards are king is $\frac{1}{221}$, Hence **Option 4** is correct

2.41. A group consists of equal number of men and women. Of this group, 20% of the men and 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is **Solution:**
Let the random variable $X \in \{0, 1\}$ represent the gender of the person. $X = 0$ denotes a female, while $X = 1$ denotes a male. Given,

$$n(X = 0) = n(X = 1) \tag{4.1}$$

$$\implies p_X(0) = p_X(1) = \frac{1}{2} \tag{4.2}$$

Let the random variable $Y \in \{0, 1\}$ represent if the person is employed. $Y = 0$ denotes unemployed, while $Y = 1$ denotes employed. Given,

$$p_{Y|X}(0|0) = 0.5 \implies p_{Y|X}(1|0) = 0.5 \tag{4.3}$$

$$p_{Y|X}(0|1) = 0.2 \implies p_{Y|X}(1|1) = 0.8 \tag{4.4}$$

To find : $p_Y(1)$

$$p_Y(1) = p_{Y|X}(1|0)p_X(0) + p_{Y|X}(1|1)p_X(1) \tag{4.5}$$

$$p_Y(1) = (0.5)(0.5) + (0.8)(0.5) \tag{4.6}$$

$$\therefore p_Y(1) = 0.65 \tag{4.7}$$

2.42. The probability that a student knows the correct answer to a multiple choice question is $\frac{2}{3}$. If the student does not know the answer, then the student guesses the answer. The probability of the guessed answer being correct is $\frac{1}{4}$. Given that the student has answered the question correctly, the conditional probability that the student knows the correct answer is

a) $\frac{2}{3}$
b) $\frac{3}{4}$
c) $\frac{5}{6}$
d) $\frac{8}{9}$

Solution:
Let the following random variables and their values denote:

$$A : \text{Knows correct answer} = 1$$
$$B : \text{Marks correct answer} = 1$$

$$\therefore \Pr(A = 1) = \frac{2}{3} \tag{2.42.1}$$
$$\Pr(B = 1|A = 1) = 1 \tag{2.42.2}$$
$$\Pr(B = 1|A = 0) = \frac{1}{4} \tag{2.42.3}$$

Applying Bayes Theorem, the value of $\Pr(B = 1)$ is :

$$\Pr(B = 1) = \Pr(B = 1|A = 1)\Pr(A = 1) + \Pr(B = 1|A = 0)\Pr(A = 0) \tag{2.42.4}$$
$$= 1 \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{3}{4} \tag{2.42.5}$$

Applying Bayes Theorem, calculating the value of $\Pr(B = 1, A = 1)$ is:

$$= \Pr(B = 1|A = 1)\Pr(A = 1) \tag{2.42.6}$$
$$= 1 \cdot \frac{2}{3} \tag{2.42.7}$$

Applying Bayes Theorem, we need to find the value of $\Pr(A = 1|B = 1)$. Upon substituting from (2.42.7) and (2.42.5), we get

$$= \frac{\Pr(B = 1, A = 1)}{\Pr(B = 1)} \tag{2.42.8}$$
$$= \frac{8}{9} \tag{2.42.9}$$

The correct answer is **Option 4**.

2.43. Consider an unbiased cubic dice with opposite faces coloured identically and each face coloured red, blue or green such that each colour appears only two times on the dice. If the dice is thrown thrice, the probability of obtaining red colour on top face of the dice at least twice is ________ . **Solution:**
Let $X \in \{0, 1, 2, 3\}$ be the random variable representing the number of times a red face is obtained. Then X is a binomial distributions with parameter:

$$p = \frac{\text{number of red coloured faces}}{\text{total number of faces}} \tag{2.43.1}$$
$$= \frac{2}{6} \tag{2.43.2}$$
$$= \frac{1}{3} \tag{2.43.3}$$

Then,

$$\Pr(X = i) = \begin{cases} {}^3C_i(p)^i(1-p)^{3-i} & i \in \{0, 1, 2, 3\} \\ 0 & \text{otherwise} \end{cases} \tag{2.43.4}$$

$$= \begin{cases} {}^3C_i(\frac{1}{3})^i(1-\frac{1}{3})^{3-i} & i \in \{0, 1, 2, 3\} \\ 0 & \text{otherwise} \end{cases} \tag{2.43.5}$$

$$F_X(i) = \begin{cases} \sum_{k=0}^{i} {}^3C_k(p)^k(1-p)^{3-k} & i \in \{0, 1, 2, 3\} \\ 0 & \text{otherwise} \end{cases} \tag{2.43.6}$$

$$\Pr(X \ge 2) = \Pr(X = 2) + \Pr(X = 3) \tag{2.43.7}$$
$$= \frac{6}{27} + \frac{1}{27} \tag{2.43.8}$$
$$= \frac{7}{27} \tag{2.43.9}$$

2.44. The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks is 5%. GIVEN that a student passes the examination, the probability that the student gets above 90% marks is
Solution:

a). $\frac{1}{18}$ c). $\frac{1}{4}$
b). $\frac{2}{9}$ d). $\frac{5}{18}$

Let A be the event that the student passes the exam and B be the event that the student gets above 90% in the exam. Thus we need to find $\Pr(B|A)$. We are given

$$\Pr(A) = \frac{1}{5} \tag{2.44.1}$$

$$\Pr(AB) = \frac{1}{20} \tag{2.44.2}$$

Thus required probability

$$= \Pr(B|A) \tag{2.44.3}$$

$$= \frac{\Pr(AB)}{\Pr(A)} \tag{2.44.4}$$

$$= \frac{1}{4} \tag{2.44.5}$$

Thus option B is the correct option.

2.45. Four red balls, four green balls and four blue balls are put in a box. Three balls are pulled out of the box at random one after another without replacement. The probability that all the three balls are red is

Solution:

Let $A, B, C \in \{0, 1\}$, where 0 denotes that pulled out ball is red, and 1 denotes that pulled out ball is not red. A denotes the first ball is pulled out of the box,B denotes the second ball is pulled out of the box,C denotes the third ball is pulled out of the box.

$$\Pr(A = 0) = \frac{4}{12} \tag{2.45.1}$$

$$\Pr(B = 0|A = 0) = \frac{3}{11} \tag{2.45.2}$$

$$\Pr(C = 0|(B = 0, A = 0)) = \frac{2}{10} \tag{2.45.3}$$

Applying Bayes Theorem to $\Pr(A = 0, B = 0)$,

$$\Pr(A = 0, B = 0) = \Pr(B = 0|A = 0)\Pr(A = 0) \tag{2.45.4}$$

using (2.45.1) and (2.45.2) ,

$$= \frac{3}{11} \cdot \frac{4}{12} \tag{2.45.5}$$

$$= \frac{1}{11} \tag{2.45.6}$$

similarly $\Pr(A = 0, B = 0, C = 0)$ can be written as,

$$= \Pr(C = 0|(B = 0, A = 0))\Pr(A = 0, B = 0) \tag{2.45.7}$$

using (2.45.3) and (2.45.6) ,

$$= \frac{2}{10} \cdot \frac{1}{11} \tag{2.45.8}$$

$$= \frac{1}{55} \tag{2.45.9}$$

2.46. Consider a company that assembles computers.The probability of a faulty assembly of any computer is p.The company therefore subjects each computer to testing process.This testing process gives the correct result for any computer with a probability of q.What is the probability of a computer being declared faulty?

a) pq+(1-p)(1-q)
b) (1-q)p
c) (1-p)q
d) pq

Solution:

Let $X_i \in \{0, 1\}$ where $\Pr(X_1 = 1)$ represents the computer is faulty before testing, $\Pr(X_2 = 1)$ represents the testing process gives the correct result.

TABLE 2.46.1

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	(1-p)(1-q)	(1-q)p
$X_2 = 1$	(1-p)q	pq

Table 2.46.1 represents the probabilities of all possible cases.The probability of a computer being declared as faulty is

$$= \Pr((X_2 = 1)(X_1 = 1)) + \Pr((X_2 = 0)(X_1 = 0)) \tag{1.1}$$

$$= pq + (1 - p)(1 - q) \tag{1.2}$$

The required option is (A).

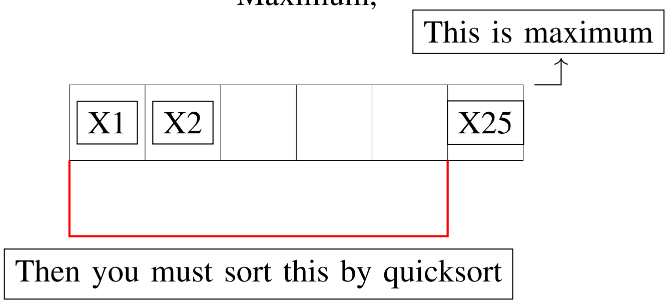
2.47. An array of 25 distinct elements is to be sorted using quicksort. Assume that the pivot element is chosen uniformly at random. The probability that the pivot element gets placed in the worst possible location in the first round of partitioning (rounded off to 2 decimal places) is —- **Solution:**

The worst possible place, the pivot element can be placed is at extreme left or extreme right. So, there are only 2 worst possible locations.

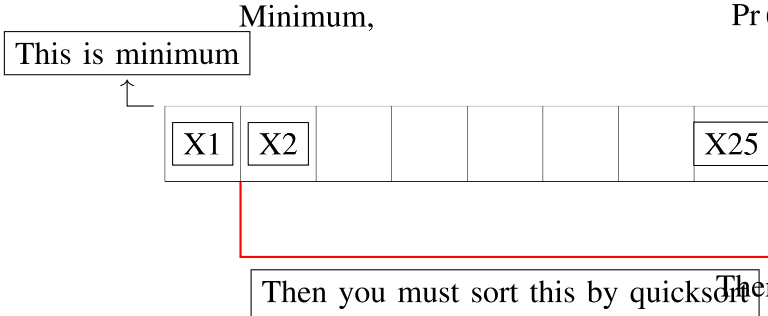
$$Pr(X1 \text{ is compared to } Xn) = \frac{2}{n}. \quad (2.47.1)$$

Total number of pivot elements = 25.
Number of worst possible location of pivot element gets placed after first round of partitioning = 2.
Probability of placing pivot element in worst possible locations $= \frac{2}{25} = 0.08$.

Maximum,



Minimum,



2.48. There are 3 red socks, 4 green socks and 3 blue socks. You choose 2 socks. The probability that they are of the same colour is

a) $\frac{1}{5}$
b) $\frac{7}{30}$
c) $\frac{1}{4}$
d) $\frac{4}{15}$

Solution: Let $X_1 \in \{0, 1, 2\}$ and $X_2 \in \{0, 1, 2\}$ be two Random Variables representing the colour of socks taken in 1^{st} draw and in 2^{nd} draw respectively. $X_1 = 0$, $X_1 = 1$, $X_1 = 2$ represent choosing Red, Green, Blue socks in the first draw respectively. Similarly, $X_2 = 0$, $X_2 = 1$, $X_2 = 2$ represent choosing Red, Green, Blue socks in the second draw respectively. Now, the probability that the socks drawn in 1^{st} draw and 2^{nd} draw are of the same colour is given by

$$\Pr(X_1 = X_2)$$

Now,

$$\Pr(X_1 = X_2) = \sum_{k=0}^{k=2} \Pr(X_1 = X_2 = k) \quad (2.48.1)$$

$$= \sum_{k=0}^{k=2} \Pr(X_1 = k, X_2 = k) \quad (2.48.2)$$

$$= \sum_{k=0}^{k=2} \Pr(X_1 = k) \Pr((X_2 = k)|(X_1 = k)) \quad (2.48.3)$$

$$= \Pr(X_1 = 0) \Pr((X_2 = 0)|(X_1 = 0)) \quad (2.48.4)$$
$$+ \Pr(X_1 = 1) \Pr((X_2 = 1)|(X_1 = 1))$$
$$+ \Pr(X_1 = 2) \Pr((X_2 = 2)|(X_1 = 2))$$

From the given information in the question,

$$\Pr(X_1 = X_2) = \left(\frac{3}{10}\right)\left(\frac{2}{9}\right) + \left(\frac{4}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{3}{10}\right)\left(\frac{2}{9}\right) \quad (2.48.5)$$

$$= \left(\frac{6}{90}\right) + \left(\frac{12}{90}\right) + \left(\frac{6}{90}\right) \quad (2.48.6)$$

$$= \frac{24}{90} = \frac{4}{15} \quad (2.48.7)$$

Therefore, the probability that the two socks are of same colour is $\frac{4}{15}$. Hence, the correct option is 4) $\frac{4}{15}$.

2.49. Suppose we uniformly and randomly select a permutation from the 20! permutations of 1,2,3,...,20. What is the probability that 2 appears at an earlier position than any other even number in the selected permutation.

(A) $\frac{1}{2}$
(B) $\frac{1}{10}$
(C) $\frac{9!}{20!}$
(D) None of the above.

Solution: Probability of at least one six

Number of dices	$n = 2$
The total no. of outcomes	36
Probability of 6 facing-up	$p = 1/6$
Probability of 6 'NOT' facing-up	$q = 5/6$
Number of sixes in the outcome	X

facing up

$$= Pr(X = 1) + Pr(X = 2) \tag{2.49.1}$$
$$= {}^2C_1 pq + {}^2C_2 p^2 q^0 \tag{2.49.2}$$
$$= {}^2C_1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + {}^2C_2 \left(\frac{1}{6}\right)^2 \tag{2.49.3}$$
$$= 2\left(\frac{5}{36}\right) + \frac{1}{36} \tag{2.49.4}$$
$$= \frac{11}{36} \tag{2.49.5}$$

2.50. Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probablity she studies computer science on Wednesday?

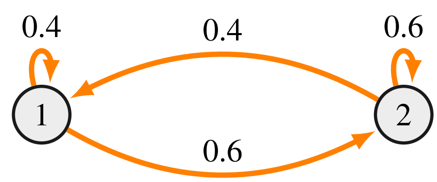
(A) 0.24
(B) 0.36
(C) 0.4
(D) 0.6

Solution: Consider the following parameters As $X_n = 0$ and $X_n = 1$ are mutually exclusive, we can easily calculate x and y.

$$x = \Pr(X_{n+1} = 0|\, X_n = 0) = 1 - \Pr(X_{n+1} = 1\,|X_n = 0) = 0.4 \tag{2.50.2}$$
$$y = \Pr(X_{n+1} = 1|\, X_n = 1) = 1 - \Pr(X_{n+1} = 0\,|X_n = 1) = 0.6 \tag{2.50.3}$$

Given that her initial state is $X_0 = 1$ ($\because$ she studies CS on Monday(n=0)).
The $\Pr(X_{n+t} = j\,|\,X_n = i)$ is given by the (i, j)th position of P^t. Therefore $\Pr(X_2 = 1|X_0 = 1)$ ($\because$ n=2 for Wednesday) is

Parameter	**Definition**	**Value**
S	State space (i.e possible states she can be in.)	$S = \{1, 2\}$, where 1 and 2 represents her studying CS or maths respectively on that day.
$\{X_0, X_1, \dots\}$	Random variables(which form a markov chain) where $X_i \in S$ represents her studying CS or maths on the ith day(i=0 for Monday)	
P	The one step state transition matrix (The elements $p_{ij} = \Pr(X_{n+1} = j\,|\,X_n = i)$)	$P = {}_{X_n}\begin{array}{c} \\ 1 \\ 2 \end{array}\overset{X_{n+1}}{\begin{array}{cc} 1 & 2 \end{array}}\begin{bmatrix} x & 0.6 \\ 0.4 & y \end{bmatrix}$ (2.50.1)



Markov Diagram

the $(1, 1)$th position of P^2.

$$P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix} \tag{2.50.4}$$

$\therefore$ The probability she studies computer science on Wednesday is $P^2_{11} = 0.4$.
(**Ans: Option (C)**)

2.51. The probability that the top and bottom cards of a randomly shuffled deck are both aces is

(A) $\frac{4}{52} \times \frac{4}{52}$
(B) $\frac{4}{52} \times \frac{3}{52}$
(C) $\frac{4}{52} \times \frac{3}{51}$
(D) $\frac{4}{52} \times \frac{4}{51}$

Solution:
Let the following random variables and their values denote:

$$A : \text{Top card is an ace} = 1$$
$$B : \text{Bottom card is an ace} = 1$$

$$\Pr(A=1) = \frac{4}{52} \quad (2.51.1)$$

$$\Pr(B=1|A=1) = \frac{3}{51} \quad (2.51.2)$$

Applying Bayes Theorem,

$$\Pr(B=1, A=1) = \Pr(B=1|A=1)\Pr(A=1) \quad (2.51.3)$$

from (2.51.1) and (2.51.2),

$$\Pr(B=1, A=1) = \frac{4}{52} \times \frac{3}{51} \quad (2.51.4)$$

The correct option is **Option (C)**.

2.52. Suppose a fair six-sided die is rolled once. If the value on the die is 1,2, or 3, the die is rolled a second time. What is the probability that the sum total of values that turn up is at least 6?

a) 10/21
b) 5/12
c) 2/3
d) 1/6

Solution:

Let us define a random variable $X \in \{0, 1\}$

X=0	Getting 1,2, or 3 on first roll
X=1	Getting the sum total of values at least 6

TABLE 2.52.1: Random Variables

Probability of getting 1,2, or 3 on first roll is given by,

$$\Pr(X=0) = \frac{3}{6} = \frac{1}{2} \quad (2.52.1)$$

$$(2.52.2)$$

Probability of getting sum total of 6 on first roll is given by,

$$\Pr(X=1) = \frac{1}{6} \quad (2.52.3)$$

$$(2.52.4)$$

Probability of getting sum total of 6 after getting 1,2,or,3 in first roll is given by,

$$\Pr(X=1|X=0) = \frac{9}{18} = \frac{1}{2} \quad (2.52.5)$$

Now,probability of getting sum total of 6 and getting 1,2,or,3 in first roll is given by,

$$\Pr(X=1, X=0) = \Pr(X=1|X=0) \times \Pr(X=0) \quad (2.52.6)$$

$$= \frac{1}{2} \times \frac{1}{2} \quad (2.52.7)$$

$$= \frac{1}{4} \quad (2.52.8)$$

X	X=0	X=1	X=1|X=0
Pr (X)	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$

TABLE 2.52.2: Probability distribution table

The probability that the sum total of values that turn up is at least 6 is given by

$$\Pr(X=1, X=0) + \Pr(X=1) = \frac{1}{4} + \frac{1}{6} \quad (2.52.9)$$

$$= \frac{5}{12} \quad (2.52.10)$$

2.53. A box contains 4 red balls and 6 black balls. Three balls are selected randomly from the box one after another, without replacement. What is the probability that the selected set contains one red ball and two black balls?

a) $\frac{1}{20}$
b) $\frac{1}{12}$
c) $\frac{3}{10}$
d) $\frac{1}{2}$

Solution:

This problem uses Hyper-Geometric distribution which involves selection of certain number of successes from a given sample without replacement.

- Number of Red balls = 4
- Number of Black balls = 6

Let M be a variable representing the number of black balls in a selection of 3 balls. M has

a
Hyper-Geometric probability mass function:

$$p_M(k) = \Pr(M = k) = \frac{{}^{K}C_k \times {}^{N-K}C_{n-k}}{{}^{N}C_n} \quad (2.53.1)$$

Here Success refers to selecting a black ball, Probability that the selected set contains 2

K	Total successes in population	6
N	Population size	6 + 4 = 10
k	Total observed successes	2
n	Number of draws	3

black balls and 1 red ball = $\Pr(M = 2)$

$$\Pr(M = 2) = \frac{{}^{K}C_2 \times {}^{N-K}C_{n-2}}{{}^{N}C_n} \quad (2.53.2)$$

$$= \frac{{}^{6}C_2 \times {}^{10-6}C_{3-2}}{{}^{10}C_3} \quad (2.53.3)$$

$$= \frac{{}^{6}C_2 \times {}^{4}C_1}{{}^{10}C_3} \quad (2.53.4)$$

$$= \frac{15 \times 4}{120} \quad (2.53.5)$$

$$= \frac{1}{2} \quad (2.53.6)$$

So the probability that the selected set of 3 balls contain 2 black balls and 1 red ball is $\frac{1}{2}$.

2.54. A box contains 2 washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently 4 bolts is

(A) $\frac{2}{315}$
(B) $\frac{1}{630}$
(C) $\frac{1}{1260}$
(D) $\frac{1}{2520}$

Solution:

Let $X \in \{0, 1, 2\}$ be the random variable such that X=0 represents that we draw 2 washers, X=1 represents that we draw 3 nuts and X=2 represents that we draw 4 bolts, continuously without replacement.

Total number of objects :

$$N = 2 + 3 + 4 = 9 \quad (2.54.1)$$

Probability of occurrence of X=0 :

$$\Pr(X = 0) = \frac{{}^{2}C_2}{{}^{9}C_2} \quad (2.54.2)$$

$$= \frac{1}{36} \quad (2.54.3)$$

Total number of objects after occurrence of X=0 :

$$N = 3 + 4 = 7 \quad (2.54.4)$$

Probability of occurrence of X=1 given that X=0 has already occurred :

$$\Pr(X = 1|X = 0) = \frac{{}^{3}C_3}{{}^{7}C_3} \quad (2.54.5)$$

$$= \frac{1}{35} \quad (2.54.6)$$

Total number of objects after occurrence of X=0 and X=1 :

$$N = 4 \quad (2.54.7)$$

Probability of occurrence of X=2 given that X=0 and X=1 has already occurred :

$$\Pr(X = 2|(X = 0, X = 1)) = \frac{{}^{4}C_4}{{}^{4}C_4} \quad (2.54.8)$$

$$= 1 \quad (2.54.9)$$

Using Multiplication law of probability, Required probability is given by :

$$\Pr(X = 0, X = 1, X = 2) = \Pr(X = 0) \times \Pr(X = 1|X = 0) \times \Pr(X = 2|(X = 0, X = 1)) \quad (2.54.10)$$

$$\implies \Pr(X = 0, X = 1, X = 2) = \frac{1}{36} \times \frac{1}{35} \times 1 \quad (2.54.11)$$

$$= \frac{1}{1260} \quad (2.54.12)$$

$\therefore$ The correct option is (C) $\frac{1}{1260}$.

2.55. A box contains 15 blue balls and 45 black balls. If two balls are selected randomly, without replacement, the probability of an outcome in which the first ball selected is a blue ball and the second ball selected is a black ball, is .....

1. $\frac{3}{16}$
2. $\frac{45}{236}$
3. $\frac{1}{4}$
4. $\frac{3}{4}$

Solution:

Let X_1 and $X_2 \in \{0, 1\}$ where 0 represents a black and 1 represents a blue ball.

a) Probability of picking a blue ball

$$\Pr(X_1 = 1) = \frac{15}{60} = \frac{1}{4} \tag{2.55.1}$$

b) Probability of picking a black ball given a blue ball is picked

$$\Pr(X_2 = 0|X_1 = 1) = \frac{45}{59} \tag{2.55.2}$$

c) Probability that first ball is blue and second ball is black

$$\Pr(X_1 = 1, X_2 = 0) = \Pr(X_1 = 1) \times \Pr(X_2 = 0|X_1 = 1) \tag{2.55.3}$$

$$= \frac{1}{4} \times \frac{45}{59} \tag{2.55.4}$$

$$= \frac{45}{236} \tag{2.55.5}$$

$\therefore$ Option 2 is correct.

2.56. An automobile plant contracted to buy shock absorbers from two suppliers X and Y. X supplies 60% and y supplies 40% of the shock absorbers. All shock absorbers are subjected to a quality test.The ones that pass the quality test are considered reliable. Of X's shock absorbers are 96% are reliable. Of Y's shock absorbers 72% are reliable The probability that a randomly chosen shock absorber which is found to be reliable is made by Y is

a) 0.288
b) 0.334
c) 0.667
d) 0.720

Solution:

Let A and B be two random variables that take values from the set {0,1}.

A:

- A=0 $\rightarrow$ shock absorber is from X
- A=1 $\rightarrow$ shock absorber is from Y

B:

- B=0 $\rightarrow$ shock absorber is not reliable
- B=1 $\rightarrow$ shock absorber is reliable

x_i	Description	P(A=x_i)
0	Shock absorber is from X	0.6
1	Shock absorber is from Y	0.4

TABLE 2.56.1: Values taken by X

Given,

$$\Pr(B = 1|A = 0) = 0.96 \tag{2.56.1}$$

$$\Pr(B = 1|A = 1) = 0.72 \tag{2.56.2}$$

Using the fact that $\Pr(E|F) = \frac{\Pr(E+F)}{\Pr(F)}$,

$$\Pr((B = 1) + (A = 0)) = \Pr(B = 1|A = 0) \times \Pr(A = 0) \tag{2.56.3}$$

$$\Pr((B = 1) + (A = 0)) = 0.576 \tag{2.56.4}$$

$$\text{Similarly}, \Pr((B = 1) + (A = 1)) = 0.288 \tag{2.56.5}$$

Since the events (A=0) and (A=1) are mutually independent and mutually exhaustive, we can say that

$$\Pr(B=1) = \Pr((B=1)+(A=0)) + \Pr((B=1)+(A=1)). \quad (2.56.6)$$

$$\implies \Pr(B=1) = 0.864 \quad (2.56.7)$$

We need to find $\Pr(A=1|B=1)$

$$\Pr(A=1|B=1) = \frac{\Pr((A=1)+(B=1))}{\Pr(B=1)} \quad (2.56.8)$$

Substituting values from (2.56.5) (2.56.7), we get

$$\Pr(A=1|B=1) = \frac{0.288}{0.864} \quad (2.56.9)$$

$$\implies \Pr(A=1|B=1) = 0.3333333 \quad (2.56.10)$$

$$\implies \Pr(A=1|B=1) = 0.334 \quad (2.56.11)$$

2.57. There are two identical locks, with two identical keys, and the keys are among the six different ones which a person carries in his pocket. In a hurry he drops one key somewhere. Then the probability that the locks can still be opened by drawing one key at random is equal to? **Solution:**

Let E_1 denote that he drops the needed key
E_2 denote that he drops an unwanted key
A denote the event of opening the locks

$$\Pr(E_1) = \frac{1}{3} \quad (2.57.1)$$

$$\Pr(E_2) = \frac{2}{3} \quad (2.57.2)$$

$$\Pr(A|E_1) = \frac{1}{5} \quad (2.57.3)$$

$$\Pr(A|E_2) = \frac{2}{5} \quad (2.57.4)$$

Hence by total probability rule,

$$\Pr(A) = \Pr(E_1) \times \Pr(A|E_1) + \Pr(E_2) \times \Pr(A|E_2) \quad (2.57.5)$$

$$= \frac{1}{3} \times \frac{1}{5} + \frac{2}{3} \times \frac{2}{5} \quad (2.57.6)$$

Hence, the probability that the locks can be opened is $\frac{1}{3}$

2.58. An urn contains four balls, each ball having equal probability of being white or black. Three black balls are added to the urn. The probability that five balls in the urn are black is

Solution:

The total number of black balls are 5
Number of black balls initially present + number of black balls added =5
So, the number of black balls initially in the urn is 5-3=2
Let X be the random variable denoting the number of black balls in the urn.
So, by binomial distribution,

$$\Pr(X=1) = p \quad (2.58.1)$$

$$\Pr(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (2.58.2)$$

$$k = 0, 1, 2, ..., n \quad (2.58.3)$$

For the given problem, $n = 4$ and $p = 0.5$, because there is equal probability for each ball of being white or black. For having exactly 2 black balls, From (2.58.3),

$$\Pr(X=2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \quad (2.58.4)$$
$$= \frac{6}{16} \quad (2.58.5)$$
$$= \frac{3}{8} \quad (2.58.6)$$

2.59. There are five bags each containing identical sets of ten distinct chocolates. One chocolate is picked from each bag. The probability that at least two chocolates are identical is

Solution:

Let $X \in \{0, 1, 2, 3, 4, 5\}$ represent the random variable, denoting the number of similar chocolates in the picked chocolates

Here, we can neglect X=1 because there can't be one similar object.

$$\Pr(X \geq 2) + \Pr(X = 0) = 1 \quad (2.59.1)$$
$$\Pr(X = 0) = \frac{10.9.8.7.6}{10^5} \quad (2.59.2)$$
$$\Pr(X = 0) = 0.3024 \quad (2.59.3)$$
$$\Pr(X \geq 2) = 1 - \Pr(X = 0) \quad (2.59.4)$$
$$= 1 - 0.3024 \quad (2.59.5)$$
$$= 0.6976 \quad (2.59.6)$$

3 Binary Channels

3.1. Consider a Binary Symmetric Channel (BSC) with probability of error being p.To transit a bit, say 1,we transmit a sequence of three 1s.The receiver will interpret the received sequence to represent 1 if at least if at least two bits are 1.The probability that the transmitted bit will be received in error is

a) $p^3 + 3p^2(1-p)$
b) p^3
c) $(1-p)^3$
d) $p^3 + p^2(1-p)$

Solution:

First of all,let the probability that transmitted bit will be received in error be X. We are given that probability of error = p. So,probability of getting no error =1 − p Also,it is given that to transmit a bit we need to send a sequence of three and for getting error at least two bits must have error.

$$X = p \times p \times p + \binom{3}{1} \times p \times p \times (1-p) \quad (3.1.1)$$
$$X = p^3 + 3 \times p^2 \times (1-p) \quad (3.1.2)$$

$$\boxed{X = p^3 + 3p^2(1-p)} \quad (3.1.3)$$

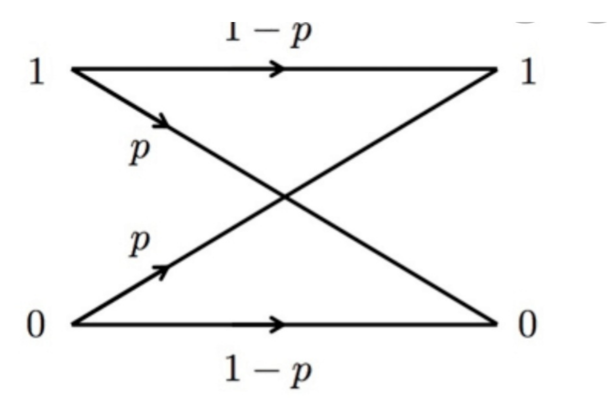


Fig. 3.1.1: BSC diagram

3.2. The input X to the binary Symmetric Channel (BSC) shown in Fig. 3.2.1 is '1' with probability 0.8. The cross-over probability is $\frac{1}{7}$. If the received bit Y=0, the conditional probability that '1' was transmitted is........

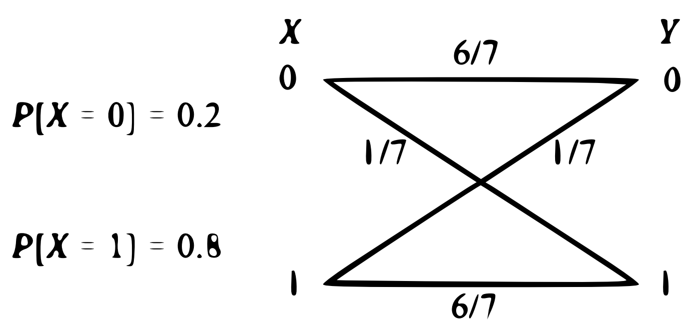


Fig. 3.2.1

Solution:

$$\Pr(X=1|Y=0)=\frac{\Pr(\{X=1\}\{Y=0\})}{\Pr(Y=0)} \tag{3.2.1}$$

$$\Pr(Y=0|X=1)=\frac{\Pr(\{X=1\}\{Y=0\})}{\Pr(X=1)} \tag{3.2.2}$$

From (3.2.2),

$$\Pr(\{X=1\}\{Y=0\})=\Pr(Y=0|X=1)\Pr(X=1) \tag{3.2.3}$$

Substituting (3.2.3) in (3.2.1),

$$\Pr(X=1|Y=0)=\frac{\Pr(Y=0|X=1)\Pr(X=1)}{\Pr(Y=0)} \tag{3.2.4}$$

Given data,

$$\Pr(Y=0|X=1)=\frac{1}{7},\Pr(Y=0|X=0)=\frac{6}{7} \tag{3.2.5}$$

$$\Pr(Y=0)=\Pr(Y=0|X=1)\Pr(X=1)+\Pr(Y=0|X=0)\Pr(X=0) \tag{3.2.6}$$

Substituting the values from (3.2.5) and the data given in the question in (3.2.6),

$$\Pr(Y=0)=\frac{2}{7} \tag{3.2.7}$$

Substituting (3.2.5), (3.2.7) and the data given in the question in (3.2.4),

$$\Pr(X=1|Y=0)=0.4 \tag{3.2.8}$$

3.3. A binary symmetric channel (BSC) has a transition probability of $\frac{1}{8}$. If the binary transmit symbol X is such that $P(X=0)=\frac{9}{10}$, then the probability of error for an optimum receiver will be

a) $\frac{7}{80}$ b) $\frac{63}{80}$ c) $\frac{9}{10}$ d) $\frac{1}{10}$

Solution:
Let random variables, $X\in\{0,1\}$ denote the bit transmitted and $Y\in\{0,1\}$ denote the output bit received.

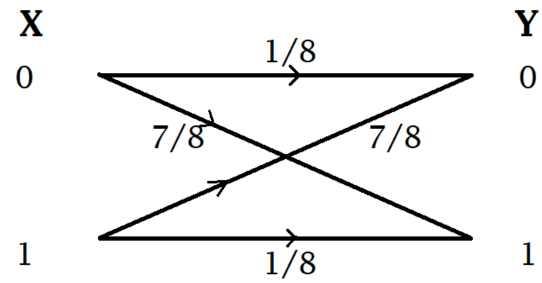


Fig. 3.3.1: Binary symmetric channel

From the given information,

$$\Pr(X=0)=\frac{9}{10} \tag{3.3.1}$$

$$\Pr(X=1)=1-\Pr(X=0)=\frac{1}{10} \tag{3.3.2}$$

Also given, transition probability $=\frac{1}{8}$. Transition probability is the probability with which the bit is transmitted correctly. That gives,

$$\Pr(Y=1|X=1)=\Pr(Y=0|X=0)=\frac{1}{8} \tag{3.3.3}$$

Probability that the bit is not transmitted correctly

$$=1-\text{transition probability}$$
$$=1-\frac{1}{8}=\frac{7}{8} \tag{3.3.4}$$

That gives,

$$\Pr(Y=0|X=1)=\Pr(Y=1|X=0)=\frac{7}{8} \tag{3.3.5}$$

Let E denote the event that bit is transmitted incorrectly. Probability of error, $\Pr(E)$

$$\Pr(E)=\Pr(X=0)\Pr(Y=1|X=0)+\Pr(X=1)\Pr(Y=0|X=1) \tag{3.3.6}$$

On substituting the values,

$$\Pr(E)=\frac{9}{10}\times\frac{7}{8}+\frac{1}{10}\times\frac{7}{8} \tag{3.3.7}$$
$$=\frac{63}{80}+\frac{7}{80} \tag{3.3.8}$$
$$=\frac{7}{8} \tag{3.3.9}$$

Answer: No option matches

3.4. A digital communication system uses a repetition code for channel encoding/decoding. During transmission, each bit is repeated three times(0 is transmitted as 000, and 1 is transmitted as 111). It is assumed that the source puts out symbols independently and with equal probability. The decoder operates as follows: In a block of three received bits, if the number of zeros exceeds the number of ones, the decoder decides in favour of a 0, and if the number of ones exceeds the number of zeros, the decoder decides in favour of a 1. Assuming a binary symmetric channel with crossover probability p = 0.1, the average probability of error is ....

Solution: Let Y be the bit sent by the sender and X be the number of 1's received by the receiver and p = 0.1 is the crossover probability

Case 1: Y = 0

$$\Pr(X = i) = \binom{n}{i} \times p^i \times (1-p)^{n-i} \tag{3.4.1}$$

When $X \geq 2$ the receiver interprets it as 1, which is an error. And by Total Probability theorem we have

$$P_1 = \frac{P(X=2) + P(X=3)}{\sum_{i=0}^{3} P(X=i)} \tag{3.4.2}$$

where P_1 is the probability of error when Y = 0

Case 2: Y = 1

$$\Pr(X = i) = \binom{n}{i} \times p^{n-i} \times (1-p)^{i} \tag{3.4.3}$$

When $X \leq 1$ the receiver interprets it as 0, which is an error. And by Total Probability theorem we have

$$P_2 = \frac{\Pr(X=0) + \Pr(X=1)}{\sum_{i=0}^{3} \Pr(X=i)} \tag{3.4.4}$$

where P_2 is the probability of error when Y = 1

$$\sum_{i=0}^{3} \Pr(X=i) = 1 \times 0.9^3 + 3 \times 0.1 \times 0.9^2 + 3 \times 0.1^2 \times 0.9 + 1 \times 0.1^3 = 1 \tag{3.4.5}$$

$$P_1 = 0.028 \tag{3.4.6}$$

$$P_2 = 0.028 \tag{3.4.7}$$

The average probability is

$$P_{avg} = \Pr(Y=0) \times P_1 + \Pr(Y=1) \times P_2 = 0.028 \tag{3.4.8}$$

	X	0	1	2	3
Y=0	$\Pr(X)$	0.729	0.243	0.027	0.001
Y=1	$\Pr(X)$	0.001	0.027	0.243	0.729

TABLE 3.4.1: Probability of number of 1's recieved

3.5. Consider the Z-channel given in Fig. 3.5.1. The input is 0 or 1 with equal probability. If the

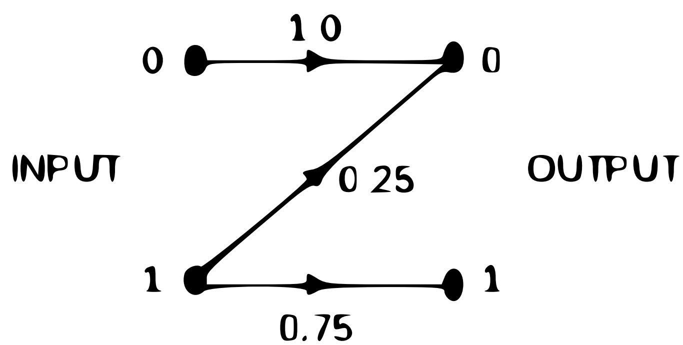
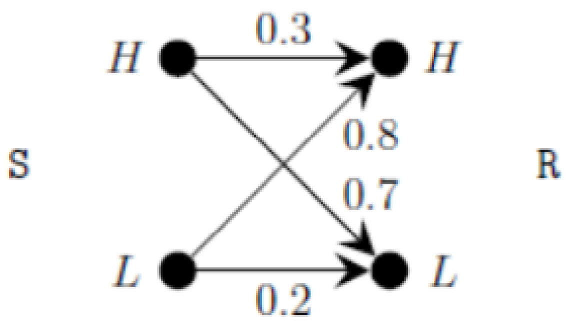


Fig. 3.5.1

output is 0, the probability that the input is also 0 equals......

3.6. A sender(S) transmits a signal,which can be one of two kinds: H and L with probabilities 0.1 and 0.9 respectively, to a receiver (R).
In the graph below,the weight of an edge(u,v) is the probability of receiving v when u is transmitted,where u,v ∈ {H,L}.For example the probability that the received signal is L given the transmitted signal is H is 0.7. If the received signal is H, the probability that the transmitted signal was H is ______ ?
Solution:
In our problem we have a binary channel which is not symmetric as crossover probabilities differ
Let $A \in \{0, 1\}$ represent the random variable, where 0 represents H being transmitted, 1 represents L being transmitted.
Let $B \in \{0, 1\}$ represent the random variable,



where 0 represents H being received, 1 represents L being received.

TABLE 3.6.1: Probability for random variables

$\Pr(A = 0)$	0.1	$\Pr(A = 1)$	0.9
$\Pr(B = 0\|A = 0)$	0.3	$\Pr(B = 0\|A = 1)$	0.8
$\Pr(B = 1\|A = 0)$	0.7	$\Pr(B = 1\|A = 1)$	0.2

Now we need to find $\Pr(A = 0|B = 0)$
Using Bayes theorem

$$\Pr(A = 0|B = 0) = \frac{\Pr(A = 0) \times \Pr(B = 0|A = 0)}{\sum_{i=0}^{1} \Pr(A = i) \times \Pr(B = 0|A = i)} \tag{3.6.1}$$

Putting in values given in question

$$\Pr(A = 0|B = 0) = \frac{1}{25} = 0.04 \tag{3.6.2}$$

The probability that transmitted signal was H is 0.04

3.7. A binary symmetric channel (BSC) has a transition probability of $\frac{1}{8}$. If the binary symbol X is such that P $(X = 0) = \frac{9}{10}$, then the probability of error for an optimum receiver will be

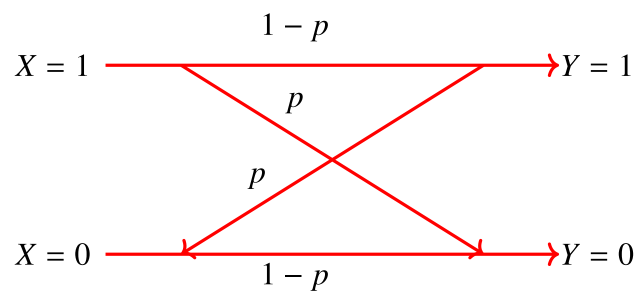
a) $\frac{7}{80}$
b) $\frac{63}{80}$
c) $\frac{63}{80}$
d) $\frac{1}{10}$

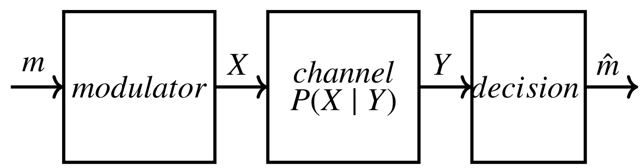
Solution: Probability of transition,p is given by

$$p = \frac{1}{8} \tag{3.7.1}$$

$$\Pr(X = 0) = \frac{9}{10} \tag{3.7.2}$$

$$\Pr(X = 1) = \frac{1}{10} \tag{3.7.3}$$





(here m and X can be considered similar)

$\therefore$ Probability of error is defined as

$$P_e = \Pr(\hat{m} \neq m) \tag{3.7.4}$$

Probability of being correct is defined as

$$P_c = 1 - P_e \tag{3.7.5}$$
$$= 1 - \Pr(\hat{m} \neq m) \tag{3.7.6}$$
$$= \Pr(\hat{m} = m) \tag{3.7.7}$$

Optimum detector maxmize P_c or equivalently minimize P_e
Probability of making correct decision, for a given received y

$$P_c = \Pr(\hat{m} = m) \tag{3.7.8}$$
$$= p(m_i \mid y)p(y) \tag{3.7.9}$$
$$= p(x_i \mid y)p(y) \tag{3.7.10}$$

Using Bayes theorem,

$$P_c = p(y \mid x_i)p(x_i) \tag{3.7.11}$$

To maximize P_c we use **Maximum a Posterior Detector (MAP)** rule , for a given Y

$$\hat{m} \implies m_i \quad if \quad \frac{p(y \mid x_i)p(x_i)}{p(y \mid x_j)p(x_j)} \geq 1 \tag{3.7.12}$$

Now , when Y = 1 then $\hat{m} = 0$ if

$$\frac{p(y=1 \mid x=0)p(x=0)}{p(y=1 \mid x=1)p(x=1)} \geq 1 \tag{3.7.13}$$
$$\implies \frac{p(y=1 \mid x=0)p(x=0)}{p(y=1 \mid x=1)p(x=1)} \tag{3.7.14}$$
$$= \frac{\frac{1}{8} \cdot \frac{9}{10}}{\frac{7}{8} \cdot \frac{1}{10}} \tag{3.7.15}$$
$$= \frac{9}{7} \geq 1 \tag{3.7.16}$$

when Y = 0 then $\hat{m} = 0$ if

$$\frac{p(y=0 \mid x=0)p(x=0)}{p(y=0 \mid x=1)p(x=1)} \geq 1 \tag{3.7.17}$$
$$\implies \frac{p(y=0 \mid x=0)p(x=0)}{p(y=0 \mid x=1)p(x=1)} \tag{3.7.18}$$
$$= \frac{\frac{7}{8} \cdot \frac{9}{10}}{\frac{1}{8} \cdot \frac{1}{10}} \tag{3.7.19}$$
$$= 63 \geq 1 \tag{3.7.20}$$

In both cases MAP detector suggest that message will be $\hat{m} = 0$
$\therefore$ probability of error

$$P_e = \Pr(\hat{m} \neq 0 \mid X=0)\Pr(X=0) + \Pr(\hat{m} \neq 1 \mid X=1)\Pr(X=1) \tag{3.7.21}$$

$$= 0 + 1 \cdot \frac{1}{10} \tag{3.7.22}$$
$$= \frac{1}{10} \tag{3.7.23}$$

So answer will be (D)

4 Independence

4.1. Two independent random variables X and Y are uniformly distributed in the interval [-1,1].The probability that max $\{X,Y\}$ is less than $\frac{1}{2}$ is

A) 3/4
B) 9/16
C) 1/4
D) 2/3

Solution:

Lemma 4.1. *CDF of the random variable X is :*

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2}(x+1) & -1 < x < 1 \\ 1 & x \geq 1 \end{cases} \tag{4.1.1}$$

Proof. Given X is uniformly distributed in [-1,1] i.e. $X \sim$ U(-1,1)
PDF of X :

$$f_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases} \tag{4.1.2}$$

For $-1 \leq x \leq 1$

$$F_X(x) = \Pr(X \leq x) \tag{4.1.3}$$
$$= \int_{-1}^{x} \frac{1}{2} dx \tag{4.1.4}$$
$$= \frac{1}{2}(x+1) \tag{4.1.5}$$

Hence (11.1.1) is proved □

Lemma 4.2. *CDF of the random variable Y is :*

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2}(x+1) & -1 < x < 1 \\ 1 & x \geq 1 \end{cases} \tag{4.1.6}$$

Proof. Given Y is uniformly distributed in [-1,1] i.e. $Y\sim$ U(-1,1)
PDF of Y :

$$f_Y(y) = \begin{cases} 0 & y \leq -1 \\ \frac{1}{2} & -1 \leq y \leq 1 \\ 0 & y \geq 1 \end{cases} \tag{4.1.7}$$

For $-1 \leq y \leq 1$

$$F_Y(y) = P(Y \leq y) \tag{4.1.8}$$
$$= \int_{-1}^{y} \frac{1}{2} dy \tag{4.1.9}$$
$$= \frac{1}{2}(y+1) \tag{4.1.10}$$

Hence (11.1.6) is proved □

Lemma 4.3.

$$\Pr\left(max\{X,Y\} < \frac{1}{2}\right) = \frac{9}{16} \tag{4.1.11}$$

Proof. $max\{X,Y\} < \frac{1}{2} \implies X < \frac{1}{2}$, $Y < \frac{1}{2}$

Given X and Y are independent,

$$Pr(X < \frac{1}{2}, Y < \frac{1}{2}) \tag{4.1.12}$$
$$= \Pr\left(X < \frac{1}{2}\right) \times \Pr\left(Y < \frac{1}{2}\right) \tag{4.1.13}$$
$$= F_X\left(\frac{1}{2}\right) \times F_Y\left(\frac{1}{2}\right) \tag{4.1.14}$$
$$= \frac{3}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \tag{4.1.15}$$
$$= \frac{9}{16} \tag{4.1.16}$$

Hence (11.1.11) is proved □

Option B is correct

4.2. Three fair cubical dice are thrown simultaneously. The probability that all three dice have the same number of dots on the faces showing up is (up to third decimal place)...........

Solution: Let

$$X_1, X_2, X_3 \in \{1, 2, 3, 4, 5, 6\} \tag{4.2.1}$$

represent the three dice.
Since, all the three are fair dice, the probability of any dice showing a particular number is given by

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & \text{i=1,2,3,4,5,6} \\ 0 & \text{otherwise} \end{cases} \tag{4.2.2}$$

If all the dice show a particular number i,

$$\implies \Pr(X_1 = X_2 = X_3 = i) \tag{4.2.3}$$

Since the events are independent,

$$\Pr(X_1 = X_2 = X_3 = i)$$
$$= \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \tag{4.2.4}$$

where i=1,2,3,4,5,6.
There are 6 faces on a cubical dice. Hence, there are six cases in which all the dice show the same number

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = X_2 = X_3 = i) \tag{4.2.5}$$

From (11.2.4),we have

$$\Pr(X_1 = X_2 = X_3)$$
$$= \sum_{i=1}^{6} \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \tag{4.2.6}$$
$$= \sum_{i=1}^{6} \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \tag{4.2.7}$$
$$= \frac{1}{36} \tag{4.2.8}$$

4.3. Given Set A = [2,3,4,5] and Set B = [11,12,13,14,15], two numbers are randomly selected,one from each set. What is probability that the sum of the two numbers equals 16?

a) 0.20 b) 0.25 c) 0.30 d) 0.33

Solution: Given,
Set A= [2,3,4,5]
Set B= [11,12,13,14,15]

Total number of element in the sample space is 20.

Let us define a random variable$X \in \{0, 1\}$

X=0	the event when A+B=16
X=1	the event when A+B $\neq$ 16

TABLE 4.3.1: Random Variables

Now, probability of selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = 1 \tag{7.1}$$

So,the probability of selecting an element from set B after selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = \frac{1}{5} \tag{7.2}$$

Therefore,
Overall probability of randomly choosing elements from set A and set B such that

$\Pr(X = 0)$is

$$\Pr(X = 0) = \Pr(A + B = 16) \tag{7.3}$$

$$\Pr(X = 0) = 1 \times \frac{1}{5} \tag{7.4}$$

$$\Pr(X = 0) = \frac{1}{5} = 0.2 \tag{7.5}$$

X	0	1
$\Pr(X)$	$\frac{1}{5}$	$\frac{4}{5}$

TABLE 4.3.2: Probability distribution table

Therefore, the correct option is (a).

4.4. Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$.The probability that $\max[X, Y]$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

4.5. A fair dice is tossed two times. The probability that the second toss result in a value that is higher than the first toss is

a) $\frac{2}{36}$ b) $\frac{2}{6}$ c) $\frac{5}{12}$ d) $\frac{1}{2}$

Solution: Given, a fair die, which is tossed twice. Let the random variable $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, represent the outcome of the number on the die in the first, second toss respectively. The probability mass function (PMF) for a fair die is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6}, & 1 \le n \le 6 \\ 0, & otherwise \end{cases} \tag{26.1}$$

Using (26.1), the cumulative distribution function (CDF) is obtained to be

$$F_{X_i}(r) = \Pr(X_i \le r) = \begin{cases} \frac{r}{6}, & 1 \le r \le 6 \\ 1, & r \ge 7 \\ 0, & otherwise \end{cases} \tag{26.2}$$

$$X_1 < X_2 \Rightarrow X_2 = k, X_1 \le k - 1 \tag{26.3}$$

$\because X_1, X_2$ are independent,

$$Pr(X_1 < X_2) = E\left[F_{X_1}(X_2 - 1)\right] \tag{26.4}$$

After unconditioning (26.4), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} p_{X_2}(k) F_{X_1}(k - 1) \tag{26.5}$$

Substituting (26.1) and (26.2), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} \frac{1}{6}\left(\frac{k-1}{6}\right) \tag{26.6}$$

On solving, we get

$$Pr(X_1 < X_2) = \frac{5}{12}\text{(option (C))} \tag{26.7}$$

TABLE 4.5.1: Cases and their theoretical probabilities

Case	$X_1 < X_2$	$X_1 > X_2$	$X_1 = X_2$
Probability	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{6}$

4.6. Consider two independent random variables X and Y with identical distributions. The variables X and Y take value 0, 1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ rrespectively. What is the conditional probability $P(X+Y = 2|X-Y = 0)$?

a) 0 b) $\frac{1}{16}$ c) $\frac{1}{6}$ d) 1

Solution: The values that the random variable X can take along with its probabilities are given by

X	0	1	2
$\Pr(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The values that the random variable Y can take along with its probabilities are given by

Y	0	1	2
$\Pr(Y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$$\Pr(X - Y = 0) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{6}{16} \tag{4.6.1}$$

$$\Pr((X + Y = 2), (X - Y = 0)) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \tag{4.6.2}$$

$$\Pr(X+Y=2 \mid X-Y=0) = \frac{\Pr((X+Y=2),(X-Y=0))}{\Pr(X-Y=0)} = \frac{\frac{1}{16}}{\frac{6}{16}} = \frac{1}{6} \tag{4.6.3}$$

4.7. Let X and Y be two statistically independent random variables uniformly distributed in the range $(-1,1)$ and $(-2,1)$ respectively. Let $Z = X+Y$, then the probability that $[Z \leqslant -2]$ is

a) zero b) $\frac{1}{6}$ c) $\frac{1}{3}$ d) $\frac{1}{12}$

Solution:
X and Y are two independent random variables. Let

$$p_X(x) = \Pr(X=x) \tag{4.7.1}$$
$$p_Y(y) = \Pr(Y=y) \tag{4.7.2}$$
$$p_Z(z) = \Pr(Z=z) \tag{4.7.3}$$

be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} p_X(x)\,dx = 1 \tag{4.7.4}$$
$$2 \times p_X(x) = 1 \tag{4.7.5}$$
$$p_X(x) = 1/2 \tag{4.7.6}$$

Similarly for Y we have,

$$\int_{-2}^{1} p_Y(y)\,dy = 1 \tag{4.7.7}$$
$$3 \times p_Y(y) = 1 \tag{4.7.8}$$
$$p_Y(y) = 1/3 \tag{4.7.9}$$

The density for X is

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{4.7.10}$$

We have ,

$$Z = X+Y \iff z = x+y \iff x = z-y \tag{4.7.11}$$

The density of X can also be represented as,

$$p_X(z-y) = \begin{cases} \frac{1}{2} & -1 \le z-y \le 1 \\ 0 & otherwise \end{cases} \tag{4.7.12}$$

and the density of Y is,

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{4.7.13}$$

The density of Z i.e. $Z = X+Y$ is given by the convolution of the densities of X and Y

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z-y)p_Y(y)\,dy \tag{4.7.14}$$

From 11.7.12 and 11.7.13 we have,
The integrand is $\frac{1}{6}$ when,

$$2 \le y \le 1 \tag{4.7.15}$$
$$-1 \le z-y \le 1 \tag{4.7.16}$$
$$z-1 \le y \le z+1 \tag{4.7.17}$$

and zero, otherwise.
Now when $-3 \le z \le -2$ them we have,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\,dy \tag{4.7.18}$$
$$= \frac{1}{6} \times (z+1-(-2)) \tag{4.7.19}$$
$$= \frac{1}{6}(z+3) \tag{4.7.20}$$

For $-2 < z \le -1$,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\,dy \tag{4.7.21}$$
$$= \frac{1}{6} \times (z+1-(-2)) \tag{4.7.22}$$
$$= \frac{1}{6}(z+3) \tag{4.7.23}$$

For $-1 < z \le 0$,

$$p_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\,dy \tag{4.7.24}$$
$$= \frac{1}{6} \times (z+1-(z-1)) \tag{4.7.25}$$
$$= \frac{1}{3} \tag{4.7.26}$$

For $0 < z \le 2$,

$$p_Z(z) = \int_{z-1}^{1} \frac{1}{6}\,dy \tag{4.7.27}$$
$$= \frac{1}{6} \times (1-(z-1)) \tag{4.7.28}$$
$$= \frac{1}{6}(2-z) \tag{4.7.29}$$

Therefore the density of Z is given by

$$p_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \le z \le -2 \\ \frac{1}{6}(z+3) & -2 < z \le -1 \\ \frac{1}{3} & -1 < z \le 0 \\ \frac{1}{6}(2-z) & 0 < z \le 2 \\ 0 & otherwise \end{cases} \tag{4.7.30}$$

The CDF of Z is defined as,

$$F_Z(z) = \Pr(Z \le z) \tag{4.7.31}$$

Now for $z \le -1$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{4.7.32}$$

$$= \int_{-3}^{z} \frac{1}{6}(z+3)\,dz \tag{4.7.33}$$

$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Bigg|_{-3}^{z} \tag{4.7.34}$$

$$= \frac{1}{6} \times \left(\left(\frac{z^2}{2} + 3z\right) - \left(\frac{9}{2} - 9\right)\right) \tag{4.7.35}$$

$$= \frac{z^2 + 6z + 9}{12} \tag{4.7.36}$$

Similarly for $z \le 0$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{4.7.37}$$

$$= \frac{1}{3} + \int_{-1}^{z} \frac{1}{3}\,dz \tag{4.7.38}$$

$$= \frac{z+2}{3} \tag{4.7.39}$$

finally for $z \le 2$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{4.7.40}$$

$$= \frac{2}{3} + \int_{0}^{z} \frac{1}{6}(2-z)\,dz \tag{4.7.41}$$

$$= \frac{2}{3} + \frac{4z - z^2}{12} \tag{4.7.42}$$

$$= \frac{8 + 4z - z^2}{12} \tag{4.7.43}$$

The CDF is as below,

$$F_Z(z) = \begin{cases} 0 & z < 3 \\ \frac{z^2+6z+9}{12} & z \le -1 \\ \frac{z+2}{3} & z \le 0 \\ \frac{8+4z-z^2}{12} & z \le 2 \\ 1 & z > 2 \end{cases} \tag{4.7.44}$$

So

$$\Pr(Z \le -2) = F_Z(2) \tag{4.7.45}$$

$$= \frac{1}{12} \tag{4.7.46}$$

i.e. option (D).
The plot for PDF of Z can be observed at figure 11.7.1 and the plot for CDF of Z is at figure 11.7.2.

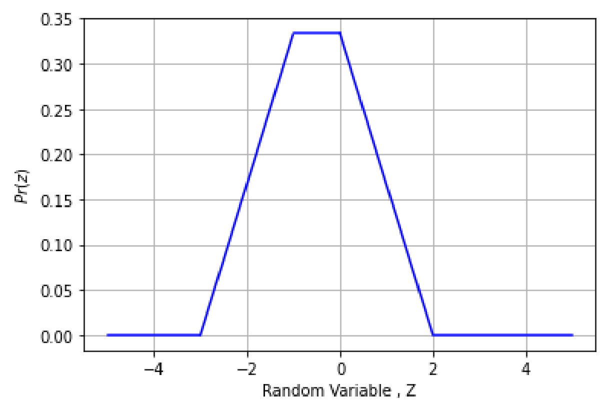


Fig. 4.7.1: The PDF of Z

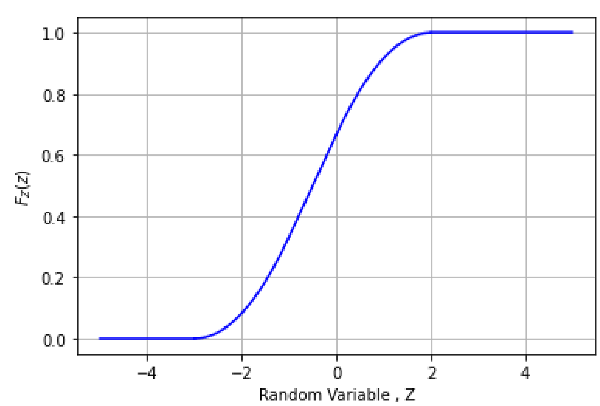


Fig. 4.7.2: The CDF of Z

4.8. Let X_1, X_2, X_3 and X_4 be independent normal random variables with zero mean and unit variance. The probability that X_4 is the smallest

among the four is.....
Solution: Required probability

$$= \Pr(X_4 = min(X_1, X_2, X_3, X_4)) \quad (4.8.1)$$
$$= \int_{-\infty}^{\infty} \Pr(X_1, X_2, X_3 > x | X_4 = x) \quad (4.8.2)$$

Since X_1, X_2, X_3 and X_4 are independent, required probability

$$= \int_{-\infty}^{\infty} (1 - F_{X_1}(x))(1 - F_{X_2}(x))(1 - F_{X_3}(x)) f_{X_4}(x) dx \quad (4.8.3)$$
$$= \int_{-\infty}^{\infty} (1 - \Phi(x))^3 \phi(x) dx \quad (4.8.4)$$

Substituting

$$u = 1 - \Phi(x) \quad (4.8.5)$$
$$du = -\phi(x) dx \quad (4.8.6)$$

we get required probability

$$= -\int_{1}^{0} u^3 du \quad (4.8.7)$$
$$= \frac{1}{4} \quad (4.8.8)$$

Note that in eq. (11.8.7) the integral is from 1 to 0 because

$$1 - \Phi(-\infty) = 1 \quad (4.8.9)$$
$$1 - \Phi(\infty) = 0 \quad (4.8.10)$$

Here $\phi(x)$ and $\Phi(x)$ represent the pdf and cdf of standard normal random variable respectively.

4.9. Let $A_1, A_2, ...A_n$ be n independent events in which the Probability of occurence of the event A_i is given by P(A_i) = 1 - $\frac{1}{\alpha^i}$, $\alpha > 1$, i = 1,2,3,....n.Then the probability that atleast one of the events occurs is

a) 1 - $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$
b) $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$
c) $\frac{1}{\alpha^n}$
d) 1 - $\frac{1}{\alpha^n}$

Solution: Let $A_1 + A_2 + A_3.... + A_n$ = S, $\Pr(S)$ = Probability of atleast one event occuring De morgan's law states that $(A+B)' = A'B'$

$$\implies \Pr(S) = 1 - \Pr(S') \quad (4.9.1)$$
$$1 - \Pr(S') = 1 - \Pr(A_1' A_2' A_3' ....A_n') \quad (4.9.2)$$

$\forall$ i $\in$ 1,2,....n
Since, A_i are independent.
$\therefore$ Complements of A_i are also independent.
$\implies$

$$\Pr(A_1' A_2' A_3' ....A_n') = \prod_{i=1}^{n} \Pr(A_i') \quad (4.9.3)$$
$$\Pr(A_i) = 1 - \frac{1}{\alpha^i} \implies \Pr(A_i') = \frac{1}{\alpha^i} \quad (4.9.4)$$

substituting (11.9.4) in (11.9.3),

$$\Pr(A_1' A_2' A_3' ....A_n') = \prod_{i=1}^{n} \frac{1}{\alpha^i} \quad (4.9.5)$$
$$\prod_{i=1}^{n} \frac{1}{\alpha^i} = \frac{1}{\alpha^{\sum_i^n i}} = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (4.9.6)$$
$$\therefore \Pr(A_1' A_2' A_3' ....A_n') = \Pr(S') = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (4.9.7)$$

from equations (11.9.2) and (11.9.7)

$$\implies \Pr(S) = 1 - \Pr(S') = 1 - \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (4.9.8)$$

$\therefore$ The correct option is **(a)**

4.10. Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $n = 1, 2, \ldots,$ then $\lim_{n\to\infty} P(\bar{X}_n \leq 1.8)$ is equal to

Solution:
Given,

$$Pr(X_1 = 1) = \frac{1}{4}, Pr(X_2 = 2) = \frac{3}{4} \quad (32.1)$$

As $X_1, X_2, \ldots,$ are identically distributed random variables, $\forall i \in \{1, 2, \ldots, n\}$

$$Pr(X_i = 1) = \frac{1}{4}, Pr(X_i = 2) = \frac{3}{4} \quad (32.2)$$

Also,

$$\because P(X_i = 1) + P(X_i = 2) = 1 \quad (32.3)$$
$$\therefore X_i \in \{1, 2\} \quad (32.4)$$

Therefore, each X_i is a bernoulli distribution with

$$p = \frac{3}{4}, q = \frac{1}{4} \tag{32.5}$$

Let

$$X = \sum_{i=1}^{n} X_i \tag{32.6}$$

be a binomial distribution. Its CDF is

$$Pr(X \le n + r) = \sum_{k=0}^{r} {}^{n}C_k p^k q^{n-k} \tag{32.7}$$

To find : $\lim_{n\to\infty} Pr(\bar{X}_n \le a)$

$$\bar{X}_n \le a \Rightarrow X \le na \tag{32.8}$$

Substituting $a(= 1.8), p, q$, we get

$$\lim_{n\to\infty} Pr(\bar{X}_n \le 1.8) = \lim_{n\to\infty} P(X \le 1.8n) \tag{32.9}$$

$$= \sum_{k=0}^{0.8n} \frac{{}^{n}C_k 3^k}{4^n} \tag{32.10}$$

On solving (32.10), we get

$$\lim_{n\to\infty} P(\bar{X}_n \le 1.8) = 1 \tag{32.11}$$

4.11. Let $\{X_n\}_{n\ge1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\ge1}$, having probability mass function

$$\Pr(Y = k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1, 2, 3 \cdots \\ 0 & otherwise \end{cases} \tag{4.11.1}$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \le 1)$ equals ....

Solution:
Given that $\{X_n\}_{n\ge1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\ge1} = \begin{cases} \frac{1}{3} & 0 \le X_n \le 3 \\ 0 & otherwise \end{cases} \tag{4.11.2}$$

So,

$$\Pr(X_n \le 1)_{n\ge1} = \frac{1}{3} \tag{4.11.3}$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \le 1) \tag{4.11.4}$$

Since, $\{X_n\}_{n\ge1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\ge1}$. And also in (11.11.4), the index of $X_i's$ depends on Y, so number of terms depends on Y, like if $Y = 1$, then there is only X_1, if $Y = 2$, then there's X_1, X_2, so required probability

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \le 1 | Y = p\right) \cdot \Pr(Y = p) \tag{4.11.5}$$

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \le 1\right) \cdot \Pr(Y = p) \tag{4.11.6}$$

$$= \sum_{p=1}^{\infty} \Pr\left(X_1, X_2, \cdots X_p \le 1\right) \cdot \Pr(Y = p) \tag{4.11.7}$$

$$= \sum_{p=1}^{\infty} \Pr(X_1 \le 1) \cdot \Pr(X_2 \le 1) \cdots \Pr\left(X_{p-1} \le 1\right) \cdot \Pr\left(X_p \le 1\right) \cdot \Pr(Y = p) \tag{4.11.8}$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{e-1}\right)\left(\frac{1}{p!}\right) \tag{4.11.9}$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{p!}\right) - 1\right] \tag{4.11.10}$$

Using Taylor's Series of e^x in (11.11.10),
Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \tag{4.11.11}$$

$$= 0.23 \tag{4.11.12}$$

4.12. Let X_1, X_2 and X_3 be independent and identically distributed random variables with $E(X_1) = 0$ and $E\left(X_1^2\right) = \frac{15}{4}$. If $\psi : (0, \infty) \to (0, \infty)$ is defined through the conditional expectation $\psi(t) = E\left(X_1^2 | X_1^2 + X_2^2 + X_3^2 = t\right), t > 0$ Then, $E(\psi((X_1 + X_2)^2))$ is equal to,
Solution: It is given that X_1, X_2 and X_3 are independent and identically distributed random

variables.

$$E\left(X_1^2|X_1^2+X_2^2+X_3^2=t\right)=E\left(X_2^2|X_1^2+X_2^2+X_3^2=t\right)$$
$$=E\left(X_3^2|X_1^2+X_2^2+X_3^2=t\right) \tag{4.12.1}$$

Now,

$$\sum_{n=1}^{3} E\left(X_n^2|X_1^2+X_2^2+X_3^2=t\right)$$
$$=E\left(X_1^2+X_2^2+X_3^2|X_1^2+X_2^2+X_3^2=t\right) \tag{4.12.2}$$
$$=t \tag{4.12.3}$$

Hence, from (11.12.1).

$$E\left(X_1^2|X_1^2+X_2^2+X_3^2=t\right)=\frac{t}{3} \tag{4.12.4}$$
$$\therefore \psi(t)=\frac{t}{3} \tag{4.12.5}$$

Hence, from (11.12.5),

$$E(\psi((X_1+X_2)^2))=E\left(\frac{(X_1+X_2)^2)}{3}\right) \tag{4.12.6}$$
$$=E\left(\frac{X_1^2+X_2^2+2X_1\times X_2}{3}\right) \tag{4.12.7}$$
$$=\frac{E(X_1^2)+E(X_2^2)+2\times E(X_1)\times E(X_2)}{3} \tag{4.12.8}$$
$$=\frac{\frac{15}{4}+\frac{15}{4}+2\times 0\times 0}{3} \tag{4.12.9}$$
$$=\frac{15}{6} \tag{4.12.10}$$
$$\therefore E(\psi((X_1+X_2)^2))=2.5 \tag{4.12.11}$$

4.13. Let $X \sim B\left(5,\frac{1}{2}\right)$ and $Y \sim U(0,1)$. The the value of:

$$\frac{\Pr(X+Y\leq 2)}{\Pr(X+Y\geq 5)}$$

is equal to? (X and Y are independent) **Solution:** Characteristic function for $X \sim B\left(5,\frac{1}{2}\right)$ will be:

$$C_X(t)=\left(\frac{e^{it}+1}{2}\right)^5 \tag{4.13.1}$$

Characteristic function for $Y \sim U(0,1)$ will be:

$$C_Y(t)=\frac{e^{it}-1}{it} \tag{4.13.2}$$

Since both X and Y are independent we can take:

$$Z=X+Y \tag{4.13.3}$$
$$C_Z(t)=C_X(t)C_Y(t) \tag{4.13.4}$$
$$C_Z(t)=\frac{(e^{it}+1)^5(e^{it}-1)}{32it} \tag{4.13.5}$$

Applying Gil-Pelaez formula:

$$F_Z(z)=\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \tag{4.13.6}$$

$$F_Z(z)=\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\frac{1}{2it}\left(\frac{(e^{it}+1)^5(e^{it}-1)e^{-itz}}{32it}\right)+\frac{1}{2it}\left(\frac{(e^{-it}+1)^5(e^{-it}-1)e^{itz}}{32it}\right)dt$$

Substituting $z=2$, the value for $\Pr(Z\leq 2)$:

$$=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\frac{8\cos 2t+2\cos 4t}{64t^2}dt+\frac{1}{\pi}\int_0^{\infty}\frac{+8\cos 3t-8\cos t-10}{64t^2}dt \tag{4.13.7}$$

Finding a general expression for integrating:

$$\int\frac{\cos ax}{x^2}dx=-\frac{\cos ax}{x}-a\int\frac{\sin ax}{x}dx+C \tag{4.13.8}$$

By applying integration by parts. Now finding the value of other integral, by substituting $u = ax$ for limits as 0 and ∞:

$$\int_0^{\infty}\frac{a\sin ax}{x}dx=\int_0^{\infty}\frac{a\sin u}{u}du \tag{4.13.9}$$
$$=\frac{a\pi}{2} \tag{4.13.10}$$

Now using the above general expressions to calculate (11.13.7) and simplifying the expression after putting the limits we get

$$= \frac{-1}{8\pi}\left(\int_0^\infty \frac{\sin 4t + 3\sin 3t + 2\sin 2t - \sin t}{t} dt\right) \tag{4.13.11}$$

$$- \frac{2(\cos t - 1)(\cos t + 1)^3}{8\pi t}\Bigg|_0^\infty + \frac{1}{2} \tag{4.13.12}$$

$$= \frac{1}{2} + \frac{-1}{8\pi} \times \frac{5\pi}{2} + 0 \tag{4.13.13}$$

$$= \frac{3}{16} \tag{4.13.14}$$

Using (11.13.10) and (11.13.8) to calculate for our second case Similarly on substituting $z = 5$, the value for $\Pr(Z \le 5)$:

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{-10\cos 3t - 8\cos 4t}{64t^2} dt + \frac{1}{\pi}\int_0^\infty \frac{-2\cos 5t + 12\cos t + 8}{64t^2} dt \tag{4.13.15}$$

$$= \frac{1}{\pi}\left(\int_0^\infty \frac{5\sin 5t + 16\sin 4t + 15\sin 3t - 6\sin t}{32} dt\right) + \frac{1}{2} + \frac{1}{\pi}\left(\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t}\Bigg|_0^\infty\right) \tag{4.13.16}$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \tag{4.13.17}$$

$$= \frac{31}{32} \tag{4.13.18}$$

The value for $\Pr(Z \ge 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \le 5) \tag{4.13.19}$$

$$= 1 - \frac{31}{32} = \frac{1}{32} \tag{4.13.20}$$

Upon substituting (11.13.14)and (11.13.20), we get:

$$\frac{\Pr(X + Y \le 2)}{\Pr(X + Y \ge 5)} = 6 \tag{4.13.21}$$

4.14. A die is thrown again and again until three sixes are obtained.Find the probability of obtaining the third six in the sixth row of a die.

5 Geometric Distribution

5.1. The probability of getting a ”head” in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a ”head” is obtained. If the tosses are independent, then the probability of getting ”head” for the first time in the fifth toss is........

Solution: Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.
$X = k$ represents $k - 1$ failures were obtained before getting 1 success. p represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \tag{5.1.1}$$

Using (5.1.1) we get

$$\Pr(X = 5) = (1-p)^{k-1} \times p$$
$$= (0.7)^4 \times 0.3 = 0.07203 \tag{5.1.2}$$

5.2. Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

a) $\frac{5}{11}$ b) $\frac{1}{2}$ c) $\frac{7}{13}$ d) $\frac{6}{11}$

Solution: Let $X \in \{1,2,3,4,5,6\}$ be the random variable representing out come of a dice. Probability of getting a six on a fair dice

$$\Pr(X = 6) = \frac{1}{6} \tag{5.2.1}$$

Probability of not getting a six on a fair dice

$$\Pr(X \neq 6) = \frac{5}{6} \tag{5.2.2}$$

The probability of some one wining in their n^{th} trail is

$$\Pr(X_n = 6 | X_k \neq 6, k = 1,2,3...,n-1) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \tag{5.2.3}$$

Let the probability of a wining the game is $\Pr(A)$
If A start’s the game then A can win on odd numbered trail(n)

$$n = 2m + 1 \tag{5.2.4}$$

Inorder for A to win B must lose in all of it’s trails until A gets a six.

Therefore,

$$\Pr(A) = \Pr(X_1 = 6) + \Pr(X_3 = 6) + \Pr(X_5 = 6) + ... \tag{5.2.5}$$

$$\Pr(A) = \left(\frac{1}{6}\right) + \left(\frac{1}{6}\left(\frac{5}{6}\right)^2\right) + \left(\frac{1}{6}\left(\frac{5}{6}\right)^4\right)... \tag{5.2.6}$$

$$= \frac{1}{6}\sum_{m=0}^{\infty}\left(\frac{5}{6}\right)^{2m} \tag{5.2.7}$$

$$= \frac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2} \tag{5.2.8}$$

$$= \frac{6}{11} \tag{5.2.9}$$

$$\implies \Pr(A) = \frac{6}{11} \tag{5.2.10}$$

Therefore, The probability that A wins the game=$\Pr(A) = \frac{6}{11}$

5.3. If a random variable X assumes only positive integral values, with the probability $P(X = x) = \frac{2}{3}(\frac{1}{3})^{x-1},\ x = 1, 2, 3, ...,$

then $E(X)$ is

a) $\frac{2}{9}$

b) $\frac{2}{3}$

c) 1

d) $\frac{3}{2}$

Solution: Let Y={0,1} be a set of random variables of a Bernoulli's distribution with 0 representing a loss and 1 a win and let $Y_i \in Y$ for i=1,2,3...,Y_i is the outcome of i^{th} try of choosing 0 or 1 from Y.

So the Random variable X is generated by assigning value of i to X where $Y_i = 1$ for the first time.

$$X = \{x : Y_{i=x} = 1, Y_{i<x} = 0\}$$
$$\implies X = \{Y_1 = 0, Y_2 = 0, Y_3 = 0, ..., Y_x = 1\}$$

For given bernouli's trail $p = \frac{2}{3}$ and $q = 1-p = \frac{1}{3}$. The given probability distribution is

$$P(X = x) = P(Y_{i=x} = 1)P(Y_{i<x} = 0)$$
$$\implies P(X = x) = p(1-p)^{x-1}$$
$$\implies P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1}$$

The expectation value of X represented by E(X) is given by

$$E(X) = \sum_{i=1}^{\infty} Pr(x = i) \times i$$

Let S=E(X),

$$\implies E(X) = S = \sum_{i=1}^{\infty} Pr(x = i) \times i \tag{5.3.1}$$

$$\implies S = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times i \tag{5.3.2}$$

$$\implies S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times i \tag{5.3.3}$$

Multiplying (5.3.2) with $\frac{1}{3}$ on both sides gives

$$\frac{1}{3}S = \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \times i \tag{5.3.4}$$

In (5.3.3) $\sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \times i$ can be written as

$\sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i-1)$

$$\implies \frac{1}{3}S = \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i-1) \tag{5.3.5}$$

(5.3.3)-(5.3.5) gives : $\frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \times (i-(i-1))$ (5.3.6)

$$\implies \frac{2}{3}S = \frac{2}{3} + \sum_{i=2}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{5.3.7}$$

$$\implies S = 1 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i} \tag{5.3.8}$$

$$\implies S = 1 + \frac{1/3}{1-\frac{1}{3}} \tag{5.3.9}$$

$$\implies S = \frac{3}{2} \tag{5.3.10}$$

The Variance Var(X) is given by $\sum x^2 P(x) - E(X)$ for the given distribution,

$$Var(X) = \sum_{i=1}^{\infty} i^2 P(x=i) - E(X) \tag{5.3.11}$$

$$\text{let } S = \sum_{i=1}^{\infty} i^2 P(x=i) = \sum_{i=1}^{\infty} i^2 \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{5.3.12}$$

$$S/3 = \sum_{i=1}^{\infty} i^2 \frac{2}{3}\left(\frac{1}{3}\right)^{i} = \sum_{i=0}^{\infty} i^2 \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{5.3.13}$$

$$= \sum_{i=1}^{\infty} (i-1)^2 \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{5.3.14}$$

(5.3.12)-(5.3.14) gives us

$$\frac{2S}{3} = \sum_{i=1}^{\infty} (i^2-(i-1)^2)\frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{5.3.15}$$

$$S = \sum_{i=1}^{\infty} (2i-1)\left(\frac{1}{3}\right)^{i-1} \tag{5.3.16}$$

$$\implies S = 3\sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} i - \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i-1} \tag{5.3.17}$$

$$\implies S = 3E(X) - \frac{1}{1-1/3} \tag{5.3.18}$$

$$\implies S = \frac{9}{2} - \frac{3}{2} = 3 \tag{5.3.19}$$

From (5.3.19) and (5.3.11) we can write

$$Var(X) = 3 - \frac{3}{2} = \frac{3}{2}$$

From (5.3.10) we can say that the expectation value of X given by E(X)=S=$\frac{3}{2}$ and $Var(X) = \frac{3}{2}$

5.4. Let X_1 and X_2 be independent geometric random variables with the same probability mass function given by $\Pr(X=k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$ Then the value of $\Pr(X_1 = 2 | X_1 + X_2 = 4)$ correct up to three decimal places is

Solution: Let

$$p_{X_i}(k) = \Pr(X_i = k) = \begin{cases} p(1-p)^{k-1} & n = 1, 2, ... \\ 0 & otherwise \end{cases} \tag{5.4.1}$$

where i=1,2

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{5.4.2}$$

$$(X_1 = 2) \cap (X_1 + X_2 = 4) = (X_1 = 2, X_2 = 2) \tag{5.4.3}$$

Thus,

$$\Pr(X_1 = 2 | X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2, X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{5.4.4}$$

Since the two events are independent,

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{\Pr(X_1 = 2)\Pr(X_2 = 2)}{\Pr(X_1 + X_2 = 4)} \tag{5.4.5}$$

Let

$$X = X_1 + X_2 \tag{5.4.6}$$

From (5.4.6),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{5.4.7}$$

$$= \sum_k \Pr(X_1 = n - k|X_2 = k)\, p_{X_2}(k) \tag{5.4.8}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n - k|X_2 = k)$$
$$= \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{5.4.9}$$

From (5.4.8) and (5.4.9),

$$p_X(n) = \sum_k p_{X_1}(n - k)p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{5.4.10}$$

where $*$ denotes the convolution operation. Substituting from (5.4.1) in (5.4.10),

$$p_X(n) = \sum_{k=1}^{n-1} p_{X_1}(n - k)p_{X_2}(k) \tag{5.4.11}$$

$$= \sum_{k=1}^{n-1}(1 - p)^{k-1} p \cdot (1 - p)^{n-k-1} p \tag{5.4.12}$$

$$= (1 - p)^{n-2} p^2 \sum_{k=1}^{n-1} 1 \tag{5.4.13}$$

$$= (n - 1)(1 - p)^{n-2} p^2 \tag{5.4.14}$$

From (5.4.14) and (5.4.1) we have

$$\Pr(X_1 = 2) = \Pr(X_2 = 2) = p(1 - p) \tag{5.4.15}$$

$$\Pr(X_1 + X_2 = 4) = 3(1 - p)^2 p^2 \tag{5.4.16}$$

Substituting in (5.4.5)

$$\Pr(X_1 = 2|X_1 + X_2 = 4) = \frac{(1 - p)^2 p^2}{3(1 - p)^2 p^2} \tag{5.4.17}$$

$$= \frac{1}{3} \tag{5.4.18}$$

5.5. Let $X_1, X_2, X_3, ...$ be a sequence of i.i.d random variables with mean 1. If N is a geometric random variable with the probability mass function $P(N = k) = \frac{1}{2^k}$; k = 1, 2, 3,... and it is independent of the X_i's, then $E(X_1 + X_2 + X_3 + ... + X_n)$ is equal to

Solution: The expectation operator,

$$E(X_1 + ... + X_n) = E(X_1) + ... + E(X_n) \tag{5.5.1}$$

We know that,

$$E(X) = \sum_{i=1}^{\infty} x_i \Pr(X = x_i) \tag{5.5.2}$$

$$= \sum_{i=1}^{\infty} k_i \Pr(X = k_i) \tag{5.5.3}$$

So now,

$$E(X_1) = k_1 \Pr(X = k_1) = 1\left(\frac{1}{2}\right) \tag{5.5.4}$$

Similarly,

$$E(X_2) = k_2 \Pr(X = k_2) = 2\left(\frac{1}{2}\right)^2 \tag{5.5.5}$$

and the pattern follows. Let

$$E(X_1 + ... + X_n) = S \tag{5.5.6}$$

By substituting (5.5.4) and (5.5.5) in (5.5.1)

$$S = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + ... \tag{5.5.7}$$

Dividing by 2 on both sides

$$\frac{S}{2} = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)^1 + 2\left(\frac{1}{2}\right)^2 + ... \tag{5.5.8}$$

Subtracting (5.5.8) from (5.5.7)

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{2}^2 + \frac{1}{2}^3 + ... \tag{5.5.9}$$

$$= \frac{1/2}{1 - 1/2} \tag{5.5.10}$$

$$= 1 \tag{5.5.11}$$

Therefore, from (5.5.6)

$$E\left(X_1 + X_2 + ... + X_n\right) = 2 \tag{5.5.12}$$

6 Binomial Distribution

6.1. A fair coin is tossed 10 times. What is the probability that ONLY the first two tosses will yield heads?

a) $\left(\frac{1}{2}\right)^2$

b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$

c) $\left(\frac{1}{2}\right)^{10}$

d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$

Solution: Let $M \sim B(n, h)$ be a random variable representing number of 'heads' in 10 tosses. So M has a binomial distribution :

$$\Pr\left(M = k\right) = {}^{n}C_k \times (h)^{n-k} \times (t)^k \tag{6.1.1}$$

Where

- n = Total number of tosses = 10
- h = Probability that 'head' appears in a toss $= \frac{1}{2}$
- t = Probability that 'tail' appears in a toss = $\frac{1}{2}$

So,

$$\Pr\left(M = k\right) = {}^{10}C_k \times \left(\frac{1}{2}\right)^{10-k} \times \left(\frac{1}{2}\right)^k \tag{6.1.2}$$

n	10
$\Pr\left(M = 2\right)$	${}^{10}C_2 \times \left(\frac{1}{2}\right)^{10-2} \times \left(\frac{1}{2}\right)^2$
Calculation	${}^{10}C_2 \times \left(\frac{1}{2}\right)^{10}$
Value	0.043945

- Number of ways of choosing 2 positions from 10 tosses = ${}^{10}C_2$
- Number of favourable outcome = 1 (Choosing FIRST and SECOND tosses as heads)
- Probability that chosen 2 'heads' are from FIRST and SECOND tosses = $\frac{1}{{}^{10}C_2}$

Probability that ONLY the first 2 tosses yield heads

$$= \Pr\left(M = 2\right) \times \frac{1}{{}^{10}C_2} \tag{6.1.3}$$

$$= {}^{10}C_2 \times \left(\frac{1}{2}\right)^{10} \times \frac{1}{{}^{10}C_2} \tag{6.1.4}$$

$$= \left(\frac{1}{2}\right)^{10} \tag{6.1.5}$$

6.2. Let the random variable X represent the number of times a fair coin needs to be tossed till two consecutive heads appear for the first time. The expectation of X is......

6.3. Let $X \in [0, 1]$ and $Y \in [0, 1]$ be two independent binary random variables. If $P(X = 0) = p$ and $P(Y = 0) = q$, then $P(X + Y \geqslant 1)$ is equal to

a) $pq + (1 - p)(1 - q)$

b) pq

c) $p(1 - q)$

d) $1 - pq$

Solution:

6.4. A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is:

a) $\frac{1}{8}$

b) $\frac{1}{2}$

c) $\frac{3}{8}$

d) $\frac{3}{4}$

6.5. A player throws a ball at a basket kept at a distance. The probability that the ball falls into the basket in a single attempt is 0.1. The player attempts to throw the ball twice. Considering each attempt to be independent, the probability that this player puts the ball into the basket only in the second attempt is.........

Solution: Let $X \in \mathbb{N}$ represent the number of times the experiment is performed.

$X = k$ represents $k - 1$ failures were obtained before getting 1 success. p represents the probability of success

$$p_X(k) = \begin{cases} (1-p)^{k-1} \times p & k \in \mathbb{N} \\ 0 & otherwise \end{cases} \quad (6.5.1)$$

Using (6.5.1) we get

$$\Pr(X=2) = (1-p)^{k-1} \times p$$
$$= (0.9) \times 0.1 = 0.09 \quad (6.5.2)$$

6.6. Shaquille O' Neal is a 60% career free throw shooter, meaning that he successfully makes 60 free throws out of 100 attempts on average. What is the probability that he will successfully make exactly 6 free throws in 10 attempts?

A) 0.2508
B) 0.2816
C) 0.2934
D) 0.6000

Solution: Let

$$X_i \in \{0,1\} \quad (6.6.1)$$

represent the i^{th} free throw, where 1 represents a successful free throw attempt and 0 represents an unsuccessful attempt. Let

$$X = \sum_{i=1}^{n} X_i \quad (6.6.2)$$

where n is the total number of free throws. Then, X has a binomial distribution with

$$\Pr(X=k) = {}^{n}C_k p^k q^{n-k} \quad (6.6.3)$$

Where,

$$p = \frac{6}{10} \quad (6.6.4)$$
$$q = 1 - p = \frac{4}{10} \quad (6.6.5)$$
$$n = 10 \quad (6.6.6)$$

from the given information. Then,

$$\Pr(X=6) = {}^{10}C_6 \left(\frac{6}{10}\right)^6 \left(\frac{4}{10}\right)^4 \quad (6.6.7)$$

On simplifying we get,

$$\Pr(X=6) = 0.2508 \quad (6.6.8)$$

Therefore, the probability that he will successfully make exactly 6 free throws in 10 attempts is 0.2508 and hence option (A) is correct.

6.7. Consider a sequence of tossing a fair coin where outcomes of tosses are independent.The probability of getting the head for the third time in the fifth toss is

(A) $\frac{5}{16}$
(B) $\frac{3}{16}$
(C) $\frac{3}{5}$
(D) $\frac{9}{16}$

Solution: Let the random variable $X \in \{0,1\}$ denotes head and tail in a toss.As both are equally probable.

$$\Pr(X=0) = \frac{1}{2} \quad (6.7.1)$$
$$\Pr(X=1) = \frac{1}{2} \quad (6.7.2)$$

Event	Description
A	nth toss is a head
B	Exactly k-1 heads in first four tosses
C	nth toss is the third head

TABLE 6.7.1: Description of events used in problem

$$\Pr(A) = \Pr(X=1) = \frac{1}{2} \quad (6.7.3)$$
$$\Pr(B) = \frac{{}^{n-1}C_{k-1}}{2^{n-1}} \quad (6.7.4)$$

$$C = AB \quad (6.7.5)$$
$$\Pr(C) = \Pr(AB) \quad (6.7.6)$$

As A and B are independent events.

$$\Pr(C) = \Pr(A)\Pr(B) \quad (6.7.7)$$
$$= \frac{1}{2} \times \frac{{}^{n-1}C_{k-1}}{2^{n-1}} \quad (6.7.8)$$
$$= \frac{{}^{n-1}C_{k-1}}{2^{n}} \quad (6.7.9)$$

Here n=5,k=3

$$\Pr(C|n=5,k=2) = \frac{^4C_2}{2^5} \tag{6.7.10}$$
$$= \frac{6}{32} \tag{6.7.11}$$

Therefore probability of getting the head for the third time in the fifth toss is $\frac{3}{16}$.

6.8. A box has ten light bulbs out of which two are defective, Two light bulbs are drawn from this box one after the other without replacement. The probability that both light bulbs drawn are not defective is

A) $\frac{8}{45}$ B) $\frac{28}{45}$ C) $\frac{16}{25}$ D) $\frac{4}{5}$

Solution: Let $X_i \in \{0,1\}$ represent the i^{th} draw, where 0 denotes a defective bulb and 1 denotes a non-defective bulb.

TABLE 6.8.1

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	2/90	16/90
$X_2 = 1$	16/90	56/90

Table 2.6.1 represents the probabilities of all possible cases when two bulbs are drawn one by one without replacement. Probability that both of the bulbs are non-defective (by substituting values from table 2.6.1)

$$= \Pr(X_2 = 1|X_1 = 1)\Pr(X_1 = 1) \tag{6.8.1}$$
$$= \frac{56}{90} \tag{6.8.2}$$
$$= \frac{28}{45} \tag{6.8.3}$$

So the correct option is (B)

6.9. Three fair dies are rolled simultaneously. The probability of getting a sum of 5 is

a) $\frac{1}{108}$
b) $\frac{1}{72}$
c) $\frac{1}{54}$
d) $\frac{1}{36}$

Solution:

Let $X_i \in \{1,2,3,4,5,6\}$, i = 1,2,3, be the random variables representing the outcome for each die. As the dies are fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{6.9.1}$$

Let X be a random variable denotes the desired outcome,

$$X = X_1 + X_2 + X_3 \tag{6.9.2}$$
$$\implies X \in \{3,4,\cdots,18\} \tag{6.9.3}$$

We have to find $P_X(n) = \Pr(X_1 + X_2 + X_3 = n)$

$$\begin{aligned} p_X(n) &= \Pr(X_1 + X_2 + X_3 = n) \\ &= \Pr(X_1 + X_2 = n - X_3) \\ &= \Sigma_k \Pr(X_1 + X_2 = n-k|X_3 = k)\, p_{X_3}(k) \end{aligned} \tag{6.9.4}$$

As X_1, X_2, X_3 are independent, After unconditioning

$$\Pr(X_1 + X_2 = n-k|X_3 = k) = \Pr(X_1 + X_2 = n-k) \tag{6.9.5}$$

Using (6.9.5) in (10.44.4)

$$\begin{aligned} p_X(n) &= \Sigma_k \Pr(X_1 + X_2 = n-k|X_3 = k)\, p_{X_3}(k) \\ &= \Sigma_k \Pr(X_1 + X_2 = n-k)\, p_{X_3}(k) \\ &= \Sigma_k(\Sigma_a \Pr(X_1 = n-k-a|X_2 = a)\Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k(\Sigma_a \Pr(X_1 = n-k-a)\Pr(X_2 = a))\, p_{X_3}(k) \\ &= \Sigma_k(\Sigma_a p_{X_1}(n-k-a)\, p_{X_2}(a))\, p_{X_3}(k) \end{aligned} \tag{6.9.6}$$

Equation (10.44.5) can be written as follows using convolution operation,

$$\begin{aligned} p_X(n) &= \Sigma_k(\Sigma_a p_{X_1}(n-k-a)\, p_{X_2}(a))\, p_{X_3}(k) \\ &= p_{X_1}(n) * p_{X_2}(n) * p_{X_3}(n) \end{aligned} \tag{6.9.7}$$

The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{6.9.8}$$

From (10.44.1) and (6.9.8),

$$P_{X_1}(z) = P_{X_2}(z) = P_{X_3}(z)$$
$$= \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{6.9.9}$$
$$= \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}, \quad |z| > 1 \tag{6.9.10}$$

upon summing up the geometric progression. From (10.44.6),

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n) * p_{X-3}(n), \tag{6.9.11}$$
$$P_X(z) = P_{X_1}(z)P_{X_2}(z)P_{X_3}(z) \tag{6.9.12}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (6.9.10) and (6.9.12),

$$P_X(z) = \left\{\frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}\right\}^3 \tag{6.9.13}$$
$$= \frac{1}{216}\frac{z^{-3}\left(1-3z^{-6}+3z^{-12}-z^{-18}\right)}{\left(1-z^{-1}\right)^3} \tag{6.9.14}$$

Using the fact that,

$$p_X(n-k) \stackrel{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{6.9.15}$$
$$nu(n) \stackrel{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{6.9.16}$$
$$n^2u(n) \stackrel{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)^3} \tag{6.9.17}$$
$$(n^2+n)u(n) \stackrel{\mathcal{H}}{\longleftrightarrow} Z\frac{2z^{-1}}{\left(1-z^{-1}\right)^2} \tag{6.9.18}$$

after some algebra, it can be shown that,

$$\begin{aligned}\frac{1}{216\times 2}&\left[\left((n-2)^2+n-2\right)u(n-2)\right.\\ &-3\left((n-8)^2+n-8\right)u(n-8)\\ &+3\left((n-14)^2+n-14\right)u(n-14)\\ &\left.-\left((n-20)^2+n-20\right)u(n-20)\right]\\ \stackrel{\mathcal{H}}{\longleftrightarrow}\ & Z\frac{1}{216}\frac{z^{-3}\left(1-3z^{-6}+3z^{-12}-z^{-18}\right)}{\left(1-z^{-1}\right)^3}\end{aligned} \tag{6.9.19}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{6.9.20}$$

From (6.9.8),(6.9.14) and (6.9.19),

$$\begin{aligned}p_X(n) = \frac{1}{216\times 2}&\left[\left((n-2)^2+n-2\right)u(n-2)\right.\\ &-3\left((n-8)^2+n-8\right)u(n-8)\\ &+3\left((n-14)^2+n-14\right)u(n-14)\\ &\left.-\left((n-20)^2+n-20\right)u(n-20)\right]\end{aligned} \tag{6.9.21}$$

From (6.9.20) and (6.9.21),

$$p_X(n) = \begin{cases} 0 & n < 3 \\ \frac{n^2-3n+2}{432} & 3 \leq n \leq 8 \\ \frac{42n-2n^2-166}{432} & 8 < n \leq 14 \\ \frac{n^2-39n+380}{432} & 14 < n \leq 18 \\ 0 & n > 18 \end{cases} \tag{6.9.22}$$

We need probability of getting sum of 5,

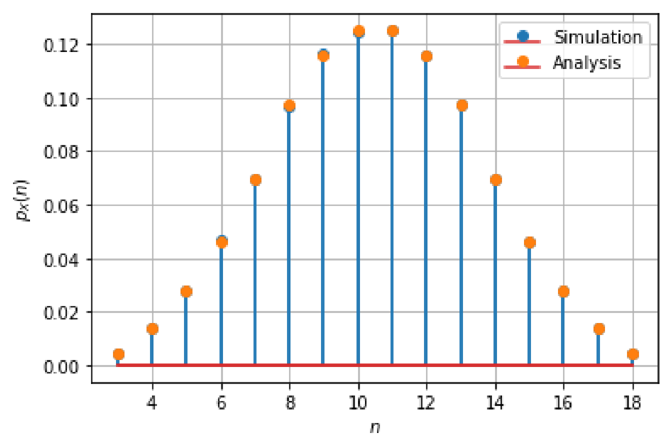


Fig. 6.9.1: Probability mass function of X (simulations are close to analysis)

$\implies$ n=5
from (6.9.22) and using n=5,

$$p_X(5) = \frac{5^2-3(5)+2}{432} \tag{6.9.23}$$
$$p_X(5) = \frac{12}{432} \tag{6.9.24}$$
$$p_X(5) = \frac{1}{36} \tag{6.9.25}$$

Therefore the probability of getting a sum of 5 when three fair dies are rolled is $\frac{1}{36}$.
Ans: Option (D)

6.10. The probability of a resistor being defective is 0.02. There are 50 such resistors in a circuit. The probability of two or more defective resistors in the circuit (round off to two decimal places) is — **Solution:** Consider, Probability of a defective resistor $= P = 0.02 = \frac{1}{50}$.
Total number of resistors = n = 50.
Let X be number of defective resistors.
By Binomial distribution,

$$Pr(X = k) = \binom{n}{k} (P)^k (1 - P)^{n-k} \tag{6.10.1}$$

$$Pr(X = 0) = \binom{50}{0} \left(\frac{1}{50}\right)^0 \left(1 - \frac{1}{50}\right)^{50-0} \tag{6.10.2}$$

$$\implies Pr(X = 0) = \left(\frac{49}{50}\right)^{50} \tag{6.10.3}$$

$$Pr(X = 1) = \binom{50}{1} \left(\frac{1}{50}\right)^1 \left(1 - \frac{1}{50}\right)^{50-1} \tag{6.10.4}$$

$$\implies Pr(X = 1) = \left(\frac{49}{50}\right)^{49} \tag{6.10.5}$$

$$\Pr(X \geq 2) = 1 - \Pr(X < 2)$$
$$\Pr(X \geq 2) = 1 - (\Pr(X = 0) + \Pr(X = 1))$$
$$\Pr(X \geq 2) = 0.2642$$

6.11. For each element in a set of size 2n, an unbiased coin is tossed. The 2n coin tosses are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is:

a) $\frac{^{2n}C_n}{4^n}$

b) $\frac{^{2n}C_n}{2^n}$

c) $\frac{1}{^{2n}C_n}$

d) $\frac{1}{2}$

Solution: The number of elements chosen is equal to the number of heads obtained by 2n coin tosses. Let X be a random variable with value of X equal to the number of heads obtained.
Probability of getting a head, $p = \frac{1}{2}$
Probability of getting a tail, $q = \frac{1}{2}$
Probability that n elements are chosen out of 2n elements is $\Pr(X = n)$
From binomial distribution we know that,

$$\Pr(X = r) = {}^{2n}C_r p^r q^{2n-r} \tag{6.11.1}$$

$$\Pr(X = n) = {}^{2n}C_n \times \left(\frac{1}{2}\right)^n \times \left(\frac{1}{2}\right)^n \tag{6.11.2}$$

$$= \frac{^{2n}C_n}{4^n} \tag{6.11.3}$$

Hence option (A) is correct.

6.12. In an industry, the probability of an accident occurring in a given month is $\frac{1}{100}$. Let $\Pr(n)$ denote the probability that there will be no accident over a period of 'n' months. Assume that the events of individual months are independent of each other. The smallest integer value of 'n' such that $\Pr(n) \leq \frac{1}{2}$ is ....(*round off to the nearest integer*).
Solution: Let A be the event of an accident occurring in a given month. So,

$$\Pr(A) = \frac{1}{100} \tag{6.12.1}$$

$$\Pr(A') = 1 - \Pr(A) \tag{6.12.2}$$

$$\Pr(A') = \frac{99}{100} \tag{6.12.3}$$

So, $\Pr(n)$ can be written as:

$$\Pr(n) = \Pr(A' \times A' \cdots A')_{A'\ n\ times} \tag{6.12.4}$$

Its given that events of individual months are independent of each other, so

$$\Pr(n) = \Pr(A') \cdot \Pr(A') \cdots \Pr(A')_{A'\ n\ times} \tag{6.12.5}$$

$$= (\Pr(A'))^n \tag{6.12.6}$$

Given:

$$\Pr(n) \leq \frac{1}{2} \tag{6.12.7}$$

So, from (6.12.6),

$$(\Pr(A'))^n \leq \frac{1}{2} \tag{6.12.8}$$

$$\implies \ln\left(\Pr\left(A'\right)\right)^n \le \ln\frac{1}{2} \quad (6.12.9)$$

$$\implies n \cdot \ln\frac{99}{100} \le \ln\frac{1}{2} \quad (6.12.10)$$

$$\implies n \ge \frac{\ln\frac{1}{2}}{\ln\frac{99}{100}} \quad (6.12.11)$$

$$\implies n \ge 68.9675 \quad (6.12.12)$$

$\therefore$ The smallest integer value of n is **69**.

6.13. A fair coin is tossed n times. The probability that the difference between number of heads and tails is $(n-3)$ is

a) 2^{-n}
b) 0
c) ${}^{n}C_{n-3}2^{-n}$
d) 2^{-n+3}

Solution:
Let number of heads be k , then number of tails are $n-k$.
Given : $|k-(n-k)| = n-3$
Case(i)

$$2k - n = n - 3$$
$$k = n - \frac{3}{2}$$

As k cannot be fractional, it's impossible.
Case(ii)

$$-(2k - n) = n - 3$$
$$k = \frac{3}{2}$$

As k cannot be fractional, it's impossible.
Thus, probability that the difference between number of heads and tails is $(n-3)$ is 0
Correct option is 2

6.14. A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is **Solution:**
Probability of selecting items follows binomial distribution with parameter for selecting defective items,

$$p = \frac{10}{100} = \frac{1}{10} \quad (6.14.1)$$

The probability of getting k defective items by selecting n items is,

$$\Pr(X = k) = \begin{cases} {}^{n}C_k p^k (1-p)^{n-k} & 0 \le k \le n \\ 0 & otherwise \end{cases} \quad (6.14.2)$$

Total no. of items chosen,

$$n = 10 \quad (6.14.3)$$

Probability of getting exactly 2 defective items,

$$\Pr(X = 2) = {}^{10}C_2 \left(\frac{1}{10}\right)^2 \left(1 - \frac{1}{10}\right)^{10-2} \quad (6.14.4)$$

$$\Pr(X = 2) = {}^{10}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^8 \quad (6.14.5)$$

$$\Pr(X = 2) = 0.1937102445 \quad (6.14.6)$$

6.15. A fair coin is tossed 20 times. The probability that 'head' will appear exactly 4 times in the first ten tosses, and 'tail' will appear exactly 4 times in the next ten tosses is....(round off to 3 decimal places)

Solution:
The probability of getting exactly 4 heads in the first 10 tosses can be calculated as

$$\Pr(H = 4, T = 6) = {}^{10}C_4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^6 \quad (6.15.1)$$

The probability of getting exactly 4 tails in the next 10 tosses can be calculated as

$$\Pr(T = 4, H = 6) = {}^{10}C_4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^6 \quad (6.15.2)$$

Since these two probabilities are independent of each other, the required probability is the product of these two

$$= {}^{10}C_4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^6 \times {}^{10}C_4 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^6 \quad (6.15.3)$$

$$= \frac{210^2}{2^{20}} = \frac{44100}{1048576} = 0.042 \quad (6.15.4)$$

So, the required probability is 0.042.

6.16. If three coins are tossed simultaneously, the probability of getting atleast one head is:

a) $\frac{1}{8}$

b) $\frac{3}{8}$

c) $\frac{1}{2}$

d) $\frac{7}{8}$

Solution:
Let X represent the number of heads obtained in a trial involving 3 tosses.
Then, X is a binomial random variable defined by: $X \sim B(n, p)$ where $n = 3$ and $p = \frac{1}{2}$ and:

$$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k} \tag{6.16.1}$$

To find:

$$\Pr(X \geq 1) \tag{6.16.2}$$
$$= 1 - \Pr(X < 1) \tag{6.16.3}$$
$$= 1 - \Pr(X = 0) \tag{6.16.4}$$
$$= 1 - {}^3C_0 p^0 (1-p)^3 \tag{6.16.5}$$
$$= \frac{7}{8} \tag{6.16.6}$$

6.17. The mean and variance, respectively of a binomial distribution for n independent trials with the probability of success as p, are

a) $\sqrt{np}$, $np(1-2p)$

b) $\sqrt{np}$, $\sqrt{np(1-p)}$

c) np, np

d) $np, np(1-p)$

Solution: Let $X_1, X_2, X_3, ..., X_n$ be the random variable for n independent trials such that

$$X = X_1 + X_2 + X_2 + X_3 + \cdots + X_n$$
$$X = \sum_{i=1}^{n} X_i$$

p = success (1) and 1 - p = failure (0) Expected Value for n trials :

$$E(X_i) = X_i \cdot p_i$$
$$E(X_i) = 1 \cdot p + 0 \cdot (1-p)$$
$$E(X_i) = p \tag{6.17.1}$$

We know that,

$$E(X) = \sum_{i=1}^{n} E(X_i)$$
$$E(X) = np \tag{6.17.2}$$

Mean of a binomial distribution for n independent trials is **np**. Now,

$$E(X_i^2) = X_i^2 \cdot p_i$$
$$E(X_i^2) = 1^2 \cdot p + 0^2 \cdot (1-p)$$
$$E(X_i^2) = p \tag{6.17.3}$$

For variance,

$$Var(X_i) = E(X_i^2) - E(X_i)^2$$
$$Var(X_i) = p - p^2 \tag{6.17.4}$$

We can add Var(X_i) to get Var(X) as these are independent trials

$$Var(X) = \sum_{i=1}^{n} Var(X_i)$$
$$Var(X) = n(p - p^2)$$
$$Var(X) = np(1-p) \tag{6.17.5}$$

Variance of a binomial distribution for n independent trials is **np(1-p)**. Hence, (4) is correct option.

6.18. A company is hiring to fill four managerial positions. The candidates are five men and three women. If every candidate is equally likely to be chosen then the probability that at least one woman is chosen is **Solution:** Let $X \in \{0, 1, 2, 3\}$ denotes the number of woman candidates chosen.

$$\Pr(X = x) = \frac{{}^3C_x \times {}^5C_{4-x}}{{}^8C_4} \tag{6.18.1}$$

X	0	1	2	3
P(X)	$\frac{{}^3C_0 \times {}^5C_4}{{}^8C_4}$	$\frac{{}^3C_1 \times {}^5C_3}{{}^8C_4}$	$\frac{{}^3C_2 \times {}^5C_2}{{}^8C_4}$	$\frac{{}^3C_3 \times {}^5C_1}{{}^8C_4}$

The complement of the event "at least one woman candidate is chosen" is "no woman candidate being chosen"

$$\Pr(X \geq 1) = 1 - \Pr(X = 0) \quad (6.18.2)$$
$$= 1 - \frac{{}^3C_0 \times {}^5C_{4-0}}{{}^8C_4} \quad (6.18.3)$$
$$= \frac{13}{14} \quad (6.18.4)$$

6.19. If a fair coin is tossed four times,what is the probability that two tails and two heads will result? **Solution:** Given question is a binomial distribution in which no of trails $n = 4$.
Let's assume a trail is succeeded if the coin turns out to be head.Since it is a fair coin probability of success is $p = 0.5$
Let X be the binomial random variable of this distribution. So $X \in \{0, 1, 2, 3, 4\}$, 0 represents 0 heads, 1 represents 1 head, 2 represents 2 heads, 3 represents 3 heads and 4 represents 4 heads in 4 trails.
From binomial distribution,

$$\Pr(\mathbf{X=r}) = {}^nC_r p^r q^{n-r} \quad (6.19.1)$$
$$= {}^nC_r p^r (1-p)^{n-r} \quad (6.19.2)$$

Probability of getting two heads and two trails will be,

$$\Pr(\mathbf{X=2}) = {}^4C_2 \times (0.5)^2 \times (1-0.5)^2 \quad (6.19.3)$$
$$= 6 \times \left(\frac{1}{4}\right) \times \left(\frac{1}{4}\right) \quad (6.19.4)$$
$$= \frac{3}{8} \quad (6.19.5)$$
$$= 0.375 \quad (6.19.6)$$

Hence,the required probability is 0.375.

6.20. A die is rolled three times The probability that exactly one odd number turns up among the three outcomes is? **Solution:**
Let **X** be the random variable such that it represents number of times an odd number appeared .
Let **Y** be the random variable such that it represents number of times an even number appeared .
Let **C** be the event that exactly one odd number appears in 3 outcomes.
$\Pr(X = m, Y = n) = \binom{m+n}{m} \times \left(\frac{1}{2}\right)^{m+n}$

$$\Pr(\mathbf{C}) = \Pr(X = 1, Y = 2) \quad (1)$$
$$\Pr(X = 1, Y = 2) = \binom{1+2}{1} \times \left(\frac{1}{2}\right)^{1+2} \quad (2)$$
$$\Pr(X = 1, Y = 2) = \binom{3}{1} \times \left(\frac{1}{2}\right)^{3} \quad (3)$$
$$\Pr(X = 1, Y = 2) = \frac{3}{8} \quad (4)$$
$$\Pr(\mathbf{C}) = \Pr(X = 1, Y = 2) \quad (5)$$
$$\Pr(\mathbf{C}) = \frac{3}{8} \quad (6)$$

6.21. A coin is picked randomly from the box and tossed .Out of the two remaining coins in the box ,one coin is then picked randomly and tossed.If the first toss results in a head,Then the probability of getting head in second toss is :

a) $\frac{2}{5}$
b) $\frac{1}{3}$
c) $\frac{1}{2}$
d) $\frac{2}{3}$

6.22. A six-faced fair dice is rolled five times. The probability (in percentage) of obtaining "ONE" at least four times is

a) 33.3
b) 3.33
c) 0.33
d) 0.0033

Solution:
Let X be the random variable denoting the number the times "ONE" is obtained when a six-faced die is rolled n-times.X follows binomial distribution.
From binomial Distribution,

$$\Pr(X = k) = {}^nC_k p^k (1-p)^{n-k} \qquad k = 0, 1, ..., n$$

For this given problem $n = 5, p = \frac{1}{6}$ for a six-faced die
The probability (in percentage) of obtaining

"ONE" at least four times is $\Pr(X \geq 4) \times 100$

$$\Pr(X \geq 4) = \sum_{k=4}^{5} \Pr(X = k)$$
$$= \Pr(X = 4) + \Pr(X = 5)$$
$$= {}^{5}C_{4}\frac{5}{6^5} + {}^{5}C_{5}\frac{1}{6^5}$$
$$= \frac{26}{6^5}$$

Probability in percentage is,

$$= \frac{26}{6^5} \times 100$$
$$= 0.334$$

Option c is correct.

6.23. A coin is tossed 4 times. What is the probability of getting heads exactly three times ?

a) $\frac{1}{4}$
b) $\frac{3}{8}$
c) $\frac{1}{2}$
d) $\frac{3}{4}$

Solution:
In an experiment of tossing a coin $n(= 4)$ times, random variable $X \in \{0, 1, 2, 3\}$ follows binomial distribution.
The binomial distribution formula is:

$$\Pr(X = k) = {}^{n}C_k \times p^k \times (1 - p)^{n-k} \quad (6.23.1)$$

Where: Let X denote the number of heads

k	total number of "successes"
p	probability of a success on an individual trial
n	number of trials = 3

TABLE 6.23.1: The binomial distribution formula

$$\Pr(X = 3) = {}^{4}C_3 \times \left(\frac{1}{2}\right)^3 \times \left(1 - \frac{1}{2}\right)^{4-3} \quad (6.23.2)$$
$$= \frac{1}{4} \quad (6.23.3)$$

Correct option is 1.

6.24. Let X be the number of heads in 4 tosses of a fair coin by Person 1 and let Y be the number of heads in 4 tosses of a fair coin by Person 2. Assume that all the tosses are independent. Then the value of $\Pr(X = Y)$ correct up to three decimal places is ________ . **Solution:** Let $X \in \{0, 1, 2, 3, 4\}$ be the random variable representing the number of heads obtained by Person 1 in 4 tosses. Similarly, Let $Y \in \{0, 1, 2, 3, 4\}$ be the random variable representing the number of heads obtained by Person 2 in 4 tosses. Then X and Y are binomial distributions with parameter:

$$p = \frac{1}{2} \quad (6.24.1)$$

Then,

$$\Pr(X = i) = \begin{cases} {}^{4}C_k (p)^k (1 - p)^{4-k} & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \quad (6.24.2)$$

$$\Pr(X = i) = \begin{cases} {}^{4}C_k (\frac{1}{2})^k (1 - \frac{1}{2})^{4-k} & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \quad (6.24.3)$$

$$\Pr(X = i) = \begin{cases} {}^{4}C_k \times (\frac{1}{2})^4 & i \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases} \quad (6.24.4)$$

Serial number	Case	Probability of the case
1	$\Pr(X = 0)$	$\frac{{}^4C_0}{16} = \frac{1}{16}$
2	$\Pr(X = 1)$	$\frac{{}^4C_1}{16} = \frac{4}{16}$
3	$\Pr(X = 2)$	$\frac{{}^4C_2}{16} = \frac{6}{16}$
4	$\Pr(X = 3)$	$\frac{{}^4C_3}{16} = \frac{4}{16}$
5	$\Pr(X = 4)$	$\frac{{}^4C_4}{16} = \frac{1}{16}$

TABLE 6.24.1: Probability distribution table of X

Similar is the distribution of Y. For finding $\Pr(X = Y)$, let $Y = y$,

$$\Pr(X = Y) = \frac{{}^4C_y}{16} \times \Pr(Y = y) \quad (6.24.5)$$

Generalizing this result,

$$\Pr(X = Y) = \sum_{y=0}^{4} \frac{{}^4C_y}{16} \times \Pr(Y = y) \quad (6.24.6)$$
$$= \sum_{y=0}^{4} \frac{{}^4C_y}{16} \times \frac{{}^4C_y}{16} \quad (6.24.7)$$

$$\Pr(X = Y) = \left(\frac{1}{16} \times \frac{1}{16}\right) + \left(\frac{4}{16} \times \frac{4}{16}\right) + \left(\frac{6}{16} \times \frac{6}{16}\right) + \left(\frac{4}{16} \times \frac{4}{16}\right) + \left(\frac{1}{16} \times \frac{1}{16}\right) \quad (6.24.8)$$

$$\Pr(X=Y) = \frac{1}{256} + \frac{16}{256} + \frac{36}{256} + \frac{16}{256} + \frac{1}{256} \tag{6.24.9}$$
$$= \frac{70}{256} \tag{6.24.10}$$
$$= \frac{35}{128} \tag{6.24.11}$$
$$= 0.2734375 \tag{6.24.12}$$

7 Exponential Distribution

7.1. Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between successive arrivals.The length of a phone call is distributed exponentially with mean 3 minutes.The probability that an arrival does not have to wait before service is

a) 0.3
b) 0.5
c) 0.7
d) 0.9

Solution:
Let X be a random variable with values equal to time between successive calls(in minutes) which is a Poisson distribution with mean of 10.

$$\implies \Pr(X=x) = \frac{e^{-10}\times 10^x}{x!} \quad (x=1,2,3,...) \tag{7.1.1}$$

Let Y be a random variable with values equal to length of a phone call which is an exponential distribution with mean 3.

$$\implies f_Y(y) = \begin{cases} \frac{e^{\frac{-y}{3}}}{3} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{7.1.2}$$

$$\implies \Pr(Y \leq y) = F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy \tag{7.1.3}$$
$$= \int_0^y \frac{e^{\frac{-y}{3}}}{3} \tag{7.1.4}$$
$$= 1 - e^{\frac{-y}{3}} \tag{7.1.5}$$

Probability that an arrival does not have to wait is $\Pr(Y \leq X)$

$$\Pr(Y \leq X) = \sum_{x=0}^{x=\infty} \Pr(Y \leq x)\Pr(X=x) \tag{7.1.6}$$
$$= \sum_{x=0}^{x=\infty}(1-e^{\frac{-x}{3}}) \times \left(\frac{e^{-10}\times 10^x}{x!}\right) \tag{7.1.7}$$
$$= e^{-10}\left(\sum_{x=0}^{x=\infty}\frac{10^x}{x!} - \sum_{x=0}^{x=\infty}\frac{10^x e^{\frac{-x}{3}}}{x!}\right) \tag{7.1.8}$$
$$= e^{-10}\left(e^{10} - \sum_{x=0}^{x=\infty}\frac{e^{\left(\log_e 10 - \frac{1}{3}\right)x}}{x!}\right) \tag{7.1.9}$$
$$= 1 - e^{-10}\left(e^{e^{\left(log_e 10 - \frac{1}{3}\right)}}\right) \tag{7.1.10}$$
$$= 0.941 \tag{7.1.11}$$

Hence option (4) is correct.

7.2. Let Z be an exponential random variable with mean 1. That is, the cumulative distribution function of Z is given by

$$F_Z(x) = \begin{cases} 1-e^{-x}, & if\ x \geq 0 \\ 0, & if\ x < 0. \end{cases} \tag{7.2.1}$$

Then Pr(Z¿2 — Z¿1), rounded off to two decimal places, is equal to **Solution:**

$$Pr(Z>2|Z>1) = \frac{Pr((Z>2),(Z>1))}{Pr(Z>1)} \tag{7.2.2}$$
$$= \frac{Pr(Z>2)}{Pr(Z>1)} \tag{7.2.3}$$
$$= \frac{1-Pr(Z\leq 2)}{1-Pr(Z\leq 1)} \tag{7.2.4}$$
$$= \frac{e^{-2}}{e^{-1}} \tag{7.2.5}$$
$$= e^{-1} = 0.3679 \tag{7.2.6}$$

7.3. The lifetime of a component of a certain type is a random variable whose probability density function is exponentially distributed with parameter 2. For a randomly picked component of this type, what is the probability that its lifetime exceeds the expected lifetime (rounded to 2 decimal places) ? **Solution:**

Given, The lifetime of a component $X \sim \exp(2)$. The probability density function (PDF) of random variable X is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases} \tag{7.3.1}$$

where: the parameter $\lambda = 2$. Since, the PDF of the random variable is exponentially distributed, Expected Lifetime $= E(X) = \frac{1}{\lambda} = \frac{1}{2} = 0.5$

$$\Pr(X > E(X)) = \int_{\frac{1}{\lambda}}^{\infty} \lambda e^{-\lambda x}\, dx \tag{7.3.2}$$

$$= -e^{-\lambda x}\Big|_{\frac{1}{\lambda}}^{\infty} \tag{7.3.3}$$

$$= \lim_{x\to\infty}(-e^{-\lambda x}) - (-e^{-\lambda \times \frac{1}{\lambda}}) \tag{7.3.4}$$

$$= \frac{1}{e} \tag{7.3.5}$$

$$= 0.36787944117 \tag{7.3.6}$$

7.4. The time to failure, in months, of lights bulbs manufactured at two plants A and B obey the exponential distributions with means 6 and 2 months respectively. Plant B produces four times as many bulbs as plant A does. Bulbs from these two plants are indistinguishable. They are mixed and sold together. Given that a bulb purchased at random is working after 12 months, What is the probability that it was manufactured in plant A?

Solution:

This problem involves Bayes theorem and Exponential distribution

- Probability that bulb is from Plant A = $\Pr(A) = \frac{1}{5}$
- Probability that bulb is from Plant B = $\Pr(B) = \frac{4}{5}$

One can use exponential distribution to find out the probability that the bulbs work after 12 months

Let X be a variable representing the lifetime of a bulb in months.

So X has a Cumulative distribution Function:

$$F_X(x,\lambda) = \begin{cases} 1 - e^{-\lambda x} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{7.4.1}$$

Let us denote that the bulbs works after 12 months with the variable W.

$\frac{1}{\lambda}$	Mean of distribution
x	Time to failure (in months)
λ_A	$\frac{1}{6}$
λ_B	$\frac{1}{2}$
$\Pr(X \leq k)$	$F_X(X,\lambda)$

$$\Pr(W \mid A) = 1 - \Pr(\text{Fails within 12 months} \mid \text{A})$$

$$= 1 - F_X(12, \lambda_A) \tag{7.4.2}$$

$$= e^{-\lambda_A \times 12} \tag{7.4.3}$$

$$\Pr(W \mid B) = 1 - \Pr(\text{Fails within 12 months} \mid \text{B})$$

$$= 1 - F_X(12, \lambda_B) \tag{7.4.4}$$

$$= e^{-\lambda_B \times 12} \tag{7.4.5}$$

From Bayes theorem,

$$\Pr(A \mid W) = \frac{Pr(A) \times Pr(W \mid A)}{Pr(A) \times Pr(W \mid A) + Pr(B) \times Pr(W \mid B)} \tag{7.4.6}$$

$$= \frac{Pr(A) \times e^{-\lambda_A \times 12}}{Pr(A) \times e^{-\lambda_A \times 12} + Pr(B) \times e^{-\lambda_B \times 12}} \tag{7.4.7}$$

Substituting the known values, we get

$$\Pr(A \mid W) = \frac{\frac{1}{5} \times e^{-2}}{\frac{1}{5} \times e^{-2} + \frac{4}{5} \times e^{-6}} \tag{7.4.8}$$

$$= 0.93173845935 \tag{7.4.9}$$

So the probability that the Bulb is manufactured in Plant A given that it works after a year is 0.93173845935.

7.5. The lifetime of two brands of bulbs X and Y are exponentially distributed with the mean life of 100 hours. Bulb X is switched on 15 hours after bulb Y has been switched on. The probability that bulb X fails before bulb Y is

(A) $\frac{15}{100}$

(B) $\frac{1}{2}$

(C) $\frac{85}{100}$

(D) 0

Solution: Let X and Y be exponential random variables which represent the lifetime of bulbs X and Y respectively, both with mean = 100. Using memorylessness property for exponential distribution, which states that :
An exponentially distributed random variable T obeys the relation

$$\Pr(T > t + s | T > s) = \Pr(T > t) \tag{7.5.1}$$

where $s, t \geq 0$
Proof : Using Complementary cumulative distributive function, we get

$$\Pr(T > t + s | T > s) = \frac{\Pr(T > t + s, T > s)}{\Pr(T > s)} \tag{7.5.2}$$

$$= \frac{\Pr(T > t + s)}{\Pr(T > s)} \tag{7.5.3}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \tag{7.5.4}$$

$$= e^{-\lambda t} \tag{7.5.5}$$

$$= \Pr(T > t) \tag{7.5.6}$$

Probability that bulb X fails before bulb Y given that bulb Y was functioning when bulb X was switched on

$$\Pr(Y > X + 15 | Y \geq 15) = \Pr(Y > X) \tag{7.5.7}$$

For both X and Y,

$$\lambda = \frac{1}{100} = 0.01 \tag{7.5.8}$$

Probability distribution function of exponential random variables is given by : For x,y ≥ 0

$$f_{\mathrm{X}}(x) = \lambda e^{-\lambda x} \tag{7.5.9}$$

$$f_{\mathrm{Y}}(y) = \lambda e^{-\lambda y} \tag{7.5.10}$$

Cumulative distribution function of exponential random variables is given by : For x ≥ 0

$$F_{\mathrm{X}}(x) = 1 - e^{-\lambda x} \tag{7.5.11}$$

$$F_{\mathrm{Y}}(x) = 1 - e^{-\lambda x} \tag{7.5.12}$$

$$\Pr(Y > X) = \int_{-\infty}^{\infty} F_{\mathrm{Y}}(x) f_{\mathrm{X}}(x) dx \tag{7.5.13}$$

$$= \int_{0}^{\infty} (1 - e^{-\lambda x}) \lambda e^{-\lambda x} \tag{7.5.14}$$

$$= \lambda \left(\frac{1}{2\lambda} e^{-2\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right) \Bigg|_0^{\infty} \tag{7.5.15}$$

$$= \left(\frac{1}{2} e^{-2\lambda x} - e^{-\lambda x} \right) \Bigg|_0^{\infty} \tag{7.5.16}$$

$$= \left(\frac{1}{2} e^{-0.02x} - e^{-0.01x} \right) \Bigg|_0^{\infty} \tag{7.5.17}$$

$$= \frac{1}{2} = 0.5 \tag{7.5.18}$$

$\therefore$The answer is option (b) $\frac{1}{2}$.

7.6. Let a random variable X follow the exponential distribution with mean 2. Define Y such that:

$$Y = [X - 2 | X > 2]$$

Then $E(Y)$ is equal to:

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) 1

(D) 2

7.7. Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $\Pr(X_1 < X_2)$ is equal to ....

Solution:
We know that,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{\lambda x} & 0 \leq x < \infty \end{cases} \tag{7.7.1}$$

Given,

$$E(X_1) = \frac{1}{\lambda} = 1 \tag{7.7.2}$$
$$\implies \lambda = 1 \tag{7.7.3}$$

Therefore,

$$f_{X_1}(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & 0 \le x < \infty \end{cases} \tag{7.7.4}$$

We know that,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^{\alpha-1}e^{\left(\frac{-x}{\beta}\right)}}{\beta^{\alpha}\Gamma(\alpha)} & 0 \le x < \infty \end{cases} \tag{7.7.5}$$

Given,

$$E(X_2) = \alpha\beta = 2 \tag{7.7.6}$$
$$V(X_2) = \alpha\beta^2 = 2 \tag{7.7.7}$$

Solving 7.7.6 and 7.7.7, we get, $\alpha = 2$, $\beta = 1$ and $\Gamma(2) = 1$
Therefore,

$$f_{X_2}(x) = \begin{cases} 0 & x < 0 \\ xe^{-x} & 0 \le x < \infty \end{cases} \tag{7.7.8}$$

Calculating the CDF of $f_{X_2}(x)$,

$$F_{X_2}(x) = \int_0^x f_{X_2}(x) \tag{7.7.9}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(\alpha, \frac{x}{\beta})}{\Gamma(\alpha)} & 0 \le x < \infty \end{cases} \tag{7.7.10}$$

For $\alpha = 2$ and $\beta = 1$
Alternately, we have CDF of X_1 and X_2 given by

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \le x < \infty \end{cases} \tag{7.7.11}$$

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{\gamma(2, x)}{\Gamma(2)} & 0 \le x < \infty \end{cases} \tag{7.7.12}$$

Thus,

$$\Pr(X_1 \le X_2) = \int_{-\infty}^{\infty} F_{X_1}(x) f_{X_2}(x) dx \tag{7.7.13}$$

$$= \int_0^{\infty} (1 - e^{-x})(xe^{-x}) dx \tag{7.7.14}$$

$$= \frac{3}{4} \tag{7.7.15}$$

$$= 0.75 \tag{7.7.16}$$

7.8. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n ($n \ge 2$) from an exponential distribution with the probability density function

$$f_X(x, \theta) = \begin{cases} e^{-(x-2\theta)}, & x > 2\theta \\ 0, & otherwise \end{cases} \tag{7.8.1}$$

where $\theta \in (0, \infty)$. If $X_{(1)} = min\{X_1, X_2, \cdots, X_n\}$ then the conditional expectation

$$E\left[\frac{1}{\theta}\left(X_{(1)} - \frac{1}{n}\right)|X_1 - X_2 = 2\right] = \_\_\_\_\_\_$$

Solution: DEFINITIONS:

a) **Completeness:** The statistic T is said to be complete for the distribution of

X if, for every measurable function g if

$$E(g(T)) = 0 \implies P(g(T) = 0) = 1 \ \forall\theta \tag{7.8.2}$$

b) **Sufficiency:** Let $f(x,\theta)$ be the joint pdf of the sample X. A statistic T is sufficient for θ iff there are functions h (does not depend on θ)and g (depends on θ) on the range of T such that

$$f(x,\theta) = g(T(x),\theta)\, h(x) \tag{7.8.3}$$

c) **Basu's Theorem:** If T(X) is complete and sufficient, and S(X) is ancillary, then S(X) and T(X) are independent for all θ.
$\implies$ complete sufficient statistic is independent of any ancillary statistic.

Given PDF of the distribution as,

$$f_X(x,\theta) = \begin{cases} e^{-(x-2\theta)}, & x > 2\theta \\ 0, & otherwise \end{cases} \tag{7.8.4}$$

Then CDF of the distribution given is,

$$F(x,\theta) = \int_{-\infty}^{x} f_X(x,\theta)\, dx \tag{7.8.5}$$

Using (7.8.4) in (7.8.5),

$$F(x,\theta) = \begin{cases} 0, & x < 2\theta \\ 1 - e^{-(x-2\theta)}, & x > 2\theta \end{cases} \tag{7.8.6}$$

As given $X_{(1)} = min\{X_1, X_2, \cdots, X_n\}$,
Let us find CDF of $X_{(1)}$,

$$\begin{aligned} F_{X_{(1)}}(x,\theta) &= \Pr(X_{(1)} \le x) \\ &= \Pr(at\, least\, one\, of\, X_1, X_2, \cdots, X_n \le x) \\ &= 1 - \Pr(X_{(1)} > x) \\ &= 1 - \Pr(X_1 > x, X_2 > x, \cdots, X_n > 1) \\ &= 1 - \Pr(X_1 > x) \cdots \Pr(X_n > x) \\ &= 1 - (1 - F(x,\theta))^n \qquad (7.8.7) \end{aligned}$$

Using (7.8.6) in (7.8.7),

$$F_{X_{(1)}}(x,\theta) = \begin{cases} 0, & x < 2\theta \\ 1 - e^{-n(x-2\theta)}, & x > 2\theta \end{cases} \tag{7.8.8}$$

Using CDF of $X_{(1)}$ to find PDF of $X_{(1)}$,

$$f_{X_{(1)}}(x,\theta) = \frac{d}{dx} F_{X_{(1)}}(x,\theta) \tag{7.8.9}$$

Using (7.8.8) in (7.8.9), PDF of $X_{(1)}$ is

$$f_{X_{(1)}}(x,\theta) = \begin{cases} ne^{-n(x-2\theta)}, & x > 2\theta \\ 0, & otherwise \end{cases} \tag{7.8.10}$$

$X_{(1)}, \cdots, X_{(n)}$ are ordered statistics of $X_1, \cdots, X_n$. Where $X_{(k)}$ is kth order statistic of $X_1, \cdots, X_n$.

$$\implies \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{(i)} \tag{7.8.11}$$

Some results that we use in future:

a) Ordered statistics are complete and sufficient statistic of X.
Proof: Let $E[g(X_{(1)})] = 0$,

$$\implies \int_{-\infty}^{\infty} g(x) f_{X_{(1)}}(x) dx = 0 \tag{7.8.12}$$

$$\int_{2\theta}^{\infty} g(x) n e^{-n(x-2\theta)} dx = 0 \tag{7.8.13}$$

$$\int_{2\theta}^{\infty} g(x) e^{-n(x-2\theta)} dx = 0 \tag{7.8.14}$$

differentiating w.r.t θ on both sides in

(7.8.14),

$$\frac{d}{dx}\int_{2\theta}^{\infty} g(x)e^{-n(x-2\theta)}dx = 0$$

$$\frac{d}{dx}\left(\int_{2\theta}^{\infty} g(x)e^{-nx}dx\right)e^{2n\theta} = 0$$

$$2ne^{2n\theta}\int_{2\theta}^{\infty} g(x)e^{-nx}dx + e^{2n\theta}(2)g(2\theta)e^{-2n\theta} = 0$$

$$2n(0) + 2g(2\theta) = 0 \implies g(2\theta) = 0$$

$\implies X_{(1)}$ is complete statistics.
Using (7.8.11) in (7.8.17)

$$f_X(x,\theta) = f(x_1,\theta)f(x_2,\theta)\cdots f(x_n,\theta) \tag{7.8.15}$$

$$= e^{-(x_1-2\theta)}e^{-(x_2-2\theta)}\cdots e^{-(x_n-2\theta)} \tag{7.8.16}$$

$$= e^{-\left(\sum_{i=1}^{n} x_i-2n\theta\right)} = e^{-\left(\sum_{i=1}^{n} x_{(i)}-2n\theta\right)} \tag{7.8.17}$$

$$= \underbrace{\prod_{j=1}^{n} e^{-(x_{(j)}-2\theta)}}_{g} \times \underbrace{(1)}_{h} \tag{7.8.18}$$

∴ Ordered statistics of X are sufficient statistics for θ.

∴ $X_{(1)}$ is complete and sufficient statistics of θ.

b) $X_1 - X_2$ is ancillary of θ.

Proof: Let U=$X_1 - X_2$ then,

$$\begin{aligned}
F_U(x) &= \Pr(X_1 - X_2 < x)\\
&= \int_{-\infty}^{\infty} \Pr(X_1 < x+k)\Pr(X_2 > k)\,dk\\
&= \int_{2\theta}^{\infty} \left(1 - e^{-(x+k-2\theta)}\right)\left(e^{-(k-2\theta)}\right)dk\\
&= \int_{2\theta}^{\infty} e^{-(k-2\theta)} - e^{-(2k+x-2\theta)}dk\\
&= \left[\frac{e^{-(k-2\theta)}}{-1} - \frac{e^{-(2k+x-2\theta)}}{-2}\right]_{2\theta}^{\infty}\\
&= (0-0) - \left(-1 + \frac{e^{-x}}{2}\right)
\end{aligned}$$

$$F_U(x) = 1 - \frac{e^{-x}}{2} \tag{7.8.19}$$

$$\implies f_U(x) = \frac{d}{dx}F_U(x) \tag{7.8.20}$$

$$= \frac{e^{-x}}{2} \tag{7.8.21}$$

∴ $U = X_1 - X_2$ is an ancillary statistic of θ.

Let U be a random variable such that $U = X_1 - X_2$.

$$E\left[\frac{1}{\theta}\left(X_{(1)} - \frac{1}{n}\right)|X_1 - X_2 = 2\right] = E\left[\frac{1}{\theta}\left(X_{(1)} - \frac{1}{n}\right)|U = 2\right] \tag{7.8.22}$$

As $X_1, X_2, \cdots, X_n$ are independent and from Basu's theorem $X_{(1)}$ and U are also independent.
As we know that if X and Y are independent then $E[X|Y] = E[X]$. Using this in

(7.8.22)

$$E\left[\frac{1}{\theta}\left(X_{(1)}-\frac{1}{n}\right)|U=2\right] = E\left[\frac{1}{\theta}\left(X_{(1)}-\frac{1}{n}\right)\right]$$
$$= \frac{1}{\theta}\left(E\left[X_{(1)}\right]-\frac{1}{n}\right) \quad (7.8.23)$$

We have to find expectation of $X_{(1)}$,

$$E\left[X_{(1)}\right] = \int_{-\infty}^{\infty} x f_{X_{(1)}}(x,\theta)dx \quad (7.8.24)$$

Using (7.8.10) in (7.8.24).

$$E\left[X_{(1)}\right] = \int_{2\theta}^{\infty} nx\,e^{-(x-2\theta)n}dx$$
$$= e^{2n\theta}\int_{2\theta}^{\infty} nx\,e^{-nx}dx \quad (7.8.25)$$

Using integration by parts in (7.8.25),

$$E\left[X_{(1)}\right] = e^{2n\theta}\int_{2\theta}^{\infty} nx\,e^{-nx}dx$$
$$= e^{2n\theta}\left(\left[nx\frac{e^{-nx}}{-n}\right]_{2\theta}^{\infty} - \int_{2\theta}^{\infty} n\frac{e^{-nx}}{-n}dx\right)$$
$$= e^{2n\theta}\left(\left[nx\frac{e^{-nx}}{-n}\right]_{2\theta}^{\infty} + \left[\frac{e^{-nx}}{-n}\right]_{2\theta}^{\infty}\right)$$
$$= e^{2n\theta}\left(2\theta e^{-2n\theta} + \frac{e^{-2n\theta}}{n}\right)$$

$$E\left[X_{(1)}\right] = 2\theta + \frac{1}{n} \quad (7.8.26)$$

Use (7.8.26) in (7.8.23),

$$E\left[\frac{1}{\theta}\left(X_{(1)}-\frac{1}{n}\right)|U=2\right] = \frac{1}{\theta}\left(E\left[X_{(1)}\right]-\frac{1}{n}\right)$$
$$= \frac{1}{\theta}\left(2\theta+\frac{1}{n}-\frac{1}{n}\right)$$

$$E\left[\frac{1}{\theta}\left(X_{(1)}-\frac{1}{n}\right)|U=2\right] = 2 \quad (7.8.27)$$

Using (7.8.27) in (7.8.22),

$$\therefore E\left[\frac{1}{\theta}\left(X_{(1)}-\frac{1}{n}\right)|X_1-X_2=2\right] = 2$$

8 Gaussian Distribution

8.1. Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $P(3V \geqslant 2U)$ is

a) $\frac{4}{9}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{5}{9}$

8.2. Consider a binary digital communication system with equally likely 0's and 1's. When binary 0 is transmitted the voltage at the detector input can lie between the level -0.25V and +0.25V with equal probability: when binary 1 is transmitted, the voltage at the detector can have any value between 0 and 1V with equal probability. If the detector has a threshold of 2.0V (i.e., if the received signal is greater than 0.2V, the bit is taken as 1), the average bit error probability is

a) 0.15 b) 0.2 c) 0.05 d) 0.5

8.3. Let X be the Gaussian random variable obtained by sampling the process at $t = t_i$ and let

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

The probability that $[X \leqslant 1]$ is ...

8.4. Let X be a zero mean unit variance Gaussian random variable. $E[|X|]$ is equal to ....

Solution: Mean $= \mu = 0$

Variance $= \sigma = 1$

Gaussian Probability Distribution Function

$$= f(x) \quad (8.4.1)$$
$$= \frac{1}{\sqrt{2\pi\sigma}}\exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \quad (8.4.2)$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) \quad (8.4.3)$$

$$E[|X|] = \int_{-\infty}^{\infty} |x| f(x) \tag{8.4.4}$$
$$= 2\int_0^{\infty} x f(x) dx \tag{8.4.5}$$
$$= 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx \tag{8.4.6}$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \exp\left(\frac{-x^2}{2}\right) dx \tag{8.4.7}$$
$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (-1) \exp(u) du \tag{8.4.8}$$

(Using substitution)

$$= \sqrt{\frac{2}{\pi}} \tag{8.4.9}$$
$$= 0.7978 \tag{8.4.10}$$

8.5. Consider a discrete-time channel $Y = X + Z$, where the additive noise Z is signal-dependent. In particular, given the transmitted symbol $X \in \{-a, +a\}$ at any instant, the noise sample Z is chosen independently from a Gaussian distribution with mean βX and unit variance. Assume a threshold detector with zero threshold at the receiver.
When $\beta = 0$, the BER was found to be $Q(a) = 1 \times 10^-8$.

$$\left(Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^{\infty} e^{\frac{-u^2}{2}} du\right.$$

, and for $v > 1$, use $Q(v) \approx e^{\frac{-v^2}{2}}\Big)$
When $\beta = -0.3$, the BER is closet to

a) 10^-7
b) 10^-6
c) 10^-4
d) 10^-2

Solution:
Given that the threshold of the detector is zero. Define a detector function g such that

$$g(Y) = \begin{cases} +a & Y > 0 \\ -a & Y < 0 \end{cases} \tag{8.5.1}$$

It is given that $X \in \{-a, a\}$ is a random variable.

$$\therefore \Pr(X = a) = \Pr(X = -a) = \frac{1}{2} \tag{8.5.2}$$

Since the noise in the signal, Z is chosen independently from a Gaussian distribution with mean $\mu = \beta X$ and unit variance, it follows that

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-\beta X)^2}{2}\right) dz \tag{8.5.3}$$
$$= \int_{-\infty}^{z-\beta X} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{8.5.4}$$
$$= \int_{\beta X - z}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz \tag{8.5.5}$$
$$= Q(\beta X - z) \tag{8.5.6}$$

Also, it is easy to see that

$$Q(-v) = 1 - Q(v) \ \forall\ v \in \mathbb{R} \tag{8.5.7}$$

The detector can record erroneous bits in the signal iff

$$X > 0\ ,\ g(Y) = -a \text{ (Call this BER}_{+a}\text{) or} \tag{8.5.8}$$
$$X < 0\ ,\ g(Y) = a \text{ (Call this BER}_{-a}\text{)} \tag{8.5.9}$$

$$\therefore \text{BER}_{+a} = \Pr(g(Y) = -a \mid X = a) \Pr(X = a) \tag{8.5.10}$$
$$= \Pr(Y < 0 \mid X = a) \Pr(X = a) \tag{8.5.11}$$
$$= \frac{1}{2} \times \Pr(X + Z < 0 \mid X = a) \tag{8.5.12}$$
$$= \frac{1}{2} \times F_Z(-a) \tag{8.5.13}$$
$$= \frac{1}{2} \times Q(\beta X + a) \text{ (From (8.5.6))} \tag{8.5.14}$$
$$= \frac{1}{2} \times Q(a(1+\beta)) \tag{8.5.15}$$

$$\text{BER}_{-a} = \Pr(g(Y) = a \mid X = -a) \Pr(X = -a) \tag{8.5.16}$$
$$= \Pr(Y > 0 \mid X = -a) \Pr(X = -a) \tag{8.5.17}$$
$$= \frac{1}{2} \times \Pr(X + Z > 0 \mid X = -a) \tag{8.5.18}$$
$$= \frac{1}{2} \times (1 - F_Z(a)) \tag{8.5.19}$$
$$= \frac{1}{2} \times (1 - Q(\beta X - a)) \quad \text{(From (8.5.6))} \tag{8.5.20}$$
$$= \frac{1}{2} \times Q(a(1+\beta)) \quad \text{(From (8.5.7))} \tag{8.5.21}$$

$$\therefore \text{BER} = \text{BER}_{+a} + \text{BER}_{-a} \tag{8.5.22}$$
$$= Q(a(1+\beta)) \tag{8.5.23}$$

When $\beta = 0$, it is given that

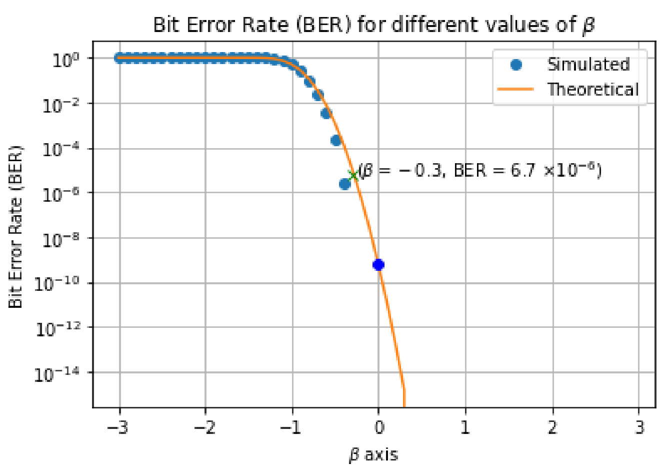


Fig. 8.5.1: Theory vs Simulated plot of BER

$$\text{BER} = Q(a) = 10^{-8} \tag{8.5.24}$$

On computing, $Q(1) \approx 0.16$. Since $Q(a) < Q(1)$, it is easy to see that $a > 1$ (as $Q(x)$ is a decreasing function)

$$\therefore e^{-a^2/2} = 10^{-8} \tag{8.5.25}$$
$$\Leftrightarrow a \approx 6.069 \tag{8.5.26}$$

When $\beta = -0.3$,

$$\text{BER} = Q(a(1+\beta)) = Q(6.069 \times (1 - 0.3)) \tag{8.5.27}$$
$$= Q(6.069 \times 0.7) \tag{8.5.28}$$
$$= Q(4.249) \tag{8.5.29}$$
$$\approx \exp(-\frac{4.249^2}{2}) \tag{8.5.30}$$
$$\approx 1.2 \times 10^{-4} \tag{8.5.31}$$

Therefore, when $\beta = -0.3$, BER is closest to 10^{-4} and option 8.5c is correct.

8.6. Suppose X and Y are two random variables such that $aX + bY$ is a normal random variable for all $a, b \in \mathbb{R}$. Consider the following statements P,Q,R and S:
(P): X is a standard normal random variable.
(Q): The conditional distribution of X given Y is normal.
(R): The conditional distribution of X given $X + Y$ is normal.
(S): X - Y has mean 0.

Which of the above statements ALWAYS hold TRUE?

a) both P and Q
b) both Q and R
c) both Q and S
d) both P and S

8.7. A random variable X takes values -1 and +1 with probabilities 0.2 and 0.8, respectively. It is transmitted across a channel which adds noise N, so that the random variable at the channel output is $Y = X+N$. The noise N is independent of X, and is uniformly distributed over the interval [-2 , 2]. The receiver makes a decision

$$\hat{X} = \begin{cases} -1, & \text{if} \quad Y \le \theta \\ +1, & \text{if} \quad Y \ge \theta \end{cases}$$

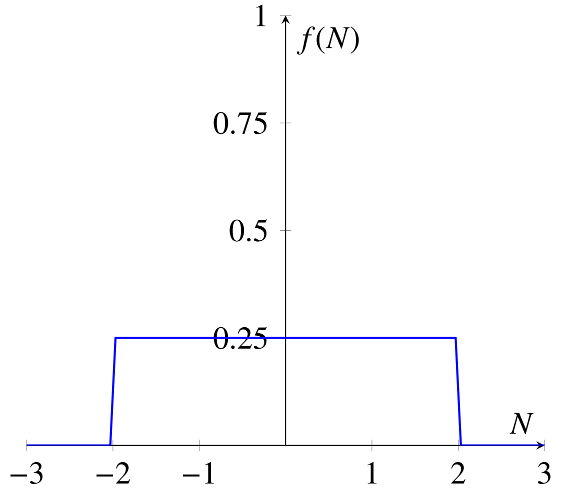
where the threshold $\theta \in [-1, 1]$ is chosen so as to minimize the probability of error $\Pr\left(\hat{X} \neq X\right)$. The minimum probability of error, rounded off to 1 decimal place, is ...

Solution:

We know that

$$X \in \{-1, +1\} \tag{8.7.1}$$
$$N \in [-2, 2] \tag{8.7.2}$$
$$Y = X + N \tag{8.7.3}$$
$$\Pr(X = -1) = 0.2 \tag{8.7.4}$$
$$\Pr(X = +1) = 0.8 \tag{8.7.5}$$

Since N is uniformly distributed
$\therefore$ the probability distribution function of N is:



The cdf of this uniform probability distribution function is

$$F_X(x) = \int_{-\infty}^{x} f(N)dN \tag{8.7.6}$$
$$= \int_{-2}^{x} \frac{1}{4}dN \tag{8.7.7}$$
$$= \frac{x+2}{4} \tag{8.7.8}$$

For $X \neq \hat{X}$ we need to check for each case. Using equation (8.7.8)

$$\therefore \Pr(N > \theta + 1) = 1 - \Pr(N < \theta + 1) \tag{8.7.9}$$
$$= 1 - F_X(\theta + 1) \tag{8.7.10}$$
$$= 1 - \frac{\theta + 3}{4} \tag{8.7.11}$$
$$= \frac{1}{4}(1 - \theta) \tag{8.7.12}$$

$$\therefore \Pr(N < \theta - 1) = F_X(\theta - 1) \tag{8.7.13}$$
$$= \frac{1}{4}(1 + \theta) \tag{8.7.14}$$

The probability of error:

$$\Pr\left(\hat{X} \neq X\right) = P(-1) \cdot P(\theta < -1 + N) + P(1) \cdot P(\theta > N + 1) \tag{8.7.15}$$

Substituting (8.7.12) and (8.7.14) in (8.7.15). We get:

$$\Pr\left(\hat{X} \neq X\right) = 0.2 \cdot \frac{1}{4}(1 - \theta) + 0.8 \cdot \frac{1}{4}(1 + \theta) \tag{8.7.16}$$

On simplifying the equation we get

$$\Pr\left(\hat{X} \neq X\right) = \frac{1}{4} + \frac{3}{20}\theta \tag{8.7.17}$$

Since this is a linear equation in θ, the minimum will occur at boundary points. Putting $\theta = +1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.4 \tag{8.7.18}$$

but on putting $\theta = -1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.1 \tag{8.7.19}$$

Hence the value of probability of error is:

$$\therefore \Pr\left(\hat{X} \neq X\right) = 0.1 \tag{8.7.20}$$

8.8. Let U and V be two independent zero mean Gaussian random variables of variances $\frac{1}{4}$ and $\frac{1}{9}$ respectively. The probability $\Pr(3V \geq 2U)$ is
a) 4/9
b) 1/2
c) 2/3
d) 5/9

Solution:
U and V are independent random variables, For V, $\mu_V = 0$, and $\sigma_V^2 = \frac{1}{9}$

$$V \sim N\left(0, \frac{1}{9}\right) \tag{8.8.1}$$

For U, $\mu_U = 0$, and $\sigma_U^2 = \frac{1}{4}$

$$U \sim N\left(0, \frac{1}{4}\right) \tag{8.8.2}$$

Let,

$$Z = 3V - 2U \tag{8.8.3}$$

$$Z \sim N\left((3\mu_V - 2\mu_U), \left((3)^2\sigma_U^2 + (2)^2\sigma_V^2\right)\right) \tag{8.8.4}$$

$$Z \sim N\left(0, 9 \times \frac{1}{9} + 4 \times \frac{1}{4}\right) \tag{8.8.5}$$

$$Z \sim N(0, 2) \tag{8.8.6}$$

For Z, $\mu = 0$, and $\sigma^2 = 2$. By Gaussian Distribution, Let X is standard normal variable,

$$X = \frac{Z - \mu}{\sigma} \tag{8.8.7}$$

$$\Pr(Z \geq 0) = \Pr(X\sigma + \mu \geq 0) \tag{8.8.8}$$

$$\Pr(Z \geq 0) = \Pr\left(X\left(\sqrt{2}\right) + 0 \geq 0\right) \tag{8.8.9}$$

$$\Pr(Z \geq 0) = \Pr(X \geq 0) \tag{8.8.10}$$

$$\Pr(Z \geq 0) = Q(0) \tag{8.8.11}$$

where $Q(x)$ is the $Q - function$,

$$Q(0) = \frac{1}{2} \tag{8.8.12}$$

$$\Pr(Z \geq 0) = \frac{1}{2} \tag{8.8.13}$$

$$\Pr(3V - 2U \geq 0) = \frac{1}{2} \tag{8.8.14}$$

$$\Pr(3V \geq 2U) = \frac{1}{2} \tag{8.8.15}$$

Option (B) is correct.

8.9. Let X be a standard normal random variable. Then $\Pr(X < 0 | \, \|\lfloor X \rfloor\| = 1)$ is equal to

a) $\dfrac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}}$

b) $\dfrac{\Phi(1) + \frac{1}{2}}{\Phi(2) + \frac{1}{2}}$

c) $\dfrac{\Phi(1) - \frac{1}{2}}{\Phi(2) + \frac{1}{2}}$

d) $\dfrac{\Phi(1) + 1}{\Phi(2) + 1}$

Solution:

$$\|\lfloor X \rfloor\| = 1 \tag{8.9.1}$$

$$\implies \lfloor X \rfloor = 1 \; or \; -1 \tag{8.9.2}$$

$$\implies X \in [1, 2) \cup [-1, 0) \tag{8.9.3}$$

Here

$\lfloor X \rfloor = greatest\ integer\ less\ than\ or\ equal\ to\ X$

Thus required probability

$$= \frac{\Pr(X \in [-1, 0))}{\Pr(X \in [1, 2) \cup [-1, 0))} \tag{8.9.4}$$

Using symmetry of standard normal random variable about y = 0, we have required probability

$$= \frac{\Pr(X \in (0, 1])}{\Pr(X \in [1, 2) \cup (0, 1]))} \tag{8.9.5}$$

$$= \frac{\Pr(X \in (0, 1])}{\Pr(X \in (0, 2))} \tag{8.9.6}$$

$$= \frac{\Pr(X < 1) - \Pr(X < 0)}{\Pr(X < 2) - \Pr(X < 0)} \tag{8.9.7}$$

$$= \frac{\Phi(1) - \Phi(0)}{\Phi(2) - \Phi(0)} \tag{8.9.8}$$

$$= \frac{\Phi(1) - \frac{1}{2}}{\Phi(2) - \frac{1}{2}} \tag{8.9.9}$$

$$= \frac{0.841 - 0.5}{0.977 - 0.5} \tag{8.9.10}$$

$$= 0.715 \tag{8.9.11}$$

Here $\Phi(x)$ represents the standard normal cumulative density function. Thus

$$X \sim \prime 1 \tag{8.9.12}$$

and

$$\Phi(x) = \int_{-\infty}^{x} f_X(x)dx \tag{8.9.13}$$

It can easily be seen that $\Phi(0) = \frac{1}{2}$, which has been used to obtain (8.9.9). (8.9.10) was obtained by consulting tables for $\Phi(x)$

8.10. Let (X, Y) have a bivariate normal distribution with the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{\pi} e^{\left(\frac{3}{2}xy - \frac{25}{32}x^2 - 2y^2\right)} \tag{8.10.1}$$

$$-\infty < x, y < \infty \tag{8.10.2}$$

Then $E(XY)$ equals

Solution:

Given probability density function for (X, Y)

$$f_{X,Y}(x,y) = \frac{1}{\pi}e^{\left(\frac{3}{2}xy - \frac{25}{32}x^2 - 2y^2\right)} \tag{8.10.3}$$

$$-\infty < x, y < \infty \tag{8.10.4}$$

Joint pdf of bivariate normal distribution $N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} \times e^{\frac{-1}{2(1-p^2)}\left[\left[\frac{(x-\mu_x)}{\sigma_x}\right]^2 + \left[\frac{(y-\mu_y)}{\sigma_y}\right]^2 - 2\rho\left[\frac{(x-\mu_x)}{\sigma_x}\frac{(y-\mu_y)}{\sigma_y}\right]\right]} \tag{8.10.5}$$

Comparing (8.10.5) and (8.10.3) we get We

μ_x	μ_y	σ_x	σ_y	ρ
0	0	1	$\frac{5}{8}$	$\frac{3}{5}$

TABLE 8.10.1: Table 1

need to find $E(XY)$

$$E(XY) = \rho\sigma_x\sigma_y + \mu_x\mu_y \tag{8.10.6}$$

Substituting values in table(8.10.1) in (8.10.6) we get

$$E(XY) = \frac{3}{8} \tag{8.10.7}$$

$$\therefore 8E(XY) = 3 \tag{8.10.8}$$

8.11. Let $X_1, X_2...$ be a sequence of independent and identically distributed random variable with

$$\Pr(X_1 = -1) = \Pr(X_1 = 1) = 1/2 \tag{8.11.1}$$

Suppose for the standard normal random variable Z,

$$\Pr(-0.1 \leq Z \leq 0.1) = 0.08. \tag{8.11.2}$$

$$\text{If } S_n = \sum_{i=1}^{n^2} X_i, \text{then} \lim_{n\to\infty} \Pr\left(S_n > \frac{n}{10}\right) =$$

a) 0.42
b) 0.46
c) 0.50
d) 0.54

Solution:

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{2}, & \text{if n= 1 or n=-1} \\ 0, & \text{otherwise} \end{cases}$$

$$\implies \mu = E(X_i) = 1/2(1-1) = 0 \tag{8.11.3}$$

$$\implies \sigma^2 = E(X_i^2) - \mu^2 = \frac{1}{2}(1+1) - 0 = 1 \tag{8.11.4}$$

Using Central Limit Theorem,we can say that for a series of random and identical variables X_i with the Mean = μ and variance = σ^2 where i $\in$ 1,2...n

$$\text{Let } \overline{X_n} \equiv \frac{\sum_{i=1}^{n} X_i}{n} \tag{8.11.5}$$

$$\text{Then} \lim_{n\to\infty} \sqrt{n}(\overline{X_n} - \mu) = N(0, \sigma^2) \tag{8.11.6}$$

$$\implies \lim_{n\to\infty} \frac{S_n}{n} = N(0,1) \tag{8.11.7}$$

$$\implies S_n = nN(0,1) \tag{8.11.8}$$

$$\implies \lim_{n\to\infty} \Pr\left(nN(0,1) > \frac{n}{10}\right) \tag{8.11.9}$$

$$\implies \lim_{n\to\infty} \Pr\left(N(0,1) > \frac{1}{10}\right) = Q(0.1) \tag{8.11.10}$$

Now using (8.11.2)

$$\implies Q(0.1) + (1 - Q(-0.1)) + 0.08 = 1 \tag{8.11.11}$$

Now as $N(0,1)$symmetric about 0

$$\implies 2 \times Q(0.1) + 0.08 = 1 \tag{8.11.12}$$

$$\implies Q(0.1) = 0.46 \tag{8.11.13}$$

$$\implies \lim_{n\to\infty} \Pr\left(S_n > \frac{n}{10}\right) = 0.46 \tag{8.11.14}$$

Hence final solution is option 2) or 0.46

8.12. Robot Ltd. wishes to maintain enough safety stock during the lead time period between starting a new production run and its completion such that the probability of satisfying the customer demand during the lead time period is 95%. The lead time periods is 5 days and daily customer demand can be assumed to follow the Gaussian (normal) distribution with mean 50 units and a standard deviation of 10 units. Using $\phi^{-1}(0.95)$ =1.64 , where ϕ represents the cumulative distribution function of the standard normal random variable, the amount of safety stock that must be maintained by Robot Ltd. to achieve this demand fulfillment probability for the lead time period is ____ units (round off to two decimal places). **Solution:**

Symbol	definition	value
X	customer demand in lead time	-
X_1	normal R.V denotes daily customer demand	-
μ	Mean of X_1	50
σ	Standard deviation of X_1	10
ϕ	CDF of standard normal R.V	-

TABLE 8.12.1: Variables and their definitions

Probability of satisfying customer demand is 0.95. Let Z be a standard normal R.V such that,

$$Z = \frac{X_1 - \mu}{\sigma} \tag{8.12.1}$$

Referring table(8.12.1) to use in (8.12.1),

$$Z = \frac{X_1 - 50}{10} \tag{8.12.2}$$

Given that,

$$\phi^{-1}(0.95) = 1.64 \tag{8.12.3}$$

$$\implies \phi(1.64) = 0.95 \tag{8.12.4}$$

$$\phi(1.64) = \Pr(Z \le 1.64) = 0.95 \tag{8.12.5}$$

$$\implies Z \le 1.64 \iff \frac{X_1 - 50}{10} \le 1.64 \tag{8.12.6}$$

$$\implies X_1 - 50 \le 1.64(10) \tag{8.12.7}$$

$$\therefore X_1 \le 66.4 \tag{8.12.8}$$

The demand in one day is independent of demand in the other day and the lead time is 5 days.

$$\implies X = 5(X_1) = 5(66.4) = 332 \tag{8.12.9}$$

Therefore the amount of safety stock that must be maintained by Robot Ltd. to achieve this demand fulfillment probability for the lead time period is 332 units.

9 Poisson Distribution

9.1. If calls arrive at a telephone exchange such that the time of arrival of any call is independent of the time of arrival of earlier or future calls, the probability distribution function of the toatl number of calls in a fixed time interval will be

a) Poisson

b) Gaussian

c) Exponential

d) Gamma

Solution:

Symbol	Description	Property	Random
T	Total time period	$T = n\Delta t$	No
n	Total Number of intervals		No
Δt	One time interval	Δt=T/n	No
k	Number of calls arrived during the time interval (0,T)		Yes
t_i	Denotes the time of arrival of each call in interval $(0, T)$		Yes
p	Probability of receiving a call at time t_i		No
λ	Average number of calls $(0, T)$	$\lambda = np$	No
e	Euler's number		No

Lets denote the fixed time interval by [0,T]. To find the probability of k number of calls during this time interval, lets divide the interval into n parts of equal length Δt. Let us denote the probability of receiving a call at a particular time t_i by p. Suppose the telephone exchange receives an average of λ calls in time interval of length T.
Hence, we have

$$np = \lambda \tag{9.1.1}$$

$$\implies p = \frac{\lambda}{n} \tag{9.1.2}$$

In Fig. 9.1.1, the interval (0,T) has been

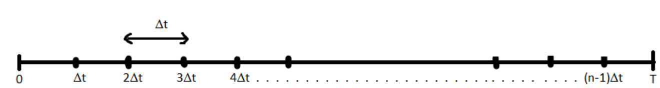


Fig. 9.1.1: Figure showing division of time intervals

divided into n equal parts, where length of each interval is Δt and the number of calls in each interval is a random variable. t_i where

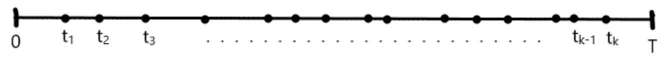


Fig. 9.1.2: Figure showing times of arrival of k calls

$i = \{1, 2, 3 \cdots k\}$ are the time of arrival of k calls in the interval $(0, T)$.

A call has probability p for arriving at $t_i, \forall i = \{1, 2, \cdots k\}$ and the probability of 1-p for not arriving at that instant.

In Binomial distribution we have certain number of intervals, i.e. n, with probability of arrival of each call as p and for a binomial random variable $X = \{0, 1 \cdots n\}$, the probability of call arriving in any k intervals is

$$\Pr(X = k) = {}^nC_k \cdot p^k \cdot (1 - p)^k \tag{9.1.3}$$

But in Poisson distribution, we essentially have infinite intervals, so $n \to \infty$. Thus, the probability expression changes to:

$$\lim_{n\to\infty} \Pr(X = k) = \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k} \tag{9.1.4}$$

$$\lim_{n\to\infty} \Pr(X = k) = \left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty} \frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^n\left(1 - \frac{\lambda}{n}\right)^{-k} \tag{9.1.5}$$

Now lets take the limit of right-hand side one term at a time. We'll do this in three steps. The first step is to find the limit of

$$\begin{aligned}\lim_{n\to\infty} \frac{n!}{(n-k)!n^k} &= \lim_{n\to\infty} \frac{n(n-1)(n-2)..(n-k+1)}{n^k}\\ &= \lim_{n\to\infty}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)....\left(\frac{n-k+1}{n}\right)\\ &= \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)...\left(1 - \frac{k-1}{n}\right)\\ &= 1 \cdot 1 \cdot 1........1\\ &= 1\end{aligned} \tag{9.1.6}$$

Now we have to find the limit of

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n \tag{9.1.7}$$

We know that the definition e is given as

$$e = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x \tag{9.1.8}$$

So, lets replace the value of $-\frac{n}{\lambda}$ by x in (9.1.7), we get

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda} \tag{9.1.9}$$

And the third part is to find the limit of

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{-k} \tag{9.1.10}$$

As n approaches infinity, this term becomes 1^{-k} which is equal to one. So,

$$\lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^{-k} = 1 \tag{9.1.11}$$

Now on substituting (9.1.6), (9.1.9) and (9.1.11) in equation (9.1.5), we get

$$\left(\frac{\lambda^k}{k!}\right)\lim_{n\to\infty}\frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^n\left(1 - \frac{\lambda}{n}\right)^{-k} = \left(\frac{\lambda^k}{k!}\right)(1)\left(e^{-\lambda}\right)(1) \tag{9.1.12}$$

This just simplifies into

$$\Pr(X = k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{9.1.13}$$

(9.1.13) is equal to probability density function of Poisson distribution, which gives us probability of k successes per period, with given parameter of λ.

$\therefore$The probability distribution function of the total number of calls in a fixed time interval will be **Poisson** distribution.
Answer: Option(A)

9.2. Let X be the Poisson random variable with parameter $\lambda = 1$. Then, the probability $\Pr(2 \leq X \leq 4)$ equals ....
Solution:
Let

$$X \in \{0, 1, 2, 3, 4, 5...\} \tag{9.2.1}$$

We know that, for a poisson random variable X with a given parameter λ, probability of $X = k$ is:

$$\Pr(X = k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{9.2.2}$$

CDF is:

$$F(X = k) = \sum_{x=0}^{k} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \tag{9.2.3}$$

And also,

$$\Pr(x < X \leq y) = F(y) - F(x) \tag{9.2.4}$$

Now by using (9.4.4),

$$\Pr(2 \leq X \leq 4) = \Pr(1 < X \leq 4) \tag{9.2.5}$$
$$= F(4) - F(1) \tag{9.2.6}$$
$$= \frac{65}{24e} - \frac{2}{e} \tag{9.2.7}$$
$$= \frac{17}{24e} \tag{9.2.8}$$

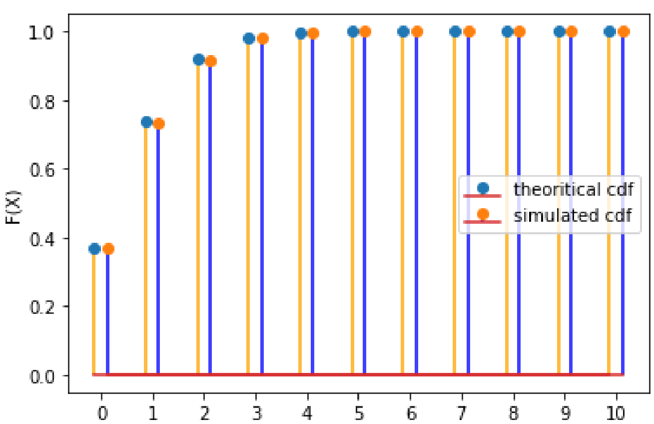


Fig. 9.2.1: Theoretical CDF vs Simulated CDF

9.3. Suppose p is the number of cars per minute passing through a certain road junction between 5 PM and 6 PM, and p has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

a) $8/(2e^3)$
b) $9/(2e^3)$
c) $17/(2e^3)$
d) $26/(2e^3)$

Solution:

Probability of Poison Distribution is,

$$\Pr(X = p) = \frac{e^{-\mu}\mu^p}{p!} \tag{9.3.1}$$

Here, p refers to no. of cars per minute, $p \in \{0, 1, 2, \ldots, \infty\}$ Mean of poison distribution,

$$\mu = 3 \tag{9.3.2}$$

$$\Pr(X = p) = \frac{e^{-3}3^p}{p!} \tag{9.3.3}$$

by Boolean logic,

TABLE 9.3.1: Table of probability of no. of cars passing per minute

p	0	1	2	3	...
$\Pr(X = p)$	$1/e^3$	$3/e^3$	$9/(2e^3)$	$9/(2e^3)$	...

$$\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \tag{9.3.4}$$

$$\Pr(X < 3) = \frac{17}{2e^3} \tag{9.3.5}$$

Option (C) is correct

9.4. Let X be the Poisson random variable with parameter $\lambda = 1$. Then, the probability $\Pr(2 \leq X \leq 4)$ equals .... **Solution:** Let

$$X \in \{0, 1, 2, 3, 4, 5...\} \tag{9.4.1}$$

We know that, for a poisson random variable X with a given parameter λ, probability of $X = k$ is:

$$\Pr(X = k) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right) \tag{9.4.2}$$

CDF is:

$$F(X = k) = \sum_{x=0}^{k} \left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \tag{9.4.3}$$

And also,

$$\Pr(x < X \leq y) = F(y) - F(x) \tag{9.4.4}$$

Now by using (9.4.4),

$$\Pr(2 \le X \le 4) = \Pr(1 < X \le 4) \quad (9.4.5)$$
$$= F(4) - F(1) \quad (9.4.6)$$
$$= \frac{65}{24e} - \frac{2}{e} \quad (9.4.7)$$
$$= \frac{17}{24e} \quad (9.4.8)$$

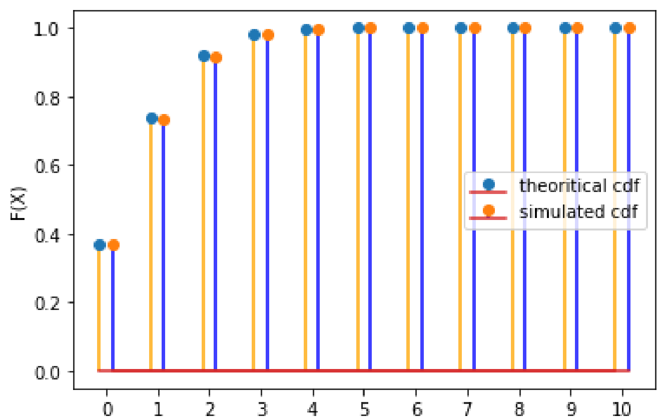


Fig. 9.4.1: Theoretical CDF vs Simulated CDF

9.5. Customers arrive at a shop according to Poisson distribution with a mean of 10 customers/hour. The manager notes that no customer arrives for the first 3 minutes after the shop opens. The probability that a customer arrives within the next 3 minutes is
Solution: Given, mean of 10 customers arrive in a time interval of 60 minutes $\iff$ mean of $\frac{t}{6}$ customers arrive in a time interval of t minutes, Customers arrive according to Poisson distribution with a mean of $\frac{t}{6}$ customers/t minutes,

$$\therefore \lambda = \frac{t}{6} \quad (9.5.1)$$

Let X denotes the number of customers in first t minutes,Y denotes the number of customers in second t minutes. according to poisson distribution,

$$\Pr(X = x) = e^{-\lambda}\frac{\lambda^x}{x!} \quad (9.5.2)$$

using (9.5.1) in (9.5.2),

$$\Pr(X = x) = e^{-\frac{t}{6}}\frac{(\frac{t}{6})^x}{x!} \quad (9.5.3)$$

the probability that a customer arrives within the next t minutes given that no customer arrives for the first t minutes after the shop opens,which can also be written as,

$$\Pr(Y \ne 0|X = 0) = \frac{\Pr(Y \ne 0, X = 0)}{\Pr(X = 0)} \quad (9.5.4)$$

As the arrival of customers in second t minutes

TABLE 9.5.1: Probability distribution for values of X and Y

	P(X)	P(Y)
0	$e^{-\frac{t}{6}}$	$e^{-\frac{t}{6}}$
1	$\frac{te^{-\frac{t}{6}}}{6}$	$\frac{te^{-\frac{t}{6}}}{6}$

does not depend on the arrival of customers in first t minutes, X and Y are independent,

$$\Pr(Y \ne 0|X = 0) = \frac{\Pr(Y \ne 0)\Pr(X = 0)}{\Pr(X = 0)} \quad (9.5.5)$$
$$= \Pr(Y \ne 0) \quad (9.5.6)$$
$$= 1 - \Pr(Y = 0) \quad (9.5.7)$$

using (9.5.3),

$$\Pr(Y \ne 0|X = 0) = 1 - e^{-\frac{t}{6}} \quad (9.5.8)$$

we need to find the probability for t=3,the required probability is given by,

$$= 1 - e^{-\frac{1}{2}} \quad (9.5.9)$$
$$= 0.3935 \quad (9.5.10)$$

9.6. Consider a single machine workstation to which jobs arrive according to a Poisson distribution with a mean arrival rate of 12 jobs/hour. The process time of the workstation is exponentially distributed with a mean of 4 minutes. The expected number of jobs at the workstation at any given point of time is . . . (*round off to the nearest integer*). **Solution:** In a Poisson process,

$$\Pr(X = x) = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^x}{x!} \quad (9.6.1)$$

If $\Delta t \to 0$ then probability of having only one Poisson job is

$$\Pr(X = 1) = \lambda\Delta t \quad (9.6.2)$$

Some assumptions:
In time interval Δt,

- Exactly one job is arrived
- or Exactly one job is completed
- or Nothing happens

Assumptions seem quite reasonable as Δt is very small then the probability of occurrence of more than one poisson job is very low.
For job arrival,

- It is distributed according to Poisson distribution.
- Its Rate parameter λ=12 jobs/hour.
- Using (9.6.2),Probability that a single job arrives in a small interval $\Delta t = \lambda \Delta t$.

For Job completions,

- Job completion time is distributed exponentially with mean of 4 minutes
- Then we can assume that no. of job completions are distributed as Poisson distribution with rate parameter $\mu = 15$ jobs/hour
- Once again using (9.6.2), Probability that a single job will be completed in a small interval $\Delta t = \mu \Delta t$

Some notations,

Parameter	**Definition**
λ	Poisson rate parameter for the arrival of jobs
μ	Poisson rate parameter for the completion of jobs
$\lambda\Delta t$	Probability that a single job arrives in a small interval Δt
$\mu\Delta t$	Probability that a single job will be completed in a small interval Δt
$P_j(t)$	probability of having j jobs at workstation at time t
π_j	steady probability of having j jobs at workstation

TABLE 9.6.1: Parameters and their definitions used in the problem

- Initial no.of jobs at workstation is 0.
- Let $P_j(t)$ denote the probability of having j jobs waiting at the workstation at the time t for this initial case.
- After a long time,probability of having j jobs becomes steady.
- Let us denote steady state probability of having j jobs as π_j.

Condition which ensures that steady state is reached is

$$\frac{dP_j(t)}{dt} = 0 \tag{9.6.3}$$

$$\lim_{\Delta t \to 0} \frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} = 0 \tag{9.6.4}$$

We can reach a state of j jobs at time $t + \Delta t$ from

- A state of $j-1$ jobs at time t with a new job arriving in the next Δt
- A state of $j+1$ jobs at time t with a job completing in the next Δt
- A state of j jobs at time t and nothing happening in the next Δt

Assuming time t is long enough for the occurrence of steady state.The above relations can be shown in probability equations as:

$$\begin{aligned} P_j(t+\Delta t) = P_{j-1}(t)\lambda\Delta t + P_{j+1}(t)\mu\Delta t \\ + P_j(t)(1 - \lambda\Delta t - \mu\Delta t) \end{aligned} \tag{9.6.5}$$

$$\begin{aligned} \frac{P_j(t+\Delta t) - P_j(t)}{\Delta t} &= P_{j-1}(t)\lambda + P_{j+1}(t)\mu \\ &- P_j(t)\lambda - P_j(t)\mu \end{aligned} \tag{9.6.6}$$

Using (9.6.4) we get,

$$\implies P_{j-1}(t)\lambda + P_{j+1}(t)\mu = P_j(t)\lambda + P_j(t)\mu \tag{9.6.7}$$

$$\pi_{j-1}\lambda + \pi_{j+1}\mu = \pi_j\lambda + \pi_j\mu \tag{9.6.8}$$

Note that the above equations are for $j \geq 1$.
For j=0 jobs at time $t+\Delta t$ we can reach it from j=1 job at time t with a job completion in the next Δt or else stay at j=0 at time t and do nothing the next Δt

$$\begin{aligned} P_0(t+\Delta t) = P_1(t)\mu\Delta t + \\ P_0(t)(1 - \lambda\Delta t) \end{aligned} \tag{9.6.9}$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = P_1(t)\mu\Delta t - P_0(t)\lambda\Delta t \tag{9.6.10}$$

Once again using (9.6.4),we will get,

$$P_0(t)\lambda\Delta t = P_1(t)\mu\Delta t \tag{9.6.11}$$

$$P_0(t)\lambda = P_1(t)\mu \tag{9.6.12}$$

$$\pi_0\lambda = \pi_1\mu \tag{9.6.13}$$

Solving (9.6.13) and (9.6.8) with appropriate j one by one,we will get P_j in terms of P_0 as

$$P_j = \left(\frac{\lambda}{\mu}\right)^j P_0 \tag{9.6.14}$$

consider $\rho = \frac{\lambda}{\mu}$.

Parameter	Definition
$E(j)$	Expected no. of jobs at workstation
ρ	$\frac{\lambda}{\mu}$

TABLE 9.6.2: Parameters and their definitions used in the problem

$$P_j = \rho^j P_0 \tag{9.6.15}$$

We can prove that (9.6.15) is indeed the solution of recursion equation (9.6.8) by using mathematical induction.
Assuming $\rho < 1$,let us calculate P_0 in terms of ρ

$$\sum_{j=0}^{\infty} P_j = 1 \tag{9.6.16}$$

$$\sum_{j=0}^{\infty} \rho^j P_0 = 1 \tag{9.6.17}$$

$$\frac{P_0}{1-\rho} = 1 \tag{9.6.18}$$

$$P_0 = 1 - \rho \tag{9.6.19}$$

This yields,

$$P_j = \rho^j (1 - \rho) \tag{9.6.20}$$

Let us calculate expected value of jobs waiting at workstation.

$$E(j) = \sum_{j=0}^{\infty} j P_j \tag{9.6.21}$$

$$E(j) = (1-\rho) \sum_{j=0}^{\infty} j \rho^j \tag{9.6.22}$$

$$\rho E(j) = (1-\rho) \sum_{j=0}^{\infty} j \rho^{j+1} \tag{9.6.23}$$

$$\rho E(j) = (1-\rho) \sum_{j=1}^{\infty} (j-1) \rho^j \tag{9.6.24}$$

Subtracting (9.6.24) from (9.6.22).we get,

$$(1-\rho)E(j) = (1-\rho) \sum_{j=1}^{\infty} \rho^j \tag{9.6.25}$$

$$E(j) = \sum_{j=1}^{\infty} \rho^j \tag{9.6.26}$$

$$E(j) = \frac{\rho}{1-\rho} \tag{9.6.27}$$

In our case $\rho = \frac{\lambda}{\mu} = \frac{12}{15} = \frac{4}{5}$. Substituting it in the (9.6.27) we get,

$$E(j) = 4 \tag{9.6.28}$$

$\therefore$ Expected no.of jobs at workstation is 4.

9.7. Cars arrive at a service station according to Poisson's distribution with a mean rate of 5 per hour.The service time per car is exponential with a mean of 10 minutes.At a steady state,the average waiting time in the queue is **Solution:** This problem can be solved using Queuing theory.But first we have to understand queuing theory.

- In queuing theory we try to determine what happens when people join in queue.
- **Parameters for measuring Queuing performance**
  a) λ = Average arrival time
  b) μ = Average service time
  c) ρ = Utilization factor
  d) L_q = Average number in the queue
  e) L = Average number in the system
  f) W_q = Average waiting time
  g) W = Average time in the system
  h) P_n = Steady state probability of exactly n customers in the system
- Typically most of the times arrivals follow poisson distribution and services follow exponential distributions.
- The given question only has one queue so we can conclude that it is a single server model and there is no limit for number of cars in the queue so we can say that it is "**M/M/1:/∞/∞/FIFO**" by kendall's notation (or) usually "**M/M/1**"
- Here '**M**' indicates the memory less property of the model first **M** is for arrival and second one for service and 1 is the number of

servers in the model and '∞' indicates the limit of the queue and second '∞' represent population and '**FIFO**' represents First-In First-Out service.

- **NOTE:** In cases where there is no limit in the queue we only take the cases where $\frac{\lambda}{\mu} < 1$. Otherwise there could be customers who will not get their service.
  The memory less property allows us to assume that one event can take place in a small interval of time. The event could be either a arrival or a service.
- **Deriving formulas :** For the time interval$(t, t+h)$, where $h \to 0$

$$\Pr(1 \text{ arrival}) = \lambda h \tag{9.7.1}$$
$$\Pr(1 \text{ service}) = \mu h \tag{9.7.2}$$
$$\Pr(\text{no arrival}) = 1 - \lambda h \tag{9.7.3}$$
$$\Pr(\text{no service}) = 1 - \mu h \tag{9.7.4}$$

$$\begin{aligned} P_n(t+h) &= P_{n-1}(t) \times \Pr(1 \text{ arrival}) \times \Pr(\text{no service}) \\ &+ P_{n+1}(t) \times \Pr(\text{no arrival}) \times \Pr(1 \text{ service}) \\ &+ P_n(t) \times \Pr(\text{no arrival}) \times \Pr(\text{no service}) \end{aligned} \tag{9.7.5}$$

$$\begin{aligned} \implies P_n(t+h) &= P_{n-1}(t)(\lambda h)(1-\mu h) \\ &+ P_{n+1}(t)(\mu h)(1-\lambda h) \\ &+ P_n(t)(1-\lambda h)(1-\mu h) \end{aligned} \tag{9.7.6}$$

Now, Neglecting higher order terms of h.

$$\begin{aligned} \implies P_n(t+h) &= P_{n-1}\lambda h + P_{n+1}\mu h \\ &+ P_n(t)(1-\lambda h-\mu h) \end{aligned} \tag{9.7.7}$$

$$\begin{aligned} \implies \frac{P_n(t+h)-P_n(t)}{h} &= P_{n-1}(t)\lambda + P_{n+1}(t)\mu \\ &- P_n(t)(\lambda+\mu) \end{aligned} \tag{9.7.8}$$

At steady state, $P_n(t+h) = P_n(t)$

$$\implies \lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu)P_n \tag{9.7.9}$$

Now, calculating $P_0(t+h)$ using (9.7.5)

$$\begin{aligned} P_0(t+h) &= P_1(t)(1-\lambda h)(\mu h) \\ &+ P_0(t)(1-\lambda h) \end{aligned} \tag{9.7.10}$$

Again, Neglecting higher order terms of h

$$\begin{aligned} \implies P_0(t+h) &= P_1(t)(\mu h) \\ &+ P_0(t)(1-\lambda h) \end{aligned} \tag{9.7.11}$$

$$\implies \frac{P_0(t+h)-P_0(t)}{h} = P_1(\mu) - P_0(\lambda) \tag{9.7.12}$$

At steady state, $P_0(t+h) = P_0(t)$

$$\implies \mu P_1 = \lambda P_0 \tag{9.7.13}$$
$$\implies P_1 = \left(\frac{\lambda}{\mu}\right) P_0 \tag{9.7.14}$$

Using (9.7.9) by substituting $n = 1$

$$\lambda P_0 + \mu P_2 = (\lambda+\mu)P_1 \tag{9.7.15}$$
$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1 \tag{9.7.16}$$

from (9.7.13) and (9.7.14)

$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \lambda P_0 \tag{9.7.17}$$
$$\implies P_2 = \left(\frac{\lambda}{\mu}\right) P_1 \tag{9.7.18}$$
$$\implies P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \tag{9.7.19}$$

We assume $\frac{\lambda}{\mu} = \rho$ and generalize P_n by (9.7.14) and (9.7.19)

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \tag{9.7.20}$$
$$\implies P_n = \rho^n P_0 \tag{9.7.21}$$

We know that sum of all probabilities equal to 1

$$\sum_{i=1}^{\infty} P_i = 1 \tag{9.7.22}$$
$$\implies P_0 + P_1 + P_2 + ... = 1 \tag{9.7.23}$$

Using (9.7.21)

$$\implies P_0 + \rho P_0 + \rho^2 P_0 + ... = 1 \tag{9.7.24}$$
$$\implies P_0\left(1+\rho+\rho^2+...\right) = 1 \tag{9.7.25}$$
$$\implies P_0\left(\frac{1}{1-\rho}\right) = 1 \tag{9.7.26}$$
$$\implies P_0 = 1-\rho \tag{9.7.27}$$
$$\therefore P_n = \rho^n(1-\rho) \tag{9.7.28}$$

The number of people in the system (L_s) is the expected value

$$L_s = \sum_{i=0}^{\infty} iP_i \tag{9.7.29}$$
$$\implies L_s = \sum_{i=0}^{\infty} i\rho^i P_0 \tag{9.7.30}$$
$$\implies L_s = \rho P_0 \sum_{i=0}^{\infty} i\rho^{i-1} \tag{9.7.31}$$
$$\implies L_s = \rho P_0 \sum_{i=0}^{\infty} \frac{d}{d\rho}\left(\rho^i\right) \tag{9.7.32}$$
$$\implies L_s = \rho P_0 \frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i \tag{9.7.33}$$
$$\implies L_s = \rho P_0 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \tag{9.7.34}$$
$$\implies L_s = \rho P_0 \frac{1}{(1-\rho)^2} \tag{9.7.35}$$

By using (9.7.27)

$$\implies L_s = \rho(1-\rho)\frac{1}{(1-\rho)^2} \tag{9.7.36}$$
$$\implies L_s = \frac{\rho}{1-\rho} \tag{9.7.37}$$

We can also say that the number of people beign served is ρ

$$\therefore L_s = L_q + \text{people beign served} \tag{9.7.38}$$
$$\implies L_s = L_q + \rho \tag{9.7.39}$$
$$\implies L_q = L_s - \rho \tag{9.7.40}$$
$$\implies L_q = \frac{\rho}{1-\rho} - \rho \tag{9.7.41}$$
$$\implies L_q = \frac{\rho^2}{1-\rho} \tag{9.7.42}$$

The relation between L_s and W_s and L_q and W_q are the Little's equation and they are related as

$$L_s = \lambda W_s \tag{9.7.43}$$
$$L_q = \lambda W_q \tag{9.7.44}$$

From the question given,

$$\lambda = 5\text{hr}^{-1} \tag{9.7.45}$$
$$\mu = \frac{1}{10}\text{min}^{-1} = 6\text{hr}^{-1} \tag{9.7.46}$$

Therefore,

$$\text{Utilization rate}(\rho) = \frac{\lambda}{\mu} = \frac{5}{6} \tag{9.7.47}$$

Average number (or) length in queue be L_q

$$L_q = \frac{\rho^2}{1-\rho} \tag{9.7.48}$$
$$= \frac{\left(\frac{5}{6}\right)^2}{1-\frac{5}{6}} \tag{9.7.49}$$
$$= \frac{25}{6} \tag{9.7.50}$$

Let the Average waiting time in queue be W_q

$$W_q = \frac{L_q}{\lambda} \tag{9.7.51}$$
$$= \frac{\frac{25}{6}}{5} \tag{9.7.52}$$
$$= \frac{5}{6}\text{hr} = 50\text{min} \tag{9.7.53}$$

The average waiting time in the queue is 50 min.

Parameter	Value
λ	5hr^{-1}
μ	6hr^{-1}
Utilization rate $(\rho) = \frac{\lambda}{\mu}$	$\frac{5}{6}$
Length in queue $\left(L_q\right) = \frac{\rho^2}{1-\rho}$	$\frac{25}{6}$
Waiting time in queue $\left(W_q\right) = \frac{L_q}{\lambda}$	$\frac{5}{6}\text{hr}$

TABLE 9.7.1: Parameters of the given question and values.

9.8. The number N of persons getting injured in a bomb blast at a busy market place is a random variable having a Poisson Distribution with parameter $\lambda(\geq 1)$. A person injured in the explosion may either suffer a minor injury requiring first aid or suffer a major injury requiring hospitalisation. Let the number of persons with minor injury be N_1 and the conditional distribution of N_1 given N is

$$\Pr(N_1 = i|N) = \frac{1}{N} \tag{9.8.1}$$

Find the expected number of persons requiring hospitalisation. **Solution:** We know,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \tag{9.8.2}$$

Also, for a Poisson Distribution:

$$\Pr(N = x) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{9.8.3}$$

where λ is the parameter
Let N_2 be the number of persons hospitalised. Let $N = a$, and $N_1 = i(i \leq a)$, then, $N_2 = a - i$
Then, from (9.8.1) and (9.8.3):

$$\Pr(N_2 = a - i) = \Pr(N_1 = i) \tag{9.8.4}$$
$$= \Pr(N_1 = i|N = a)\Pr(N = a) \tag{9.8.5}$$
$$= \frac{1}{a}\frac{e^{-\lambda}\lambda^a}{a!} \tag{9.8.6}$$

Thus,

$$E(N_2) = \sum_{a=0}^{\infty}\sum_{i=0}^{a}(a-i) \times \frac{1}{a}\frac{e^{-\lambda}\lambda^a}{a!} \tag{9.8.7}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\sum_{i=0}^{a}\frac{a-i}{a} \tag{9.8.8}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\left(a - \frac{(a+1)}{2}\right) \tag{9.8.9}$$
$$= \sum_{a=0}^{\infty}\frac{e^{-\lambda}\lambda^a}{a!}\frac{a-1}{2} \tag{9.8.10}$$
$$= \frac{e^{-\lambda}}{2}\left[\sum_{a=0}^{\infty}\frac{a\lambda^a}{a!} - \sum_{a=0}^{\infty}\frac{\lambda^a}{a!}\right] \tag{9.8.11}$$
$$= \frac{e^{-\lambda}}{2}\left[\lambda\sum_{a=1}^{\infty}\frac{\lambda^{a-1}}{(a-1)!} - \sum_{a=0}^{\infty}\frac{\lambda^a}{a!}\right] \tag{9.8.12}$$
$$= \frac{e^{-\lambda}}{2}\left[\lambda e^{\lambda} - e^{\lambda}\right] \tag{9.8.13}$$
$$= \frac{\lambda - 1}{2} \tag{9.8.14}$$

9.9. Suppose customers arrive at an ATM facility according to Poisson process with rate 5 customers per hour. The probability (rounded off to two decimal places) that no customer arrives at the ATM facility from 1:00pm to 1:18pm. **Solution:** Given, Poisson rate

$$\lambda = 5 \tag{9.9.1}$$

The time interval is given as 1:00 pm to 1:18 pm Then, the length of the interval

$$\tau = \frac{18}{60} - \frac{0}{60} \tag{9.9.2}$$
$$= \frac{3}{10} \tag{9.9.3}$$

Thus, if X is the number of arrivals in that interval, we can write

$$X \sim Poisson(\lambda\tau) = Poisson\left(\frac{3}{2}\right) \tag{9.9.4}$$

We know that, if $X(n)$ has a Poisson distribution whose parameter is k then

$$\Pr(X = n) = \left(\frac{k^n e^{-k}}{n!}\right) \tag{9.9.5}$$

CDF is:

$$F(X = n) = \sum_{x=0}^{n} \left(\frac{k^n e^{-k}}{n!} \right) \tag{9.9.6}$$

And also,

$$\Pr(x < X \leq y) = F(y) - F(x) \tag{9.9.7}$$

Given,

$$n = 0 \tag{9.9.8}$$

So from (9.9.7)

$$\Pr(X = 0) = F(0) \tag{9.9.9}$$

Therefore, the probability that no customer arrives at the ATM facility from 1:00pm to 1:18pm is
$\Pr(X = 0)$

$$= \frac{e^{\frac{-3}{2}} \left(\frac{3}{2}\right)^0}{0!} \tag{9.9.10}$$
$$= e^{-3/2} \tag{9.9.11}$$
$$\sim 0.22 \tag{9.9.12}$$

9.10. Consider an amusement park where visitors are arriving according to a Poisson process with rate 1. Upon arrival, a visitor spends a random amount of time in the park and then departs. The time spent by the visitors is independent of one another, as well as of the arrival process and have common probability density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & otherwise \end{cases} \tag{9.10.1}$$

If at a given point, there are 10 visitors in the park, and p is the probability that there will be exactly two more arrivals before the next departure, then $\frac{1}{p}$ equals..... **Solution:**
According to the question, we want the following events to occur in order:

a) First visitor, P_1 arrives while no one leaves
b) Second visitor P_2 arrives while no one leaves
c) One or more person leaves before the third visitor P_3 arrives

Let the above events be E_1, E_2 and E_3 respectively. Thus the required probability

Symbol	Representation
X_1	Arrival time of P_1
$X_1 + X_2$	Arrival time of P_2
$X_1 + X_2 + X_3$	Arrival time of P_3
$Y_1, ..., Y_{10}$	Departure times of the 10 people in park currently
$X_1 + Y_{11}$	Departure time of P_1
$X_1 + X_2 + Y_{12}$	Departure time of P_2

TABLE 9.10.1: Notations

$$= \Pr(E_1 E_2 E_3) \tag{9.10.2}$$
$$= \Pr(E_1)\Pr(E_2|E_1)\Pr(E_3|E_1E_2) \tag{9.10.3}$$

First we present the following result which shall be useful later. For $n > 0$,

$$\int_0^\infty xe^{-nx} dx = \frac{1}{n^2} \tag{9.10.4}$$

The above can be derived using integration by parts as follows

$$\int_0^\infty xe^{-nx} dx = -\frac{xe^{-nx}}{n}\bigg|_0^\infty + \frac{1}{n}\int_0^\infty e^{-nx} dx \tag{9.10.5}$$
$$= -\frac{e^{-nx}}{n^2}\bigg|_0^\infty \tag{9.10.6}$$
$$= \frac{1}{n^2} \tag{9.10.7}$$

Next we note that X_1, X_2 and X_3 are identical random variables having Poisson distribution with rate 1. Thus for $i \in \{1, 2, 3\}$,

$$\lambda = 1 * X_i = X_i \tag{9.10.8}$$
$$k = 1 \tag{9.10.9}$$

$$\implies f_{X_i}(x) = \begin{cases} \frac{x^1 e^{-x}}{1!} = xe^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{9.10.10}$$

Also $Y_1, ..., Y_{12}$ are identical random variables.

Thus for $i \in \{1, ..., 12\}$, as given in question,

$$f_{Y_i}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \quad (9.10.11)$$

$$\implies F_{Y_i}(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \quad (9.10.12)$$

Now we find $\Pr(E_1)$, $\Pr(E_2|E_1)$ and $\Pr(E_3|E_1E_2)$ in order to find the required probability from eq (9.10.3).

$$\Pr(E_1) = \Pr(Y_1, ..., Y_{10} > X_1) \quad (9.10.13)$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ..., Y_{10} > x|X_1 = x) \quad (9.10.14)$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{10} f_{X_1}(x) dx \quad (9.10.15)$$

$$= \int_{0}^{\infty} xe^{-11x} dx \quad (9.10.16)$$

$$= \frac{1}{121} \quad (9.10.17)$$

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2|Y_1, ..., Y_{10} > X_1) \quad (9.10.18)$$

Using memoryless property of exponential random variable,

$$\Pr(E_2|E_1) = \Pr(Y_1, ..., Y_{11} > X_2) \quad (9.10.19)$$

$$= \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{11} > x|X_2 = x) \quad (9.10.20)$$

$$= \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{11} f_{X_2}(x) dx \quad (9.10.21)$$

$$= \int_{0}^{\infty} xe^{-12x} dx \quad (9.10.22)$$

$$= \frac{1}{144} \quad (9.10.23)$$

$$\Pr(E_3|E_1E_2) = \Pr(min(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12}) < X_1 + X_2 + X_3|Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2) \quad (9.10.24)$$

We can simplify and write $\Pr(E_3|E_1E_2) =$

$$1 - \Pr(Y_1, ..., Y_{10}, X_1 + Y_{11}, X_1 + X_2 + Y_{12} > X_1 + X_2 + X_3|Y_1, ..., Y_{10}, X_1 + Y_{11} > X_1 + X_2) \quad (9.10.25)$$

Using memoryless property of exponential random variable,

$$\Pr(E_3|E_1E_2) = 1 - \Pr(Y_1, ..., Y_{12} > X_3) \quad (9.10.26)$$

$$= 1 - \int_{-\infty}^{\infty} \Pr(Y_1, ...Y_{12} > x|X_3 = x) \quad (9.10.27)$$

$$= 1 - \int_{-\infty}^{\infty} (1 - F_{Y_1}(x))^{12} f_{X_3}(x) dx \quad (9.10.28)$$

$$= 1 - \int_{0}^{\infty} xe^{-13x} dx \quad (9.10.29)$$

$$= 1 - \frac{1}{169} \quad (9.10.30)$$

$$= \frac{168}{169} \quad (9.10.31)$$

Thus on substituting values in (9.10.3),

$$\Pr(E_1E_2E_3) = \frac{1}{121} \times \frac{1}{144} \times \frac{168}{169} \quad (9.10.32)$$

$$= 5.7 \times 10^{-5} \quad (9.10.33)$$

9.11. Let X and Y be two independent Poisson random variables with parameters 1 and 2 respectively. Then, $\Pr(X = 1|X + Y = 4)$ is

A) 0.426
B) 0.293
C) 0.395
D) 0.512

Solution: Given, $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$. The probability mass functions (PMFs) of random variables X and Y are given by:

$$p_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{for } x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases} \quad (9.11.1)$$

$$p_Y(y) = \begin{cases} \frac{e^{-\mu}\mu^y}{y!}, & \text{for } y = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases} \quad (9.11.2)$$

where: the parameters $\lambda = 1$ and $\mu = 2$. As X and Y are independent, we have for $k \geq 0$, the distribution function $p_{X+Y}(k)$ is a convolution of distribution functions $p_X(k)$ and $p_Y(k)$:

$$p_{X+Y}(k) = \Pr(X+Y=k) = \Pr(Y=k-X) \tag{9.11.3}$$

$$= \sum_i \Pr(Y=k-i|X=i) \times p_X(i) \tag{9.11.4}$$

After unconditioning, as X and Y are independent:

$$\Pr(Y=k-i|X=i) = \Pr(Y=k-i) = p_Y(k-i) \tag{9.11.5}$$

$$p_{X+Y}(k) = p_Y(k) * p_X(k) \tag{9.11.6}$$

$$= \sum_{i=0}^{k} p_Y(k-i) \times p_X(i) \tag{9.11.7}$$

$$= \sum_{i=0}^{k} e^{-\mu} \frac{\mu^{k-i}}{(k-i)!} e^{-\lambda} \frac{\lambda^i}{i!} \tag{9.11.8}$$

$$= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \mu^{k-i} \lambda^i \tag{9.11.9}$$

$$= e^{-(\mu+\lambda)} \frac{1}{k!} \sum_{i=0}^{k} {}^kC_i \mu^{k-i} \lambda^i \tag{9.11.10}$$

$$= \frac{(\mu+\lambda)^k}{k!} \times e^{-(\mu+\lambda)} \tag{9.11.11}$$

Hence, $X + Y \sim \mathcal{P}(\mu + \lambda)$.

$$\Pr(X=1|X+Y=4) = \frac{\Pr(X=1, Y=3)}{\Pr(X+Y=4)} \tag{9.11.12}$$

$$= \frac{\Pr(X=1) \times \Pr(Y=3)}{\Pr(X+Y=4)} \tag{9.11.13}$$

$$= \frac{\dfrac{e^{-1} \times 1^1}{1!} \times \dfrac{e^{-2} \times 2^3}{3!}}{\dfrac{e^{-3} \times 3^4}{4!}} \tag{9.11.14}$$

$$= 4 \times \frac{(1)(2)^3}{(3)^4} \tag{9.11.15}$$

$$= \frac{32}{81} \tag{9.11.16}$$

$$= 0.39506172839 \tag{9.11.17}$$

10 Random Variables

10.1. Let X be a random variable having the distribution function :

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \leq x < 1 \\ \frac{1}{3} & 1 \leq x < 2 \\ \frac{1}{2} & 2 \leq x < \frac{11}{3} \\ 1 & x \geq \frac{11}{3} \end{cases}$$

Then $E(X)$ is equal to :

Solution:

Definition 2 (Heaviside step function)**.** *Heaviside step function $u(x)$ is*

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Using the Heaviside step function $u(x)$, a function $F(t)$ can be obtained whose output is $f(t)$ for the interval $[a, b)$ and 0 everywhere else

$$F(t) = f(t)[u(t-a) - u(t-b)] \tag{1}$$

Definition 3 (Dirac delta function)**.** *Dirac delta function is the derivative of the Heaviside step* *function* $u(x)$

$$\delta(x) = \frac{du(x)}{dx} \tag{2}$$

An important property of the Dirac delta function is

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \tag{3}$$

Using Definition 2, we get CDF $F(x)$ in terms of Heaviside step function $u(x)$

$$F(x) = \frac{1}{4}\{u(x)-u(x-1)\}+\frac{1}{3}\{u(x-1)-u(x-2)\}+\frac{1}{2}\left\{u(x-2)-u\left(x-\frac{11}{3}\right)\right\}+u\left(x-\frac{11}{3}\right)$$

$$\implies F(x) = \frac{u(x)}{4}+\frac{u(x-1)}{12}+\frac{u(x-2)}{6}+\frac{u\left(x-\frac{11}{3}\right)}{2} \tag{4}$$

Differentiating (4) and using Definition 3, we obtain PDF $f(x)$

$$f(x) = \frac{\delta(x)}{4}+\frac{\delta(x-1)}{12}+\frac{\delta(x-2)}{6}+\frac{\delta\left(x-\frac{11}{3}\right)}{2} \tag{5}$$

Using (5), we can state that Random Variable X is discrete and it takes values at the points where $f(x) \to \infty$

$$\therefore X \in \left\{0, 1, 2, \frac{11}{3}\right\} \tag{6}$$

To obtain the PMF($p_X(k)$) we use the formula

$$(p_X(k)) = \lim_{x\to k}\int_k^x f(x)dx \tag{7}$$

Definition 4 (PMF of Random Variable X)**.** *The PMF*($p_X(k)$) *using* (7) *is :*

$$(p_X(k)) = \begin{cases} \frac{1}{4} & \text{if } k=0 \\ \frac{1}{12} & \text{if } k=1 \\ \frac{1}{6} & \text{if } k=2 \\ \frac{1}{2} & \text{if } k=\frac{11}{3} \\ 0 & \text{otherwise} \end{cases}$$

To obtain $E(x)$ we use the formula

$$E(X) = \sum x \times (p_X(k)) \tag{8}$$

Therefore, using PMF we get

$$E(X) = \left(0\times\frac{1}{4}\right)+\left(1\times\frac{1}{12}\right)+\left(2\times\frac{1}{6}\right)+\left(\frac{11}{3}\times\frac{1}{2}\right)$$

$$\implies E(X) = 2.25$$

10.2. Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} ae^{-2y} & 0<x<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

The value of a is

(A) 4
(B) 2
(C) 1
(D) 0.5

Solution: Let X and Y be continuous random variables as mentioned in the question above.

For a joint PDF we know that,

$$\int_{x=-\infty}^{x=\infty} f_X(x,y)\,\mathrm{d}x = 1$$

Similarly,

$$\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=y} f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$$

Now,

$$\int_{y=-\infty}^{y=0}\int_{x=-\infty}^{x=y} f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{y=0}^{y=\infty}\int_{x=0}^{x=y} a\times e^{-2y}\,\mathrm{d}x\,\mathrm{d}y = 1 \tag{10.2.1}$$

$$0 + \int_{y=0}^{y=\infty}\left(\int_{x=0}^{x=y} a\times e^{-2y}\,\mathrm{d}x\right)\mathrm{d}y = 1 \tag{10.2.2}$$

$$\int_{y=0}^{y=\infty}\left(a\times y\times e^{-2y}\right)\mathrm{d}y = 1 \tag{10.2.3}$$

$$a\times\left(\frac{-y}{2}\times e^{-2y} - \frac{e^{-2y}}{4}\right)_0^\infty = 1 \tag{10.2.4}$$

$$a\times\frac{1}{4} = 1 \tag{10.2.5}$$

$$a = 4 \tag{10.2.6}$$

Therefore, the answer is (A).
Marginal density of X is,

$$f_X(X) = \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\,\mathrm{d}y \tag{10.2.7}$$

$$= \int_{y=-\infty}^{y=0} f_{XY}(x,y)\,\mathrm{d}y + \int_{y=0}^{y=\infty} 4\times e^{-2y}\,\mathrm{d}y \tag{10.2.8}$$

$$= 0 + 2 \tag{10.2.9}$$

$$= 2 \tag{10.2.10}$$

Marginal density of Y is,

$$f_Y(Y) = \int_{x=-\infty}^{x=\infty} f_{XY}(x,y)\,\mathrm{d}x \tag{10.2.11}$$

$$= \int_{x=-\infty}^{x=0} f_{XY}(x,y)\,\mathrm{d}x + \int_{x=0}^{x=\infty} 4\times e^{-2y}\,\mathrm{d}x \tag{10.2.12}$$

$$= 0 + \left(4y\times e^{-2y}\right)_0^\infty \tag{10.2.13}$$

$$= 4ye^{-2y} \qquad \forall y\epsilon\,(0,\infty) \tag{10.2.14}$$

$$\tag{10.2.15}$$

10.3. A fair die is rolled twice independently. Let X and Y denote the outcomes of the first and second roll, respectively. Then $E(X+Y\mid(X-Y)^2=1)$ equals

Solution:
X and Y are two independent random variables that can take the values 1, 2, 3, 4, 5, 6.

$$\Pr(X=k) = \frac{1}{6}, 1\le k\le 6 \tag{10.3.1}$$

$$\Pr(Y=k) = \frac{1}{6}, 1\le k\le 6 \tag{10.3.2}$$

Lemma 10.1. *The PMF of X+Y is given by*

$$\Pr(X+Y=n) = \begin{cases} \frac{n-1}{36} & ,2\le n\le 7\\ \frac{13-n}{36} & ,8\le n\le 12\end{cases} \tag{10.3.3}$$

Proof. Using convolution for discrete random variables,
$\Pr(X+Y=n)$

$$= \sum_{k=n-6}^{n-1}\Pr(X=k, Y=n-k), 1\le k\le 6 \tag{10.3.4}$$

Since X and Y are independent,

$$= \sum_{k=n-6}^{n-1}\Pr(X=k)\times\Pr(Y=n-k), 1\le k\le 6 \tag{10.3.5}$$

$$= \sum_{k=n-6}^{n-1} \frac{1}{36}, 1 \le k \le 6 \tag{10.3.6}$$

$$= \begin{cases} \frac{n-1}{36} & , 2 \le n \le 7 \\ \frac{13-n}{36} & , 8 \le n \le 12 \end{cases} \tag{10.3.7}$$

□

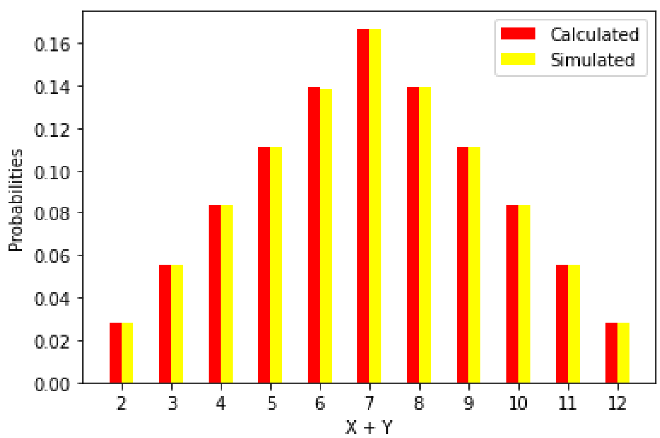


Fig. 10.3.1: Plot of PMF for X+Y

Lemma 10.2. *The PMF of X-Y is given by*

$$\Pr(X - Y = n) = \begin{cases} \frac{n+6}{36} & , -5 \le n \le 0 \\ \frac{6-n}{36} & , 1 \le n \le 5 \end{cases} \tag{10.3.8}$$

Proof. Using convolution for discrete random variables,
$\Pr(X - Y = n)$

$$= \sum_{k=n+1}^{n+6} \Pr(X = k, Y = k - n), 1 \le k \le 6 \tag{10.3.9}$$

Since X and Y are independent,

$$= \sum_{k=n+1}^{n+6} \Pr(X = k) \times \Pr(Y = k - n), 1 \le k \le 6 \tag{10.3.10}$$

$$= \sum_{k=n+1}^{n+6} \frac{1}{36}, 1 \le k \le 6 \tag{10.3.11}$$

$$= \begin{cases} \frac{n+6}{36} & , -5 \le n \le 0 \\ \frac{6-n}{36} & , 1 \le n \le 5 \end{cases} \tag{10.3.12}$$

□

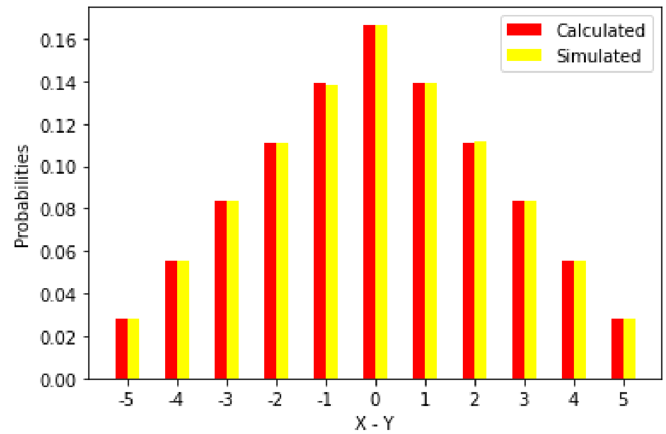


Fig. 10.3.2: Plot of PMF for X-Y

$$E\left(X + Y \mid (X - Y)^2 = 1\right)$$
$$= \sum n \times \Pr\left(X + Y = n \mid (X - Y)^2 = 1\right) \tag{10.3.13}$$

$$= \sum n \times \frac{\Pr\left(X + Y = n, (X - Y)^2 = 1\right)}{\Pr\left((X - Y)^2 = 1\right)} \tag{10.3.14}$$

$$= \sum n \times \frac{\Pr(X + Y = n, (X - Y) = 1)}{\Pr((X - Y) = 1)} \times \Pr\left((X - Y) = 1 | (X - Y)^2 = 1\right) + \sum n \times \frac{\Pr(X + Y = n, (X - Y) = -1)}{\Pr((X - Y) = -1)} \times \Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right) \tag{10.3.15}$$

$$= \frac{\Pr\left((X - Y) = 1 \mid (X - Y)^2 = 1\right)}{\Pr((X - Y) = 1)} \times \sum n \times \Pr(X + Y = n, (X - Y) = 1) + \frac{\Pr\left((X - Y) = -1 \mid (X - Y)^2 = 1\right)}{\Pr((X - Y) = -1)} \times \sum n \times \Pr(X + Y = n, (X - Y) = -1) \tag{10.3.16}$$

Using equations (10.3.7) and (10.3.12) in (10.3.16)
We get,

$$E\left(X+Y \mid (X-Y)^2=1\right)$$
$$= \left(\frac{\frac{1}{2}}{\frac{5}{36}}\right)\times\left(\frac{35}{36}\right)+\left(\frac{\frac{1}{2}}{\frac{5}{36}}\right)\times\left(\frac{35}{36}\right) \quad (10.3.17)$$
$$= 7 \quad (10.3.18)$$

10.4. Let $\Omega = (0,1]$ be the sample space and let $P(.)$ be a probability function defined by

$$P((0,x]) = \begin{cases} x/2, & 0 < x < 1/2 \\ x, & 1/2 \le x \le 1 \end{cases} \quad (10.4.1)$$

Then $P(\{\frac{1}{2}\}) =$

Solution: Given that, the CDF of the given random variable is

$$F_X(x) = \begin{cases} x/2, & 0 < x < \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1 \end{cases}$$

that means probability of the random variable being m is

$$\Pr(X=m) = F_X(m) - \lim_{t\to m^-} F_X(t) \quad (10.4.2)$$

Hence the probability value at $X = \frac{1}{2}$ is

$$\Pr(X=1/2) = F_X\left(\frac{1}{2}\right) - \lim_{t\to\frac{1}{2}^-} F_X(t) \quad (10.4.3)$$
$$= \frac{1}{2} - \lim_{x\to\frac{1}{2}^-}\frac{x}{2} \quad (10.4.4)$$
$$= \frac{1}{2}-\frac{1}{4} \quad (10.4.5)$$
$$= \frac{1}{4} = 0.25 \quad (10.4.6)$$

10.5. $p_X(x) = Me^{-2|x|} + Ne^{-3|x|}$ is the probability density function for the real random variable X, Over the entire x axis. M and N are both positive real numbers. Find The equation relating M and N

Solution: We know that $p_X(x) \ge 0$.

Theorem 10.1. *The integral of probability density function over the continuous random variable is equal to 1.*

$$\int_{-\infty}^{\infty} p_X(x)\, dx = 1 \quad (10.5.1)$$

$$\int_{-\infty}^{\infty}\left(Me^{-2|x|}+Ne^{-3|x|}\right)dx = 1 \quad (10.5.2)$$

(Since $Me^{-2|x|}+Ne^{-3|x|}$ is an even function)

$$2\int_{0}^{\infty}\left(Me^{-2|x|}+Ne^{-3|x|}\right)dx = 1 \quad (10.5.3)$$

$$2\int_{0}^{\infty}\left(Me^{-2x}+Ne^{-3x}\right)dx = 1 \quad (10.5.4)$$

$$2\left(M\frac{e^{-2x}}{-2}+N\frac{e^{-3x}}{-3}\right)\Big|_0^{\infty} dx = 1 \quad (10.5.5)$$

$$2\left(0-\left(\frac{M}{-2}+\frac{N}{-3}\right)\right) = 1 \quad (10.5.6)$$

$$2\left(\frac{M}{2}+\frac{N}{3}\right) = 1 \quad (10.5.7)$$

The equation when rearranged properly gives us the desired connection beteen M and N which is $3M+2N=3$.

10.6. The probability density function(PDF) of a random variable X is as shown in Fig. 10.6.1. The corresponding cumulative distribution function (CDF) has the form

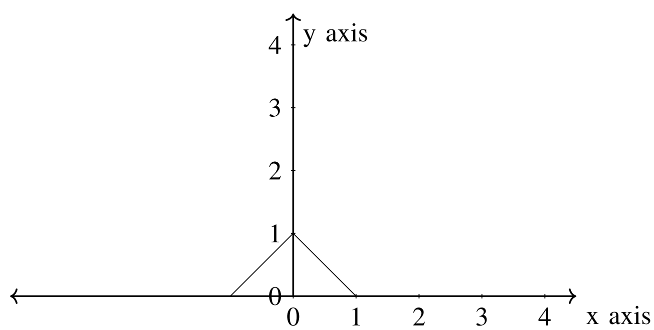


Fig. 10.6.1: PDF Graph

Solution: The given PDF graph can be represented by the function $f(x)$.

$$f(x) = \begin{cases} 0 & |x| > 1 \\ 1-|x| & |x| \le 1 \end{cases} \quad (10.6.1)$$

The CDF can be expressed as

$$F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} \int_{-1}^{x}(1+x)dx & x \in [-1,0) \\ \int_{0}^{x}(1-x)dx & x \in (0,1] \end{cases} \quad (10.6.2)$$

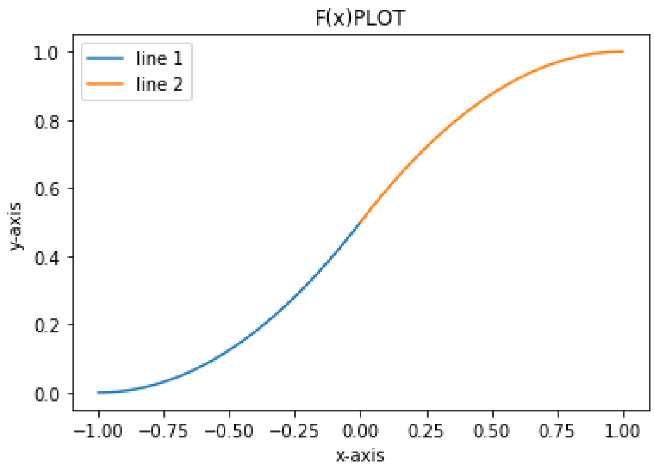


Fig. 10.6.2: CDF graph

$$F(x) = \begin{cases} 0 & |x| > 1 \\ \frac{x^2}{2} + x + \frac{1}{2} & -1 \le x < 0 \\ \frac{-x^2}{2} + x + \frac{1}{2} & 0 \le x < 1 \end{cases} \quad (10.6.3)$$

and plotted in Fig. 10.6.2.

10.7. A continuous random variable X has a probability density function $f(x) = e^{-x}, 0 < x < \infty$. Then $P(X > 1)$ is

a) 0.368 b) 0.5 c) 0.632 d) 1.0

Solution:
Given,

$$f(x) = e^{-x}, 0 < x < \infty \quad (10.7.1)$$

We have to find $\Pr(X > 1)$,

$$\Pr(X > 1) = \int_1^\infty f(x)\, dx \quad (10.7.2)$$

Using (10.7.1) in (10.7.2)

$$\Pr(X > 1) = \int_1^\infty e^{-x}\, dx \quad (10.7.3)$$
$$= \left[-e^{-x}\right]_1^\infty \quad (10.7.4)$$
$$= (-e^{-\infty}) - (-e^{-1}) \quad (10.7.5)$$
$$= e^{-1} \quad (10.7.6)$$
$$= \frac{1}{e} \quad (10.7.7)$$
$$\implies \Pr(X > 1) = 0.368 \quad (10.7.8)$$

Finding the probability using uniform distribution,
Let $F_X(x)$ be the cumulative distribution function of random variable X.

$$F_X(x) = \int_0^x f(x)\, dx \quad (10.7.9)$$

$F_X(x)$ can be obtained from the uniform distribution of a random variable U on (0,1) and let U=e^{-x}.

$$0 < U < 1 \quad (10.7.10)$$

As for random variable X also,

$$0 < F_X(x) < 1 \quad (10.7.11)$$

This similarity between U and $F_X(x)$ is used to generate the random variable X from U.

$$F_X(x) = \Pr(X < x) \quad (10.7.12)$$
$$= \Pr(-\log_e U < x) \quad (10.7.13)$$
$$= \Pr(U < e^{-x}) \quad (10.7.14)$$
$$= F_U(e^{-x}) \quad (10.7.15)$$

From uniform distribution,

$$F_U(x) = x, 0 < x < 1 \quad (10.7.16)$$

In the figure 10.7.1,orange colour graph represents the pdf of the random variable X and blue colour graph represents the cdf of the random variable X. Using (10.7.16) in (10.7.15), Cumulative distribution function (CDF) of random variable X is,

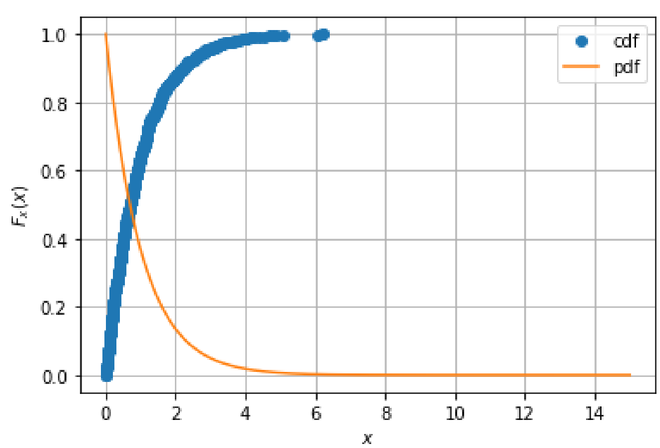


Fig. 10.7.1: CDF of random variable X

$$F_X(x) = \Pr(X < x) \quad (10.7.17)$$
$$= 1 - e^{-x}, 0 < x < \infty \quad (10.7.18)$$

Now we have to find $\Pr(X > 1)$,

$$\Pr(X > 1) = 1 - \Pr(X < 1) \tag{10.7.19}$$

Using (10.7.18),

$$\Pr(X > 1) = 1 - (1 - e^{-1}) \tag{10.7.20}$$
$$\Pr(X > 1) = e^{-1} \tag{10.7.21}$$
$$\implies \Pr(X > 1) = 0.368 \tag{10.7.22}$$

10.8. A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.8.1}$$

If the expected value $E[X] = \frac{2}{3}$, then $Pr[X < 0.5]$ is..........

Solution:

We know that the total probability is one,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{10.8.2}$$

Using (10.41.1) in (10.41.3),

$$\int_0^1 (a + bx)\,dx = 1 \tag{10.8.3}$$
$$\left[ax + \frac{bx^2}{2}\right]_0^1 = 1 \tag{10.8.4}$$
$$\left(a + \frac{b}{2}\right) - 0 = 1 \tag{10.8.5}$$
$$\implies a + \frac{b}{2} = 1 \tag{10.8.6}$$

We know that expectation value of X,

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx \tag{10.8.7}$$

Using $E(X) = \frac{2}{3}$ and (10.41.1) in (10.8.7), we get

$$\frac{2}{3} = \int_0^1 x(a + bx)\,dx \tag{10.8.8}$$
$$= \int_0^1 ax + bx^2\,dx \tag{10.8.9}$$
$$= \left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 \tag{10.8.10}$$
$$= \frac{a}{2} + \frac{b}{3} - 0 \tag{10.8.11}$$
$$\implies \frac{a}{2} + \frac{b}{3} = \frac{2}{3} \tag{10.8.12}$$

By solving (10.8.6) and (10.8.12), we get

$$a = 0 \text{ and } b = 2. \tag{10.8.13}$$

Using values of a and b in (10.41.1), we get

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & otherwise \end{cases} \tag{10.8.14}$$

The graph of PDF of X is 10.8.1

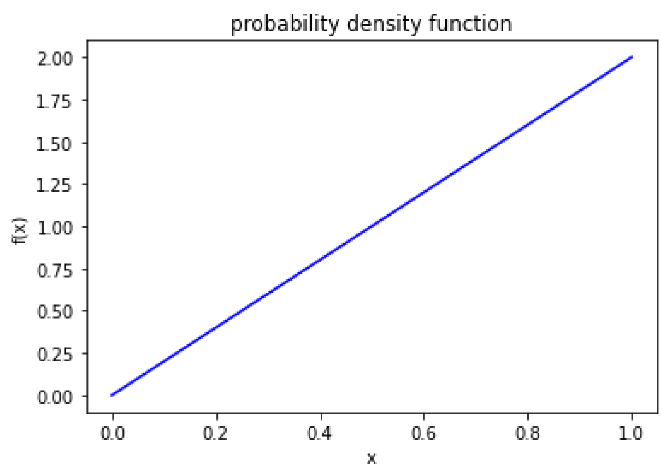


Fig. 10.8.1: Probability Density Function (PDF) of X

Let $F_X(x)$ be the cumulative distribution function of random variable X.

$$F_X(x) = \int_{-\infty}^{x} f(x)\,dx \tag{10.8.15}$$

$F_X(x)$ can be obtained from the uniform distribution of a random variable U on (0,1) and let U=X^2.

$$0 < U < 1 \tag{10.8.16}$$

As for random variable X also,

$$0 < F_X(x) < 1 \tag{10.8.17}$$

This similarity between U and $F_X(x)$ is used to generate the random variable X from U.

$$F_X(x) = \Pr(X < x) \tag{10.8.18}$$
$$= \Pr\left(\sqrt{U} < x\right) \tag{10.8.19}$$
$$= \Pr\left(U < x^2\right) \tag{10.8.20}$$
$$= F_U(x^2) \tag{10.8.21}$$

From uniform distribution,
The graph of Probability Density Function (PDF) of U is 10.8.2

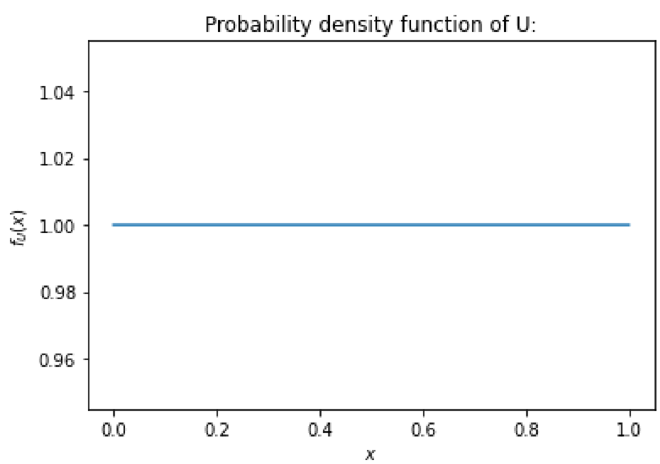


Fig. 10.8.2: Probability Density Function (PDF) of U

$$F_U(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{10.8.22}$$

Using (10.8.22) in (10.8.21),
Cumulative distribution function (CDF) of random variable X is,

$$F_X(x) = \Pr(X < x) = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{10.8.23}$$

The graph of Cumulative distribution function (CDF) of random variable X is 10.8.3
Now we have to find $\Pr(X < 0.5)$,Using

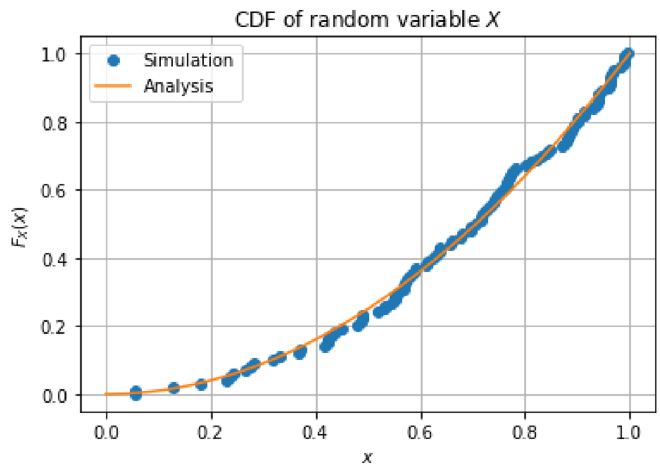


Fig. 10.8.3: Cumulative Density Function (CDF)

(10.8.23),

$$\Pr(X < 0.5) = (0.5)^2 \tag{10.8.24}$$
$$\implies \Pr(X < 0.5) = 0.25 \tag{10.8.25}$$

10.9. Let X be a random variable with a probability density function

$$f(x) = \begin{cases} 0.2 & |x| \le 1 \\ 0.1 & 1 \le |x| \le 4 \\ 0 & otherwise \end{cases} \tag{10.9.1}$$

Find $\Pr(0.5 < X \le 5)$
Solution:
We know, if X is a continuous random variable, and its p.d.f is given by $f(x)$, then we define the c.d.f $F(x)$ as:

$$F(x) = \Pr(X \le x) \tag{10.9.2}$$

and is given by:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx \tag{10.9.3}$$

$f(x)$ is a valid p.d.f because:

a) The area under the curve of the p.d.f is 1, i.e:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{10.9.4}$$

b) $f(x) \ge 0$ for all $x \in \mathbb{R}$

Since $f(x)$ is a valid p.d.f, from (10.9.3), we get the following c.d.f:

$$F(x) = \begin{cases} 0 & x \leq -4 \\ 0.1(x+4) & -4 \leq x \leq -1 \\ 0.3 + 0.2(x+1) & -1 \leq x \leq 1 \\ 0.7 + 0.1(x-1) & 1 \leq x \leq 4 \\ 1 & 4 \leq x \end{cases} \tag{10.9.5}$$

Thus,

$$\Pr(0.5 \leq X \leq 5) = F(5) - F(0.5) = 0.4 \tag{10.9.6}$$

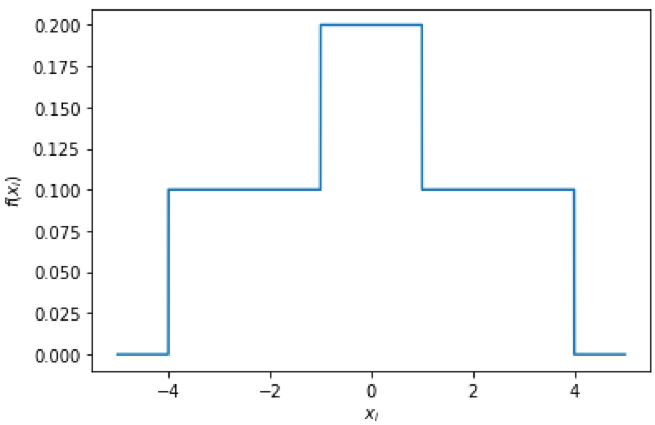


Fig. 10.9.1: plot of $f(x)$- p.d.f

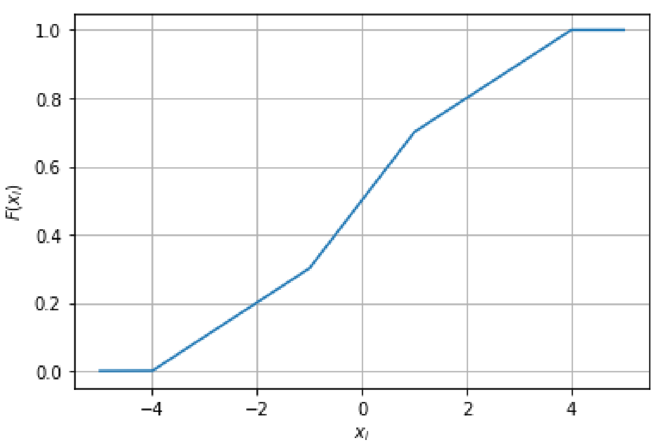


Fig. 10.9.2: plot of $F(x)$- c.d.f

10.10. Consider two identically distributed zero-mean random variables U and V. Let the cumulative distribution functions of U and 2V be F(x) and G(x) respectively. Then,for all values of x

a) $F(x) - G(x) \leqslant 0$

b) $F(x) - G(x) \geqslant 0$

c) $(F(x) - G(x))x \leqslant 0$

d) $(F(x) - G(x))x \geqslant 0$

Solution: If X is a random variable, the cumulative distribution functions of U and 2V can be written in terms of X as

$$F(x) = \Pr(X \leq x) \tag{10.10.1}$$

$$G(x) = \Pr(2X \leq x) \tag{10.10.2}$$

Or,

$$G(x) = \Pr(X \leq x/2) \tag{10.10.3}$$

Using 10.10.1 in 10.10.3, we can see that

$$G(x) = F(x/2) \tag{10.10.4}$$

So,

$$F(x) - G(x) = F(x) - F(x/2) \tag{10.10.5}$$

As F is Cumulative Distribution Function, it is non-decreasing.
That means for $x \geq y$, $F(x) \geq F(y)$.

Using this, we can form the following table:

Case	$F(x) - F(x/2)$	$(F(x) - F(x/2))x$
$x \geq 0$	≥ 0	≥ 0
$x \leq 0$	≤ 0	≥ 0

TABLE 10.10.1

From the table we can see that for any value of x,

$$(F(x) - F(x/2))x \geq 0 \tag{10.10.6}$$

Or, using 10.10.4,

$$(F(x) - G(x))x \geq x \tag{10.10.7}$$

10.11. A probability density function is of the form
$$p(x) = Ke^{-\alpha|x|}, x \in (-\infty, \infty)$$
The value of K is

a) 0.5 b) 1 c) 0.5α d) α

10.12. The probability density function (PDF) of a random variable X is as shown in Fig. 10.12.1.

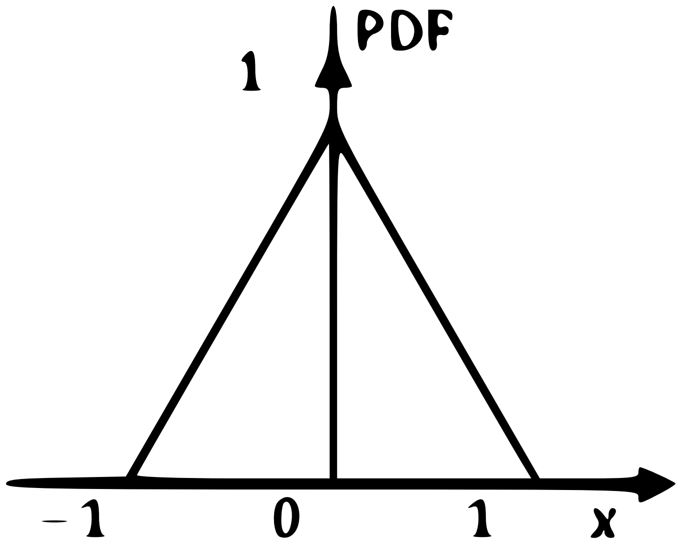


Fig. 10.12.1

The corresponding cumulative distribution function (CDF) has the form

a) Fig. 10.12.2 c) Fig. 10.12.4

b) Fig. 10.12.3 d) Fig. 10.12.5

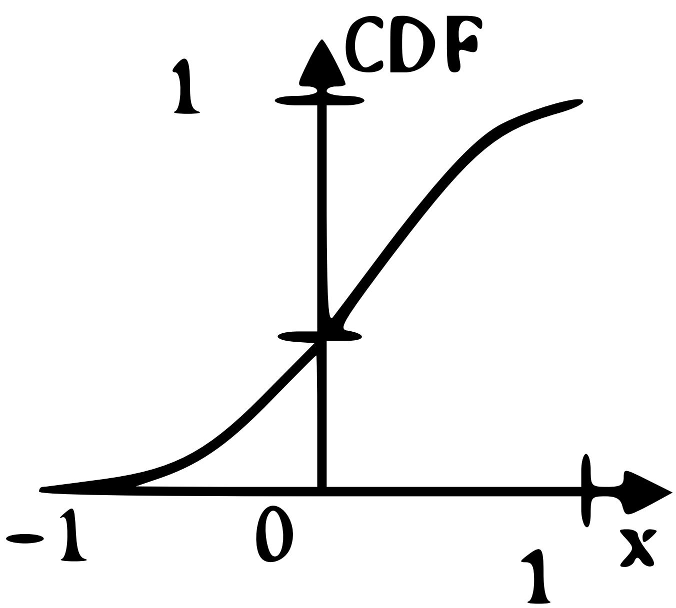


Fig. 10.12.2

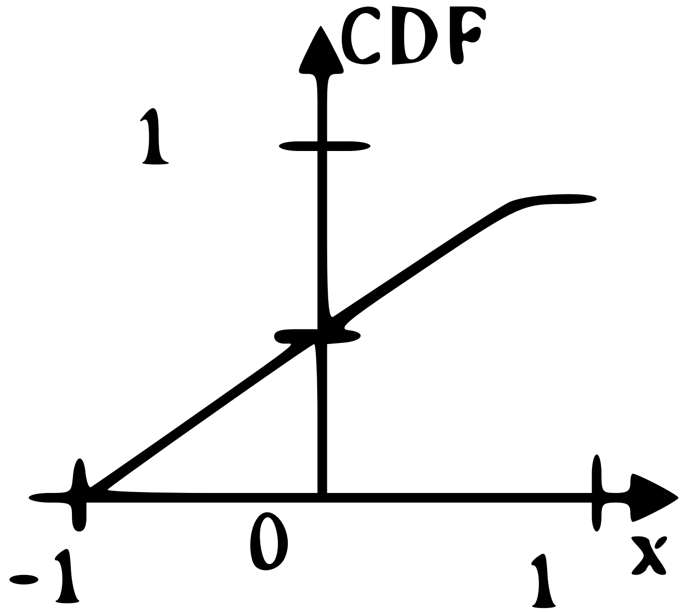


Fig. 10.12.3

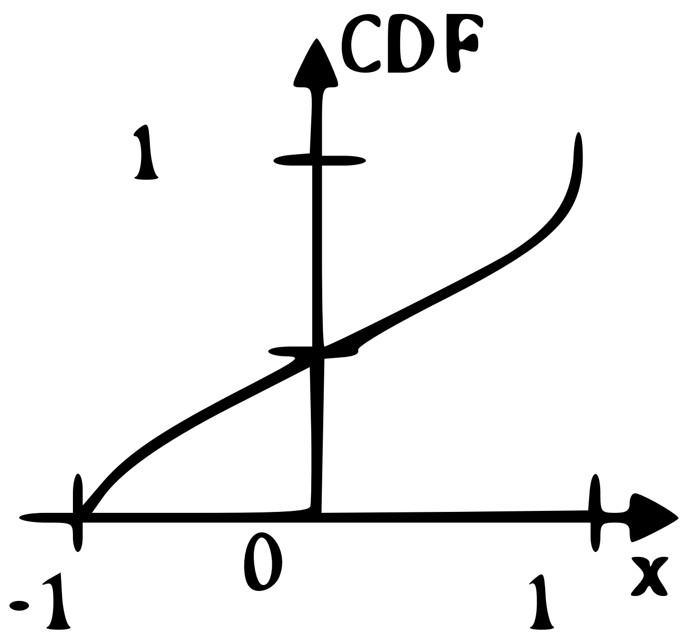


Fig. 10.12.4

10.13. The distribution function $f_x(x)$ of a random variable X is shown in Fig. 10.13.1. The probability that X=1 is

a) Zero c) 0.55

b) 0.25 d) 0.30

10.14. Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leq x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,the value of c is equal to ________________

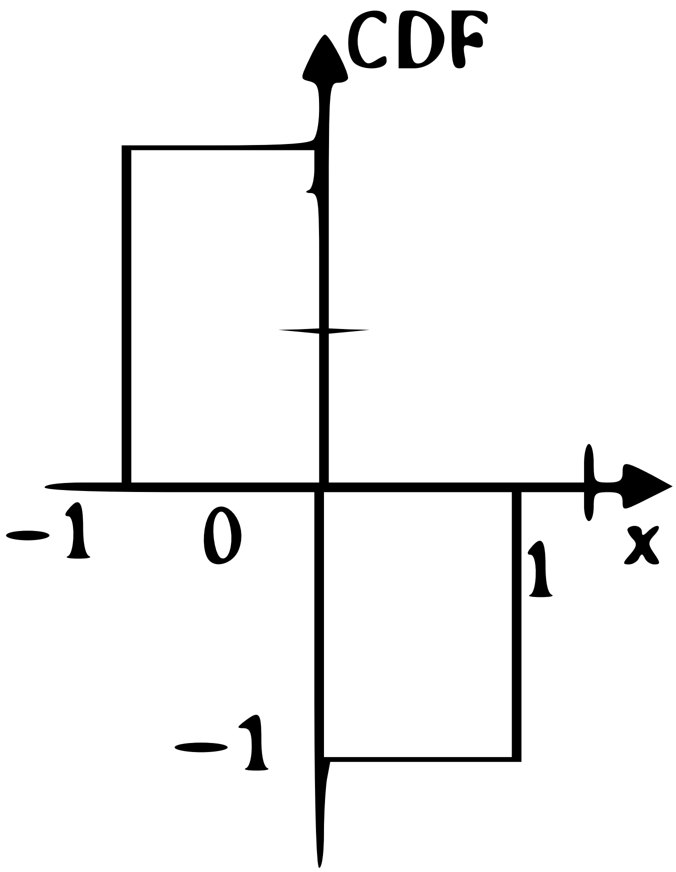


Fig. 10.12.5

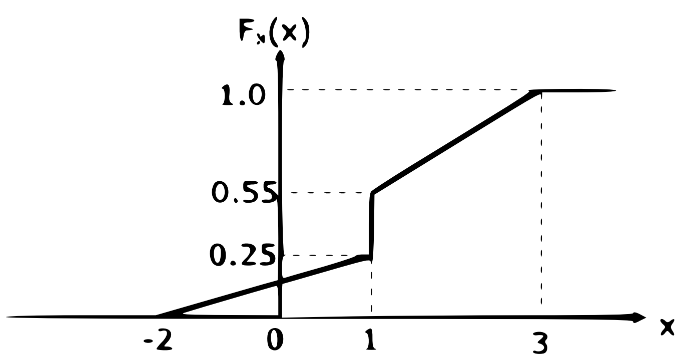


Fig. 10.13.1

Solution: For a probability density function of a continuous random variable,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \tag{10.14.1}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{0}^{1/2} f_X(x)\,dx + \int_{1/2}^{1} f_X(x)\,dx \tag{10.14.2}$$

$$= \frac{1}{2}(x)(x)\Big|_{x=\frac{1}{2}} + \int_{1/2}^{1} c(2x-1)^2\,dx \tag{10.14.3}$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + c\left(\frac{4x^3}{3} - 2x^2 + x\right)\Bigg]_{1/2}^{1} \tag{10.14.4}$$

$$= \frac{1}{8} + c\left(\frac{1}{3} - \frac{1}{6}\right) \tag{10.14.5}$$

$$= \frac{1}{8} + \frac{c}{6} \tag{10.14.6}$$

from (10.14.1) and (10.14.6) we get

$$1 = \frac{1}{8} + \frac{c}{6} \tag{10.14.7}$$

$$\therefore c = \frac{21}{4} \tag{10.14.8}$$

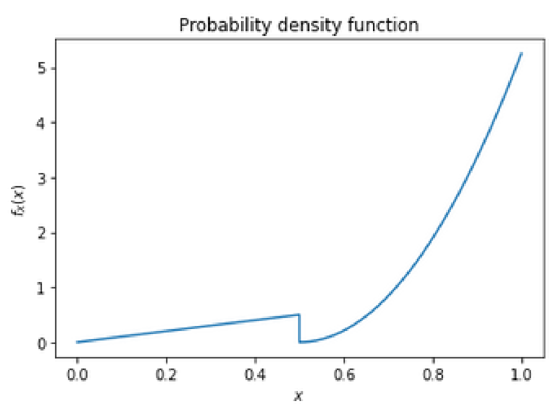


Fig. 10.14.1: Graph of $f_X(x)$

$$F_X(x) = f_X(X <= x) = \int_{-\infty}^{x} f_X(x)\,dx \tag{10.14.9}$$

from $f_X(x)$ and equation (10.14.9),

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 \le x \le \frac{1}{2} \\ \frac{1}{8} + \frac{7}{8}(2x-1)^3 & \frac{1}{2} \le x \le 1 \\ 1 & x > 1 \end{cases}$$

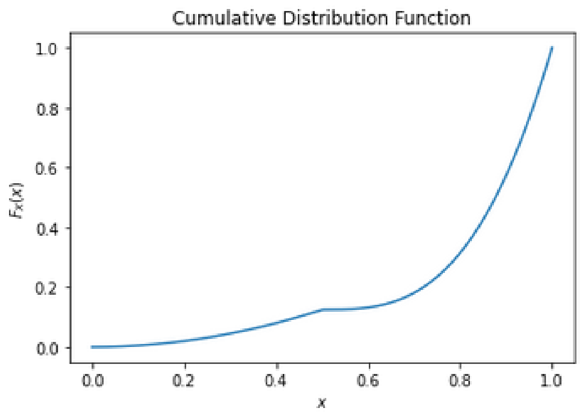
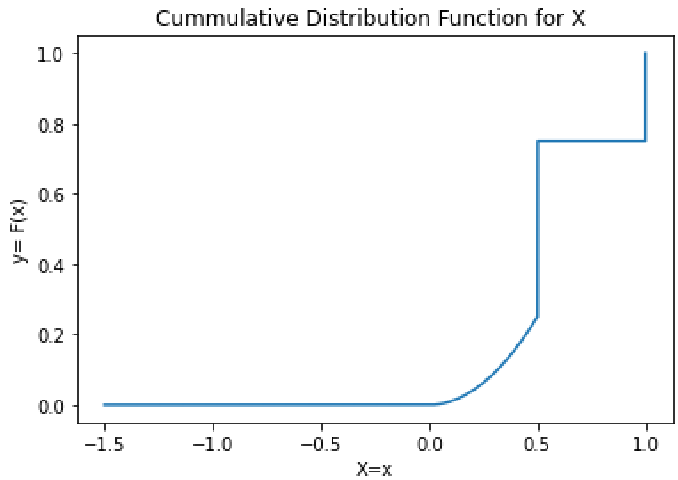


Fig. 10.14.2: Graph of $F_X(x)$



10.15. Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq \ x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leq \ x < 1 \\ 1 & x \geq \ 1. \end{cases}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to ________

Solution:

$$P(a<x<b) = F(b) - F(a) \tag{10.15.1}$$

We want,

$$S = P(\frac{1}{4}<X<1) \tag{10.15.2}$$

$$S = F(1) - F(\frac{1}{4}) \tag{10.15.3}$$

$$S = \frac{3}{4} - \frac{1^2}{4^2} \tag{10.15.4}$$

$$S = \frac{11}{16} \tag{10.15.5}$$

Hence, P($\frac{1}{4}$ < X < 1) is equal to $\frac{11}{16}$

10.16. Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed,then $P(X_1 < X_2)$ is equal to ________

Solution:

a) Given that X_1 is an exponential random variable. Let the P.D.F of X_1 be

$$p_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases} \tag{10.16.1}$$

C.D.F of x_1 is :

$$\begin{aligned} F_{X_1}(x_1) &= \int_{-\infty}^{x_1} p_{X_1}(x_1)dx_1 \\ &= \int_{-\infty}^{0} p_{X_1}(x_1)dx_1 + \int_{0}^{x_1} p_{X_1}(x_1)dx_1 \\ &= \int_{-\infty}^{0} 0 \times dx_1 + \int_{0}^{x_1} \lambda e^{-\lambda x_1} dx_1 \\ &= 1 - e^{-\lambda x_1} \end{aligned} \tag{10.16.2}$$

$$\text{As mean} = \lambda \tag{10.16.3}$$

$$\text{Given that mean} = 1 \tag{10.16.4}$$

$$\text{so } \lambda = 1 \tag{10.16.5}$$

b) Given that X_2 is an gamma random variable.Let the P.D.F of X_2 be:

$$p_{X_2}(x_2) = \begin{cases} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} & x_2 \geq 0 \\ 0 & x_2 < 0 \end{cases} \tag{10.16.6}$$

$$\text{Since mean} = \frac{b}{a} = 2 \tag{10.16.7}$$

$$\text{Also,variance} = \frac{b}{a^2} = 2 \tag{10.16.8}$$

From (10.16.7) and (10.16.8)

$$b = 2, a = 1 \tag{10.16.9}$$

Since the total probability of X_2 is 1 so,

$$\int_{-\infty}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{10.16.10}$$

$$\int_{-\infty}^{0} p_{X_2}(x_2)dx_2 + \int_{0}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{10.16.11}$$

$$\int_{-\infty}^{0} 0 \times dx_2 + \int_{0}^{\infty} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 = 1 \tag{10.16.12}$$

$$\frac{a^b}{\Gamma(b)} \int_{0}^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = 1 \tag{10.16.13}$$

$$\int_{0}^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = \frac{\Gamma(b)}{a^b} \tag{10.16.14}$$

now substituting $a + \lambda$ for a in (10.16.14) gives

$$\int_{0}^{\infty} x_2^{b-1} e^{-(a+\lambda)x_2} dx_2 = \frac{\Gamma(b)}{(a+\lambda)^b} \tag{10.16.15}$$

Now we have to find $P(X_1 < X_2)$

c) Given that X_1 and X_2 are independent random variables,so

$$P(X_1 < X_2|X_2) = F_{X_1}(X_2) = 1 - e^{-\lambda X_2} \tag{10.16.16}$$

Now,

$$P(X_1 < X_2) = \int_{0}^{\infty} F_{X_1}(X_2) \times p_{X_2}(x_2)dx_2 \tag{10.16.17}$$

from (10.16.6),(10.16.16)

$$P(X_1 < X_2) = \int_{0}^{\infty} (1 - e^{-\lambda X_2}) \times \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 \tag{10.16.18}$$

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \int_{0}^{\infty} x_2^{b-1}(e^{-ax_2} - e^{-(a+\lambda)x_2})dx_2 \tag{10.16.19}$$

from (10.16.14) and (10.16.15)

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \left(\frac{\Gamma(b)}{a^b} - \frac{\Gamma(b)}{(a+\lambda)^b} \right) \tag{10.16.20}$$

$$P(X_1 < X_2) = 1 - \frac{a^b}{(a+\lambda)^b} \tag{10.16.21}$$

$$P(X_1 < X_2) = 1 - \left(\frac{a}{a+\lambda} \right)^b \tag{10.16.22}$$

from (10.16.5) and (10.16.9)

$$P(X_1 < X_2) = 1 - \left(\frac{1}{1+1} \right)^2 \tag{10.16.23}$$

$$P(X_1 < X_2) = 1 - \frac{1}{4} = \frac{3}{4} \tag{10.16.24}$$

Common Data for the next two Questions :

10.17. Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given $X = x$, is uniform on the interval $(x-1, x+1)$. Suppose $E(X) = 1$ and $Var(X) = \frac{5}{3}$. The mean of the random variable Y is

a) $\frac{1}{2}$

b) 1

c) $\frac{3}{2}$

d) 2

Solution: We know that,

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \tag{10.17.1}$$

Given that $f_{Y|X=x}(y)$ is uniform over the interval (x-1,x+1).

$$\Rightarrow f_{Y|X=x}(y) = \begin{cases} \frac{1}{2} & y \in (x-1, x+1) \\ 0 & \text{otherwise} \end{cases} \tag{10.17.2}$$

Given $E(X) = 1$

$$\Rightarrow \int_{-\infty}^{\infty} x f_X(x)dx = 1 \tag{10.17.3}$$

Now consider $E(Y|X = x)$,

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy \tag{10.17.4}$$

From (10.17.2) it simplifies to,

$$\Rightarrow E(Y|X=x) = \int_{-\infty}^{x-1} y f_{Y|X=x}(y)dy + \int_{x-1}^{x+1} y f_{Y|X=x}(y)dy + \int_{x+1}^{\infty} y f_{Y|X=x}(y)dy \tag{10.17.5}$$

$$\Rightarrow E(Y|X=x) = \int_{x-1}^{x+1} y\left(\frac{1}{2}\right)dy \tag{10.17.6}$$
$$= x \tag{10.17.7}$$

Now we can write ,

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X=x) f_X(x)dx \tag{10.17.8}$$
$$= \int_{-\infty}^{\infty} x f_X(x)dx \tag{10.17.9}$$
$$= E(X) \tag{10.17.10}$$

From (10.17.3) we get

$$E(Y) = 1. \tag{10.17.11}$$

10.18. The variance of the random variable Y is

a) $\frac{1}{2}$

b) $\frac{2}{3}$

c) 1

d) 2

Solution:

$$Var(Y|X=x) = \int_{-\infty}^{\infty} (y - E(Y))^2 f_{Y|X=x}(y)dy \tag{10.18.1}$$
$$= \int_{x-1}^{x+1} (y-1)^2 \left(\frac{1}{2}\right)dy \tag{10.18.2}$$

$$Var(Y) = \int_{-\infty}^{\infty} Var(Y|X=x) f_X(x)dx \tag{10.18.3}$$
$$= \left(\frac{1}{2}\right)\int_{x-1}^{x+1}\left(y^2 - 2y + 1\right)dy \tag{10.18.4}$$
$$= \left(\frac{1}{2}\right)\left(\frac{6x^2+2}{3} + 2 - 4x\right) \tag{10.18.5}$$
$$= x^2 - 2x + \frac{4}{3} \tag{10.18.6}$$

$$Var(Y) = \int_{-\infty}^{\infty}\left(x^2 - 2x + \frac{4}{3}\right) f_X(x)dx \tag{10.18.7}$$
$$= \int_{-\infty}^{\infty} x^2 f_X(x)dx - 2\int_{-\infty}^{\infty} x f_X(x)dx + \tag{10.18.8}$$
$$\frac{4}{3}\int_{-\infty}^{\infty} f_X(x)dx \tag{10.18.9}$$

$$f_X(x)dx = 1 \tag{10.18.10}$$
$$Var(X) = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{5}{3} \tag{10.18.11}$$
$$E(x) = \int_{-\infty}^{\infty} x f_X(x)dx = 1 \tag{10.18.12}$$
$$\tag{10.18.13}$$

From (10.18.9),(10.18.11),(10.18.12)and (10.18.13) we get

$$Var(Y) = \frac{5}{3} - 2 + \frac{4}{3} \tag{10.18.14}$$
$$= 1 \tag{10.18.15}$$

∴ **Option C is true**

10.19. Let the random variable X have the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

Then $P(2 \le X < 4)$ is equal to ______

Solution:
Given F(X) is the CDF of the random variable X.

$P(2 \le X < 4)$ will be the sum of all the probabilities of values the random variable X can take in $[2, 4)$.
So it is the difference between CDF values of the random variable X at X=4- and at X=2-.
Therefore,

$$P(2 \le X < 4) = \lim_{X \to 4-} F(X) - \lim_{X \to 2-} F(X) \tag{10.19.1}$$

$$= \lim_{X \to 4-} 1 - \lim_{X \to 2-} \frac{3}{5} \tag{10.19.2}$$

$$= 1 - \frac{3}{5} \tag{10.19.3}$$

$$= \frac{2}{5} = 0.4 \tag{10.19.4}$$

Hence,$P(2 \le X < 4) = 0.4$.

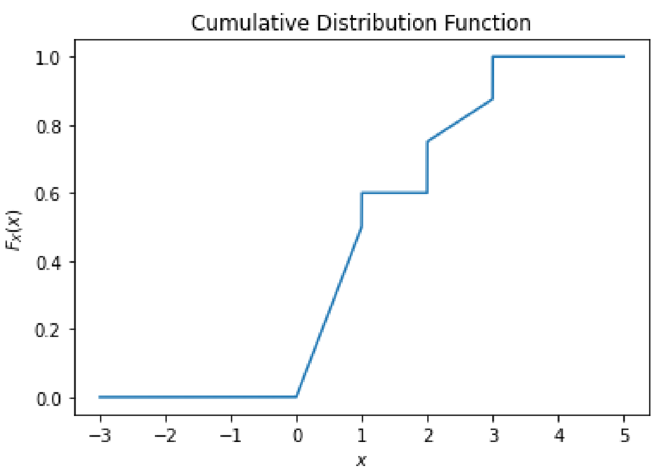


Fig. 10.19.1: CDF of X

10.20. Let X be a random variable having the distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le \ x < 1 \\ \frac{1}{3} & 1 \le \ x < 2 \\ \frac{1}{2} & 2 \le \ x < \frac{11}{3} \\ 1 & x \ge \ \frac{11}{3}. \end{cases}$$

Then $E(X)$ is equal to _______________

10.21. Let $\Omega = (0, 1]$ be the sample space and let $P(\cdot)$ be a probability function defined by

$$P((0, x]) = \begin{cases} \frac{x}{2} & 0 \le \ x < \frac{1}{2} \\ x & \frac{1}{2} \le \ x \le \ 1. \end{cases}$$

Then $P\left(\{\frac{1}{2}\}\right)$ is equal to _______________

10.22. Suppose the random variable U has uniform distribution on $[0, 1]$ and $X = -2 \log U$. The density of X is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0, 2] \\ 0 & \text{otherwise.} \end{cases}$

10.23. Suppose X is a real-valued random variable.Which of the following values **CANNOT** be attained by $E[X]$ and $E[X^2]$, respectively?

a) 0 *and* 1

b) 2 *and* 3

c) $\frac{1}{2}$ *and* $\frac{1}{3}$

d) 2 *and* 5

Solution: We know that

$$var(X) = E[(X - E[X])^2] \tag{10.23.1}$$

$$var(X) = E[X^2] - (E[X])^2 \tag{10.23.2}$$

For uniform distribution in the interval $[a, b]$

$$var(X) = \frac{(b-a)^2}{12} \tag{10.23.3}$$

For uniform distribution,$(b - a)^2 \ge 0$
By definition of variance,it is average value of $(X - E[X])^2$.
Since $(X - E[X])^2 \ge 0$,average $E[(X - E[X])^2] \ge 0$.

$$\therefore var(X) \ge 0 \tag{10.23.4}$$

$$\therefore E[X^2] - (E[X])^2 \ge 0 \tag{10.23.5}$$

a) $E[X] = 0$ and $E[X^2] = 1$

$$E[X^2] - (E[X])^2 = 1 - 0 \tag{10.23.6}$$

$$= 1 \tag{10.23.7}$$

$$\therefore E[X^2] - (E[X])^2 \ge 0 \tag{10.23.8}$$

$$\therefore E[X] = 0 \text{ and } E[X^2] = 1 \text{ can be attained}$$

b) $E[X] = \frac{1}{2}$ and $E[X^2] = \frac{1}{3}$

$$E[X^2] - (E[X])^2 = \frac{1}{3} - \frac{1}{4} \tag{10.23.9}$$

$$= \frac{1}{12} \tag{10.23.10}$$

$$\therefore E[X^2] - (E[X])^2 \ge 0 \tag{10.23.11}$$

$\therefore\ E[X] = \frac{1}{2}$ and $E[X^2] = \frac{1}{3}$ can be attained

c) $E[X] = 2$ and $E[X^2] = 3$

$$E[X^2] - (E[X])^2 = 3 - 4 \quad (10.23.12)$$
$$= -1 \quad (10.23.13)$$
$$\therefore E[X^2] - (E[X])^2 \leq 0 \quad (10.23.14)$$

$\therefore\ E[X] = 2$ and $E[X^2] = 3$ cannot be attained

d) $E[X] = 2$ and $E[X^2] = 5$

$$E[X^2] - (E[X])^2 = 5 - 4 \quad (10.23.15)$$
$$= 1 \quad (10.23.16)$$
$$\therefore E[X^2] - (E[X])^2 \geq 0 \quad (10.23.17)$$

$\therefore\ E[X] = 2$ and $E[X^2] = 5$ can be attained

$\therefore\ E[X] = 2$ and $E[X^2] = 3$ cannot be attained

10.24. Let $\Omega = (0, 1]$ be the sample space and let $P(.)$ be a probability function defined by

$$P((0, x]) = \begin{cases} \frac{x}{2} & 0 \leqslant x < \frac{1}{2} \\ x & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

Then $P\left(\{\frac{1}{2}\}\right)$ is equal to.......

10.25. Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leqslant x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leqslant x < 1 \\ 1 & x \geqslant 1 \end{cases}$$

Then $P(\frac{1}{4} < x < 1)$ is equal to........

Solution:

We know that,

$$P(p < X < q) = F(q^-) - F(p) \quad (10.25.1)$$
$$P\left(\frac{1}{4} < X < 1\right) = F(1^-) - F\left(\frac{1}{4}\right) \quad (10.25.2)$$
$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \quad (10.25.3)$$
$$= \frac{11}{16} \quad (10.25.4)$$
$$= 0.6875 \quad (10.25.5)$$

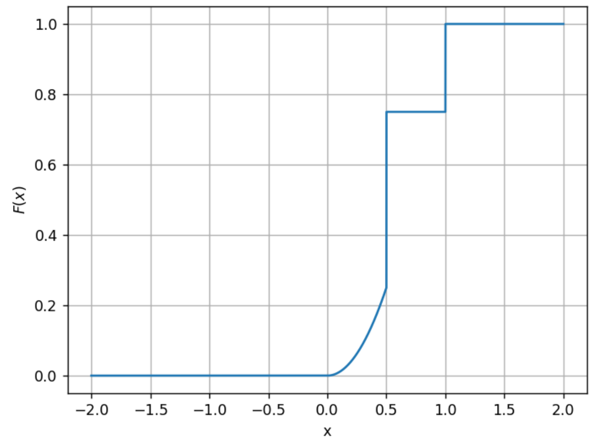


Fig. 10.25.1: The figure depicts the CDF of X

10.26. Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $P(X_1 < X_2)$ is equal to.....

10.27. Suppose the random variable U has uniform distribution on $[0, 1]$ and $X = -2 \log U$. The density of X is

a) $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

d) $f(x) = \begin{cases} \frac{1}{2} & x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$

Solution: U - uniformly distributed random variable on $\in [0,1]$. Probability density function of U is:

$$f_U(u) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (10.27.1)$$

X is given by :

$$X = -2\ln(U) \tag{10.27.2}$$
$$\implies 0 \le X \le \infty \tag{10.27.3}$$

CDF of X is defined as

$$F_X(x) = \Pr(X \le x) \tag{10.27.4}$$
$$= \Pr(-2\ln(U) \le x) \tag{10.27.5}$$
$$= \Pr(\ln(U) \ge (-x)/2) \tag{10.27.6}$$
$$= \Pr(U \ge \exp(-x/2)) \tag{10.27.7}$$
$$= 1 - \Pr(U \le exp(-x/2)) \tag{10.27.8}$$
$$= 1 - exp(-x/2) \tag{10.27.9}$$

where x ∈ [0,∞]
PDF of X :

$$f_X(x) = \frac{d(F_X(x))}{dx} \tag{10.27.10}$$
$$= \frac{1}{2}exp((-x)/2) \tag{10.27.11}$$

we have

$$0 \le X \le \infty \tag{10.27.12}$$

$$f_X(x) = \begin{cases} \frac{1}{2}exp(\frac{-x}{2}) & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{10.27.13}$$

∴ answer will be option (3)

10.28. Suppose X is a real-valued random variable. Which of the following values CANNOT be attained by $E[X]$ and $E[X^2]$, respectively?

a) 0 and 1

c) $\frac{1}{2}$ and $\frac{1}{3}$

b) 2 and 3

d) 2 and 5

Solution: The variance of a distribution is given by

$$\sigma^2 = E[X^2] - E[X]^2 \tag{10.28.1}$$

As variance is always positive,

$$E[X^2] - E[X]^2 \ge 0 \tag{10.28.2}$$

is a necessary condition for any real valued random variable. Computing the value of $E[X^2] - E[X]^2$ for the options, we have

(A) 0 and 1

$$\implies E[X^2] - E[X]^2 = 1 - 0^2 = 1 \ge 0 \tag{10.28.3}$$

(B) 2 and 3

$$\implies E[X^2] - E[X]^2 = 3 - 2^2 = -1 \le 0 \tag{10.28.4}$$

(C) $\frac{1}{2}$ and $\frac{1}{3}$

$$\implies E[X^2] - E[X]^2 = \frac{1}{3} - \frac{1}{2}^2 = \frac{1}{12} \ge 0 \tag{10.28.5}$$

(D) 2 and 5

$$\implies E[X^2] - E[X]^2 = 5 - 2^2 = 1 \ge 0 \tag{10.28.6}$$

10.29. Probability density function $p(x)$ of a random variable x is as shown below. The value of α is

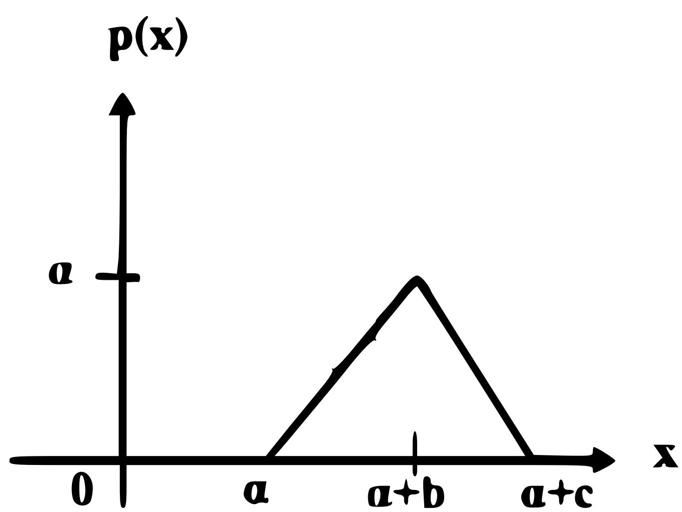


Fig. 10.29.1

a) $\frac{2}{c}$

c) $\frac{2}{(b+c)}$

b) $\frac{1}{c}$

d) $\frac{1}{(b+c)}$

10.30. Let the probability density function of random variable,X,be given as:

$f_x(x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}u(-x)$

where u(x) is the unit step function.Then the value of a and Prob$\{X \leq 0\}$, respectively,are:

(A) $2,\frac{1}{2}$

(B) $4,\frac{1}{2}$

(C) $2,\frac{1}{4}$

(D) $4,\frac{1}{4}$

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1. \tag{10.30.1}$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\infty} f_x(x)\,dx = 1 \tag{10.30.2}$$

$$\int_{-\infty}^{0} ae^{4x}\,dx + \int_{0}^{\infty} \frac{3}{2}e^{-3x}\,dx = 1 \tag{10.30.3}$$

The expression (10.30.3) was written from (10.30.2) since,

$$u(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Simplifying (10.30.3) we have:

$$\int_{-\infty}^{0} ae^{4x}\,dx + \int_{0}^{\infty} \frac{3}{2}e^{-3x}\,dx = 1$$

$$\implies a\left[\frac{e^{4x}}{4}\right]_{-\infty}^{0} + \frac{3}{2}\left[\frac{e^{-3x}}{-3}\right]_{0}^{\infty} = 1 \tag{10.30.4}$$

$$\implies a\left[\frac{1}{4} - 0\right] - \frac{1}{2}[0-1] = 1 \tag{10.30.5}$$

$$\implies \frac{a}{4} + \frac{1}{2} = 1 \implies a = 2 \tag{10.30.6}$$

Therefore,

$$f_x(x) = \begin{cases} \frac{3}{2}e^{-3x}, & \text{for } x \geq 0 \\ 2e^{4x}, & \text{for } x < 0 \end{cases} \tag{10.30.7}$$

The plot for PDF of X can be observed at figure 10.30.1

The CDF of X is defined as follows:

$$F_X(x) = \Pr(X \leq x) \tag{10.30.8}$$

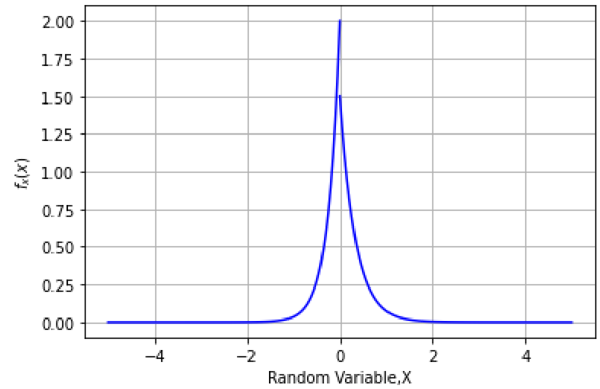


Fig. 10.30.1: The PDF of X

Now for $x < 0$,

$$\Pr(X \leq x) = \int_{-\infty}^{x} f_x(x)\,dx \tag{10.30.9}$$

$$= \int_{-\infty}^{x} 2e^{4x}\,dx \tag{10.30.10}$$

$$= 2\left[\frac{e^{4x}}{4}\right]_{-\infty}^{x} \tag{10.30.11}$$

$$= 2\left[\frac{e^{4x}}{4} - 0\right] \tag{10.30.12}$$

$$= \frac{e^{4x}}{2} \tag{10.30.13}$$

Similarly for $x \geq 0$,

$$\Pr(X \leq x) = \int_{-\infty}^{x} f_x(x)\,dx \tag{10.30.14}$$

$$= \int_{-\infty}^{0} 2e^{4x}\,dx + \int_{0}^{x} \frac{3}{2}e^{-3x}\,dx \tag{10.30.15}$$

$$= 2\left[\frac{e^{4x}}{4}\right]_{-\infty}^{0} + \left[\frac{-e^{-3x}}{2}\right]_{0}^{x} \tag{10.30.16}$$

$$= 2\left[\frac{1}{4} - 0\right] - \frac{1}{2}\left[e^{-3x} - 1\right] \tag{10.30.17}$$

$$= 1 - \frac{e^{-3x}}{2} \tag{10.30.18}$$

The CDF of X is as below:

$$F_X(x) = \begin{cases} 1 - \frac{e^{-3x}}{2}, & \text{for } x \geq 0 \\ \frac{e^{4x}}{2}, & \text{for } x < 0 \end{cases} \tag{10.30.19}$$

The plot for CDF of X can be observed at figure 10.30.1.

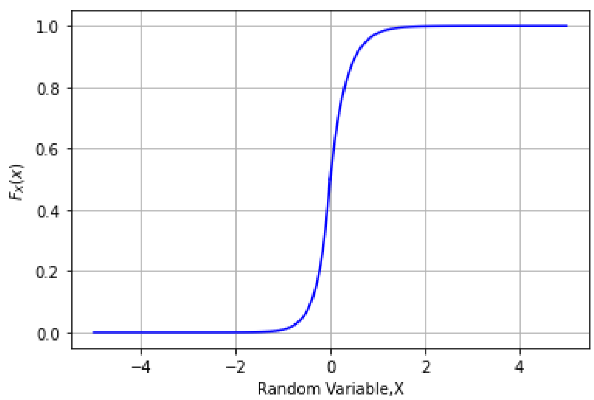


Fig. 10.30.2: The CDF of X

$$\therefore \Pr(X \le 0) = F_X(0) = \frac{1}{2} \tag{10.30.20}$$

10.31. Suppose X_i for $i = 1,2,3$ are independent and identically distributed random variables whose probability mass functions are $\Pr(X_i = 0) = \Pr(X_i = 1) = \frac{1}{2}$ for $i = 1,2,3$. Define another random variable $Y = X_1X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $\Pr(Y = 0|X_3 = 0) =$

Solution:

For

$$\because Y = (X_1X_2) \oplus X_3 = 0 \tag{10.31.1}$$
$$\implies X_1X_2 = X_3 \tag{10.31.2}$$

$$\Pr(Y = 0|X_3 = 0) = \frac{\Pr(Y = 0, X_3 = 0)}{\Pr(X_3 = 0)} \tag{10.31.3}$$
$$= \frac{\Pr(X_1X_2 = X_3, X_3 = 0)}{\Pr(X_3 = 0)} \tag{10.31.4}$$

$$\Pr(X_3 = 0) = \frac{1}{2} \tag{10.31.5}$$

if $X_3 = 0$, from (10.31.2)

$$X_1X_2 = 0 \tag{10.31.6}$$

The random variables are independent of each other:

$\Pr(X_1 = 0, X_2 = 0)$	$\Pr(X_1 = 0) \cdot \Pr(X_2 = 0)$	0.25
$\Pr(X_1 = 1, X_2 = 0)$	$\Pr(X_1 = 1) \cdot \Pr(X_2 = 0)$	0.25
$\Pr(X_1 = 0, X_2 = 1)$	$\Pr(X_1 = 0) \cdot \Pr(X_2 = 1)$	0.25

TABLE 10.31.1: Probabilities

$$\Pr(X_1X_2 = 0) = \Pr(X_1 = 0, X_2 = 0) + \Pr(X_1 = 0, X_2 = 1) + \Pr(X_1 = 1, X_2 = 0) \tag{10.31.7}$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \tag{10.31.8}$$

$$\Pr(Y = 0, X_3 = 0) = \Pr(X_1X_2 = X_3 = 0) \tag{10.31.9}$$
$$= \Pr(X_1X_2 = 0) \cdot \Pr(X_3 = 0) \tag{10.31.10}$$
$$= \frac{3}{4} \cdot \frac{1}{2} \tag{10.31.11}$$
$$= \frac{3}{8} \tag{10.31.12}$$

Upon substituting (10.31.12) and (10.31.5) in (10.31.3)

$$\Pr(Y = 0|X_3 = 0) = \frac{3}{4} = 0.75 \tag{10.31.13}$$

10.32. A continuous random variable X has a probability density function

$$f(x) = e^{-x}, \text{where, } 0 < x < \infty. \tag{10.32.1}$$

Then $\Pr(X > 1)$ is ? **Solution:**
x is uniform with

$$0 < x < \infty. \tag{10.32.2}$$

$$f(x) = e^{-x} \text{ is uniform, with } 0 < f(x) < 1. \tag{10.32.3}$$

Let,

$$F_X(x) \text{ be the cumulative distribution function of X.} \tag{10.32.4}$$

$$\text{As, } 0 < x < \infty, F_X(x) = 0 \text{ for } x < 0 \tag{10.32.5}$$

$$F_X(x) = \Pr(X \le x) = \int_0^x f(x)\,dx = \int_0^x e^{-x}dx \tag{10.32.6}$$
$$= [-e^{-x}]_0^x = (-e^{-x}) - \left(-e^0\right) = 1 - e^{-x} \tag{10.32.7}$$

$$\Pr(X > 1) = 1 - F_X(1) \tag{10.32.8}$$
$$= 1 - \left(1 - e^{-1}\right) = 0.368 \tag{10.32.9}$$

10.33. Assume that the duration in minutes of a telephone conversation follows the exponential distribution f(x) = $\frac{1}{5}e^{-\frac{x}{5}}$, x ≥ 0. The probability that the conversation will exceed five minutes is...

a) $\frac{1}{e}$
b) 1 - $\frac{1}{e}$
c) $\frac{1}{e^2}$
d) 1 - $\frac{1}{e^2}$

Solution:
Let X be a Random variable defined,that denotes the duration of a telephonic conversation in minutes.
So, X ∈ [0,∞)
Given, $f_X(x) = \frac{1}{5}e^{-\frac{x}{5}}$
Let CDF of X be $F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$
$$= \int_{-\infty}^{0} f_X(t)\,dt + \int_{0}^{x} f_X(t)\,dt$$
$$F_X(x) = \int_{0}^{x} f_X(t)\,dt \because f_X(x) = 0 \forall x < 0$$
$$\therefore F_X(x) = \int_{0}^{x} \frac{1}{5}e^{-\frac{t}{5}}\,dt$$
$$\implies F_X(x) = 1 - e^{-\frac{x}{5}} \qquad (1)$$

$$F_X(x) = \Pr(X \le x)$$
$$\Pr(X > 5) = 1 - \Pr(X \le 5)$$
$$\implies \Pr(X > 5) = 1 - F_X(5)$$
$$= 1 - (1 - e^{-\frac{5}{5}})$$
$$= e^{-\frac{5}{5}}$$
$$= e^{-1} = \frac{1}{e}$$

10.34. Suppose Y is distributed uniformly in the open interval (1,6). The probability that the polynomial $3x^2+6xY+3Y+6$ has only real roots is (rounded off to 1 decimal place) **Solution:** Given, Y has a uniform distribution in the interval (1,6). This implies, the probability density function of Y,

$$f(y) = \begin{cases} \frac{1}{6-1} = \frac{1}{5} & (1 < y < 6) \\ 0 & \text{otherwise} \end{cases} \qquad (10.34.1)$$

From this, cumulative distribution function of Y,

$$F_Y(y) = \begin{cases} \frac{y-1}{5} & (1 < y < 6) \\ 0 & y \le 1 \\ 1 & y \ge 6 \end{cases} \qquad (10.34.2)$$

Given polynomial: $3x^2+(6Y)x+(3Y+6)$ Comparing it with the form: ax^2+bx+c
Here, a=3; b=6Y; c=3$Y+6$ Condition for real roots,

$$b^2 - 4ac \ge 0 \qquad (10.34.3)$$
$$(6Y)^2 - 4(3)(3Y+6) \ge 0 \qquad (10.34.4)$$
$$Y^2 - Y - 2 \ge 0 \qquad (10.34.5)$$
$$(Y-2)(Y+1) \ge 0 \qquad (10.34.6)$$

$$\therefore Y \le -1, Y \ge 2 \qquad (10.34.7)$$

Probability that the given polynomial has real roots is,

$$P(Y \le -1) + P(Y \ge 2) = F_Y(-1) + 1 - F_Y(2^-) \qquad (10.34.8)$$
$$= 0 + 1 - \left(\frac{2-1}{5}\right) \qquad (10.34.9)$$
$$= 0.8 \qquad (10.34.10)$$

10.35. Let X be a continuous random variable denoting the temperature measured. The range of temperature is [0,100] degree Celsius and let probability density function of X be $f(x)$=0.01 for $0 \le X \le 100$.
The mean of X is ?

(A) 2.5
(B) 5.0
(C) 25.0
(D) 50.0

Solution: Given X is a continuous random variable. The probability density function of X is $f(x)$

$$f(x) = \begin{cases} 0.01 & 0 \le x \le 100 \\ 0 & \text{otherwise} \end{cases} \qquad (10.35.1)$$

Mean of the random variable X is μ

$$\mu = \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x = \int_{0}^{100} x\,(0.01)\, \mathrm{d}x \quad (10.35.2)$$

$$= (0.01) \int_{0}^{100} x\, \mathrm{d}x = (0.01) \left. \frac{x^2}{2} \right|_0^{100} \quad (10.35.3)$$

$$= 50.0 \text{ degree Celsius} \quad (10.35.4)$$

10.36. The PDF of a Gaussian random variable X is given by $P_X(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-4)^2}{18}}$.The probability of the event X = 4 is

a) $\frac{1}{2}$

b) $\frac{1}{3\sqrt{2\pi}}$

c) 0

d) $\frac{1}{4}$

Solution: Given PDF function is

$$P_X(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-4)^2}{18}} \quad (10.36.1)$$

Since continuous probability functions are defined for an infinite number of points over a continuous interval, the probability at a single point is always zero.

$$\Pr(x) = \lim_{\delta \to 0} \int_{x}^{x+\delta} \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-4)^2}{18}} dx \quad (10.36.2)$$

$$= 0 \quad (10.36.3)$$

Hence the probability is 0.

10.37. Probability density function $p(x)$ of random variable x is as shown below. The value of a is

A) $\frac{2}{c}$
B) $\frac{1}{c}$
C) $\frac{2}{(b+c)}$
D) $\frac{1}{(b+c)}$

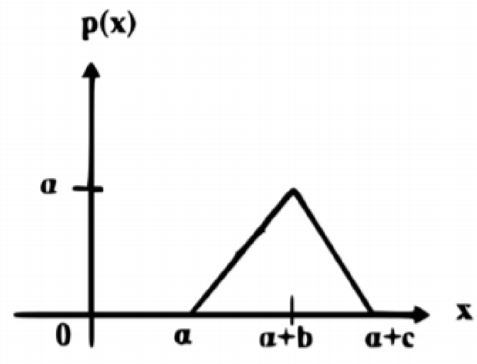


Fig. 10.37.1: PDF

Solution:
Let Y_1 and Y_2 be two independent and identically distributed (IID) uniform random variables.
Let X be a random variable such that

$$X = Y_1 + Y_2 \quad (10.37.1)$$

Let

$$p_{Y_1}(y) = \Pr(Y_1 = y) \quad (10.37.2)$$
$$p_{Y_2}(y) = \Pr(Y_2 = y) \quad (10.37.3)$$
$$p_X(x) = \Pr(X = x) \quad (10.37.4)$$

be the probability densities of random variables Y_1, Y_2 and X.
Y_1 and Y_2 lie in the range $\left(\frac{-c}{4}, \frac{c}{4}\right)$, therefore, the PDF for Y_1 and Y_2,

$$p_{Y_1}(y) = p_{Y_2}(y) = \begin{cases} \frac{2}{c} & \frac{-c}{4} \le y \le \frac{c}{4} \\ 0 & \text{otherwise} \end{cases} \quad (10.37.5)$$

The density of X is obtained by convolution of Y_1 and Y_2

$$p_X(x) = p_{Y_1}(x) * p_{Y_2}(x) \quad (10.37.6)$$

where $*$ denotes the convolution operation. Since convolution operation is time invariant,

$$p_X(x-t) = p_{Y_1}(x-t) * p_{Y_2}(x) = p_{Y_1}(x) * p_{Y_2}(x-t) \quad (10.37.7)$$

On time shifting Y_1 by shifting factor $t = a + \frac{c}{2}$,

$$p_X\left(x - \left(a + \frac{c}{2}\right)\right) = p_{Y_1}\left(x - \left(a + \frac{c}{2}\right)\right) * p_{Y_2}(x) \quad (10.37.8)$$

Thus, the PDF of time shifted X obtained by convolution is,

$$p_x = \begin{cases} \frac{4}{c^2}(x-a) & a \le x \le a + \frac{c}{2} \\ \frac{4}{c^2}(a+c-x) & a + \frac{c}{2} \le x \le a + c \\ 0 & \text{otherwise} \end{cases} \tag{10.37.9}$$

On comparing the parameters of PDF of time shifted X with that in the question, we have

$$b = \frac{c}{2} \tag{10.37.10}$$

$$a = \frac{2}{c} \tag{10.37.11}$$

Answer : Option A

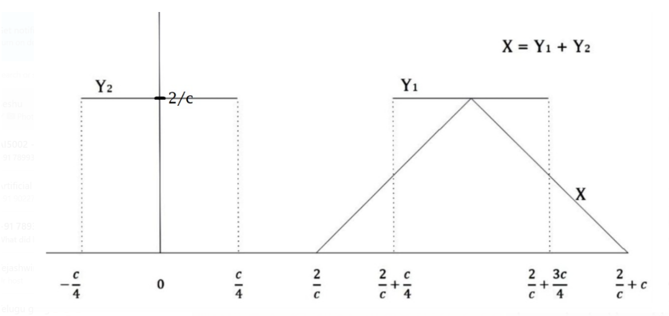


Fig. 10.37.2: PDF of time shifted X

The following are some observations:

a) The sum of two equally distributed random variables will lead to a triangular probability density
b) The two uniformly distributed random variables lie in the range $\left(\frac{-c}{4}, \frac{c}{4}\right)$ and $\left(\frac{2}{c}+\frac{c}{4}, \frac{2}{c}+\frac{3c}{4}\right)$.
$\because X = Y_1 + Y_2$ the range of X is thus $\left(\frac{2}{c}, \frac{2}{c}+c\right)$
c) On time shifting Y_1 to the right by a factor $a + \frac{c}{2}$, the convoluted PDF of X also shifts by the same factor without any change in it's width.

Fig 10.37.3 and Fig 10.37.4 are the plots of PDF and CDF obtained by taking c=2

10.38. Let X be a random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{10.38.1}$$

Then $P\left(\frac{1}{4} < X < 1\right)$ is equal to

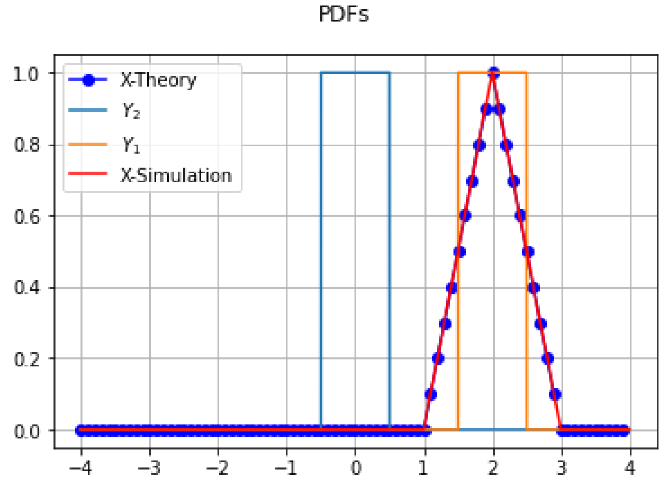


Fig. 10.37.3: PDF of Y_1, Y_2 and X

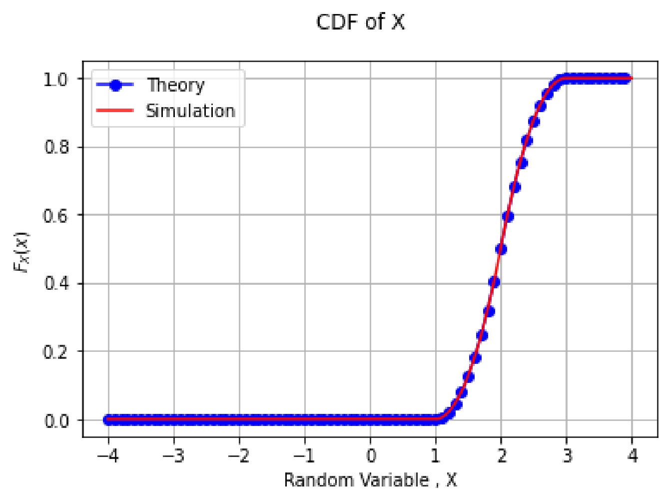


Fig. 10.37.4: CDF of X

Solution:

$$P\left(\frac{1}{4} < X < 1\right) = F\left(1^-\right) - F\left(\frac{1}{4}\right) \tag{10.38.2}$$

$$= \frac{3}{4} - \left(\frac{1}{4}\right)^2 \tag{10.38.3}$$

$$= \frac{11}{16} \tag{10.38.4}$$

10.39. A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

The probability density function of $Y = 3X + 2$

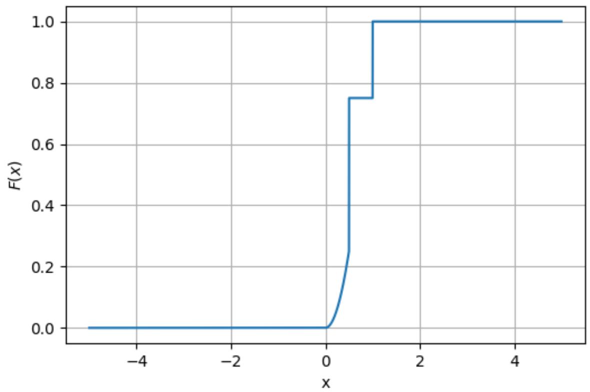


Fig. 10.38.1: The CDF of X

is

a)

$$f(y) = \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

b)

$$f(y) = \begin{cases} \frac{2}{5}e^{-\frac{2}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

c)

$$f(y) = \begin{cases} \frac{3}{5}e^{-\frac{3}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

d)

$$f(y) = \begin{cases} \frac{4}{5}e^{-\frac{4}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases}$$

Solution: Given $Y = 3X + 2$
CDF of Y,

$$\begin{aligned} F_y(Y) &= \Pr(Y \le y) \\ &= \Pr\left(X \le \frac{y-2}{3}\right) \\ &= F_x\left(\frac{y-2}{3}\right) \end{aligned}$$

Thus, pdf of Y ,

$$\begin{aligned} f_Y(y) &= \frac{1}{3} f_X\left(\frac{y-2}{3}\right) \\ &= \frac{1}{3} \times \begin{cases} \frac{3}{5}e^{-\frac{3}{5}\left(\frac{y-2}{3}\right)} & y > 2 \\ 0 & y \le 2 \end{cases} \\ &= \begin{cases} \frac{1}{5}e^{-\frac{1}{5}(y-2)} & y > 2 \\ 0 & y \le 2 \end{cases} \end{aligned}$$

Hence, correct option is 1.

10.40. Let the probability density function of a random variable X be

$$f(x) = \begin{cases} x & 0 \leqslant x < \frac{1}{2} \\ c(2x-1)^2 & \frac{1}{2} \leqslant x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Then value of c is equal to ...

Solution: We know that,

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1 \tag{10.40.1}$$

$$\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{\frac{1}{2}} f_x(x)\,dx + \int_{\frac{1}{2}}^{1} f_x(x)\,dx + \int_{1}^{\infty} f_x(x)\,dx = 1 \tag{10.40.2}$$

$$\int_{0}^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} c(2x-1)^2\,dx = 1 \tag{10.40.3}$$

$$\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + c\left[\frac{(2x-1)^3}{6}\right]_{\frac{1}{2}}^{1} = 1 \tag{10.40.4}$$

$$\frac{1}{8} + \frac{c}{6} = 1 \tag{10.40.5}$$

$$c = \frac{21}{4} \tag{10.40.6}$$

$$\therefore \text{ Required value of } c = \frac{21}{4}$$

10.41. Let the random variable X have the distribution function $F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \le x < 1 \\ \frac{3}{5} & if\ 1 \le x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \le x < 3 \\ 1 & if\ x \ge 3 \end{cases}$

Then $\Pr(2 \le x < 4)$ is equal to

Solution:

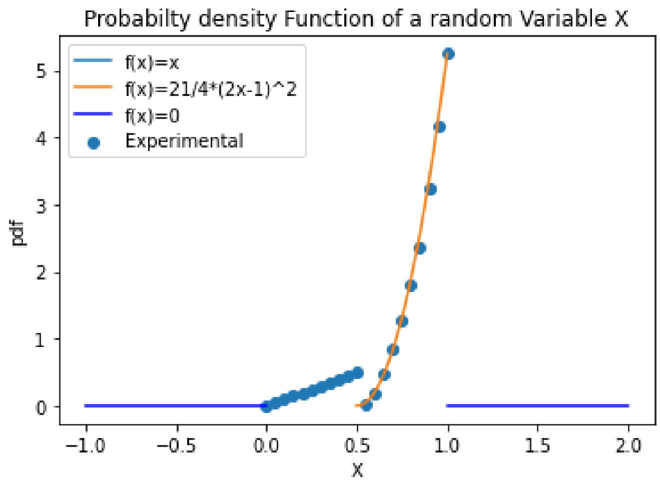


Fig. 10.40.1: Experimental and Theoritical pdf of X

Given,

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{x}{2} & if\ 0 \leq x < 1 \\ \frac{3}{5} & if\ 1 \leq x < 2 \\ \frac{1}{2} + \frac{x}{8} & if\ 2 \leq x < 3 \\ 1 & if\ x \geq 3 \end{cases} \tag{10.41.1}$$

We need to find $\Pr(2 \leq x < 4)$,which is also can be written as

$$\Pr(2 \leq x < 4) = \Pr(x < 4) - \Pr(x < 2) \tag{10.41.2}$$

$$= F(X = 4^{-}) - F(X = 2^{-}) \tag{10.41.3}$$

Using (10.41.1) in (10.41.3),

$$\Pr(2 \leq x < 4) = 1 - \frac{3}{5} \tag{10.41.4}$$

$$= \frac{2}{5} \tag{10.41.5}$$

$$= 0.4 \tag{10.41.6}$$

10.42. Let X and Y have joint probability function given by

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq x \leq 1 - y, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

If f_Y denotes the marginal probability density function of Y, then $f_Y(1/2) =$?

Solution:

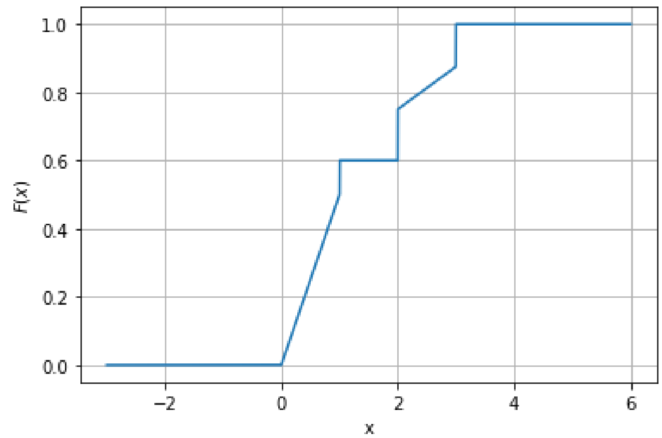


Fig. 10.41.1: cdf of random variable X

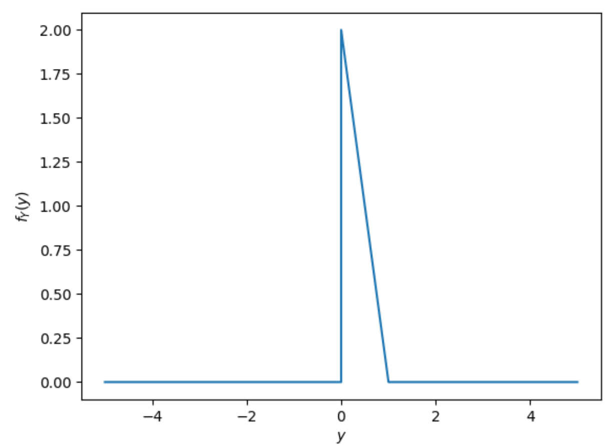


Fig. 10.42.1: Marginal PDF

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y).dx \tag{23.1}$$

$$\implies f_Y(y) = \begin{cases} 0 + \int_0^{1-y} 2.dx & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{23.2}$$

$$\implies f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{23.3}$$

$$\therefore f_Y(1/2) = 1 \tag{23.4}$$

10.43. Let X be a random variable with probability mass function $p(n) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1}$ $n = 1, 2 \ldots$ Then $E[X - 3|X > 3]$ is $\ldots$

Solution:
Given

$$\Pr(X=n)=\begin{cases}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} & n=1,2\ldots\\ 0 & otherwise\end{cases} \tag{10.43.1}$$

Using the linearity of the expectation operator:

$$E[X-3\mid X>3]=E[X\mid X>3]-3 \tag{10.43.2}$$

Now ,

$$E[X\mid X>3]=\sum_{x=1}^{\infty}x\Pr(X=x\mid X>3) \tag{10.43.3}$$

$$=\sum_{x=1}^{\infty}x\frac{\Pr(X=x,X>3)}{\Pr(X>3)} \tag{10.43.4}$$

Calculating $\Pr(X>3)$

$$\Pr(X>3)=1-\Pr(X\le 3) \tag{10.43.5}$$

$$=1-\sum_{x'=1}^{3}\Pr(X=x') \tag{10.43.6}$$

$$=1-\sum_{x'=1}^{3}\left(\frac{3}{4}\right)^{x'-1}\left(\frac{1}{4}\right) \tag{10.43.7}$$

$$=\frac{27}{64} \tag{10.43.8}$$

Also,

$$\Pr(X=x,X>3)=\begin{cases}\Pr(X=x) & x>3\\ 0 & x\le 3\end{cases} \tag{10.43.9}$$

Substituting (10.43.8) and (10.43.9) in (10.43.4) we get

$$E[X\mid X>3]=\sum_{x=1}^{3}0+\sum_{x=4}^{\infty}\left[x\frac{\Pr(X=x)}{\frac{27}{64}}\right] \tag{10.43.10}$$

$$=\frac{64}{27}\sum_{x=4}^{\infty}\left[x\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{x-1}\right] \tag{10.43.11}$$

$$=\frac{16}{27}\sum_{x=4}^{\infty}\left[x\left(\frac{3}{4}\right)^{x-1}\right] \tag{10.43.12}$$

Let

$$S=\sum_{x=4}^{\infty}\left[x\left(\frac{3}{4}\right)^{x-1}\right] \tag{10.43.13}$$

Multiplying ((10.43.13)) with $\frac{3}{4}$ on both sides gives

$$\frac{3}{4}S=\sum_{x=4}^{\infty}x\frac{1}{4}\left(\frac{3}{4}\right)^{x} \tag{10.43.14}$$

From (10.43.14) and(10.43.13)

$$S=4\left(\frac{3}{4}\right)^{3}+5\left(\frac{3}{4}\right)^{4}+6\left(\frac{3}{4}\right)^{5}\ldots \tag{10.43.15}$$

$$\frac{3}{4}S=0\left(\frac{3}{4}\right)^{3}+4\left(\frac{3}{4}\right)^{4}+5\left(\frac{3}{4}\right)^{5}+\ldots \tag{10.43.16}$$

subtracting (10.43.14) from (10.43.13) we get

$$\frac{S}{4}=4\left(\frac{3}{4}\right)^{3}+\left(\frac{3}{4}\right)^{4}+\left(\frac{3}{4}\right)^{5}+\left(\frac{3}{4}\right)^{6}+\ldots \tag{10.43.17}$$

$$=4\left(\frac{3}{4}\right)^{3}+\sum_{x=4}^{\infty}\left(\frac{3}{4}\right)^{x} \tag{10.43.18}$$

$$=\frac{189}{64} \tag{10.43.19}$$

Substituting vale of S in (10.43.12) we get

$$E[X|X>3]=7 \tag{10.43.20}$$

Thus putting this in (10.43.2)

$$E[X-3|X>3]=4 \tag{10.43.21}$$

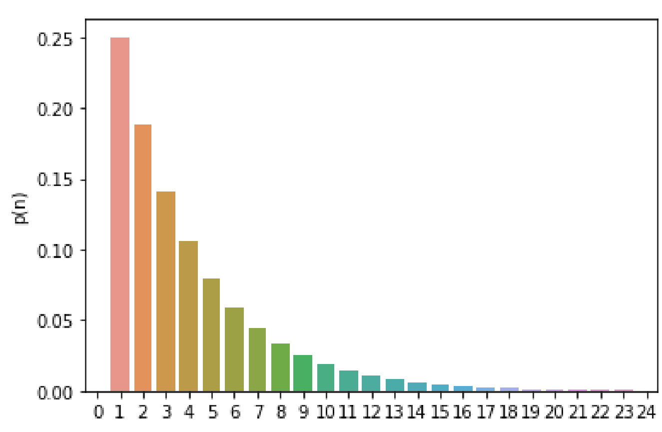


Fig. 10.43.1: PMF of X

10.44. Consider the trinomial distribution with the probability mass function

$$\Pr(X = x, Y = y) = \left(\frac{7!}{x!y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$

where $x \geq 0, y \geq 0$ *and* $x + y \leq 7$. Then $E(Y|X = 3)$ is equal to

Solution: Probability mass function of a trinomial distribution is :

$$\Pr(X = x, Y = y) = \left(\frac{7!}{x!y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$

$$= \left(\frac{7!}{x!(7-x)!}\frac{(7-x)!}{y!(7-x-y)!}\right)(0.6)^x(0.2)^y(0.2)^{7-x-y}$$

$$\Pr(X = x, Y = y) = {}^7C_x{}^{7-x}C_y(0.6)^x(0.2)^y(0.2)^{7-x-y} \quad (10.44.1)$$

Using (10.44.1), $\Pr(X = x)$ is

$$\Pr(X = x) = \sum_{y=0}^{7-x}\Pr(X = x, Y = y)$$

$$= {}^7C_x(0.6)^x\sum_{y=0}^{7-x}{}^{7-x}C_y(0.2)^y(0.2)^{7-x-y}$$

$$= {}^7C_x(0.6)^x(0.4)^{7-x}$$

$$\Pr(X = x) = {}^7C_x(0.6)^x(0.4)^{7-x} \quad (10.44.2)$$

We have to find $E[Y|X = 3]$,

$$E[Y|X = 3] = \sum_{y=0}^{4} y\Pr(Y = y|X = 3) \quad (10.44.3)$$

$$E[Y|X = 3] = \sum_{y=0}^{4} y\left(\frac{\Pr(X = 3, Y = y)}{\Pr(X = 3)}\right) \quad (10.44.4)$$

By taking X=3 in (10.44.1) and (10.44.2) to use in (10.44.4),

$$E[Y|X = 3] = \sum_{y=0}^{4} y\left(\frac{\Pr(X = 3, Y = y)}{\Pr(X = 3)}\right)$$

$$= \sum_{y=0}^{4} y\left(\frac{{}^7C_3{}^4C_y(0.6)^3(0.2)^y(0.2)^{4-y}}{{}^7C_3(0.6)^3(0.4)^4}\right)$$

$$= \sum_{y=0}^{4} y\left(\frac{{}^4C_y(0.2)^4}{(0.4)^4}\right)$$

$$E[Y|X = 3] = \sum_{y=0}^{4}\frac{y({}^4C_y)}{16} \quad (10.44.5)$$

We know that,

$${}^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right) \quad (10.44.6)$$

Using (10.44.6) in (10.44.5),

$$E[Y|X = 3] = \frac{1}{16}\sum_{y=0}^{4} y({}^4C_y) \quad (10.44.7)$$

$$= \frac{1}{16}\sum_{y=1}^{4} y\left(\frac{4}{y}\right)({}^3C_{y-1}) \quad (10.44.8)$$

$$= \frac{1}{4}\sum_{k=0}^{3}({}^3C_k) \quad (10.44.9)$$

$$= \frac{1}{4}(1+1)^3 = \frac{1}{4}(8) \quad (10.44.10)$$

$$E[Y|X = 3] = 2 \quad (10.44.11)$$

Therefore the value of $E[Y|X = 3] = 2$.

10.45. If a random variable X assumes only positive integral values ,with the probability

$$P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1}, x = 1, 2, 3, ..., \quad (10.45.1)$$

then $E(X)$ is

a) $\frac{2}{9}$

b) $\frac{2}{3}$

c) 1

d) $\frac{3}{2}$

Solution: Given that random variable X assumes only positive integral values and its probability is:

$$P(X = x) = \frac{2}{3}\left(\frac{1}{3}\right)^{x-1} \quad (10.45.2)$$

The expectation value E(X) is given by

$$E(X) = \sum_{i=1}^{\infty} i \times P(X = i) \tag{10.45.3}$$

Let $E(X) = S$
so,

$$S = \sum_{i=1}^{\infty} i \times P(X = i) \tag{10.45.4}$$

$$\implies S = \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{10.45.5}$$

$$\implies S = \frac{2}{3} + \sum_{i=2}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} \tag{10.45.6}$$

As

$$\sum_{i=2}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i-1} = \sum_{i=1}^{\infty} (i+1) \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{10.45.7}$$

Now substituting (10.45.7) in (10.45.6)

$$\implies S = \frac{2}{3} + \sum_{i=1}^{\infty} (i+1) \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{10.45.8}$$

$$\implies S = \frac{2}{3} + \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} + \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{10.45.9}$$

Dividing with 3 on both sides in (10.45.5) gives

$$\frac{S}{3} = \sum_{i=1}^{\infty} i \times \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{10.45.10}$$

Now substituting (10.45.10) in (10.45.9) gives

$$\implies S = \frac{2}{3} + \frac{S}{3} + \sum_{i=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{i} \tag{10.45.11}$$

$$\implies \frac{2S}{3} = \frac{2}{3} + \frac{2}{3}\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i} \tag{10.45.12}$$

$$\implies \frac{2S}{3} = \frac{2}{3}\left(1 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i}\right) \tag{10.45.13}$$

$$\implies S = 1 + \sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^{i} \tag{10.45.14}$$

$$\implies S = 1 + \frac{\frac{1}{3}}{1 - \frac{1}{3}} \tag{10.45.15}$$

$$\implies S = 1 + \frac{1}{2} = \frac{3}{2} \tag{10.45.16}$$

$$\implies E(X) = S = \frac{3}{2} \tag{10.45.17}$$

$\therefore$ **Option D is correct**

10.46. Let the cumulative distribution function of the random variable X be given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ (1+x)/2 & 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Then $\Pr(X = 1/2) =$? **Solution:**
Given,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ \frac{(1+x)}{2} & 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{24.1}$$

$$\Pr(X = 1/2) = \Pr(X \le 1/2) - \Pr(X < 1/2) \tag{24.2}$$

$$\implies \Pr(X = 1/2) = F_X\left(\frac{1}{2}\right) - F_X\left(\frac{1}{2}^{-}\right) \tag{24.3}$$

Using (24.1) in (24.3),

$$\implies \Pr(X = 1/2) = \frac{(1 + 1/2)}{2} - (1/2) \tag{24.4}$$

$$\implies \Pr(X = 1/2) = (3/4) - (1/2) \tag{24.5}$$

$$\therefore \Pr(X = 1/2) = 1/4 \tag{24.5}$$

The cdf plot of random variable X is as shown in Fig. 10.46.1

10.47. Let $\Omega = (0, 1]$ be the sample space and let $P(.)$ be a probability distribution given by

$$P((0, x]) = \begin{cases} \frac{x}{2} & 0 \le x < \frac{1}{2} \\ x & \frac{1}{2} \le x \le 1 \end{cases} \tag{10.47.1}$$

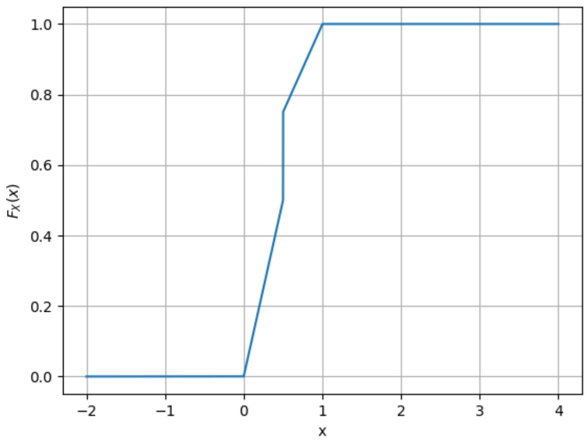


Fig. 10.46.1: cdf plot of random variable X

Find $P\left(\frac{1}{2}\right)$
Solution:
CDF of X is defined as,

$$F_X(x) = \Pr(X \leq x) \tag{10.47.2}$$

$\because x > 0$

$$F_X(x) = P((0, x]) \tag{10.47.3}$$

Thus, CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < \frac{1}{2} \\ x & \frac{1}{2} \leq x \leq 1 \\ 1 & x \geq 1 \end{cases} \tag{10.47.4}$$

$$\Pr\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}^{-}\right) \tag{10.47.5}$$

$$= \frac{1}{2} - \frac{1/2}{2} \tag{10.47.6}$$

$$= \frac{1}{4} \tag{10.47.7}$$

The plot of CDF is given in the Figure 10.47.1

10.48. Let X be a random variable having probability density function

$$f(x) = \begin{cases} \frac{3}{13}(1-x)(9-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.48.1}$$

Then $\frac{4}{3}E[X(X^2 - 15X + 27)]$ equals — (round of to two decimal places).

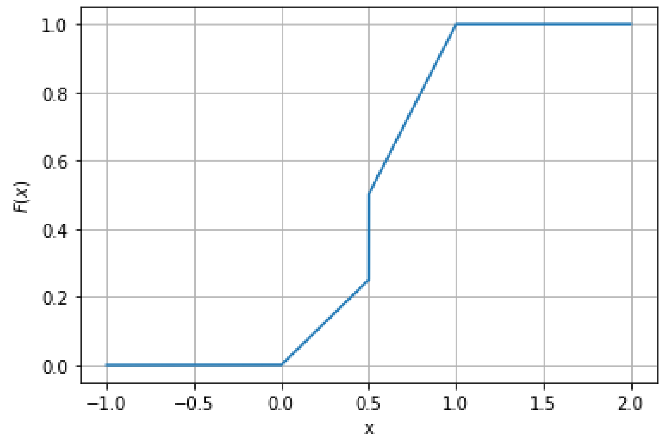


Fig. 10.47.1: CDF of X

Solution:
Let X be the random variable. To find

$$\frac{4}{3}E[X(X^2 - 15X + 27)] \tag{10.48.2}$$

Let,

$$g(X) = X(X^2 - 15X + 27) \tag{10.48.3}$$

$$= X^3 - 15X^2 + 27X \tag{10.48.4}$$

Then for random variable X we have that,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx \tag{10.48.5}$$

The probability distribution of X is,

$$f(x) = \begin{cases} \frac{3}{13}(1-x)(9-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.48.6}$$

Using 10.48.6 we have,

$$E[g(X)] = 0 + \int_{0}^{1} g(x)f(x)\,dx + 0 \tag{10.48.7}$$

Where ,

$$f(x) = \frac{3}{13}(1-x)(9-x) \text{ and} \tag{10.48.8}$$

$$g(x) = x^3 - 15x^2 + 27x \tag{10.48.9}$$

Using Integration by substitution let,

$$t = x^3 - 15x^2 + 27x$$

$$dt = 3x^2 - 30x + 27$$

$$= 3(1-x)(9-x)$$

The corresponding limits are,

$$\text{For } x = 0 \implies t = 0^3 - 15 \times 0^2 + 27 \times 0 = 0 \tag{10.48.10}$$

$$\text{For } x = 1 \implies t = 1^3 - 15 \times 1^2 + 27 \times 1 = 13 \tag{10.48.11}$$

Therefore we have,

$$E[g(X)] = \frac{1}{13}\int_0^{13} t\,dt \tag{10.48.12}$$

$$= \frac{1}{13} \times \left(\frac{t^2}{2}\right)\Bigg|_0^{13} \tag{10.48.13}$$

$$= \frac{1}{13} \times \frac{13^2}{2} \tag{10.48.14}$$

$$= \frac{13}{2} \tag{10.48.15}$$

Thus,

$$\frac{4}{3}E[g(X)] = \frac{4}{3} \times \frac{13}{2} \tag{10.48.16}$$

$$= \frac{26}{3} \tag{10.48.17}$$

$$= 8.67 \text{ (rounded off)} \tag{10.48.18}$$

Therefore,

$$\frac{4}{3}E[X(X^2 - 15X + 27)] = 8.67 \tag{10.48.19}$$

10.49. Let X be a random variable with uniform distribution on the interval $(-1, 1)$ and $Y = (X+1)^2$. Then the probability density function $f(y)$ of Y, over the interval (0,4), is

a) $\frac{3\sqrt{y}}{16}$

b) $\frac{1}{4\sqrt{y}}$

c) $\frac{1}{6\sqrt{y}}$

d) $\frac{1}{\sqrt{y}}$

Solution: We know that, since $Y = (X + 1)^2$,

$$F_Y(y) = 0 \ \forall\ y < 0 \tag{10.49.1}$$

Therefore, for $y \geq 0$,

$$F_Y(y) = \Pr\left((x+1)^2 \leq y\right) \tag{10.49.2}$$

$$= \Pr\left(-\sqrt{y} - 1 \leq x \leq \sqrt{y} - 1\right) \tag{10.49.3}$$

$$= \Pr\left(-\sqrt{y} - 1 \leq x \leq \sqrt{y} - 1\right) \tag{10.49.4}$$

$$= F_X(\sqrt{y} - 1) - F_X(-\sqrt{y} - 1) \tag{10.49.5}$$

Since X is a uniform random variable in $(-1, 1)$,

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.49.6}$$

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{x}{2} + \frac{1}{2} & -1 < x < 1 \\ 1 & x \geq 1 \end{cases} \tag{10.49.7}$$

Using (10.49.7) in (10.49.5), and using the fact that

$$-\sqrt{y} - 1 \leq -1 \ \forall\ y \geq 0, \tag{10.49.8}$$

we get

$$F_Y(y) = \begin{cases} F_X(\sqrt{y} - 1) - 0 & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{10.49.9}$$

$$= \begin{cases} 0 & y < 0 \\ \frac{\sqrt{y}}{2} & 0 \leq y \leq 4 \\ 1 & y > 4 \end{cases} \tag{10.49.10}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} & 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases} \tag{10.49.11}$$

Therefore, **option 2** is correct. Fig. 10.49.1 shows a theoretical vs simulated plot of the PDF of random variable Y.

11 Independence

11.1. Two independent random variables X and Y are uniformly distributed in the interval [-1,1].The probability that max $\{X,Y\}$ is less than $\frac{1}{2}$ is

A) 3/4

B) 9/16

C) 1/4

D) 2/3

Solution:

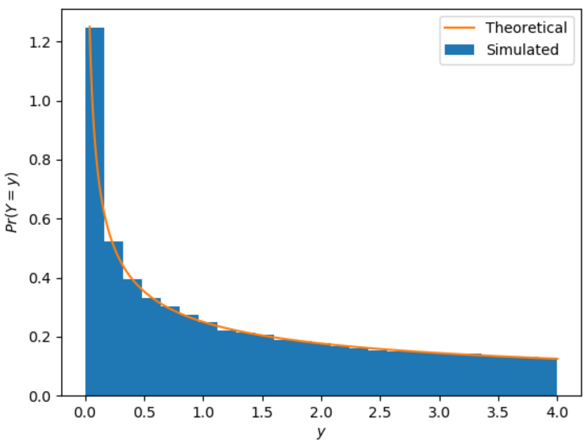


Fig. 10.49.1: The PDF of Y

Lemma 11.1. *CDF of the random variable X is :*

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2}(x+1) & -1 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{11.1.1}$$

Proof. Given X is uniformly distributed in [-1,1] i.e. $X \sim$ U(-1,1)
PDF of X :

$$f_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2} & -1 \le x \le 1 \\ 0 & x \ge 1 \end{cases} \tag{11.1.2}$$

For $-1 \le x \le 1$

$$F_X(x) = \Pr(X \le x) \tag{11.1.3}$$
$$= \int_{-1}^{x} \frac{1}{2} dx \tag{11.1.4}$$
$$= \frac{1}{2}(x+1) \tag{11.1.5}$$

Hence (11.1.1) is proved □

Lemma 11.2. *CDF of the random variable Y is :*

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2}(x+1) & -1 < x < 1 \\ 1 & x \ge 1 \end{cases} \tag{11.1.6}$$

Proof. Given Y is uniformly distributed in [-1,1] i.e. $Y \sim$ U(-1,1)
PDF of Y :

$$f_Y(y) = \begin{cases} 0 & y \le -1 \\ \frac{1}{2} & -1 \le y \le 1 \\ 0 & y \ge 1 \end{cases} \tag{11.1.7}$$

For $-1 \le y \le 1$

$$F_Y(y) = P(Y \le y) \tag{11.1.8}$$
$$= \int_{-1}^{y} \frac{1}{2} dy \tag{11.1.9}$$
$$= \frac{1}{2}(y+1) \tag{11.1.10}$$

Hence (11.1.6) is proved □

Lemma 11.3.

$$\Pr\left(max\{X,Y\} < \frac{1}{2}\right) = \frac{9}{16} \tag{11.1.11}$$

Proof. $max\{X,Y\} < \frac{1}{2} \implies X < \frac{1}{2}, Y < \frac{1}{2}$
Given X and Y are independent,

$$Pr(X < \frac{1}{2}, Y < \frac{1}{2}) \tag{11.1.12}$$
$$= \Pr\left(X < \frac{1}{2}\right) \times \Pr\left(Y < \frac{1}{2}\right) \tag{11.1.13}$$
$$= F_X\left(\frac{1}{2}\right) \times F_Y\left(\frac{1}{2}\right) \tag{11.1.14}$$
$$= \frac{3}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \tag{11.1.15}$$
$$= \frac{9}{16} \tag{11.1.16}$$

Hence (11.1.11) is proved □

Option B is correct

11.2. Three fair cubical dice are thrown simultaneously. The probability that all three dice have the same number of dots on the faces showing up is (up to third decimal place)...........
Solution: Let

$$X_1, X_2, X_3 \in \{1,2,3,4,5,6\} \tag{11.2.1}$$

represent the three dice.
Since, all the three are fair dice, the probability of any dice showing a particular number is given by

$$\Pr(X = i) = \begin{cases} \frac{1}{6} & \text{i=1,2,3,4,5,6} \\ 0 & \text{otherwise} \end{cases} \tag{11.2.2}$$

If all the dice show a particular number i,

$$\implies \Pr(X_1 = X_2 = X_3 = i) \quad (11.2.3)$$

Since the events are independent,

$$\Pr(X_1 = X_2 = X_3 = i) = \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \quad (11.2.4)$$

where i=1,2,3,4,5,6.
There are 6 faces on a cubical dice. Hence, there are six cases in which all the dice show the same number

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = X_2 = X_3 = i) \quad (11.2.5)$$

From (11.2.4),we have

$$\Pr(X_1 = X_2 = X_3) = \sum_{i=1}^{6} \Pr(X_1 = i)\Pr(X_2 = i)\Pr(X_3 = i) \quad (11.2.6)$$

$$= \sum_{i=1}^{6} \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \quad (11.2.7)$$

$$= \frac{1}{36} \quad (11.2.8)$$

11.3. Given Set A = [2,3,4,5] and Set B = [11,12,13,14,15], two numbers are randomly selected,one from each set. What is probability that the sum of the two numbers equals 16?

a) 0.20 b) 0.25 c) 0.30 d) 0.33

Solution: Given,
Set A= [2,3,4,5]
Set B= [11,12,13,14,15]

Total number of element in the sample space is 20.

Let us define a random variable$X \in \{0, 1\}$

X=0	the event when A+B=16
X=1	the event when A+B $\neq$ 16

TABLE 11.3.1: Random Variables

Now, probability of selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = 1 \quad (7.1)$$

So,the probability of selecting an element from set B after selecting an element from set A such that $\Pr(X = 0)$ is

$$\Pr(X = 0) = \Pr(A + B = 16) = \frac{1}{5} \quad (7.2)$$

Therefore,
Overall probability of randomly choosing elements from set A and set B such that $\Pr(X = 0)$is

$$\Pr(X = 0) = \Pr(A + B = 16) \quad (7.3)$$

$$\Pr(X = 0) = 1 \times \frac{1}{5} \quad (7.4)$$

$$\Pr(X = 0) = \frac{1}{5} = 0.2 \quad (7.5)$$

X	0	1
$\Pr(X)$	$\frac{1}{5}$	$\frac{4}{5}$

TABLE 11.3.2: Probability distribution table

Therefore, the correct option is (a).

11.4. Two independent random variables X and Y are uniformly distributed in the interval $[-1, 1]$.The probability that $\max[X, Y]$ is less than $\frac{1}{2}$ is

a) $\frac{3}{4}$ b) $\frac{9}{16}$ c) $\frac{1}{4}$ d) $\frac{2}{3}$

11.5. A fair dice is tossed two times. The probability that the second toss result in a value that is higher than the first toss is

a) $\frac{2}{36}$ b) $\frac{2}{6}$ c) $\frac{5}{12}$ d) $\frac{1}{2}$

Solution: Given, a fair die, which is tossed twice. Let the random variable $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, represent the outcome of the number on the die in the first, second toss respectively. The probability mass function (PMF) for a fair die is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6}, & 1 \leq n \leq 6 \\ 0, & otherwise \end{cases} \quad (26.1)$$

Using (26.1), the cumulative distribution function (CDF) is obtained to be

$$F_{X_i}(r) = \Pr(X_i \le r) = \begin{cases} \frac{r}{6}, & 1 \le r \le 6 \\ 1, & r \ge 7 \\ 0, & otherwise \end{cases} \quad (26.2)$$

$$X_1 < X_2 \Rightarrow X_2 = k, X_1 \le k-1 \quad (26.3)$$

$\because X_1, X_2$ are independent,

$$Pr(X_1 < X_2) = E\left[F_{X_1}(X_2 - 1)\right] \quad (26.4)$$

After unconditioning (26.4), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} p_{X_2}(k) F_{X_1}(k-1) \quad (26.5)$$

Substituting (26.1) and (26.2), we get

$$Pr(X_1 < X_2) = \sum_{k=1}^{6} \frac{1}{6}\left(\frac{k-1}{6}\right) \quad (26.6)$$

On solving, we get

$$Pr(X_1 < X_2) = \frac{5}{12}(\text{option (C)}) \quad (26.7)$$

TABLE 11.5.1: Cases and their theoretical probabilities

Case	$X_1 < X_2$	$X_1 > X_2$	$X_1 = X_2$
Probability	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{6}$

11.6. Consider two independent random variables X and Y with identical distributions. The variables X and Y take value 0, 1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ rrespectively. What is the conditional probability $P(X+Y = 2|X-Y = 0)$?

a) 0 b) $\frac{1}{16}$ c) $\frac{1}{6}$ d) 1

Solution: The values that the random variable X can take along with its probabilities are given by

X	0	1	2
$\Pr(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The values that the random variable Y can take along with its probabilities are given by

Y	0	1	2
$\Pr(Y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$$\Pr(X - Y = 0) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{6}{16} \quad (11.6.1)$$

$$\Pr((X + Y = 2), (X - Y = 0)) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \quad (11.6.2)$$

$$\Pr(X + Y = 2 \mid X - Y = 0) = \frac{\Pr((X + Y = 2), (X - Y = 0))}{\Pr(X - Y = 0)} = \frac{\frac{1}{16}}{\frac{6}{16}} = \frac{1}{6} \quad (11.6.3)$$

11.7. Let X and Y be two statistically independent random variables uniformly distributed in the range $(-1, 1)$ and $(-2, 1)$ respectively. Let $Z = X + Y$, then the probability that $[Z \leqslant -2]$ is

a) zero b) $\frac{1}{6}$ c) $\frac{1}{3}$ d) $\frac{1}{12}$

Solution:
X and Y are two independent random variables. Let

$$p_X(x) = \Pr(X = x) \quad (11.7.1)$$

$$p_Y(y) = \Pr(Y = y) \quad (11.7.2)$$

$$p_Z(z) = \Pr(Z = z) \quad (11.7.3)$$

be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} p_X(x)\, dx = 1 \quad (11.7.4)$$

$$2 \times p_X(x) = 1 \quad (11.7.5)$$

$$p_X(x) = 1/2 \quad (11.7.6)$$

Similarly for Y we have,

$$\int_{-2}^{1} p_Y(y)\, dy = 1 \quad (11.7.7)$$

$$3 \times p_Y(y) = 1 \quad (11.7.8)$$

$$p_Y(y) = 1/3 \quad (11.7.9)$$

The density for X is

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & otherwise \end{cases} \tag{11.7.10}$$

We have ,

$$Z = X + Y \iff z = x + y \iff x = z - y \tag{11.7.11}$$

The density of X can also be represented as,

$$p_X(z-y) = \begin{cases} \frac{1}{2} & -1 \le z - y \le 1 \\ 0 & otherwise \end{cases} \tag{11.7.12}$$

and the density of Y is,

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \le y \le 1 \\ 0 & otherwise \end{cases} \tag{11.7.13}$$

The density of Z i.e. $Z = X + Y$ is given by the convolution of the densities of X and Y

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z-y)p_Y(y)\,dy \tag{11.7.14}$$

From 11.7.12 and 11.7.13 we have,
The integrand is $\frac{1}{6}$ when,

$$2 \le y \le 1 \tag{11.7.15}$$
$$-1 \le z - y \le 1 \tag{11.7.16}$$
$$z - 1 \le y \le z + 1 \tag{11.7.17}$$

and zero, otherwise.
Now when $-3 \le z \le -2$ them we have,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\,dy \tag{11.7.18}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{11.7.19}$$
$$= \frac{1}{6}(z+3) \tag{11.7.20}$$

For $-2 < z \le -1$,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\,dy \tag{11.7.21}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{11.7.22}$$
$$= \frac{1}{6}(z+3) \tag{11.7.23}$$

For $-1 < z \le 0$,

$$p_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\,dy \tag{11.7.24}$$
$$= \frac{1}{6} \times (z + 1 - (z-1)) \tag{11.7.25}$$
$$= \frac{1}{3} \tag{11.7.26}$$

For $0 < z \le 2$,

$$p_Z(z) = \int_{z-1}^{1} \frac{1}{6}\,dy \tag{11.7.27}$$
$$= \frac{1}{6} \times (1 - (z-1)) \tag{11.7.28}$$
$$= \frac{1}{6}(2-z) \tag{11.7.29}$$

Therefore the density of Z is given by

$$p_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \le z \le -2 \\ \frac{1}{6}(z+3) & -2 < z \le -1 \\ \frac{1}{3} & -1 < z \le 0 \\ \frac{1}{6}(2-z) & 0 < z \le 2 \\ 0 & otherwise \end{cases} \tag{11.7.30}$$

The CDF of Z is defined as,

$$F_Z(z) = \Pr(Z \le z) \tag{11.7.31}$$

Now for $z \le -1$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{11.7.32}$$
$$= \int_{-3}^{z} \frac{1}{6}(z+3)\,dz \tag{11.7.33}$$
$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Bigg|_{-3}^{z} \tag{11.7.34}$$
$$= \frac{1}{6} \times \left(\left(\frac{z^2}{2} + 3z\right) - \left(\frac{9}{2} - 9\right)\right) \tag{11.7.35}$$
$$= \frac{z^2 + 6z + 9}{12} \tag{11.7.36}$$

Similarly for $z \le 0$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{11.7.37}$$
$$= \frac{1}{3} + \int_{-1}^{z} \frac{1}{3}\,dz \tag{11.7.38}$$
$$= \frac{z+2}{3} \tag{11.7.39}$$

finally for $z \le 2$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{11.7.40}$$
$$= \frac{2}{3} + \int_0^z \frac{1}{6}(2-z)\,dz \tag{11.7.41}$$
$$= \frac{2}{3} + \frac{4z - z^2}{12} \tag{11.7.42}$$
$$= \frac{8 + 4z - z^2}{12} \tag{11.7.43}$$

The CDF is as below,

$$F_Z(z) = \begin{cases} 0 & z < 3 \\ \frac{z^2+6z+9}{12} & z \le -1 \\ \frac{z+2}{3} & z \le 0 \\ \frac{8+4z-z^2}{12} & z \le 2 \\ 1 & z > 2 \end{cases} \tag{11.7.44}$$

So

$$\Pr(Z \le -2) = F_Z(2) \tag{11.7.45}$$
$$= \frac{1}{12} \tag{11.7.46}$$

i.e. option (D).
The plot for PDF of Z can be observed at figure 11.7.1 and the plot for CDF of Z is at figure 11.7.2.

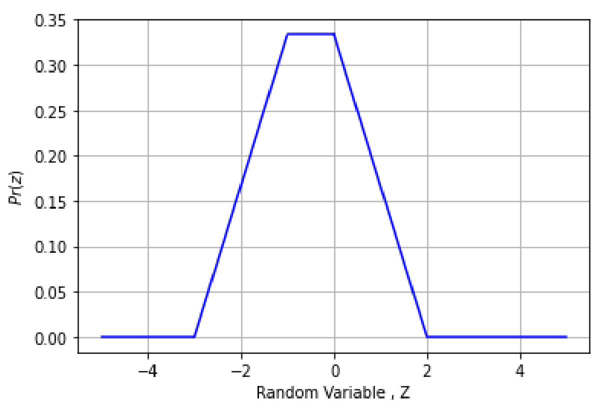


Fig. 11.7.1: The PDF of Z

11.8. Let X_1, X_2, X_3 and X_4 be independent normal random variables with zero mean and unit variance. The probability that X_4 is the smallest among the four is.....

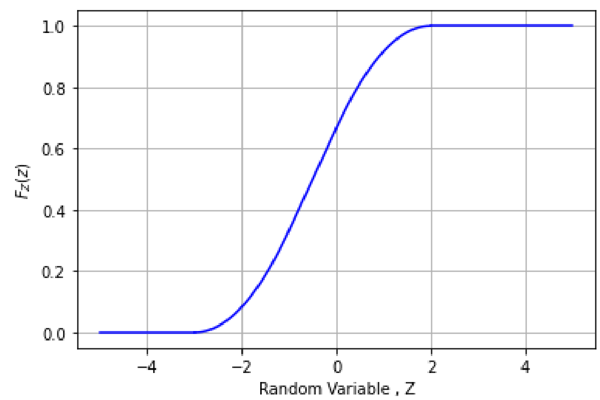


Fig. 11.7.2: The CDF of Z

Solution: Required probability

$$= \Pr(X_4 = min(X_1, X_2, X_3, X_4)) \tag{11.8.1}$$
$$= \int_{-\infty}^{\infty} \Pr(X_1, X_2, X_3 > x | X_4 = x) \tag{11.8.2}$$

Since X_1, X_2, X_3 and X_4 are independent, required probability

$$= \int_{-\infty}^{\infty} (1 - F_{X_1}(x))(1 - F_{X_2}(x))(1 - F_{X_3}(x)) f_{X_4}(x) dx \tag{11.8.3}$$

$$= \int_{-\infty}^{\infty} (1 - \Phi(x))^3 \phi(x) dx \tag{11.8.4}$$

Substituting

$$u = 1 - \Phi(x) \tag{11.8.5}$$
$$du = -\phi(x) dx \tag{11.8.6}$$

we get required probability

$$= -\int_1^0 u^3 du \tag{11.8.7}$$
$$= \frac{1}{4} \tag{11.8.8}$$

Note that in eq. (11.8.7) the integral is from 1 to 0 because

$$1 - \Phi(-\infty) = 1 \tag{11.8.9}$$
$$1 - \Phi(\infty) = 0 \tag{11.8.10}$$

Here $\phi(x)$ and $\Phi(x)$ represent the pdf and cdf of standard normal random variable respectively.

11.9. Let $A_1, A_2, ...A_n$ be n independent events in which the Probability of occurence of the event A_i is given by P(A_i) = 1 - $\frac{1}{\alpha^i}$, $\alpha > 1$, i = 1,2,3,....n.Then the probability that atleast one of the events occurs is

a) 1 - $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

b) $\frac{1}{\alpha^{\frac{n(n+1)}{2}}}$

c) $\frac{1}{\alpha^n}$

d) 1 - $\frac{1}{\alpha^n}$

Solution: Let $A_1 + A_2 + A_3 .... + A_n$ = S,
$\Pr(S)$ = Probability of atleast one event occuring De morgan's law states that $(A+B)' = A'B'$

$$\implies \Pr(S) = 1 - \Pr(S') \quad (11.9.1)$$

$$1 - \Pr(S') = 1 - \Pr(A_1'A_2'A_3'....A_n') \quad (11.9.2)$$

$\forall$ i $\in$ 1,2,....n
Since, A_i are independent.
$\therefore$ Complements of A_i are also independent.
$\implies$

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \Pr(A_i') \quad (11.9.3)$$

$$\Pr(A_i) = 1 - \frac{1}{\alpha^i} \implies \Pr(A_i') = \frac{1}{\alpha^i} \quad (11.9.4)$$

substituting (11.9.4) in (11.9.3),

$$\Pr(A_1'A_2'A_3'....A_n') = \prod_{i=1}^{n} \frac{1}{\alpha^i} \quad (11.9.5)$$

$$\prod_{i=1}^{n} \frac{1}{\alpha^i} = \frac{1}{\alpha^{\sum_i^n i}} = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (11.9.6)$$

$$\therefore \Pr(A_1'A_2'A_3'....A_n') = \Pr(S') = \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (11.9.7)$$

from equations (11.9.2) and (11.9.7)

$$\implies \Pr(S) = 1 - \Pr(S') = 1 - \frac{1}{\alpha^{\frac{n(n+1)}{2}}} \quad (11.9.8)$$

$\therefore$ The correct option is **(a)**

11.10. Let $X_1, X_2, \dots,$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = \frac{1}{4}$ and $P(X_1 = 2) = \frac{3}{4}$. If $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, for $n = 1, 2, \dots,$ then $\lim_{n\to\infty} P(\bar{X}_n \leq 1.8)$ is equal to

Solution:

Given,

$$Pr(X_1 = 1) = \frac{1}{4}, Pr(X_2 = 2) = \frac{3}{4} \quad (32.1)$$

As $X_1, X_2, \dots,$ are identically distributed random variables, $\forall i \in \{1, 2, \dots, n\}$

$$Pr(X_i = 1) = \frac{1}{4}, Pr(X_i = 2) = \frac{3}{4} \quad (32.2)$$

Also,

$$\because P(X_i = 1) + P(X_i = 2) = 1 \quad (32.3)$$

$$\therefore X_i \in \{1, 2\} \quad (32.4)$$

Therefore, each X_i is a bernoulli distribution with

$$p = \frac{3}{4}, q = \frac{1}{4} \quad (32.5)$$

Let

$$X = \sum_{i=1}^{n} X_i \quad (32.6)$$

be a binomial distribution. Its CDF is

$$Pr(X \leq n + r) = \sum_{k=0}^{r} {}^nC_k p^k q^{n-k} \quad (32.7)$$

To find : $\lim_{n\to\infty} Pr(\bar{X}_n \leq a)$

$$\bar{X}_n \leq a \Rightarrow X \leq na \quad (32.8)$$

Substituting $a(= 1.8), p, q,$ we get

$$\lim_{n\to\infty} Pr(\bar{X}_n \leq 1.8) = \lim_{n\to\infty} P(X \leq 1.8n) \quad (32.9)$$

$$= \sum_{k=0}^{0.8n} \frac{{}^nC_k 3^k}{4^n} \quad (32.10)$$

On solving (32.10), we get

$$\lim_{n\to\infty} P(\bar{X}_n \leq 1.8) = 1 \quad (32.11)$$

11.11. Let $\{X_n\}_{n\geq1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on [0,3]. Let Y be a random variable, independent of $\{X_n\}_{n\geq1}$, having probability mass function

$$\Pr(Y = k) = \begin{cases} \frac{1}{(e-1)k!} & k = 1, 2, 3 \cdots \\ 0 & otherwise \end{cases} \quad (11.11.1)$$

Then $\Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1)$ equals

....

Solution:
Given that $\{X_n\}_{n\geq 1}$ is having a uniform distribution on [0,3], so probability can be written as

$$\Pr(X_n)_{n\geq 1} = \begin{cases} \frac{1}{3} & 0 \leq X_n \leq 3 \\ 0 & otherwise \end{cases} \tag{11.11.2}$$

So,

$$\Pr(X_n \leq 1)_{n\geq 1} = \frac{1}{3} \tag{11.11.3}$$

Required probability

$$= \Pr(max\{X_1, X_2, \cdots X_Y\} \leq 1) \tag{11.11.4}$$

Since, $\{X_n\}_{n\geq 1}$ is a sequence of independent variables and Y is also independent of $\{X_n\}_{n\geq 1}$. And also in (11.11.4), the index of $X_i's$ depends on Y, so number of terms depends on Y, like if $Y = 1$, then there is only X_1, if $Y = 2$, then there's X_1, X_2, so required probability

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1 | Y = p\right) \cdot \Pr(Y = p) \tag{11.11.5}$$

$$= \sum_{p=1}^{\infty} \Pr\left(max\{X_1, X_2, \cdots X_p\} \leq 1\right) \cdot \Pr(Y = p) \tag{11.11.6}$$

$$= \sum_{p=1}^{\infty} \Pr\left(X_1, X_2, \cdots X_p \leq 1\right) \cdot \Pr(Y = p) \tag{11.11.7}$$

$$= \sum_{p=1}^{\infty} \Pr(X_1 \leq 1) \cdot \Pr(X_2 \leq 1) \cdots \Pr\left(X_{p-1} \leq 1\right) \cdot \Pr\left(X_p \leq 1\right) \cdot \Pr(Y = p) \tag{11.11.8}$$

$$= \sum_{p=1}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{e-1}\right)\left(\frac{1}{p!}\right) \tag{11.11.9}$$

$$= \left(\frac{1}{e-1}\right)\left[\sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p \left(\frac{1}{p!}\right) - 1\right] \tag{11.11.10}$$

Using Taylor's Series of e^x in (11.11.10),

Required probability

$$= \frac{e^{1/3}}{e-1} - \frac{1}{e-1} \tag{11.11.11}$$
$$= 0.23 \tag{11.11.12}$$

11.12. Let X_1, X_2 and X_3 be independent and identically distributed random variables with $E(X_1) = 0$ and $E\left(X_1^2\right) = \frac{15}{4}$. If $\psi : (0, \infty) \rightarrow (0, \infty)$ is defined through the conditional expectiation $\psi(t) = E\left(X_1^2 | X_1^2 + X_2^2 + X_3^2 = t\right), t > 0$ Then, $E(\psi((X_1 + X_2)^2))$ is equal to,
Solution: It is given that X_1, X_2 and X_3 are independent and identically distributed random variables.

$$E\left(X_1^2 | X_1^2 + X_2^2 + X_3^2 = t\right) = E\left(X_2^2 | X_1^2 + X_2^2 + X_3^2 = t\right) = E\left(X_3^2 | X_1^2 + X_2^2 + X_3^2 = t\right) \tag{11.12.1}$$

Now,

$$\sum_{n=1}^{3} E\left(X_n^2 | X_1^2 + X_2^2 + X_3^2 = t\right) = E\left(X_1^2 + X_2^2 + X_3^2 | X_1^2 + X_2^2 + X_3^2 = t\right) \tag{11.12.2}$$
$$= t \tag{11.12.3}$$

Hence, from (11.12.1).

$$E\left(X_1^2 | X_1^2 + X_2^2 + X_3^2 = t\right) = \frac{t}{3} \tag{11.12.4}$$
$$\therefore \psi(t) = \frac{t}{3} \tag{11.12.5}$$

Hence, from (11.12.5),

$$E(\psi((X_1+X_2)^2)) = E\left(\frac{(X_1+X_2)^2)}{3}\right) \quad (11.12.6)$$

$$= E\left(\frac{X_1^2+X_2^2+2X_1\times X_2}{3}\right) \quad (11.12.7)$$

$$= \frac{E(X_1^2)+E(X_2^2)+2\times E(X_1)\times E(X_2)}{3} \quad (11.12.8)$$

$$= \frac{\frac{15}{4}+\frac{15}{4}+2\times 0\times 0}{3} \quad (11.12.9)$$

$$= \frac{15}{6} \quad (11.12.10)$$

$$\therefore E(\psi((X_1+X_2)^2)) = 2.5 \quad (11.12.11)$$

11.13. Let $X \sim B\left(5, \frac{1}{2}\right)$ and $Y \sim U(0,1)$. The the value of:

$$\frac{\Pr(X+Y\le 2)}{\Pr(X+Y\ge 5)}$$

is equal to? (X and Y are independent) **Solution:** Characteristic function for $X \sim B\left(5, \frac{1}{2}\right)$ will be:

$$C_X(t) = \left(\frac{e^{it}+1}{2}\right)^5 \quad (11.13.1)$$

Characteristic function for $Y \sim U(0,1)$ will be:

$$C_Y(t) = \frac{e^{it}-1}{it} \quad (11.13.2)$$

Since both X and Y are independent we can take:

$$Z = X+Y \quad (11.13.3)$$

$$C_Z(t) = C_X(t)C_Y(t) \quad (11.13.4)$$

$$C_Z(t) = \frac{(e^{it}+1)^5(e^{it}-1)}{32it} \quad (11.13.5)$$

Applying Gil-Pelaez formula:

$$F_Z(z) = \frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\left(e^{-itz}C_Z(t)\right)}{t}dt \quad (11.13.6)$$

$$F_Z(z) = \frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{1}{2it}\left(\frac{(e^{it}+1)^5(e^{it}-1)e^{-itz}}{32it}\right) + \frac{1}{2it}\left(\frac{(e^{-it}+1)^5(e^{-it}-1)e^{itz}}{32it}\right)dt$$

Substituting $z = 2$, the value for $\Pr(Z \le 2)$:

$$= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \frac{8\cos 2t+2\cos 4t}{64t^2}dt + \frac{1}{\pi}\int_0^\infty \frac{+8\cos 3t-8\cos t-10}{64t^2}dt \quad (11.13.7)$$

Finding a general expression for integrating:

$$\int \frac{\cos ax}{x^2}dx = -\frac{\cos ax}{x} - a\int \frac{\sin ax}{x}dx + C \quad (11.13.8)$$

By applying integration by parts. Now finding the value of other integral, by substituting $u = ax$ for limits as 0 and ∞:

$$\int_0^\infty \frac{a\sin ax}{x}dx = \int_0^\infty \frac{a\sin u}{u}du \quad (11.13.9)$$

$$= \frac{a\pi}{2} \quad (11.13.10)$$

Now using the above general expressions to calculate (11.13.7) and simplifying the expression after putting the limits we get

$$= \frac{-1}{8\pi}\left(\int_0^\infty \frac{\sin 4t+3\sin 3t+2\sin 2t-\sin t}{t}dt\right) \quad (11.13.11)$$

$$- \left.\frac{2(\cos t-1)(\cos t+1)^3}{8\pi t}\right|_0^\infty + \frac{1}{2} \quad (11.13.12)$$

$$= \frac{1}{2}+\frac{-1}{8\pi}\times\frac{5\pi}{2}+0 \quad (11.13.13)$$

$$= \frac{3}{16} \quad (11.13.14)$$

Using (11.13.10) and (11.13.8) to calculate for our second case Similarly on substituting $z = 5$, the value for $\Pr(Z \le 5)$:

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{-10\cos 3t - 8\cos 4t}{64t^2}dt + \frac{1}{\pi}\int_0^\infty \frac{-2\cos 5t + 12\cos t + 8}{64t^2}dt \tag{11.13.15}$$

$$= \frac{1}{\pi}\left(\int_0^\infty \frac{5\sin 5t + 16\sin 4t + 15\sin 3t - 6\sin t}{32}dt\right) + \frac{1}{2} + \frac{1}{\pi}\left(\frac{16(\cos t - 1)(\cos t)(\cos t + 1)^3}{32t}\Bigg|_0^\infty\right) \tag{11.13.16}$$

$$= \frac{1}{2} + \frac{1}{\pi} \times \frac{15\pi}{32} + 0 \tag{11.13.17}$$

$$= \frac{31}{32} \tag{11.13.18}$$

The value for $\Pr(Z \ge 5)$:

$$\Pr(Z > 5) = 1 - \Pr(Z \le 5) \tag{11.13.19}$$

$$= 1 - \frac{31}{32} = \frac{1}{32} \tag{11.13.20}$$

Upon substituting (11.13.14)and (11.13.20), we get:

$$\frac{\Pr(X + Y \le 2)}{\Pr(X + Y \ge 5)} = 6 \tag{11.13.21}$$

11.14. A die is thrown again and again until three sixes are obtained.Find the probability of obtaining the third six in the sixth row of a die.

12 Integral Transforms

12.1. Let X_n denote the sum of points obtained when n fair dice are rolled together. The expectation and variance of X_n are

a) $\frac{7}{2}n$ and $\frac{35}{12}n^2$ respectively.

b) $\frac{7}{2}n$ and $\frac{35}{12}n$ respectively.

c) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively.

d) None of the above

Solution: We know, when one dice is rolled probability i.e $\Pr(X_1 = r)$ for all r in {1,2,3,4,5,6} is equal to p

$$p = \frac{1}{6} \tag{12.1.1}$$

Let Y_i denote the value obtained on ith dice when n dices are rolled , therefore

$$X_n = \sum_{i=1}^{n} Y_i \tag{12.1.2}$$

Now i will calculate expectation value of value obtained when one dice is rolled using below formula;

$$E(Y_i) = E(X_1) = \sum_{r=1}^{6}(r \times p) \tag{12.1.3}$$

$$= \frac{1}{6} \times \sum_{r=1}^{6} r \tag{12.1.4}$$

$$= \frac{7}{2}. \tag{12.1.5}$$

a) Since the Expectation value of a sum of independent events is the sum of their expectation. So,

$$E(X_n) = \sum_{i=1}^{n} E(Y_i) \tag{12.1.6}$$

$$= \sum_{i=1}^{n} \frac{7}{2} = \frac{7}{2}n \tag{12.1.7}$$

b) By Using the following formula ,we can calculate variance of X_1 ,

$$V(X_1) = (E(X_1)^2) - (E(X_1))^2 \tag{12.1.8}$$

$$\sum_{i=1}^{k} r^2 = \frac{k \times (k+1) \times (2(k)+1)}{6} \tag{12.1.9}$$

Now calculating E(X_1^2),by using (12.1.9)

$$E(X_1^2) = \sum_{r=1}^{6}(r^2 \times p) \tag{12.1.10}$$

$$= \frac{1}{6} \times \sum_{r=1}^{6} r^2 \tag{12.1.11}$$

$$= \frac{91}{6} \tag{12.1.12}$$

By using (12.1.5),(12.1.8) and (12.1.12)

$$V(X_1) = V(Y_i) \tag{12.1.13}$$

$$= \frac{35}{12} \tag{12.1.14}$$

Variance of sum can be calculated by using following formula,

$$V(X_n) = V(\sum_{i=0}^{n} Y_i) \tag{12.1.15}$$

$$= \sum_{i=1}^{n} V(Y_i) + \sum_{1\le i\ne j\le n} \text{Cov}(Y_i, Y_j) \tag{12.1.16}$$

Since Co-variance of independent random variables is zero.So,

$$V(X_n) = \sum_{i=1}^{n} V(Y_i) + 0 \tag{12.1.17}$$

$$= \frac{35}{12}n \tag{12.1.18}$$

Hence option(B) is correct.

12.2. Consider that X and Y are independent continuous valued random variables with uniform PDF given by $X \sim U(2,3)$ and $Y \sim U(1,4)$. Then $\Pr(Y \le X)$ is equal to .....
Solution:

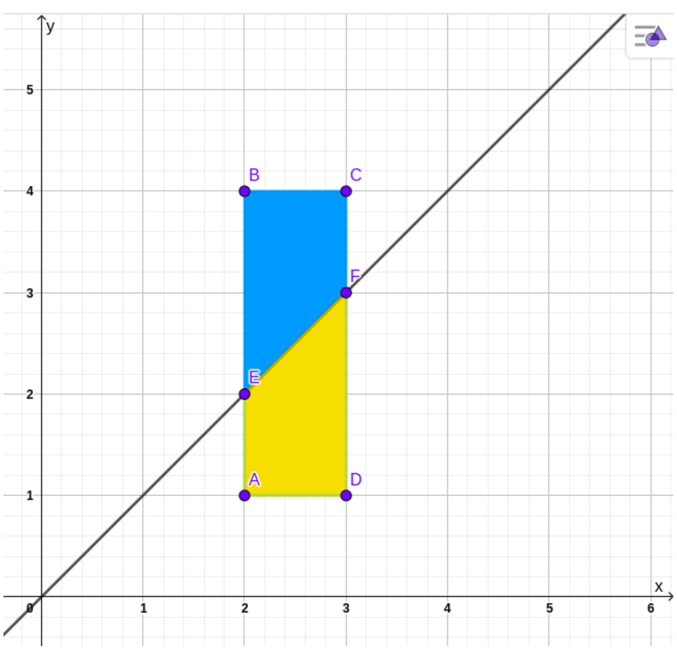


Fig. 12.2.1: Probability Distribution of (X, Y)

In figure 12.2.1, rectangle ABCD represents sample space of (X, Y). $Y \le X$ for any point (X, Y) if and only if the point lies on or below line EF. Therefore

$$\Pr(Y \le X) = \frac{Area\ of\ AEFD}{Area\ of\ ABCD} \tag{12.2.1}$$

$$= \frac{1}{2} \tag{12.2.2}$$

Alternately, we have PDF and CDF of X and Y given by

$$f_X(x) = \begin{cases} 1 & 2 \le x \le 3 \\ 0 & otherwise \end{cases} \tag{12.2.3}$$

$$F_X(x) = \begin{cases} 0 & x < 2 \\ x-2 & 2 \le x \le 3 \\ 1 & x > 3 \end{cases} \tag{12.2.4}$$

$$f_Y(x) = \begin{cases} 1 & 1 \le x \le 4 \\ 0 & otherwise \end{cases} \tag{12.2.5}$$

$$F_Y(x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{3} & 1 \le x \le 4 \\ 1 & x > 4 \end{cases} \tag{12.2.6}$$

Thus

$$\Pr(Y \le X) = \int_{-\infty}^{\infty} F_Y(x) f_X(x) dx \tag{12.2.7}$$

$$= \int_{2}^{3} \frac{x-1}{3} dx \tag{12.2.8}$$

$$= \frac{1}{2} \tag{12.2.9}$$

12.3. Given Set A = {2,3,4,5} and Set B = {11,12,13,14,15}, two numbers are randomly selected, one from each set. What is probability that the sum of the two numbers equals 16?
Solution:
Let $X_1 \in \{2,3,4,5\}$ and $X_2 \in \{11,12,13,14,15\}$ be the random variables such that X_1 represents the number chosen from set A and X_2 the number chosen from set B.
Then, the probability mass functions are

$$p_{X_1}(n) = \Pr(X_1 = n) = \begin{cases} \frac{1}{4} & 2 \le n \le 5 \\ 0 & otherwise \end{cases} \tag{12.3.1}$$

$$p_{X_2}(n) = \Pr(X_2 = n) = \begin{cases} \frac{1}{5} & 11 \le n \le 15 \\ 0 & otherwise \end{cases} \tag{12.3.2}$$

Let X be the random variable denoting the sum (X=X_1+X_2). Then, X can take the values $\{13, 14, 15, 16, 17, 18, 19, 20\}$.

$$p_X(n) = \Pr(X_1 + X_2 = n) \tag{12.3.3}$$
$$= \Pr(X_1 = n - X_2) \tag{12.3.4}$$
$$= \Sigma_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{12.3.5}$$

As X_1, X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) \tag{12.3.6}$$

from (12.3.5) and (12.3.6)

$$p_X(n) = \Sigma_k p_{X_1}(n-k) p_{X_2}(n) = p_{X_1}(n) * p_{X_2}(n) \tag{12.3.7}$$

where * denotes the convolution operator.
As,

$$p_X(n) = \Sigma_k p_{X_1}(n-k) p_{X_2}(k) \tag{12.3.8}$$
$$= \frac{1}{5} \sum_{k=11}^{15} p_{X_1}(n-k) \tag{12.3.9}$$
$$= \frac{1}{5} \sum_{k=n-15}^{n-11} p_{X_1}(k) \tag{12.3.10}$$

Since $p_{X_1}(k) = 0$ for $k < 2, k > 5$
Therefore, we get

$$p_x(n) = \begin{cases} 0 & n \le 12 \\ \frac{1}{5} \sum_{k=2}^{n-11} p_{X_1}(k) & 2 \le n - 11 \le 5 \\ \frac{1}{5} \sum_{k=n-15}^{5} p_{X_1}(k) & 2 \le n - 15 \le 5 \\ 0 & n > 20 \end{cases} \tag{12.3.11}$$

Therefore, from (12.3.1) we get

$$p_x(n) = \begin{cases} 0 & n \le 12 \\ \frac{n-12}{20} & 13 \le n \le 16 \\ \frac{21-n}{20} & 17 \le n \le 20 \\ 0 & n > 20 \end{cases} \tag{12.3.12}$$

Required probability is the probability of the sum of numbers selected from the sets, one from each set to be 16.

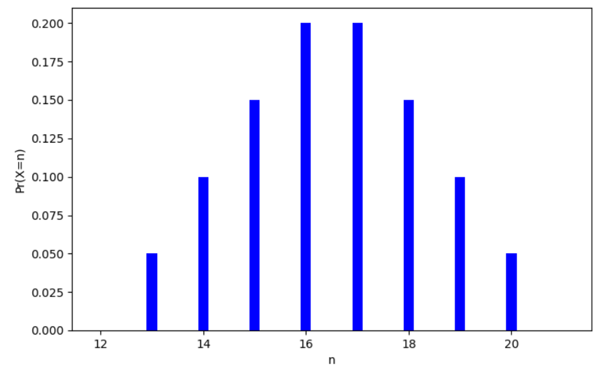


Fig. 12.3.1: Probability mass function of X

Therefore from (12.3.12),

$$p_X(16) = \left(\frac{16 - 12}{20}\right) \tag{12.3.13}$$
$$\implies p_X(16) = \frac{4}{20} \tag{12.3.14}$$
$$\implies \Pr(X_1 + X_2 = 16) = \frac{1}{5} \tag{12.3.15}$$
$$\therefore \Pr(X_1 + X_2 = 16) = 0.2 \tag{12.3.16}$$

12.4. Let X and Y be two statistically independent random variables uniformly distributed in the range (-1, 1) and (-2, 1) respectively. Let $Z = X + Y$, then the probability that $[Z \le -2]$ is
(A) 0
(B) $\frac{1}{6}$
(C) $\frac{1}{3}$
(D) $\frac{1}{12}$

12.5. Two die are thrown. What is the probability that sum of numbers on the two dice is eight

(a) $\frac{5}{36}$

(b) $\frac{5}{18}$

(c) $\frac{1}{4}$

(d) $\frac{1}{3}$

Solution: Let X be a discrete random variable which denotes the sum obtained on two dice and $X_1 \in \{1, 6\}$ be a discrete random variable denoting the outcome on a single die.

$$\Pr(X = n) = \begin{cases} 0, & \text{if } n < 1 \\ \frac{n-1}{36}, & \text{if } 1 \le n-1 \le 6 \\ \frac{13-n}{26}, & \text{if } 1 < n-6 \le 6 \\ 0, & \text{if } n > 12 \end{cases} \tag{12.5.1}$$

$$\text{Required probability} = \Pr(X = 8)$$

$$\text{So, from (12.5.1)}, \Pr(X = 8) = \frac{5}{36}$$

12.6. Four fair six-sided dice are rolled.The probability that sum of results being 22 is $\frac{X}{1296}$.the value of X is **Solution:**
Let $X_i \in \{1,2,3,4,5,6\}$, i = 1,2,3,4 be the random variables representing the outcome for each die. As the dies are fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{12.6.1}$$

Let X be a random variable denotes the desired outcome,

$$X = X_1 + X_2 + X_3 + X_4 \tag{12.6.2}$$

$$\implies X \in \{4, 5, \cdots, 24\} \tag{12.6.3}$$

We have to find $P_X(n) = \Pr(X_1 + X_2 + X_3 + X_4 = n)$ The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{12.6.4}$$

From (12.6.1) and (12.6.4),

$$P_{X_1}(z) = P_{X_2}(z) = P_{X_3}(z) = P_{X_3}(z)$$

$$= \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{12.6.5}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{12.6.6}$$

upon summing up the geometric progression.From convolution

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n) * p_{X_3}(n) * p_{X_4}(n), \tag{12.6.7}$$

$$P_X(z) = P_{X_1}(z)P_{X_2}(z)P_{X_3}(z)p_{X_4}(z) \tag{12.6.8}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (12.6.6) and (12.6.8),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^4 \tag{12.6.9}$$

$$= \frac{1}{1296}\frac{z^{-4}\left(1 - 4z^{-6} + 6z^{-12} - 4z^{-24} + z^{-24}\right)}{\left(1 - z^{-1}\right)^4} \tag{12.6.10}$$

Using the fact that,

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{12.6.11}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{12.6.12}$$

$$n^2u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3} \tag{12.6.13}$$

$$(n^2 + n)u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{2z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{12.6.14}$$

$$(n^3 + 3n^2 + 2n)u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{6z^{-1}}{\left(1 - z^{-1}\right)^4} \tag{12.6.15}$$

after some algebra, it can be shown that,

$$\begin{aligned} &\frac{1}{1296 \times 6}\Big[\Big((n-3)^3 + 3(n-3)^2 + 2(n-3)\Big)u(n-3) \\ &- 4\Big((n-9)^3 + 3(n-9)^2 + 2(n-9)\Big)u(n-9) \\ &+6\Big((n-15)^3 + 3(n-15)^2 + 2(n-15)\Big)u(n-15) \\ &-4\Big((n-21)^3 + 3(n-21)^2 + 2(n-21)\Big)u(n-21) \\ &+\Big((n-27)^3 + 3(n-27)^2 + 2(n-27)\Big)u(n-27)\Big] \\ &\overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{1296}\frac{z^{-4}\left(1 - 4z^{-6} + 6z^{-12} - 4z^{-24} + z^{-24}\right)}{\left(1 - z^{-1}\right)^4} \end{aligned} \tag{12.6.16}$$

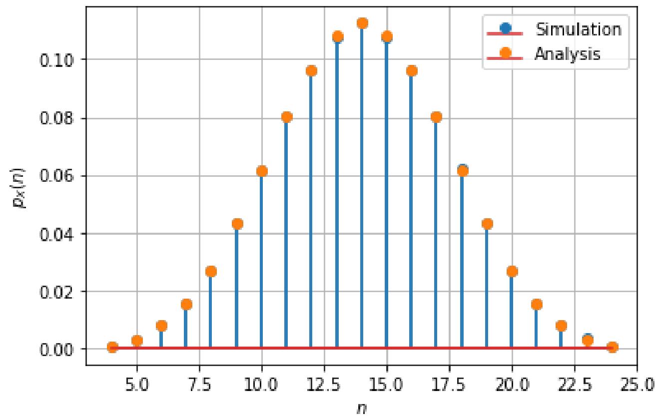


Fig. 12.6.1: Probability of getting sum of 22

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{12.6.17}$$

From (12.6.4),(12.6.10) and (12.6.16),

$$\begin{aligned} p_X(n) = &\frac{1}{1296 \times 6} \times \\ &\Big[\Big((n-3)^3 + 3(n-3)^2 + 2(n-3)\Big) u(n-3) \\ &- 4\Big((n-9)^3 + 3(n-9)^2 + 2(n-9)\Big) u(n-9) \\ &+6\Big((n-15)^3 + 3(n-15)^2 + 2(n-15)\Big) u(n-15) \\ &-4\Big((n-21)^3 + 3(n-21)^2 + 2(n-21)\Big) u(n-21) \\ &+ \Big((n-27)^3 + 3(n-27)^2 + 2(n-27)\Big) u(n-27)\Big] \end{aligned} \tag{12.6.18}$$

From (12.6.17) and (12.6.18),

$$p_X(n) = \begin{cases} 0 & n < 4 \\ \frac{n^3-6n^2+11n-6}{7776} & 4 \leq n \leq 9 \\ \frac{90n^2-3n^3-753n+2010}{7776} & 9 < n \leq 15 \\ \frac{3n^3-162n^2+2769n-14370}{7776} & 15 < n \leq 21 \\ \frac{-n^3+78n^2-2027n+17550}{7776} & 21 < n \leq 24 \\ 0 & 24 < n \end{cases} \tag{12.6.19}$$

We need probability of getting sum of 22,
$\implies$ n=22

from (12.6.19) and using n=22,

$$p_X(22) = \frac{-(22)^3 + 78(22)^2 - 2027(22) + 17550}{7776} \tag{12.6.20}$$

$$p_X(22) = \frac{60}{7776} \tag{12.6.21}$$

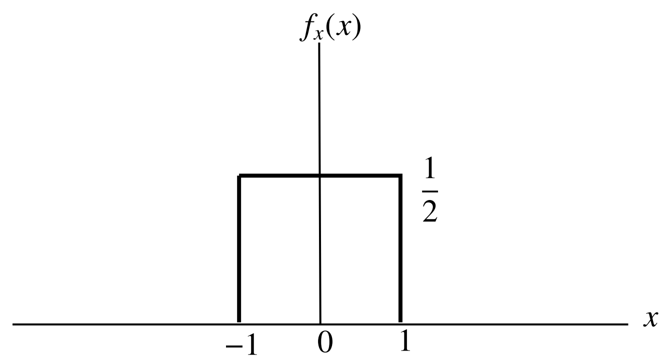
$$p_X(22) = \frac{10}{1296} \tag{12.6.22}$$

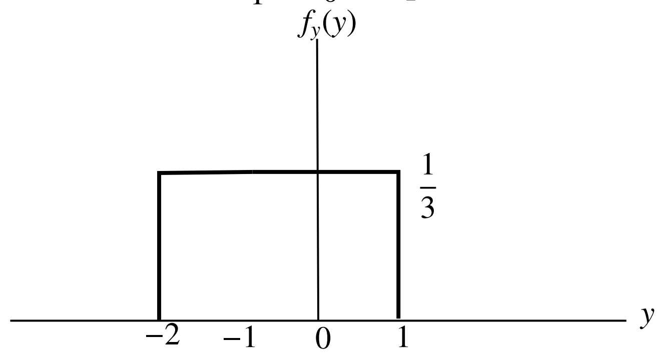
12.7. Let X and Y be two statistically independent random variables uniformly distributed in the ranges $(-1, 1)$ and $(-2, 1)$ respectively. Let $Z = X + Y$. then the probability that $[Z \leq -2]$ is

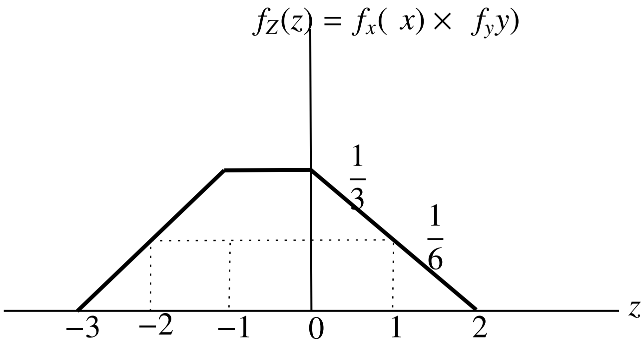
a) zero
b) $\frac{1}{6}$
c) $\frac{1}{3}$
d) $\frac{1}{12}$

Solution:
The pdf of $Z(= X + Y)$ will be convolution of pdf of X and pdf of Y as shown below.

$$f_x(x) \times f_y(y) = f_z(z) \tag{12.7.1}$$







Now

$$\Pr(Z \leq z) = \int_{-\infty}^{z} f_Z(z) dz \tag{12.7.2}$$

$$\Pr(Z \leq -2) = \int_{-\infty}^{-2} f_Z(z) dz \tag{12.7.3}$$

$$= \text{Area } [z \leq -2] \tag{12.7.4}$$

$$= \frac{1}{2} \times \frac{1}{6} \times 1 = \frac{1}{12} \tag{12.7.5}$$

Hence (D) is correct option.

12.8. A single die is thrown twice. What is the probability that the sum is neither 8 or 9?

(a) $\frac{1}{9}$
(b) $\frac{5}{36}$
(c) $\frac{1}{4}$
(d) $\frac{3}{4}$

Solution:
Let $X \in \{0, 1\}$ be the random variable, where X=0 represents that we get sum to be 8 or 9 and X=1 represents that we get sum between 2 and 12 except 8 and 9.
Total number of possible outcomes is :

$$N = {}^6C_1 \times {}^6C_1 = 36 \tag{12.8.1}$$

Probability that the sum is neither 8 or 9

$$\Pr(X = 1) = 1 - \Pr(X = 0) \tag{12.8.2}$$

Only 9 outcomes are favourable to the occurrence of X=0 .
Probability of getting sum 8 or 9 is :

$$\Pr(X = 0) = \frac{9}{36} = \frac{1}{4} \tag{12.8.3}$$

Substituting value in (12.8.2) , we get

$$\Pr(X = 1) = 1 - \frac{1}{4} = \frac{3}{4} \tag{12.8.4}$$

Hence, the correct option is (d) $\frac{3}{4}$

12.9. The characteristic function of a random variable X is given by

$$\phi_X(t) = \begin{cases} \frac{\sin t \cos t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases} \tag{12.9.1}$$

Then $P\left(|X| \leq \frac{3}{2}\right) =$ **Solution:**
The pdf is given by

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(t) e^{-jxt} dt \tag{12.9.2}$$

If

$$g(x) \overset{\mathcal{H}}{\longleftrightarrow} FG(t) \tag{12.9.3}$$

$$\implies G(t) \overset{\mathcal{H}}{\longleftrightarrow} Fg(-x) \tag{12.9.4}$$

where $\left(\overset{\mathcal{H}}{\longleftrightarrow} F\right)$ represents Fourier transform and

$$G(t) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi xt} dx \tag{12.9.5}$$

we know that the Fourier transform of rectangular function is sinc function

$$rect\left(\frac{x}{\tau}\right) \overset{\mathcal{H}}{\longleftrightarrow} F\tau sinc(t\tau) \tag{12.9.6}$$

from (12.9.4) we get

$$\tau sinc(t\tau) \overset{\mathcal{H}}{\longleftrightarrow} Frect\left(-\frac{x}{\tau}\right) \tag{12.9.7}$$

$$\implies rect\left(-\frac{x}{\tau}\right) = \int_{-\infty}^{\infty} \tau \frac{\sin \pi t\tau}{\pi t\tau} e^{-j2\pi xt} dt \tag{12.9.8}$$

substituting $\tau = \frac{2}{\pi}$ and changing $2\pi x \to x$ we get

$$\frac{1}{4} rect\left(\frac{-x}{4}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin 2t}{2t}\right) e^{-jxt} dt \tag{12.9.9}$$

So

$$f_X(x) = \frac{1}{4} rect\left(\frac{-x}{4}\right) \quad (12.9.10)$$

$$P\left(|X| \leq \frac{3}{2}\right) = \int_{\frac{-3}{2}}^{\frac{3}{2}} \frac{1}{4} dx \quad (12.9.11)$$

$$= \frac{3}{4} \quad (12.9.12)$$

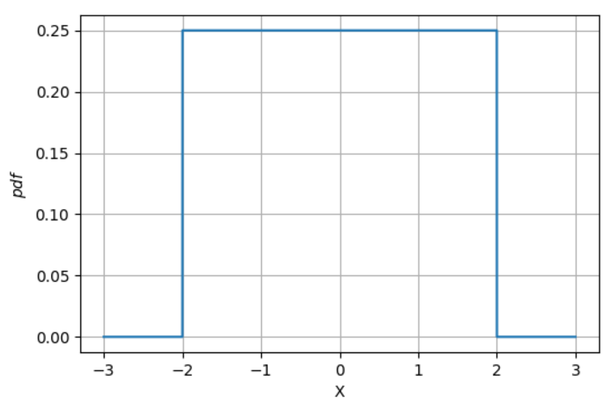


Fig. 12.9.1: $f_X(x)$

12.10. Let X be a random variable with characteristic function $\phi_X(\cdot)$ such that $\phi_X(2\pi) = 1$.Let $\mathbb{Z}$ denote the set of integers.Then $P(X \in \mathbb{Z})$ is equal to ... **Solution:** We know that,

$$\phi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x)\, dx \quad (12.10.1)$$

$$\phi_X(2\pi) = \int_{\mathbb{R}} e^{2\pi ix} f_X(x)\, dx \quad (12.10.2)$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx + i \int_{\mathbb{R}} \sin(2\pi x) f_X(x)\, dx \quad (12.10.3)$$

$$\because \phi_X(2\pi) = 1, \int_{\mathbb{R}} \sin(2\pi x) f_X(x)\, dx = 0 \quad (12.10.4)$$

$$1 = \phi_X(2\pi) \quad (12.10.5)$$

$$= \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx \quad (12.10.6)$$

Assume that $\cos(2\pi x) \neq 1$. This implies that $\cos(2\pi x) < 1 \ \forall\ x \in \mathbb{R}$.

$$\therefore 1 = \int_{\mathbb{R}} \cos(2\pi x) f_X(x)\, dx \quad (12.10.7)$$

$$< \int_{\mathbb{R}} 1 \cdot f_X(x)\, dx \quad (12.10.8)$$

$$< \int_{\mathbb{R}} f_X(x)\, dx \quad (12.10.9)$$

$$< 1. \quad \text{(Contradiction)}$$

Hence, our assumption that $\cos(2\pi x) \neq 1$ is incorrect.

$$\therefore \cos(2\pi x) = 1, \text{ for all } X = x \quad (12.10.10)$$

$$\Rightarrow X \in \mathbb{Z} \quad (12.10.11)$$

$$\Rightarrow \Pr(X \in \mathbb{Z}) = 1 \quad (12.10.12)$$

13 Two Dimensions

13.1. Let X and Y be random variables having the joining probability density function

$$f(x,y)=\begin{cases}\frac{1}{\sqrt{2\pi y}}e^{\frac{-1}{2y}(x-y)^2} & -\infty<x<\infty, 0<y<1\\ 0 & \text{otherwise}\end{cases} \tag{13.1.1}$$

The Variance of the random variable X is

a) $\frac{1}{12}$

b) $\frac{1}{4}$

c) $\frac{7}{12}$

d) $\frac{5}{12}$

Solution: Variance of the random variable X is

$$V(X)=E\left(X^2\right)-(E(X))^2 \tag{13.1.2}$$

Lemma 13.1.

$$\int_{-\infty}^{\infty} xe^{-\frac{1}{2y}(x-y)^2}\,dx = \sqrt{2\pi}y^{\frac{3}{2}} \tag{13.1.3}$$

Proof.

$$\int_{-\infty}^{\infty} xe^{-\frac{1}{2y}(x-y)^2}\,dx \tag{13.1.4}$$

$$=\int_{-\infty}^{\infty}(x-y)\,e^{-\frac{1}{2y}(x-y)^2}\,dx + y\int_{-\infty}^{\infty}e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{13.1.5}$$

$$=0+\sqrt{2\pi}y^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{y}\sqrt{2\pi}}e^{\frac{-1}{2(\sqrt{y})^2}(x-y)^2}\,dx \tag{13.1.6}$$

$$=\sqrt{2\pi}y^{\frac{3}{2}}\lim_{x_0\to-\infty}Q\left(\frac{x_0-y}{\sqrt{y}}\right) \tag{13.1.7}$$

$$=\sqrt{2\pi}y^{\frac{3}{2}} \tag{13.1.8}$$

□

Lemma 13.2.

$$E(X)=\frac{1}{2} \tag{13.1.9}$$

Proof.

$$E(X)=\int_0^1\int_{-\infty}^{\infty}xf_{XY}(x,y)\,dx\,dy \tag{13.1.10}$$

$$=\int_0^1\int_{-\infty}^{\infty}x\frac{1}{\sqrt{2\pi y}}e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{13.1.11}$$

$$=\int_0^1\frac{1}{\sqrt{2\pi y}}\left(\int_{-\infty}^{\infty}xe^{-\frac{1}{2y}(x-y)^2}\,dx\right)dy \tag{13.1.12}$$

From (13.1.12) and (13.1.8),

$$E(X)=\int_0^1 y\,dy \tag{13.1.13}$$

$$E(X)=\frac{1}{2} \tag{13.1.14}$$

□

Lemma 13.3.

$$\int_{-\infty}^{\infty}x^2e^{-\frac{1}{2y}(x-y)^2}\,dx = \sqrt{2\pi}y^{\frac{3}{2}}(y+1) \tag{13.1.15}$$

Proof.

$$\int_{-\infty}^{\infty}x^2e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{13.1.16}$$

$$=\left(\sqrt{\frac{\pi}{2}}y^{\frac{3}{2}}(y+1)\left(1-2Q\left(\frac{x-y}{\sqrt{y}}\right)\right)-ye^{\frac{-(x-y)^2}{2y}}(x+y)\right)]_{-\infty}^{\infty} \tag{13.1.17}$$

$$=0-\sqrt{2\pi}y^{\frac{3}{2}}(y+1)\,Q\left(\frac{x-y}{\sqrt{y}}\right)]_{-\infty}^{\infty}-0 \tag{13.1.18}$$

$$=\sqrt{2\pi}y^{\frac{3}{2}}(y+1) \tag{13.1.19}$$

□

Lemma 13.4.

$$E\left(X^2\right)=\frac{5}{6} \tag{13.1.20}$$

Proof.

$$E\left(X^2\right)=\int_0^1\int_{-\infty}^{\infty}x^2f_{XY}(x,y)\,dx\,dy \tag{13.1.21}$$

$$=\int_0^1\int_{-\infty}^{\infty}x^2\frac{1}{\sqrt{2\pi y}}e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{13.1.22}$$

$$=\int_0^1\frac{1}{\sqrt{2\pi y}}\left(\int_{-\infty}^{\infty}x^2e^{-\frac{1}{2y}(x-y)^2}\,dx\right)dy \tag{13.1.23}$$

From (13.1.23) and (13.1.19), we get

$$E\left(X^2\right) = \int_0^1 y\,(y+1)\,dy \tag{13.1.24}$$

$$= \left(\frac{y^3}{3} + \frac{y^2}{2}\right)]_0^1 \tag{13.1.25}$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \tag{13.1.26}$$

□

From (13.1.14) and (13.1.26), we get

$$V(X) = E\left(X^2\right) - (E(X))^2 \tag{13.1.27}$$

$$= \frac{5}{6} - \frac{1}{4} \tag{13.1.28}$$

$$= \frac{7}{12} \tag{13.1.29}$$

Therefore, the answer is (C)$\frac{7}{12}$

13.2. Let X and Y be two random variables having the joint probability density function

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3}|Y = \frac{3}{4}\right)$ is equal to ______________

a) $\frac{5}{9}$ b) $\frac{2}{3}$ c) $\frac{7}{9}$ d) $\frac{8}{9}$

Solution: We have

$$\Pr\left(X \leq \frac{2}{3}\middle|Y = \frac{3}{4}\right) = \frac{\Pr\left(X \leq \frac{2}{3}, Y = \frac{3}{4}\right)}{\Pr\left(Y = \frac{3}{4}\right)} \tag{13.2.1}$$

So we have to consider: $\Pr(X \leq 2/3, Y = 3/4)$ and $\Pr(Y = 3/4)$. They are both lines as can be seen in the Python plot. (In black and in both black and purple respectively)

Hence instead of integrating over area in the XY plane, we have to integrate over these line segments:

$$L_1 : 0 < X \leq 2/3, Y = 3/4 \tag{13.2.2}$$

$$L_2 : 0 < X < 3/4, Y = 3/4 \tag{13.2.3}$$

As stated before, L_1 segment is shown in black while L_2 segment is shown in black and purple.

Therefore, we have:

$$\Pr\left(X \leq \frac{2}{3}\middle|Y = \frac{3}{4}\right) = \frac{\int\limits_{L_1} f_{XY}(x,y)\,dx}{\int\limits_{L_2} f_{XY}(x,y)\,dx} \tag{13.2.4}$$

$$= \frac{\int\limits_0^{2/3} 2\,dx}{\int\limits_0^{3/4} 2\,dx} \tag{13.2.5}$$

$$\Pr\left(X \leq \frac{2}{3}\middle|Y = \frac{3}{4}\right) = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9} \tag{13.2.6}$$

13.3. Two random variables X and Y are distributed according to

$$f_{x,y}(x,y) = \begin{cases} (x+y) & 0 \leqslant x \leqslant 10 \leqslant y \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

The probability $P(X + Y \leqslant 1)$ is ....

13.4. Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $P(Y > max(X, -X)) =$

a) $\dfrac{1}{2}$ c) $\dfrac{1}{4}$

b) $\dfrac{1}{3}$ d) $\dfrac{1}{6}$

Solution: pdf of X is :

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{13.4.1}$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \tag{13.4.2}$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{13.4.3}$$

pdf of Y is :

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{13.4.4}$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{13.4.5}$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{13.4.6}$$

cdf of Y is:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy \tag{13.4.7}$$

$$= \int_{-1}^{y} \frac{2\sqrt{1-y^2}}{\pi}dy \tag{13.4.8}$$

$$= \frac{2}{\pi}\left[\frac{\sin^{-1} y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4}\right] \tag{13.4.9}$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \tag{13.4.10}$$

$$= \frac{2}{\pi}\left(\sin^{-1} X + X\sqrt{1-X^2}\right) \tag{13.4.11}$$

Integrating our probability over all of X we get the value of $E[\Pr(-x < Y < x)]$ as

$$= \int_{-\infty}^{\infty} f_X(x) \Pr(-x < Y < x)\, dx \tag{13.4.12}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) dx \tag{13.4.13}$$

Substituting

$$u = \sin^{-1} x + x\sqrt{1-x^2} \tag{13.4.14}$$

$$\frac{du}{dx} = 2\sqrt{1-x^2} \tag{13.4.15}$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2}du \tag{13.4.16}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{u^2}{4}\right)\Big|_0^{\frac{\pi}{2}} \tag{13.4.17}$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \tag{13.4.18}$$

$$= \frac{4\cdot\pi^2}{\pi^2\cdot 16} \tag{13.4.19}$$

$$= \frac{1}{4} \tag{13.4.20}$$

Common Data for the next two Questions :

13.5. Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, \\ & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The variance of the random variable X is

a) $\frac{1}{12}$

b) $\frac{1}{4}$

c) $\frac{7}{12}$

d) $\frac{5}{12}$

13.6. The covariance between the random variables X and Y

a) $\frac{1}{3}$

b) $\frac{1}{4}$

c) $\frac{1}{6}$

d) $\frac{1}{12}$

13.7. Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

The value of a is

a) 4

b) 2

c) 1

d) 0.5

13.8. The value of $E(X|Y=2)$ is

a) 4

b) 3

c) 2

d) 1

13.9. Let X and Y be two random variables having the joint probability density function

$$f(x,y)=\begin{cases}2 & 0<x<y<1\\ 0 & \text{otherwise}\end{cases}$$

Then the conditional probability $P(X \leqslant \frac{2}{3}|Y=\frac{3}{4})$ is equal to

a) $\frac{5}{9}$

b) $\frac{2}{3}$

c) $\frac{7}{9}$

d) $\frac{8}{9}$

13.10. Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y)=\begin{cases}2 & 0<x+y<1, x>0, y>0\\ 0 & \text{elsewhere.}\end{cases}$$

$P(X+Y<\frac{1}{2})$ is

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) $\frac{3}{4}$

d) 1

Solution: Given X and Y be two continuous random variables with the joint probability density function

$$f(x,y)=\begin{cases}2 & 0<x+y<1, x>0, y>0\\ 0 & \text{elsewhere}\end{cases} \tag{13.10.1}$$

we know that

$$P((x,y)\in A)=\int\int_A f(x,y)\,dxdy \quad A\in\mathbb{R}^2 \tag{13.10.2}$$

from given information
for positive x and y

$$0<x+y<\frac{1}{2}\Rightarrow 0<x<\frac{1}{2}-y \tag{13.10.3}$$

so using eq(0.0.3)

$$P\left(x+y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-y} f(x,y)dxdy \tag{13.10.4}$$

$$=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}-y} 2 \quad dxdy=\int_0^{\frac{1}{2}}\left(2x \quad \Big|_0^{\frac{1}{2}-y}\right)dy \tag{13.10.5}$$

$$=\int_0^{\frac{1}{2}} 2\left(\frac{1}{2}-y\right) \quad dy=2\left(\frac{1}{2}y-\frac{y^2}{2}\right)\Big|_0^{\frac{1}{2}} \tag{13.10.6}$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{4} \tag{13.10.7}$$

Therefore

$$P\left(X+Y<\frac{1}{2}\right)=\frac{1}{4} \tag{13.10.8}$$

volume under the graph which contains the region

$$X+Y<\frac{1}{2} \quad \text{gives us} \quad P\left(X+Y<\frac{1}{2}\right) \tag{13.10.9}$$

$$P\left(X+Y<\frac{1}{2}\right)=\text{Area of the base . height} \tag{13.10.10}$$

Area of the base triangle is

$$\frac{1}{2}.height.base=\frac{1}{2}.\frac{1}{2}.\frac{1}{2} \tag{13.10.11}$$

$$\text{volume} = \text{Area . height} = \frac{1}{8}.2=\frac{1}{4} \tag{13.10.12}$$

13.11. $E(X|Y=\frac{1}{2})$

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) 1

d) 2

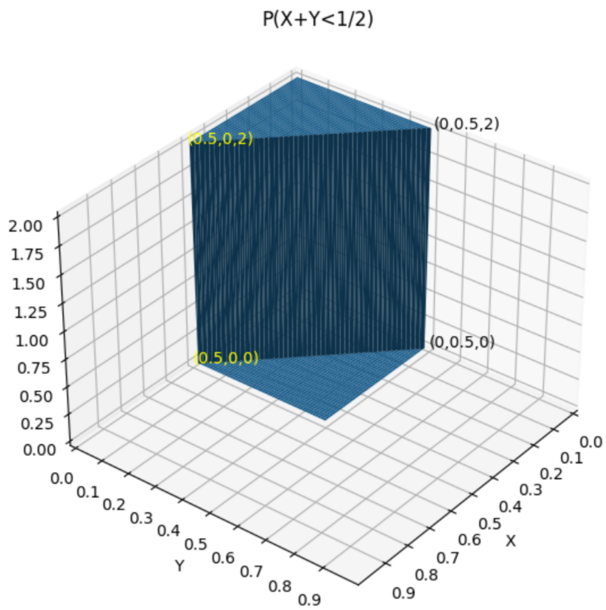


Fig. 13.10.1: $P\left(x+y<\frac{1}{2}\right)$

Solution: Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2 & 0<x+y<1, x>0, y>0 \\ 0 & \text{elsewhere.} \end{cases} \tag{13.11.1}$$

Then $E\left(X|Y=\frac{1}{2}\right)$ is Given X and Y are two continuous random variables with joint probability density function,

$$f(x,y) = \begin{cases} 2 & 0<x+y<1, x>0, y>0 \\ 0 & \text{elsewhere.} \end{cases} \tag{13.11.2}$$

We know that,
$0<x+y<1 \implies 0<y<1-x$ for $0<x<1$.

Then,

$$f_X(x) = \int f_{XY}(x,y)\,dy \tag{13.11.3}$$
$$= \int_0^{1-x} (2)dy \tag{13.11.4}$$
$$= 2(1-x) \tag{13.11.5}$$
$$\implies f_X(x) = \begin{cases} 2(1-x) & 0\le x<1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.11.6}$$

Similarly,
$0<x+y<1 \implies 0<x<1-y$ for $0<y<1$
Then,

$$f_y(y) = \int f_{XY}(x,y)\,dx \tag{13.11.7}$$
$$= \int_0^{1-y} (2)dx \tag{13.11.8}$$
$$= 2(1-y) \tag{13.11.9}$$
$$\implies f_Y(y) = \begin{cases} 2(1-y) & 0\le y<1 \\ 0 & \text{otherwise.} \end{cases} \tag{13.11.10}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{13.11.11}$$
$$= \begin{cases} \frac{2}{2(1-y)} & \text{if } 0\le x+y<1 \\ 0 & \text{otherwise} \end{cases} \tag{13.11.12}$$

Then,

$$E(X|Y=y) = \int_{-\infty}^{\infty} (x)\left(\frac{1}{1-y}\right)dx \tag{13.11.13}$$
$$= \frac{1}{1-y}\int_0^{1-y}(x)dx \tag{13.11.14}$$
$$= \frac{1}{1-y}\left[\frac{x^2}{2}\right]_0^{1-y} \tag{13.11.15}$$
$$\therefore E(X|Y=y) = \frac{1-y}{2} \tag{13.11.16}$$
$$\implies E\left(X|Y=\frac{1}{2}\right) = \frac{1-\frac{1}{2}}{2} \tag{13.11.17}$$
$$\therefore E\left(X|Y=\frac{1}{2}\right) = \frac{1}{4} \tag{13.11.18}$$

13.12. The joint probability density function of two random variables X and Y is given as

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \leqslant x \leqslant 1 0 \leqslant x \leqslant 1 \\ 0 & \text{elsewhere} \end{cases}$$

$E(X)$ and $E(Y)$ are, respectively,

a) $\frac{2}{5}$ and $\frac{3}{5}$ c) $\frac{3}{5}$ and $\frac{6}{5}$

b) $\frac{3}{5}$ and $\frac{3}{5}$ d) $\frac{4}{5}$ and $\frac{6}{5}$

Solution: For a continuous joint probability distribution $E(X)$
and $E(Y)$ are obtained using the following equations
(13.12.1) and (13.12.2)

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \cdot f(x,y)\, dx\, dy \qquad (13.12.1)$$

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \cdot f(x,y)\, dx\, dy \qquad (13.12.2)$$

Using equation (13.12.1) $E(X)$ is calculated as

$$E(X) = \int_0^1\int_0^1 x\,\frac{6}{5}\left(x+y^2\right) dx\, dy \; + 0$$

$$= \int_0^1 \frac{6}{5}\left(\int_0^1 x^2\, dx\right) + \frac{6}{5}y^2\left(\int_0^1 x\, dx\right) dy$$

$$= \int_0^1 \frac{6}{5}\left(\frac{1}{3}\right) + \frac{6}{5}y^2\left(\frac{1}{2}\right) dy$$

$$= \frac{2}{5}\int_0^1 dy + \frac{3}{5}\int_0^1 y^2\, dy$$

$$= \frac{2}{5} + \frac{3}{5}\left(\frac{1}{3}\right)$$

$$E(X) = \frac{3}{5}$$

Using equation (13.12.2) $E(Y)$ is calculated as

$$E(Y) = \int_0^1\int_0^1 y\,\frac{6}{5}\left(x+y^2\right) dx\, dy \; + 0$$

$$= \int_0^1 \frac{6}{5}x\left(\int_0^1 y\, dy\right) + \frac{6}{5}\left(\int_0^1 y^3\, dy\right) dx$$

$$= \int_0^1 \frac{6}{5}x\left(\frac{1}{2}\right) + \frac{6}{5}\left(\frac{1}{4}\right) dx$$

$$= \frac{3}{5}\int_0^1 x\, dx \; + \; \frac{3}{10}\int_0^1 dx$$

$$= \frac{3}{5}\left(\frac{1}{2}\right) + \frac{3}{10}$$

$$E(Y) = \frac{3}{5}$$

$$\therefore E(X) = \frac{3}{5} \text{ and } E(Y) = \frac{3}{5}$$

Hence the answer is **option b**

13.13. Two random variables X and Y are distributed according to

$$f_{XY}(x,y) = \begin{cases} x+y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} \qquad (13.13.1)$$

The probability $P(X+Y \leq 1)=$ **Solution:**

13.14. Two random variables X and Y are distributed according to

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{13.14.1}$$

The probability $\Pr(X+Y \leq 1)$ is **Solution:**

$$Pr(X+Y \leq 1) = \int_0^1 \int_0^{1-y} f_{X,Y}(x,y)\, dx\, dy \tag{13.14.2}$$

$$= \int_0^1 \int_0^{1-y} (x+y)\, dx\, dy \tag{13.14.3}$$

$$= \int_0^1 \left(\left(\frac{x^2}{2} + xy \right) \Big|_0^{1-y} \right) dy \tag{13.14.4}$$

$$= \int_0^1 \left(\frac{1-y^2}{2} \right) dy \tag{13.14.5}$$

$$= \left(\frac{y}{2} - \frac{y^3}{6} \right) \Big|_0^1 dy \tag{13.14.6}$$

$$= \frac{1}{3} \tag{13.14.7}$$

Therefore, required probability is $= \frac{1}{3}$

13.15. Let X and Y be continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{13.15.1}$$

Then $E(X|Y=2)$ is ...**Solution:** Given X and Y are two continuous random variables with joint probability density function,

$$f(x,y) = \begin{cases} ae^{-2y} & 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{13.15.2}$$

We know that,
$0 < x < y < \infty \implies x < y < \infty$ for $0 < x < \infty$.
Then,

$$f_X(x) = \int f_{XY}(x,y)\, dy \tag{13.15.3}$$

$$= \int_x^\infty ae^{-2y} dy \tag{13.15.4}$$

$$= \left[\frac{ae^{-2y}}{(-2)} \right]_x^\infty \tag{13.15.5}$$

$$= \frac{-a}{2} \left[e^{-2y} \right]_x^\infty \tag{13.15.6}$$

$$= \frac{-a}{2} [0 - e^{-2x}] \tag{13.15.7}$$

$$\implies f_X(x) = \begin{cases} \frac{a}{2} e^{-2x} & 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{13.15.8}$$

Similarly,
$0 < x < y < \infty \implies 0 < x < y$ for $0 < y < \infty$
Then,

$$f_y(y) = \int f_{XY}(x,y)\, dx \tag{13.15.9}$$

$$= \int_0^y ae^{-2y} dx \tag{13.15.10}$$

$$= ae^{-2y} [x]_0^y \tag{13.15.11}$$

$$= aye^{-2y} \tag{13.15.12}$$

$$\implies f_Y(y) = \begin{cases} aye^{-2y} & 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{13.15.13}$$

Therefore ,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{13.15.14}$$

$$= \frac{ae^{-2y}}{aye^{-2y}} \tag{13.15.15}$$

$$= \frac{1}{y} \tag{13.15.16}$$

$$\implies f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases} \tag{13.15.17}$$

Then,

$$E(X|Y=y) = \int_{-\infty}^{\infty} (x) f_{X|Y}(x|y)\, dx \quad (13.15.18)$$

$$= \int_0^y (x)\left(\frac{1}{y}\right) dx \quad (13.15.19)$$

$$= \frac{1}{y}\int_0^y (x) dx \quad (13.15.20)$$

$$= \frac{1}{y}\left[\frac{x^2}{2}\right]_0^y \quad (13.15.21)$$

$$= \frac{1}{y}\left(\frac{y^2}{2}\right) \quad (13.15.22)$$

$$= \frac{y}{2} \quad (13.15.23)$$

$$\implies E(X|Y=y) = \frac{y}{2} \quad (13.15.24)$$

$$\therefore E(X|Y=2) = 1 \quad (13.15.25)$$

13.16. Let Z be the vertical coordinate, between -1 and 1, of a point chosen uniformly at random on the surface of a unit sphere in R^3. Then, $\Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right)$ is

Solution: The equation of the sphere can be written as : $x^2 + y^2 + z^2 = 1$. Now,

$$\Pr\left(\frac{-1}{2} \le z \le 0\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (13.16.1)$$

$$\Pr\left(0 \le z \le \frac{1}{2}\right) = \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (13.16.2)$$

$$\therefore \Pr\left(\frac{-1}{2} \le z \le \frac{1}{2}\right) = 2 \times \Pr\left(0 \le z^2 \le \frac{1}{4}\right) \quad (13.16.3)$$

$$\Pr\left(0 \le z^2 \le \frac{1}{4}\right) = \Pr\left(\frac{3}{4} \le x^2 + y^2 \le 1\right) \quad (13.16.4)$$

Taking, $x^2 + y^2 = r^2$. (13.16.5)

$$\Pr\left(\frac{3}{4} \le r^2 \le 1\right) = \frac{1}{4} \quad (13.16.6)$$

$\left(\text{Since, } r^2 \text{ is uniform between 0 and 1}\right)$

$$\therefore \Pr\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right) = 2 \times \frac{1}{4} = \frac{1}{2} \quad (13.16.7)$$

13.17. Let (X,Y) be a random vector such that, for any $y > 0$, the conditional probability density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = ye^{-yx}, x > 0.$$

If the marginal probability density function of Y is

$$g(y) = ye^{-y}, y > 0$$

then $E(Y|x = 1) =$

Solution: Given, the conditional probability density function of X given $Y = y$,

$$f_{X|Y=y}(x) = ye^{-yx}, x > 0 \quad (13.17.1)$$

and, the marginal probability density function of Y,

$$g(y) = ye^{-y}, y > 0 \quad (13.17.2)$$

let the joint probability density function of (X,Y) be $f_{X,Y}(x, y)$. We know that,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{g(y)} \quad (13.17.3)$$

using (13.17.1) and (13.17.2) in (13.17.3),

$$f_{X,Y}(x, y) = y^2 e^{-y(x+1)}, x, y > 0 \quad (13.17.4)$$

let the marginal probability density function of X be $f_X(x)$, as we know ,

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x, y)\, dy \quad (13.17.5)$$

using (13.17.4) in (13.17.5),

$$f_X(x) = \int_0^{\infty} y^2 e^{-y(x+1)}\, dy \quad (13.17.6)$$

$$= \frac{2}{(x+1)^3}, x > 0 \quad (13.17.7)$$

The conditional probability density function of Y given $X = x$ is given by,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} \quad (13.17.8)$$

using (13.17.4) and (13.17.7) in (13.17.8),

$$f_{Y|X=x}(y) = \frac{y^2 e^{-y(x+1)}(x+1)^3}{2}, x, y > 0 \quad (13.17.9)$$

The conditional probability density function of Y given $X = 1$ is given by,

$$f_{Y|X=1}(y) = 4y^2 e^{-2y}, y > 0 \quad (13.17.10)$$

We need to find $E(Y|X = 1)$ which is given by,

$$E(Y|X = 1) = \int_0^\infty y f_{Y|X=1}(y)\, dy \quad (13.17.11)$$

using (13.17.10) in (13.17.11),

$$E(Y|X = 1) = \int_0^\infty 4y^3 e^{-2y}\, dy \quad (13.17.12)$$

$$= \left[\frac{-e^{-2y}(8y^3 + 12y^2 + 12y + 6)}{4}\right]_0^\infty \quad (13.17.13)$$

$$= \frac{3}{2} \quad (13.17.14)$$

13.18. Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{\pi}, & \text{if} \quad x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $\Pr(Y > \max(X, -X))$ is

Solution:

The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^\infty f(x, y) dy \quad (13.18.1)$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \quad (13.18.2)$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \quad (13.18.3)$$

$$f_Y(y) = \int_{-\infty}^\infty f(x, y) dx \quad (13.18.4)$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx \quad (13.18.5)$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \quad (13.18.6)$$

The cdf of Y is:

$$F_Y(y) = \int_{-\infty}^y f_Y(y) dy \quad (13.18.7)$$

$$= \int_{-1}^y \frac{2\sqrt{1-y^2}}{\pi} dy \quad (13.18.8)$$

$$= \frac{2}{\pi}\left(\frac{\sin^{-1} y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4}\right) \quad (13.18.9)$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \quad (13.18.10)$$

$$= \frac{2}{\pi}\left(\sin^{-1} X + X\sqrt{1-X^2}\right) \quad (13.18.11)$$

Integrating our probability over all of X we get the value of $E[\Pr(-x < Y < x)]$:

$$= \int_{-\infty}^\infty f_X(x) \Pr(-x < Y < x)\, dx \quad (13.18.12)$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) dx \quad (13.18.13)$$

Substituting

$$u = \sin^{-1} x + x\sqrt{1-x^2} \quad (13.18.14)$$

$$\frac{du}{dx} = 2\sqrt{1-x^2} \quad (13.18.15)$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2} du \quad (13.18.16)$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{u^2}{4}\bigg|_0^{\frac{\pi}{2}}\right) \quad (13.18.17)$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \quad (13.18.18)$$

$$= \frac{4 \cdot \pi^2}{\pi^2 \cdot 16} \quad (13.18.19)$$

$$= \frac{1}{4} \quad (13.18.20)$$

The probability for:

$$\Pr(Y > \max(X, -X)) = \frac{1}{4} \quad (13.18.21)$$

13.19. Let X and Y be two continuous random variables with the joint probability density function

$$f(x,y) = \begin{cases} 2, & 0 < x+y < 1, x > 0, y > 0, \\ 0, & elsewhere. \end{cases} \tag{13.19.1}$$

$E\left(X \mid Y = \frac{1}{2}\right)$ is
a) $1/4$
b) $1/2$
c) 1
d) 2

Solution:

13.20. Let X, Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} e^{-y}(1-e^{-x}) \text{ if } 0 < x < y < \infty \\ e^{-x}(1-e^{-y}) \text{ if } 0 < y \leq x < \infty \end{cases}$$

Then The value of $E[X+Y]$ is **Solution:** Let $g(X,Y) = X+Y$ We know that,

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) dxdy$$

Then,

$$E[X+Y] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x+y) f_{X,Y}(x,y)\, dxdy$$
$$= \int_{0}^{+\infty}\int_{0}^{+\infty} (x+y) f_{X,Y}(x,y)\, dxdy$$
$$= \int_{0}^{+\infty}\left(\int_{0}^{+\infty} x f_{X,Y}(x,y)\, dx + \int_{0}^{+\infty} y f_{X,Y}(x,y)\, dx\right) dy$$

First we will calculate the $\int_0^{+\infty} y f_{X,Y}(x,y)\, dx$, $\int_0^{+\infty} x f_{X,Y}(x,y)\, dx$ seperately.
consider,

$$\int_{0}^{+\infty} y f_{X,Y}(x,y)\, dx$$
$$= \int_{0}^{y} ye^{-y}(1-e^{-x})\, dx + \int_{y}^{+\infty} ye^{-x}(1-e^{-y})\, dx$$
$$= (ye^{-y})(y + e^{-y} - 1) + y(1-e^{-y})e^{-y}$$
$$= y^2 e^{-y}$$

So,

$$\int_{0}^{+\infty} y f_{X,Y}(x,y)\, dx = y^2 e^{-y} \tag{37.1}$$

Now consider,

$$\int_{0}^{+\infty} x f_{X,Y}(x,y)\, dx$$
$$= \int_{0}^{y} xe^{-y}(1-e^{-x})\, dx + \int_{y}^{+\infty} xe^{-x}(1-e^{-y})$$
$$= e^{-y}\left(\frac{y^2}{2} + e^{-y}(y+1) - 1\right) + (1-e^{-y})(e^{-y}(y+1))$$
$$= \frac{y^2 e^{-y}}{2} + ye^{-y}$$

So,

$$\int_{0}^{+\infty} x f_{X,Y}(x,y)\, dx = \frac{y^2 e^{-y}}{2} + ye^{-y} \tag{37.2}$$

From Eq 37.1 and 37.2

$$E[X+Y] = \int_{0}^{+\infty}\left(\frac{y^2e^{-y}}{2} + ye^{-y} + y^2e^{-y}\right) dy$$
$$= \int_{0}^{+\infty}\left(\frac{3}{2}y^2e^{-y} + ye^{-y}\right) dy$$
$$= \left(-\frac{3}{2}(y^2+2y+2)e^{-y} + (-e^{-y}(y+1))\right)\Big|_0^{+\infty}$$
$$= \frac{3}{2} \times 2 + 1$$
$$= 4$$

So,

$$E[X+Y] = 4$$

13.21. Let (X, Y) be a two-dimensional random variable such that $E(X) = E(Y) = 1/2$, $Var(X) = Var(Y) = 1$ and $Cov(X,Y) = 1/2$. Then, $P(|X-Y| > 6)$ is
a) less than $1/6$
b) equal to $1/2$
c) equal to $1/3$
d) greater than $1/2$

Solution:
Given,

$$E(X) = E(Y) = 3 \tag{13.21.1}$$

$$Var(X) = Var(Y) = 1 \tag{13.21.2}$$

$$Cov(X,Y) = 1/2 \tag{13.21.3}$$

Now,

$$Var(X) = E(X^2) - (E(X))^2 \tag{13.21.4}$$

Substituting given values, we get,

$$1 = E(X^2) - 3^2 \tag{13.21.5}$$

So,

$$E(X^2) = 10 \tag{13.21.6}$$

Similarly for Y,

$$E(Y^2) = 10 \tag{13.21.7}$$

Also,

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{13.21.8}$$

Substituting given values, we get,

$$1/2 = E(XY) - (3)(3) \tag{13.21.9}$$

So,

$$E(XY) = 19/2 \tag{13.21.10}$$

Let Z be a random variable defined as

$$Z = X - Y \tag{13.21.11}$$

Then using (13.21.1),

$$E(Z) = E(X - Y) = E(X) - E(Y) = 0 \tag{13.21.12}$$

Now, using (13.21.12)

$$Var(Z) = E(Z^2) - (E(Z))^2 = E(Z^2) \tag{13.21.13}$$

$$Var(Z) = E((X - Y)^2) \tag{13.21.14}$$

$$Var(Z) = E(X^2) + E(Y^2) - 2E(XY) \tag{13.21.15}$$

Using (13.21.6), (13.21.7) and (13.21.10),

$$Var(Z) = 10 + 10 - 2 \times 19/2 \tag{13.21.16}$$

$$Var(Z) = 1 \tag{13.21.17}$$

Theorem 13.1. *(Chebychev's Inequality) Let T be an arbitrary random variable, with finite mean E(T), then for all a > 0,*

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{13.21.18}$$

Proof. Let A be a non-negative random variable and $a > 0$ be any real number. Define a new random variable B by

$$B = \begin{cases} a & A \geq a \\ 0 & A < a \end{cases} \tag{13.21.19}$$

Then clearly $B \leq A$ and by monotonicity,

$$E(B) \leq E(A) \tag{13.21.20}$$

$$E(B) = a \Pr(B = a) + 0 \Pr(B = 0) \tag{13.21.21}$$

$$E(B) = a \Pr(A \geq a) \tag{13.21.22}$$

By (13.21.20) and (13.21.22),

$$a \Pr(A \geq a) \leq E(A) \tag{13.21.23}$$

$$\Pr(A \geq a) \leq \frac{E(A)}{a} \tag{13.21.24}$$

Set $A = (T - E(T))^2$. Then,

$$\Pr(|T - E(T)| \geq a) = \Pr\left(A \geq a^2\right) \tag{13.21.25}$$

Using (13.21.24),

$$\Pr(|T - E(T)| \geq a) \leq \frac{E(A)}{a^2} \tag{13.21.26}$$

$$\Pr(|T - E(T)| \geq a) \leq \frac{E(T - E(T))^2}{a^2} \tag{13.21.27}$$

$$\Pr(|T - E(T)| \geq a) \leq \frac{Var(T)}{a^2} \tag{13.21.28}$$

□

Applying Chebychev's Inequality for Z with $a = 6$, we get,

$$\Pr(|Z - E(Z)| \geq 6) \leq \frac{Var(Z)}{6^2} \tag{13.21.29}$$

Using (13.21.12) and (13.21.17),

$$\Pr(|Z - 0| \geq 6) \leq \frac{1}{36} \tag{13.21.30}$$

As $Z = X - Y$,

$$\Pr(|X - Y| \geq 6) \leq \frac{1}{36} \tag{13.21.31}$$

13.22. The variable x takes a value between 0 and 10 with uniform probability distribution. The variable y takes a value between 0 and 20 with uniform probability distribution.The probability that sum of variables $(x + y)$ being greater than 20 is

13.23. Let (X,Y) be the coordinates of a point chosen at random inside the disc $x^2 + y^2 \leq r^2$ where $r \geq 0$. The probability that $Y \geq mX$ is

(a) $\frac{1}{2^r}$

(b) $\frac{1}{2^m}$

(c) $\frac{1}{2}$

(d) $\frac{1}{2^{r+m}}$

Solution:

We know that the point (X, Y) satisfies the equation

$$x^2 + y^2 \leq r^2 \tag{13.23.1}$$

Let a random variable $Z \in \{0, 1\}$ denote the possible outcomes of the experiment

Equation satisfied by (X,Y)	Z
$y - mx < 0$	0
$y - mx \geq 0$	1

TABLE 13.23.1: Outcome of the Experiment

The coordinates (X, Y) can be parametrized as follows:

$$X = a \sin \theta \tag{13.23.2}$$
$$Y = a \cos \theta \tag{13.23.3}$$

where $a \in [0, r]$ and $\theta \in [0, 2\pi]$.

$$Y \geq mX \tag{13.23.4}$$
$$\implies a \sin \theta \geq ma \cos \theta \tag{13.23.5}$$

This gives two cases for an arbitrary value of m (as seen in fig 13.23.1):

a) when $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$, from 13.23.1,

$$\tan \theta \geq m \tag{13.23.6}$$
$$\implies \theta \in [\tan^{-1} m, \pi/2] \tag{13.23.7}$$

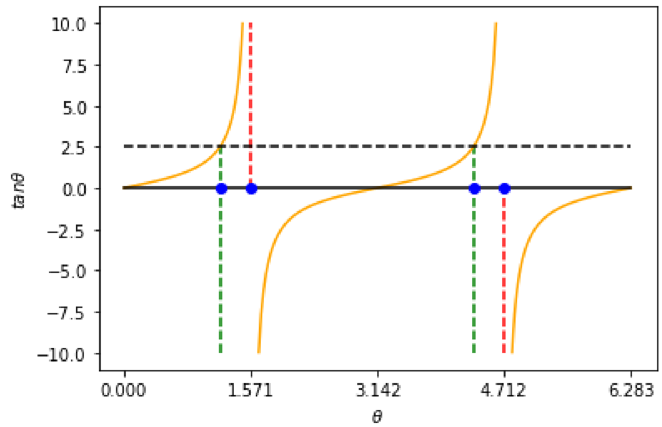


Fig. 13.23.1: $tan\theta$ with $m = 2.5$

b) similarly, when $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\tan \theta \leq m \tag{13.23.8}$$
$$\implies \theta \in [\pi/2, \pi + \tan^{-1} m] \tag{13.23.9}$$

$$\therefore \theta \in [\tan^{-1} m, \pi + \tan^{-1} m] \tag{13.23.10}$$

θ will have a uniform probability distribution function:

$$f(\theta) = \begin{cases} 0 & \text{if } \theta < 0 \\ \frac{1}{2\pi} & \text{if } 0 \leq \theta \leq 2\pi \\ 0 & \text{if } \theta > 2\pi \end{cases}$$

The shaded region of figure 13.23.2 represents the required probability.

$$\Pr\left(\arctan m \leq \theta \leq \tan^{-1} m + \pi\right)$$
$$= \int_{\tan^{-1} m}^{\pi+\tan^{-1} m} f(\theta)\, d\theta \tag{13.23.11}$$

$$= \int_{\tan^{-1} m}^{\pi+\tan^{-1} m} \frac{1}{2\pi}\, d\theta \tag{13.23.12}$$
$$= \frac{\pi}{2\pi} \tag{13.23.13}$$
$$= \frac{1}{2} \tag{13.23.14}$$

$\therefore$ option (c) is correct.

13.24. Let $X \sim B(5, \frac{1}{2})$ and $Y \sim U(0, 1)$. Then $\frac{P(X+Y\leq 2)}{P(X+Y\geq 5)}$

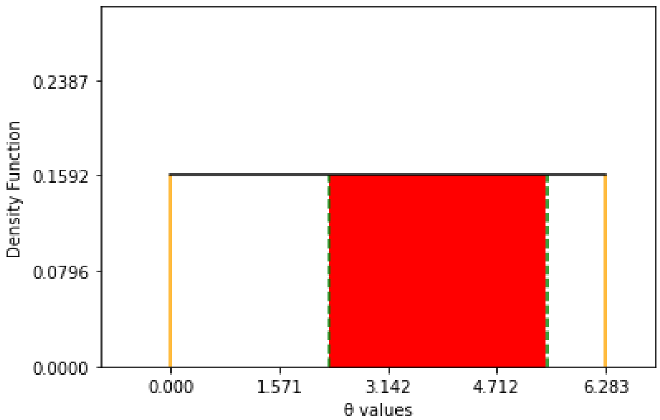


Fig. 13.23.2: Distribution function of θ

is equal to (where $B(n,p)$: Binomial distribution with n trials and success probability p; $n \in \{1,2,\dots\}$ and $p \in (0,1)$ U(a,b) : Uniform distribution on the interval $(a,b), -\infty < a < b < \infty$)

Solution: Given X is a Binomial Random Variable with 5 trails and success probability $p = 0.5$ and Y is a Continuous Random Variable over the interval $(0,1)$. So, $X \in \{0,1,2,3,4,5\}$ and $Y = U(0,1)$ Since X and Y are Independent Random Variables,

$$\Pr(X+Y \le 2) = \Pr(X = a, Y \le 2-a) \tag{13.24.1}$$

$$= \sum_{a=0}^{a=2} \Pr(X=a)\Pr(Y \le 2-a) \tag{13.24.2}$$

$$\Pr(X+Y \le 2) = \Pr(X=0)\Pr(Y \le 2) + \Pr(X=1)\Pr(Y \le 1) + \Pr(X=2)\Pr(Y \le 0) \tag{13.24.3}$$

Since X is a Binomial Random Variable,

$$\Pr(X=k) = \begin{cases} {}^nC_k p^{n-k}(1-p)^k & 0 \le k \le 5 \\ 0 & otherwise \end{cases} \tag{13.24.4}$$

Substituting the values of n = 5 and p = $\frac{1}{2}$ in (13.24.4), we get

$$\Pr(X=k) = {}^5C_k \left(\frac{1}{2}\right)^{5-k} \left(\frac{1}{2}\right)^k = {}^5C_k \left(\frac{1}{2}\right)^5$$

Also, the Cumulative Distribution Function of Y is defined as

$$CDF(Y) = F_Y(a) = \Pr(Y \le a) = \begin{cases} 0 & a \le 0 \\ a & 0 < a < 1 \\ 1 & a \ge 1 \end{cases} \tag{13.24.5}$$

By substituting the probability values from (13.24.4) and (13.24.5) in (13.24.3), we get

$$\Pr(X+Y \le 2) = {}^5C_0 \left(\frac{1}{2}\right)^5 (1) + {}^5C_1 \left(\frac{1}{2}\right)^5 (1) + {}^5C_2 \left(\frac{1}{2}\right)^5 (0) \tag{13.24.6}$$

$$= (1)\left(\frac{1}{32}\right) + (5)\left(\frac{1}{32}\right) + 0 \tag{13.24.7}$$

$$= \left(\frac{1}{32}\right) + \left(\frac{5}{32}\right) \tag{13.24.8}$$

$$= \frac{6}{32} \tag{13.24.9}$$

$$\Pr(X+Y \le 2) = \frac{3}{16} \tag{13.24.10}$$

Now,

$$\Pr(X+Y \ge 5) = 1 - \Pr(X+Y < 5) \tag{13.24.11}$$

$$= 1 - [\Pr(X+Y \le 5) - \Pr(X+Y = 5)] \tag{13.24.12}$$

But, as Y is a Continuous Random Variable over $(0,1)$, so $\Pr(Y=k) = 0 \ \forall$ k $\in [0,1]$. Therefore considering all possible cases,

$$\Pr(X+Y=5) = \Pr(X=4)\Pr(Y=1) + \Pr(X=5)\Pr(Y=0) \tag{13.24.13}$$

$$= \Pr(X=4)(0) + \Pr(X=5)(0) \tag{13.24.14}$$

$$= 0 + 0 \tag{13.24.15}$$

$$\Pr(X+Y=5) = 0 \tag{13.24.16}$$

Hence, by substituting (13.24.16) in (13.24.12), we get

$$\Pr(X+Y\geq 5)=1-[\Pr(X+Y\leq 5)-0] \quad (13.24.17)$$

$$\Pr(X+Y\geq 5)=1-\Pr(X+Y\leq 5) \quad (13.24.18)$$

$$\Pr(X+Y\geq 5)=1-\Pr(X=a,Y\leq 5-a) \quad (13.24.19)$$

$$=1-\left[\sum_{a=0}^{a=5}\Pr(X=a)\Pr(Y\leq 5-a)\right] \quad (13.24.20)$$

$$=1-[\Pr(X=0)\Pr(Y\leq 5)+\Pr(X=1)\Pr(Y\leq 4)+\Pr(X=2)\Pr(Y\leq 3)+\Pr(X=3)\Pr(Y\leq 2)+\Pr(X=4)\Pr(Y\leq 1)+\Pr(X=5)\Pr(Y\leq 0)] \quad (13.24.21)$$

By substituting the probability values from (13.24.4) and (13.24.5) in (13.24.21), we get

$$\Pr(X+Y\geq 5)=1-[{}^5C_0\left(\frac{1}{2}\right)^5(1)+{}^5C_1\left(\frac{1}{2}\right)^5(1)+{}^5C_2\left(\frac{1}{2}\right)^5(1)+{}^5C_3\left(\frac{1}{2}\right)^5(1)+{}^5C_4\left(\frac{1}{2}\right)^5(1)+{}^5C_5\left(\frac{1}{2}\right)^5(0)] \quad (13.24.22)$$

$$\Pr(X+Y\geq 5)=1-\left(\frac{1}{2}\right)^5[{}^5C_0+{}^5C_1+{}^5C_2+{}^5C_3+{}^5C_4] \quad (13.24.23)$$

$$=1-\left(\frac{1}{32}\right)[1+5+10+10+5] \quad (13.24.24)$$

$$=1-\left(\frac{1}{32}\right)[31]=\frac{1}{32} \quad (13.24.25)$$

Hence, $\Pr(X+Y\leq 2)=\frac{3}{16}$ and $\Pr(X+Y\geq 5)=\frac{1}{32}$.

$$\therefore \frac{\Pr(X+Y\leq 2)}{\Pr(X+Y\geq 5)}=\frac{\frac{3}{16}}{\frac{1}{32}}=6.$$

$$\therefore \frac{\Pr(X+Y\leq 2)}{\Pr(X+Y\geq 5)}=6$$

Hence, the required ratio is 6 .

14 Markov Chains

14.1. Two players A, and B alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is:

(A) $\frac{5}{11}$
(B) $\frac{1}{2}$
(C) $\frac{7}{13}$
(D) $\frac{6}{11}$

Solution:

- Given the die is fair.
- So, for any given throw by A or B:

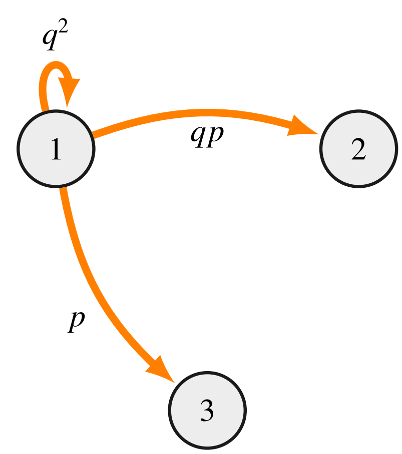
  The probability of getting 6 = $\frac{1}{6}$ = p (say)

  The probability of NOT getting 6 = $\frac{5}{6}$ = q (say)

Constraint: A starts the game and A should win.

$\implies$ Until A wins, both A and B cannot win.

Corresponding Markov chain is:



State	Corresponding description
1	A and B both lose.
2	A loses and B wins.
3	WINNER

Table 1: State description table

Definition 5. *2 and 3 are ABSORBING states. Transition/Stochastic matrix is: T*

$$\begin{array}{cc} & \textit{(To)} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \textit{(From)} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} q^2 & qp & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T \end{array}$$

Theorem 14.1. *Every transition matrix can be partitioned as* $\left(\begin{array}{c|c} Q & P \\ \hline O & J \end{array}\right)$
where Q arise from transition probabilities between non-absorbing states
R arises from Transition probability from non-absorbing state to absorbing state.
O = Null matrix, J = Identity matrix
T^k *approaches* $\overline{T}$ *as k increases.* $\overline{T}$ *is the limiting matrix and* $\overline{T} = \begin{pmatrix} O & NP \\ O & J \end{pmatrix}$
where N is the fundamental matrix. $N = (I - Q)^{-1}$

Definition 6. *The matrix D = NP gives the probability of ending up in the absorbing states (2 and 3) when the chain starts from a non-absorbent state 1.*

$$T = \left(\begin{array}{c|cc} q^2 & qp & p \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \tag{14.1.1}$$

$$Q = \left(q^2\right), P = \begin{pmatrix} qp & p \end{pmatrix}, O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14.1.2}$$

$$N = \left(1 - q^2\right)^{-1} = \left(\frac{1}{1 - q^2}\right) \tag{14.1.3}$$

$$D = NP = \begin{pmatrix} \frac{qp}{1 - q^2} & \frac{p}{1 - q^2} \end{pmatrix} \tag{14.1.4}$$

$$\Pr(\text{A wins}) = D_2 = \frac{p}{1 - q^2} = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11} \tag{14.1.5}$$

$$\Pr(\text{A wins}) = \frac{6}{11}$$

OPTION D is correct

14.2. A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

Solution: Given that the coin is tossed until a head appears on an odd toss.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{14.2.1}$$

Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$, such that:

TABLE 14.2.1: States and their notations

Notation	State
$S = 1$	Odd try
$S = 2$	Even try
$S = 3$	Loss
$S = 4$	Success

The state transition matrix for the Markov chain is:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{2.0.1}$$

The transient states are 1,2 and the absorbing states are 3 and 4. The standard form of the matrix is;

$$P = \begin{array}{c} \\ A \\ N \end{array} \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{2.0.2}$$

where,
Now, we convert the transition matrix to this standard form.

$$P = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{2.0.3}$$

TABLE 14.2.2: Notations and their meanings

Notation	Meaning
A	Absorbing states
N	Non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other sub-matrices

From (2.0.3),

$$R = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{2.0.4}$$

The limiting matrix for absorbing Markov chain is,

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{2.0.5}$$

where,

$$F = (I - Q)^{-1} \tag{2.0.6}$$

is called the fundamental matrix of P.
On solving we get,

$$\bar{P} = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0.667 & 0.333 & 0 & 0 \end{bmatrix} \end{array} \tag{2.0.7}$$

An element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability to the state j, starting from the state i.
Let $\Pr(A)$ be the probability that the first head is obtained on an odd toss. Then,

$$\Pr(A) = p_{14} \tag{14.2.2}$$
$$= 0.667 \tag{14.2.3}$$
$$\therefore \Pr(A) = \frac{2}{3} \tag{14.2.4}$$

14.3. A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) $\frac{3}{4}$

Solution: Given, a fair coin is tossed till heads turns up.

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{47.1}$$

Let us define a Markov chain $\{X_0, X_1, X_2 \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$ where $n \in \{0, 1, 2, \dots\}$,

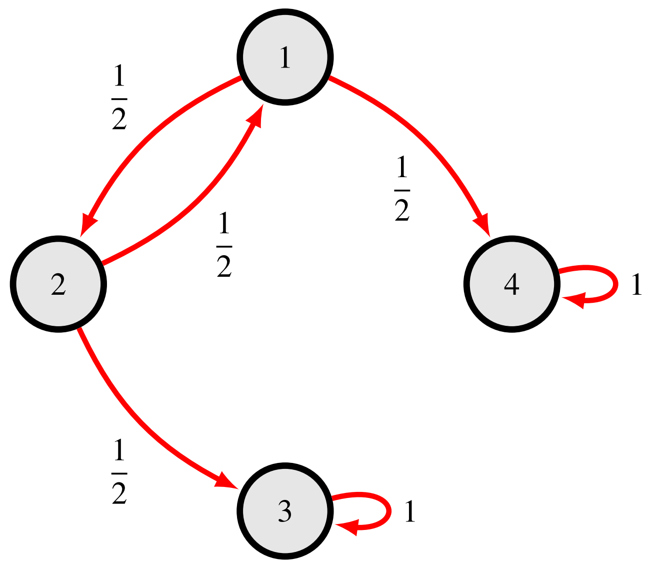
TABLE 14.3.1: Definition of Random Variables

R.V	Value=0	Value=1
X	$N_{tosses} = 2k$	$N_{tosses} = 2k - 1$
Y	H	T

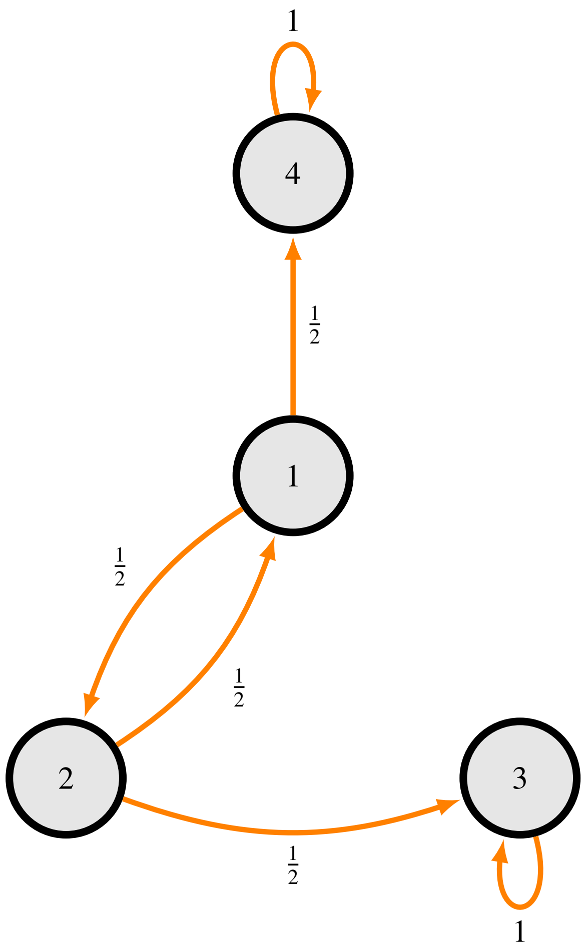
TABLE 14.3.2: Markov states and Notations

Notation	State
$S = 1$	$(X, Y) = (0, 1)$
$S = 2$	$(X, Y) = (1, 1)$
$S = 3$	$(X, Y) = (0, 0)$
$S = 4$	$(X, Y) = (1, 0)$

Markov chain diagram



Markov chain diagram



Definition 7. *The standard form of a state transition matrix is,*

$$\vec{P} = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} \vec{I} & \vec{O} \\ \vec{R} & \vec{Q} \end{bmatrix} \end{array} \tag{47.2}$$

where,

TABLE 14.3.3: Notations and their meanings

Notation	Meaning
A	Absorbing states (3,4)
N	Non-absorbing states (1,2)
$\vec{I}$	Identity matrix
$\vec{O}$	Zero matrix
$\vec{R}, \vec{Q}$	Other sub-matrices

Corollary 14.2. *The state transition matrix for the above Markov chain is,*

$$\vec{P} = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{47.3}$$

From (47.3),

$$\vec{R} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \vec{Q} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{47.4}$$

Definition 8. *The limiting matrix for absorbing Markov chain is,*

$$\vec{\bar{P}} = \begin{bmatrix} \vec{I} & \vec{O} \\ \vec{F}\vec{R} & \vec{O} \end{bmatrix} \tag{47.5}$$

where,

$$\vec{F} = (\vec{I} - \vec{Q})^{-1} \tag{47.6}$$

is called the fundamental matrix of $\vec{P}$.

Corollary 14.3. *Limiting Matrix of the Markov chain under observation is,*

$$\vec{\bar{P}} = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \end{array} \tag{47.7}$$

Definition 9. *A element $\bar{p}_{ij}$ of $\vec{\bar{P}}$ denotes the absorption probability in state j, starting from state i.*

Corollary 14.4. *The required probability is,*

$$P = \bar{p}_{14} \tag{47.8}$$

From (47.7) *and* (47.8)*,*

$$P = \frac{2}{3} \tag{47.9}$$

Therefore, option 3 is correct.

14.4. **Step 1.** Flip a coin twice.

Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.

Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.

Step 4. If the outcomes are (TAILS, TAILS),

then go to Step 1.
The probability that the output of the experiment is Y is (upto two decimal places)......

Solution: Given, a fair coin is tossed is tossed two times. Let's define a Markov chain $\{X_n, n = 0, 1, 2, \dots\}$, where $X_n \in S = \{1, 2, 3\}$, such that

TABLE 14.4.1: States and their notations

Notation	State
$S = 1$	getting $\{TT\}$
$S = 2$	getting output Y
$S = 3$	getting output N

The state transition matrix for the Markov chain is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{14.4.1}$$

Clearly, the state 1 are transient, while $2, 3$ are absorbing. The standard form of a state transition matrix is

$$P = \begin{array}{c} \\ A \\ N \end{array} \begin{array}{c} \begin{matrix} A & N \end{matrix} \\ \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \tag{14.4.2}$$

where, Converting (14.4.1) to standard form,

TABLE 14.4.2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

we get

$$P = \begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 2 & 3 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \end{array} \tag{14.4.3}$$

From (14.4.2),

$$R = \begin{bmatrix} 0.25 & 0.5 \end{bmatrix}, Q = \begin{bmatrix} 0.25 \end{bmatrix} \tag{104.5}$$

The limiting matrix for absorbing Markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \tag{14.4.4}$$

where,

$$F = (I - Q)^{-1} \tag{14.4.5}$$

is called the fundamental matrix of P.
On solving, we get

$$\bar{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 2 & 3 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.33 & 0.17 & 0 \end{bmatrix} \end{array} \tag{14.4.6}$$

A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state j, starting from state i. Then, the absorption probability in state 2 (i.e getting output Y) starting from state 1 is $\bar{p}_{12}$.

$$\therefore \bar{p}_{12} = 0.33 \,\text{(correct upto 2 decimal places)} \tag{14.4.7}$$

14.5. Players A and B take turns to throw a fair dice with six faces. If A is the first player to throw, then the probability of B being the first one to get a six is — (round of to two decimal places).

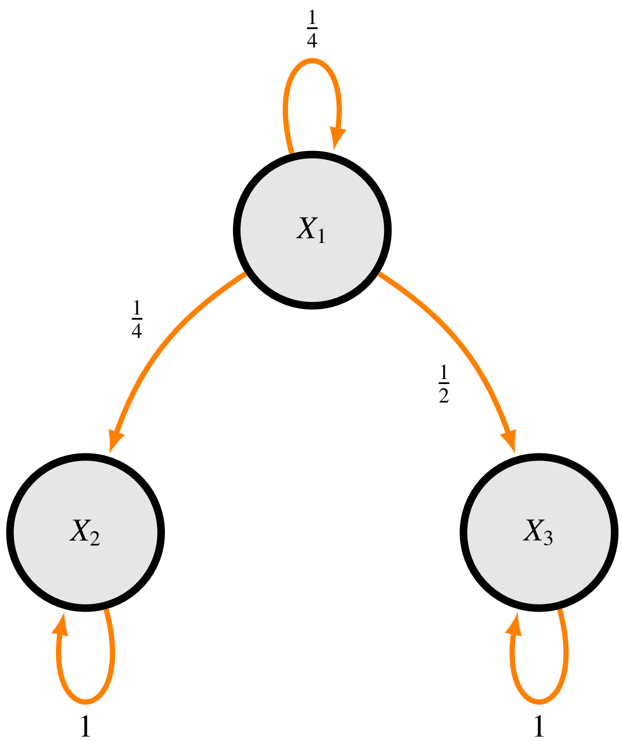
Solution:
Let the random variable X represent which player gets six first. That is $X = 0$ when A gets a six first and $X = 1$ when B gets six first.
Let another random variable Y represent getting a six on the dice. $Y = 1$ for six and $Y = 0$ for any other number.
Let N be the number of turns until we get a six.

$$\Pr(Y = 0) = \frac{5}{6} \tag{14.5.1}$$

$$\Pr(Y = 1) = \frac{1}{6} \tag{14.5.2}$$

The event success is when B gets a six for first time and failure is when neither A nor B

Markov chain diagram



gets six. Let p denote probability of success

$$p = \Pr(Y = 1) \tag{14.5.3}$$
$$\Pr(Y = 0) = 1 - p \tag{14.5.4}$$
$$p = \frac{1}{6} \tag{14.5.5}$$

To get $X = 1$ in N turns we have to get $N - 1$ failures for B and N failures for A and finally one success for B. Therefore the geometric distribution is,

$$f(N) = (1-p)^{n-1} \times p \times (1-p)^n \tag{14.5.6}$$
$$= (1-p)^{2n-1} \times p \tag{14.5.7}$$
$$= \left(\frac{5}{6}\right)^{2n-1} \times \frac{1}{6} \tag{14.5.8}$$

The result has been summarized in table 14.5.1.

Thus the total probability is sum of these

No. of turns	Probability
1	$5^1/6^2$
2	$5^3/6^4$
$\vdots$ n	$\vdots$ $5^{2n-1}/6^{2n}$
$\vdots$	$\vdots$

TABLE 14.5.1: Summary of turns

individual probabilities i.e.

$$\Pr(X = 1) = \sum_{N=1}^{\infty} f(N) \tag{14.5.9}$$
$$= \frac{5}{6^2} + \frac{5^3}{6^4} + \ldots + \frac{5^{2n-1}}{6^{2n}} + \ldots \tag{14.5.10}$$
$$= \frac{5}{6^2} \times \left(1 + \frac{5^2}{6^2} + \frac{5^4}{6^4} + \ldots\right) \tag{14.5.11}$$

By Using sum of infinite GP we have,

$$\Pr(X = 1) = \frac{5}{6^2} \times \left(\frac{1}{1 - \frac{25}{36}}\right) \tag{14.5.12}$$
$$= \frac{5}{36} \times \frac{36}{11} \tag{14.5.13}$$
$$= \frac{5}{11} = 0.45 \tag{14.5.14}$$

14.6. A fair coin is tossed till a head appears for the first time. The probability that the number of requried tosses is odd,is

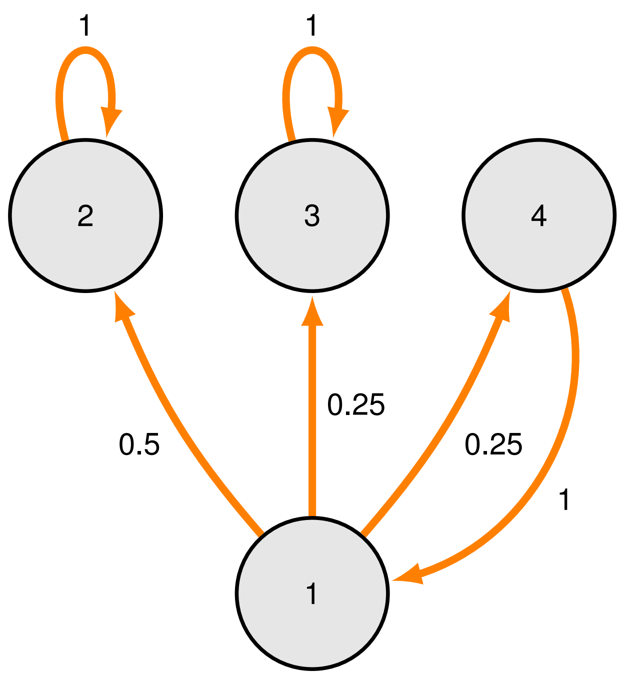
a) $\frac{1}{3}$
b) $\frac{1}{2}$
c) $\frac{2}{3}$
d) $\frac{3}{4}$

Solution:

14.7. Consider the experiment with the following steps.

a) Flip a coin twice.
b) If the outcomes are (TAILS, HEADS) then output Y and stop.
c) If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
d) If the outcomes are (TAILS, TAILS), then

Fig. 14.7.1: **Markov chain diagram**



go to Step 1.

The probability that the output of the experiment is Y is (upto two decimal places)......

Solution: Given a fair coin is flipped twice. Let us define a Markov chain with states {1,2,3,4}, such that

State	Events
1	Event of tossing a fair coin twice
2	Event of obtaining 'N' as the output
3	Event of obtaining 'Y' as the output
4	Event of obtaining (TAIL,TAIL) as the output

TABLE 14.7.1: Representation of different events

We know that when a fair coin is tossed,

$$\Pr(\text{HEAD}) = 1/2 \text{ and} \tag{14.7.1}$$

$$\Pr(\text{TAIL}) = 1/2. \tag{14.7.2}$$

Then,

The state transition matrix (P) for the Markov chain is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{14.7.3}$$

By the definition of transient and absorbing states, we can say that 1,4 are transient states whereas 2,3 are absorbing.

Then, the canonical form of the transition matrix is,

$$P = \begin{array}{c} \\ 2 \\ 3 \\ 1 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 1 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \tag{14.7.4}$$

The canonical form divides the transition matrix into four sub-matrices based on the states as listed below.

$$\begin{array}{r} \\ \text{Absorbing} \\ \text{Non-Absorbing} \end{array} \begin{array}{c} \begin{array}{cc} \text{Absorbing} & \text{Non-Absorbing} \end{array} \\ \left[\begin{array}{c|c} \text{I} & \text{O} \\ \hline \text{A} & \text{B} \end{array}\right] \end{array}$$

where,

Variable	Type of Matrix
I	Identity matrix
O	Zero matrix
A, B	Some matrices

TABLE 14.7.2: Representation of different matrices

and From (14.7.4),

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{bmatrix} \tag{14.7.5}$$

The fundamental matrix F for the absorbing Markov chain is defined as

$$F = (I - B)^{-1} \tag{14.7.6}$$

Then,

$$F = \begin{bmatrix} 1 & -\frac{1}{4} \\ -1 & 1 \end{bmatrix}^{-1} \tag{14.7.7}$$

$$\implies F = \begin{bmatrix} 1.33 & 0.33 \\ 1.33 & 1.33 \end{bmatrix} \tag{14.7.8}$$

Therefore,

$$FA = \begin{bmatrix} 0.67 & 0.33 \\ 0.67 & 0.33 \end{bmatrix} \tag{14.7.9}$$

Then the limiting matrix for the markov chain is

$$\bar{P} = \begin{bmatrix} I & O \\ FA & O \end{bmatrix} \tag{14.7.10}$$

where the element p_{ij} of $\bar{P}$ represents the probability of absorption in state j, when the initial state is i.

$$\therefore \bar{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 1 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.67 & 0.33 & 0 & 0 \\ 0.67 & 0.33 & 0 & 0 \end{bmatrix} \end{array} \tag{14.7.11}$$

Therefore,

$$\text{Req. Probability } = p_{13} = 0.33 \tag{14.7.12}$$

14.8. Consider a simple symmetric random walk on integers, where from every state i you move to i-1 and i+1 with the probability half each. Then which of the following are true?

a) The random walk is aperiodic
b) The random walk is irreducible
c) The random walk is null recurrent
d) The random walk is positive recurrent

Solution:

Definition 10 (Aperiodicity)**.** *A random walk defined by a Markov chain having state space S and state transition matrix P ,is said to be aperiodic if there exists self-transition in the chain such that*

$$p_{ii}^{n} > 0 \quad for \quad i \in S, n \in Z^{+} \tag{14.8.1}$$

Definition 11 (Irreducibility)**.** *A random walk defined by a Markov chain having state space S and state transition matrix P ,is said to be irreducible if all states communicate with each other such that*

$$p_{ij}^{n} > 0 \quad for \quad i, j \in S, n \in Z^{+} \tag{14.8.2}$$

Definition 12 (Positive and Null Recuurancy)**.** *A random walk defined by a Markov chain having state space S ,is said to be positive recurrent if the excepted time to return to state i $\forall$ $i \in S$ is finite such that*

$$E(\tau_{ii}) < \infty \tag{14.8.3}$$

and,is said to be null recurrent if the excepted time to return to state i $\forall$ $i \in S$ is infinite such that

$$E(\tau_{ii}) = \infty \tag{14.8.4}$$

Let us define a Markov Chain for the given simple symmetric random walk with states $\{i-1, i, i+1\}$.

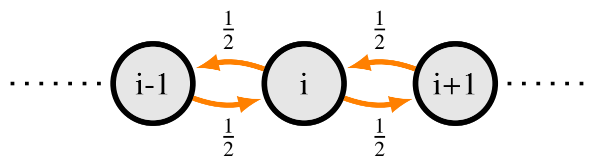


Fig. 14.8.1: **Markov chain diagram**

State transition matrix P can be defined as:

$$P = \begin{array}{c} \\ i-1 \\ i \\ i+1 \end{array} \begin{array}{c} \begin{array}{ccc} i-1 & i & i+1 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \end{bmatrix} \end{array} \tag{14.8.5}$$

a) From State Transition Matrix P,

$$p_{mm} = 0 \tag{14.8.6}$$

where

$$m = \{i-1, i, i+1\} \tag{14.8.7}$$

$\because$ There is no self-transition in the chain.
$\therefore$ Random Walk is not aperiodic.

b) From State Transition Matrix P,

$$p_{mn} > 0 \tag{14.8.8}$$

where

$$m, n = \{i-1, i, i+1\} \tag{14.8.9}$$

$\because$ All states communicate with each other .
$\therefore$ Random Walk is irreducible.

c) Let $p = \frac{1}{2}$ be the probability to move from state i to state $i+1$ and $q = \frac{1}{2}$ be the probability to move from state i to state $i-1$.

Then,the excepted time of getting back to i $\forall$ i is given by

$$E(\tau_{ii}) = \frac{1}{|p-q|} \tag{14.8.10}$$
$$= \frac{1}{0} \tag{14.8.11}$$
$$= \infty \tag{14.8.12}$$

$\therefore$ Random Walk is null recurrent .

Hence, $\boxed{Options \quad (2), (3)}$ are true .

Fig. 14.8.2: Random Walk on Integers

15 Random Process

15.1. $X(t)$ is a random process with a constant mean value of 2 and the autocorrelation function

$$R_x(\tau) = 4\left(e^{-0.2|\tau|} + 1\right). \tag{15.1.1}$$

Let Y and Z be the random variables obtained by sampling $X(t)$ at $t = 2$ and $t = 4$ respectively. Let $W = Y - Z$. The variance of W is

a) 13.36
b) 9.36
c) 2.64
d) 8.00

Solution: Let $t_1 = 2, t_2 = 4$ such that

$$Y = X(t_1), Z = X(t_2) \tag{15.1.2}$$

From the given information,

$$E[Y] = E[Z] = 0, \tag{15.1.3}$$
$$\implies \sigma_W^2 = E[Y-Z]^2 \tag{15.1.4}$$
$$= E\left[X^2(t_1)\right] + E\left[X^2(t_2)\right] - 2E[X(t_1)X(t_2)] \tag{15.1.5}$$
$$= 2E\left[X^2(t_1)\right] - 2E[X(t_1)X(t_1+2)] \tag{15.1.6}$$
$$= 2R_X(0) - 2R_X(2) \tag{15.1.7}$$

$\because$

$$\implies R_X(\tau) = E[X(t)X(t+\tau)]$$
$$\implies R_X(0) = E\left[X^2(t)\right] \tag{15.1.8}$$

From (15.1.1), (15.1.7) and (15.1.8),

$$\sigma_W^2 = 2.64 \tag{15.1.9}$$

So,option 3 is correct.

15.2. Let Y and Z be the random variables obtained by sampling $X(t)$ at $t = 2$ and $t = 4$ respectively. Let W = Y-Z. The variance of W is

a) 13.36 b) 9.36 c) 2.64 d) 8.00

15.3. Consider the random process $X(t) = U + Vt$, where U is a zero-mean Gaussian random variable and V is a random variable distributed between 0 and 2. Assume that U and V are statistically independent. The mean value of the random process at t=2 is........ **Solution:** Here U is a gaussian random variable of mean 0 and Let us consider V is uniformly distributed random variable in $(0, 2)$.

Random Variable	U	V	X(t)
Expected Value	0	1	t

TABLE 15.3.1: Random Variables and Expected Values

From Table 15.3.1 we can deduce that,

$$E[X(t)] = E[U + Vt] \tag{15.3.1}$$
$$E[X(t)] = E[U] + tE[V] \tag{15.3.2}$$
$$E[X(t)] = 0 + t \tag{15.3.3}$$
$$E[X(t)] = t \tag{15.3.4}$$
$$E[X(2)] = 2 \tag{15.3.5}$$

$\therefore$ mean of random process X(t) at 2 is 2.

15.4. $X(t)$ is a random process with a constant mean value of 2 and the autocorrelation function:

$$R_x(\tau) = 4\left[e^{-0.2|\tau|} + 1\right] \tag{15.4.1}$$

Let X be the Gaussian Random Variable obtained by sampling the process at $t = t_i$, and let

$$Q(\alpha) = \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}\, dy \tag{15.4.2}$$

Find $\Pr(X \leq 1)$

(A) $1 - Q(0.5)$ (B) $Q(0.5)$
(C) $Q(\frac{1}{2\sqrt{2}})$ (D) $1 - Q(\frac{1}{2\sqrt{2}})$

Solution:

X is a normal random variable defined by

$$X \sim N\left(2, \sigma_x^2\right) \tag{15.4.3}$$

Thus, from (15.4.1):

$$Var(X) = \sigma_x^2 = R_x(0) \tag{15.4.4}$$

$$\sigma_x^2 = 8 \tag{15.4.5}$$

$$\sigma_x = 2\sqrt{2} \tag{15.4.6}$$

Converting X to a standard normal random variable using :

$$Z = \frac{X - \mu_x}{\sigma_x} \tag{15.4.7}$$

$$\Pr(X \leq 1) \tag{15.4.8}$$

$$= \Pr\left(\frac{X-2}{2\sqrt{2}} \leq \frac{1-2}{2\sqrt{2}}\right) \tag{15.4.9}$$

$$= \Pr\left(Z \leq \frac{-1}{2\sqrt{2}}\right) \tag{15.4.10}$$

where Z is a standard normal random variable defined by $Z \sim N(0,1)$

Due to symmetry of the bell curve graph:

$$\Pr\left(Z \leq \frac{-1}{2\sqrt{2}}\right) = \Pr\left(Z \geq \frac{1}{2\sqrt{2}}\right) \tag{15.4.11}$$

From (15.4.2),

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = \Pr(Z \geq \alpha) \tag{15.4.12}$$

Thus,

$$\Pr\left(Z \geq \frac{1}{2\sqrt{2}}\right) = Q\left(\frac{1}{2\sqrt{2}}\right) \tag{15.4.13}$$

16 Convergence

16.1. Consider a binomial random variable X. If $X_1, X_2, ..., X_n$ are independent and identically distributed samples from the distribution of **X** with sum Y=$\Sigma_{i=1}^{n} X_i$, then the distribution of **Y** as $n \to \infty$ can be approximated as

a) Exponential
b) Bernoulli
c) Binomial
d) Normal

Solution: Given a binomial random variable **X**

$$\Rightarrow X \sim B(r,p) \tag{16.1.1}$$

also given that $X_1, X_2, ..., X_n$ are independent and identically distributed samples

$$\Rightarrow X_1 = X_2 = ... = X_n = X \sim B(r,p) \tag{16.1.2}$$

also given that

$$Y = \sum_{i=1}^{n} X_i \tag{16.1.3}$$

We know that the characteristic equation of binomial trials with n elements is

$$\phi_X(t) = \left(1 - p + pe^{it}\right)^n \tag{16.1.4}$$

consider two sets of Bernoulli trials containing $r_1 \& r_2$ elements respectively where both trials have the same probability 'p'$(X \sim B(r_1,p), Y \sim (r_2,p))$. Now considering both as a whole set

$$B(r_1,p) + B(r_2,p) = \Phi_{X+Y}(t) \tag{16.1.5}$$

$$= \Phi_X(t) \times \Phi_Y(t)$$

$$= \left(1-p+pe^{it}\right)^{r_1} \times \left(1-p+pe^{it}\right)^{r_2} \tag{16.1.6}$$

$$= \left(1-p+pe^{it}\right)^{r_1+r_2} \tag{16.1.7}$$

$$= B(r_1+r_2,p)$$

$$\therefore B(r_1,p) + B(r_2,p) = B(r_1+r_2,p) \tag{16.1.8}$$

using this recursively we get

$$Y = B(rn,p) \tag{16.1.9}$$

$\Rightarrow$ using standard formulae

$$\text{mean of Y } \mu_Y = nrp$$
$$\text{and variance } \sigma_Y^2 = nrp(1-p) \tag{16.1.10}$$

By central limit theorem(CLT)

$$Z_n = \sqrt{n}\left(\frac{\frac{Y}{n} - \mu_Y}{\sigma_Y}\right)$$
$$= \frac{Y - n\mu_Y}{\sqrt{n}\sigma_Y} \tag{16.1.11}$$

$\lim_{n\to\infty} Z_n \sim N(0,1)$

Which is a normal distribution

$\therefore$ the correct answer is option D

16.2. Let $\{X_j\}$ be a sequence of independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = \frac{1}{4}$

and let $Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2$. Then Y_n converges, in probability, to ________ . **Solution:**
A sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in probability, to a random variable Y if

$$\lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \tag{16.2.1}$$

Similarly, a sequence of random variables $Y_1, Y_2, Y_3 \dots$ converges, in mean square, to a random variable Y if

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{16.2.2}$$

A random variable converges, in probability, to a value if it converges, in mean square, to the same particular value by Markov's Inequality. Proof for this is: For any $\epsilon > 0$

$$\Pr(|Y_n - Y| \geq \epsilon) = \Pr(|Y_n - Y|^2 \geq \epsilon^2) \tag{16.2.3}$$

$$\Pr(|Y_n - Y| \geq \epsilon) \leq \frac{E|Y_n - Y|^2}{\epsilon^2} \quad \text{(by Markov's Inequality)} \tag{16.2.4}$$

$$\lim_{n\to\infty} E(|Y_n - Y|^2) = 0 \tag{16.2.5}$$

$$0 \leq \lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) \leq \frac{0}{\epsilon^2} \tag{16.2.6}$$

$$\lim_{n\to\infty} \Pr(|Y_n - Y| \geq \epsilon) = 0 \quad \forall \epsilon > 0 \tag{16.2.7}$$

Given in the question that $\{X_j\}$ is a sequence of random variables with

$$\Pr(X_j = 1) = \frac{1}{4} \tag{16.2.8}$$

$$\Pr(X_j = 0) + \Pr(X_j = 1) = 1 \tag{16.2.9}$$

$$\Pr(X_j = 0) = 1 - \frac{1}{4} = \frac{3}{4} \tag{16.2.10}$$

$$X_j \in \{0, 1\} \tag{16.2.11}$$

Since $0^2 = 0$ and $1^2 = 1$,

$$X_j^2 = X_j \quad \forall j \in \{1, 2, \dots, n\} \tag{16.2.12}$$

Thus,

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \tag{16.2.13}$$

$$= \frac{1}{n}\sum_{j=1}^{n} X_j \tag{16.2.14}$$

$$\Pr(Y_n = y) = {}^{n}C_{ny} \left(\frac{1}{4}\right)^{ny} \left(\frac{3}{4}\right)^{n-ny} \tag{16.2.15}$$

Let us assume

$$k = ny \tag{16.2.16}$$

$$k \in \{0, 1, 2, \dots, n-1, n\} \tag{16.2.17}$$

$$\Pr(Y_n = \frac{k}{n}) = {}^{n}C_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{16.2.18}$$

$$E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = E\left(Y_n^2 - \frac{1}{2}Y_n + \frac{1}{16}\right) \tag{16.2.19}$$

$$= E(Y_n^2) - \frac{1}{2}E(Y_n) + \frac{1}{16} \tag{16.2.20}$$

$$E(Y_n^2) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \Pr\left(Y_n = \frac{k}{n}\right) \tag{16.2.21}$$

$$= \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) {}^{n}C_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{16.2.22}$$

$$E(Y_n^2) = 0 + \frac{1}{n^2} \times n \left(\frac{1}{4}\right)^{1} \left(\frac{3}{4}\right)^{n-1} + \sum_{k=2}^{n} \left(\frac{k}{n}\right)^2 \times \frac{n(n-1)}{k(k-1)} \times {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{16.2.23}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \sum_{k=2}^{n} \left(\frac{k}{k-1}\right) {}^{n-2}C_{k-2} \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{n-k} \tag{16.2.24}$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) + \frac{n-1}{n}\left(\sum_{k=2}^{n} \frac{1}{k-1}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \quad (16.2.25)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{n} \times \frac{1}{16}\left(\sum_{k=2}^{n} {}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k-2}\left(\frac{3}{4}\right)^{(n-2)-(k-2)}\right) + \frac{1}{n}\left(\sum_{k=2}^{n} \frac{n-1}{k-1}{}^{n-2}C_{k-2}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k}\right) \quad (16.2.26)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\sum_{j=0}^{n-2} {}^{n-2}C_{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-2)-j}\right) + \frac{1}{4n}\left(\sum_{k=2}^{n} {}^{n-1}C_{k-1}\left(\frac{1}{4}\right)^{k-1}\left(\frac{3}{4}\right)^{(n-1)-(k-1)}\right) \quad (16.2.27)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n}\left(\frac{1}{4}+\frac{3}{4}\right)^{n-2} + \frac{1}{4n}\left(\sum_{j=1}^{n-1} {}^{n-1}C_{j}\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-1)-j}\right) \quad (16.2.28)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n}\left(\left(\frac{1}{4}+\frac{3}{4}\right)^{n-1} - \left(\frac{3}{4}\right)^{n-1}\right) \quad (16.2.29)$$

$$E(Y_n^2) = \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} + \frac{n-1}{16n} + \frac{1}{4n} - \frac{1}{4n}\left(\frac{3}{4}\right)^{n-1} \quad (16.2.30)$$

$$= \frac{1}{16} + \frac{3}{16n} \quad (16.2.31)$$

$$E(Y_n) = \sum_{k=0}^{n} \frac{k}{n}\Pr\left(Y_n = \frac{k}{n}\right) \quad (16.2.32)$$

$$= \sum_{k=0}^{n} \left(\frac{k}{n}\right){}^{n}C_k\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \quad (16.2.33)$$

$$= 0 + \sum_{k=1}^{n} \frac{k}{n} \times \frac{n}{k} \times {}^{n-1}C_{k-1}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{n-k} \quad (16.2.34)$$

$$= \frac{1}{4}\sum_{j=0}^{n-1} {}^{n-1}C_j\left(\frac{1}{4}\right)^{j}\left(\frac{3}{4}\right)^{(n-1)-j} \quad (16.2.35)$$

$$= \frac{1}{4}\left(\frac{1}{4}+\frac{3}{4}\right)^{n-1} \quad (16.2.36)$$

$$= \frac{1}{4} \quad (16.2.37)$$

Using equations (16.2.31) and (16.2.37) in (16.2.20),

$$E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \frac{1}{16} + \frac{3}{16n} - \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \quad (16.2.38)$$

$$= \frac{3}{16n} \quad (16.2.39)$$

$$\lim_{n\to\infty} E\left(\left|Y_n - \frac{1}{4}\right|^2\right) = \lim_{n\to\infty} \frac{3}{16n} \quad (16.2.40)$$

$$= \frac{3}{16}\lim_{n\to\infty}\frac{1}{n} \quad (16.2.41)$$

$$= 0 \quad (16.2.42)$$

Thus, Y_n converges, in mean square, to $\frac{1}{4}$ and hence Y_n converges, in probability, to $\frac{1}{4}$.

16.3. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having uniform distribution on (0,2). For $n \geq 1$, let $Z_n = -\log_e\left(\prod_{i=1}^{n}(2 - X_i)\right)^{\frac{1}{n}}$. Then, as $n \to \infty$, the sequence $\{Z_n\}_{n\geq 1}$ converges almost surely to _____ (Round of to 2 decimal places).

Solution:

Simplifying Z_n, we have

$$Z_n = -\log_e\left(\prod_{i=1}^{n}(2-X_i)\right)^{\frac{1}{n}} \tag{16.3.1}$$
$$= -\frac{1}{n}\cdot\log_e\left(\prod_{i=1}^{n}(2-X_i)\right) \tag{16.3.2}$$
$$= \sum_{i=1}^{n}\left((-\log_e(2-X_i))\cdot\frac{1}{n}\right) \tag{16.3.3}$$
$$= E\left(-\log_e(2-X_i)\right) \tag{16.3.4}$$

Let X and Z be random variables. X follows a uniform distribution from 0 to 2.

$$X \sim \mathcal{U}[0,2], \tag{16.3.5}$$
$$\text{and let } Z = -\log_e(2-X) \tag{16.3.6}$$

The sequence X_n converges in distribution to X. i.e.

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \tag{16.3.7}$$

From **The Law of Large Numbers**, we have that for large n, $Z_n = E\left(-\log_e(2-X_i)\right)$ should be close to $E\left(-\log_e(2-X)\right) = E(Z)$. i.e.

$$\Pr\left(\lim_{n\to\infty} Z_n = E(Z)\right) = 1 \tag{16.3.8}$$

If $\Pr(\lim_{n\to\infty} Y_n = Y)$=1, we say that Y_n almost surely converges to Y. Therefore, by (16.3.8) as $n\to\infty$, Z_n almost surely converges to $E(Z)$.

The CDF of Z is defined as

$$F_Z(z) = \Pr(Z \le z) \tag{16.3.9}$$
$$= \Pr(-\log_e(2-X) \le z) \tag{16.3.10}$$
$$= \Pr(\log_e(2-X) \ge -z) \tag{16.3.11}$$
$$= \Pr(2-X \ge \exp(-z)) \tag{16.3.12}$$
$$= \Pr(X \le 2-\exp(-z)) \tag{16.3.13}$$
$$= F_X(2-\exp(-z)) \tag{16.3.14}$$

The CDF for X ($F_X(x)$), a uniform distribution on $(0,2)$ is given by

$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{x}{2} & 0\le x\le 2 \\ 1 & x>2 \end{cases} \tag{16.3.15}$$

Substituting the above in (16.3.14),

$$F_X(2-\exp(-z)) = \begin{cases} 0 & 2-\exp(-z)<0 \\ 1-\frac{\exp(-z)}{2} & 0\le 2-\exp(-z)\le 2 \\ 1 & 2-\exp(-z)>2 \end{cases} \tag{16.3.16}$$

After some algebra, the above conditions yield

$$F_Z(z) = \begin{cases} 0 & z<-\log_e(2) \\ 1-\frac{\exp(-z)}{2} & z\ge-\log_e(2) \end{cases} \tag{16.3.17}$$

$$\implies f_Z(z) = \frac{\mathrm{d}(F_Z(z))}{\mathrm{d}z} = \begin{cases} 0 & z<-\log_e(2) \\ \frac{\exp(-z)}{2} & z\ge-\log_e(2) \end{cases} \tag{16.3.18}$$

Now calculating the expectation value for Z,

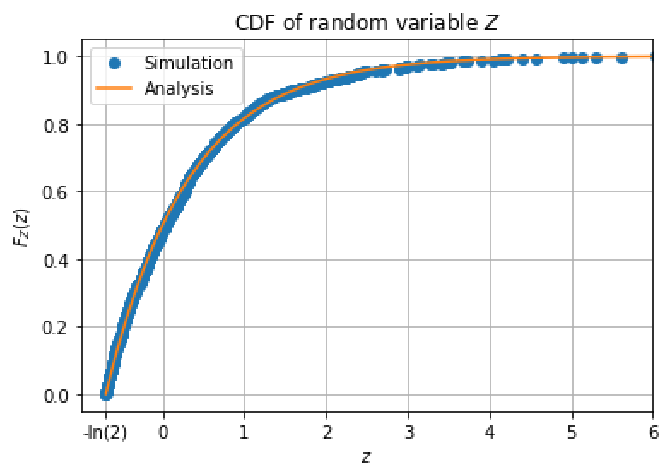


Fig. 16.3.1: $F_Z(z)$

we have

$$E(Z) = \int_{-\ln 2}^{\infty} z\, f_Z(z)\, dz \tag{16.3.19}$$
$$= \int_{-\ln 2}^{\infty} \frac{z\,e^{-z}}{2}\, dz \tag{16.3.20}$$
$$= \left[\frac{-(z+1)\,e^{-z}}{2}\right]_{-\ln 2}^{\infty} \tag{16.3.21}$$
$$= 1-\ln(2) \tag{16.3.22}$$
$$\approx 0.3068 \tag{16.3.23}$$

17 Statistics

17.1. Consider a communication scheme where the binary valued signal X satisfies $P\{X = +1\} = 0.75$ and $P\{X = -1\} = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance σ^2. The received signal Y is fed to the threshold detector. The output of the threshold detector $\hat{X}$ is:

$$\hat{X} = \begin{cases} +1 & Y > \tau \\ -1 & Y \leqslant \tau \end{cases}$$

To achieve minimum probability of error $P\{\hat{X} \neq X\}$, the thresholds τ should be

a) strictly positive

b) zero

c) strictly negative

d) strictly positive, zero or strictly negative depending on the nonzero value of σ^2

17.2.

Common Data for the following two Questions :

Let X be a random variable with probability density function $f \in \{f_0, f_1\}$, where

$$f_0(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For testing the null hypothesis $H_0 : f \equiv f_0$ against the alternative hypothesis $H_1 : f \equiv f_1$ at level of significance $\alpha = 0.19$, the power of the most powerful test is

a) 0.729

b) 0.271

c) 0.615

d) 0.385

17.3. The variance of the random variable X is

a) $\frac{1}{12}$

b) $\frac{1}{4}$

c) $\frac{7}{12}$

d) $\frac{5}{12}$

17.4. The covariance between the random variables X and Y is

a) $\frac{1}{3}$

b) $\frac{1}{4}$

c) $\frac{1}{6}$

d) $\frac{1}{12}$

17.5. A screening test is carried out to detect a certain disease. It is found that 12% of the positive reports and 15% of the negative reports are incorrect. Assuming that the probability of a person getting positive report is 0.01, the probability that a person tested gets an incorrect report is . . .

Solution: Let $X \in \{0, 1\}$ represent the random variable, where 0 represents the case where a person gets a positive report while 1 represents the case where a person gets a negative report. From the question,

$$\Pr(X = 0) = 0.01 \tag{17.5.1}$$

$$\Pr(X = 0) + \Pr(X = 1) = 1 \tag{17.5.2}$$

$$\Pr(X = 1) = 1 - 0.01 = 0.99 \tag{17.5.3}$$

Let $Y \in \{0, 1\}$ represent the random variable, where 0 represents a correct report whereas 1 represents an incorrect report.

$$\Pr(Y = 1|X = 0) = 12\% = 0.12 \tag{17.5.4}$$

$$\Pr(Y = 1|X = 1) = 15\% = 0.15 \tag{17.5.5}$$

Then, from total probability theorem,

$$\Pr(Y = 1) = \Pr(Y = 1, X = 0) + \Pr(Y = 1, X = 1) \tag{17.5.6}$$

Using Bayes theorem,

$$\Pr(Y = 1) = \Pr(Y = 1|X = 0) \times \Pr(X = 0) + \Pr(Y = 1|X = 1) \times \Pr(X = 1) \tag{17.5.7}$$

$$\Pr(Y = 1) = 0.12 \times 0.01 + 0.15 \times 0.99 \tag{17.5.8}$$

$$= 0.0012 + 0.1485 \tag{17.5.9}$$

$$= 0.1497 \tag{17.5.10}$$

17.6. A diagnostic test for a certain disease is 90% accurate. That is, the probability of a person having (respectively, not having) the disease tested positive (respectively, negative) is 0.9. Fifty percent of the population has the disease. What is the probability that a randomly chosen person has the disease given that the person tested negative?

Solution: Let X and Y be two Bernoulli random variables such that X,Y∈{0, 1} and as given fifty percent of the population has the disease, the probability mass function of X is

$$p_X(n) = \Pr(X = n) = \begin{cases} 0.5 & n = 1 \\ 0.5 & n = 0 \\ 0 & otherwise \end{cases} \tag{17.6.1}$$

where X denotes the health status of a person(X=1 if person is healthy and X=0 if person is diseased) and Y denotes the diagnostic test result (Y=1 if it is positive and Y=0 if it is negative).

Given the probabilities of,

$$\Pr(Y = 1|X = 0) = 0.9 \tag{17.6.2}$$
$$\Pr(Y = 0|X = 1) = 0.9 \tag{17.6.3}$$

we need to find $\Pr(X = 0|Y = 0)$,

$$\Pr(X = 0|Y = 0) = \frac{\Pr(X = 0 \cap Y = 0)}{\Pr(Y = 0)} \tag{17.6.4}$$

$$\Pr(X = 0|Y = 0) = \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\Pr(Y = 0)} \tag{17.6.5}$$

$$\Pr(Y = 0) = \Pr(Y = 0|X = 1)\Pr(X = 1) + \Pr(Y = 0|X = 0)\Pr(X = 0) \tag{17.6.6}$$

Using (17.6.1),(17.6.2) and (17.6.3) in (17.6.6),

$$\Pr(Y = 0) = 0.9(0.5) + (1 - 0.9)0.5$$
$$\Pr(Y = 0) = 0.5 \tag{17.6.7}$$

Using (17.6.1),(17.6.2) and (17.6.7) in (17.6.5)

$$\Pr(X = 0|Y = 0) = \frac{(1 - 0.9)0.5}{0.5} \tag{17.6.8}$$
$$\Pr(X = 0|Y = 0) = 0.1 \tag{17.6.9}$$

17.7. The probability density function of a random variable X is

$$f(x) = \begin{cases} \frac{1}{\lambda}e^{\left(-\frac{x}{\lambda}\right)}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{17.7.1}$$

where $\lambda > 0$. For testing the hypothesis $H_0 : \lambda = 3$ against $H_1 : \lambda = 5$, a test is given as "Reject H_0 if $X \ge 4.5$".The probability of type 1 error and power of the test are respectively:

a) 0.1353 and 0.4966 c) 0.2021 and 0.4493

b) 0.1827 and 0.379 d) 0.2231 and 0.4066

Solution:

Definition 13. *A type 1 error occurs if the null hypothesis H_0 is rejected even if it is true.*

Definition 14. *The probability that the alternative hypothesis H_1 is true is defined to be Power of a given test.*

Given,

$$f_X(x) = \begin{cases} \frac{1}{\lambda}e^{\left(-\frac{x}{\lambda}\right)}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{17.7.2}$$

Let cumulative distribution function be $F_X(x)$ for a given λ. Hence,

$$F_X(x) = \int_{-\infty}^{x} f_X(a)da \tag{17.7.3}$$

From the probability density function,

$$\implies F_X(4.5) = \int_{-\infty}^{x} f_X(a)da \tag{17.7.4}$$
$$= \int_{0}^{4.5} \frac{1}{\lambda}e^{\left(-\frac{a}{\lambda}\right)}da \tag{17.7.5}$$
$$= 1 - e^{-\frac{4.5}{\lambda}} \tag{17.7.6}$$

We need the probability for $X \ge 4.5$,hence required probability is,

$$1 - F_X(4.5) = e^{-\frac{4.5}{\lambda}} \tag{17.7.7}$$

From (17.7.7) we get probability that the given null hypothesis(H_0) is true is,

$$e^{-\frac{4.5}{3}} = 0.2231. \tag{17.7.8}$$

∴ The **probability of type 1 error is 0.2231**. From (17.7.7),we get the required probability that the given alternative hypothesis(H_1) is true is,

$$e^{-\frac{4.5}{5}} = 0.4066 \tag{17.7.9}$$

$\therefore$ The **power of the test is 0.4066**

17.8. Let $Y_1, Y_2, ..., Y_{15}$ be a random sample of size 15 from the probability density function

$$f_y(y) = 3(1-y)^2, 0 < y < 1 \tag{Eq:1}$$

Use the central limit theorem to approximate $P\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right)$ **Solution:**
The **central limit theorem** states that whenever a random sample of size n is taken from any distribution with mean and variance, then the sample mean will be approximately normally distributed with mean and variance. The larger the value of the sample size, the better the approximation to the normal.

$$Z_n = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \tag{1.1}$$

From equation 1.1

$$\bar{Y} = Z_n\left(\frac{\sigma}{\sqrt{n}}\right) + \mu \tag{1.2}$$

$$\Pr\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right) = \Pr\left(\frac{1}{8} < Z_n\left(\frac{\sigma}{\sqrt{n}}\right) + \mu < \frac{3}{8}\right) \tag{1.3}$$

$$= \Pr\left(\frac{\frac{1}{8} - \mu}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8} - \mu}{\frac{\sigma}{\sqrt{n}}}\right) \tag{1.4}$$

$\bar{Y}$:Mean of the randomly selected 15 variables

$$\bar{Y} = \frac{Y_1 + Y_2 + ..Y_{15}}{15} \tag{1.5}$$

Mean of probability density function is

$$\mu = \int_{-\infty}^{\infty} y f(y) dy \tag{1.6}$$

$$= \int_0^1 y \times 3(1-y)^2 dy \tag{1.7}$$

$$= \frac{1}{4} \tag{1.8}$$

Variance of probability density function is

$$\sigma^2 = E[y^2] - (E[y])^2 \tag{1.9}$$

$$= \left(\int_0^1 y^2 f(y) dy\right) - \left(\frac{1}{4}\right)^2 \tag{1.10}$$

$$\int_0^1 y^2 f(y) dy = \int_0^1 y^2 \times 3(1-y)^2 dy \tag{1.11}$$

$$= 3\int_0^1 (y - y^2)^2 dy \tag{1.12}$$

$$= \frac{1}{10} \tag{1.13}$$

Substituting equation 1.13 in equation 1.10

$$\sigma^2 = \frac{1}{10} - \frac{1}{16} \tag{1.14}$$

$$= \frac{3}{80} \tag{1.15}$$

Using Q function in equation 1.4 we have,

$$\Pr\left(\frac{1}{8} < \bar{Y} < \frac{3}{8}\right) = \Pr\left(\frac{\frac{1}{8} - \mu}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= \Pr\left(\frac{\frac{1}{8} - \mu(y)}{\frac{\sigma}{\sqrt{n}}} < Z_n < \frac{\frac{3}{8} - \mu(y)}{\frac{\sigma}{\sqrt{n}}}\right) \tag{1.16}$$

$$= Q\left(\frac{\frac{-1}{8}}{\sqrt{\frac{3}{80}}}\right) - Q\left(\frac{\frac{1}{8}}{\sqrt{\frac{3}{80}}}\right) \tag{1.17}$$

$$= 1 - 2Q\left(\frac{\frac{1}{8}}{\sqrt{\frac{3}{80}}}\right) \tag{1.18}$$

$$= 1 - 2Q(0.645) \tag{1.19}$$

$$= 0.9938 \tag{1.20}$$

17.9. Let X be a non-constant positive Random Variable such that $E(X) = 9$.
Then which of the following statements is True?
a) $E\left(\frac{1}{X+1}\right) > 0.1$ and $\Pr(X \geq 10) \leq 0.9$
b) $E\left(\frac{1}{X+1}\right) < 0.1$ and $\Pr(X \geq 10) \leq 0.9$
c) $E\left(\frac{1}{X+1}\right) > 0.1$ and $\Pr(X \geq 10) > 0.9$
d) $E\left(\frac{1}{X+1}\right) < 0.1$ and $\Pr(X \geq 10) > 0.9$
Solution:
Given, for X > 0 ,$E(X) = 9$, $E\left(\frac{1}{X+1}\right)$ can be

estimated by Jensens's Inequality.
pre - requisites:
In general, $\phi(X)$ is a convex function iff:

$$\frac{d^2\phi}{dX^2} \geq 0$$

Jensen's Inequality:
In the context of probability theory, it is generally stated in the following form: if X is a random variable and ϕ is a convex function, then

$$\phi(E(X)) \leq E(\phi(X)) \tag{1}$$

So for $\phi(X) = \dfrac{1}{X+1}$,

$$\frac{d\phi}{dX} = -\frac{1}{(X+1)^2}$$
$$\frac{d^2\phi}{dX^2} = \frac{2}{(X+1)^3} \implies \frac{d^2\phi}{dX^2} \geq 0, (\because X > 0) \tag{2}$$

by eq (1) and (2)

$$E\left(\frac{1}{X+1}\right) \geq \frac{1}{E(X)+1}$$
$$\implies E\left(\frac{1}{X+1}\right) \geq \frac{1}{9+1}$$
$$\implies E\left(\frac{1}{X+1}\right) \geq 0.1 \tag{3}$$

$\Pr(X \geq 10)$ can be estimated by Markov's Inequality.
Markov's Inequality: If X is a non-negative random variable and a > 0, then the probability that X is at least a is at most the expectation of X divided by a.
Mathematically,

$$\Pr(X \geq a) \leq \frac{E(X)}{a} \tag{4}$$

by (4) for a = 10

$$\Pr(X \geq 10) \leq \frac{E(X)}{10}$$
$$\implies \Pr(X \geq 10) \leq \frac{9}{10}$$
$$\therefore \Pr(X \geq 10) \leq 0.9 \tag{5}$$

So, from (3) and (5)

$$\boxed{\textbf{Option 1 is the Correct Answer}}$$

17.10. Let $X_1, X_2, X_3, \cdots$ be a sequence of i.i.d uniform $(0,1)$ random variables. Then the value of

$$\lim_{n\to\infty} \Pr(-\ln(1-X_1) - \cdots - \ln(1-X_n) > n) \tag{17.10.1}$$

is equal to
Solution:

$$f_{X_i}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{17.10.2}$$

Let $Y_1, Y_2, \cdots,$ be another sequence of random variables where $Y_i = -\ln(1-X_i), i = 1, 2, 3, \cdots$

$$f_{Y_i}(x) = \frac{f_{X_i}(x)}{\frac{dY_i}{dX_i}} \tag{17.10.3}$$
$$f_{Y_i}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{17.10.4}$$

From the above probability function, we have all Y_i's to be exponential random variables.

$$Y_i \sim \text{Exp}(1) \tag{17.10.5}$$
$$\Rightarrow \mu = 1, \sigma^2 = 1 \tag{17.10.6}$$

The required probability is

$$\lim_{n\to\infty} \Pr\left(\sum_{i=1}^{n} Y_i > n\right) \tag{17.10.7}$$
$$= \lim_{n\to\infty} \Pr\left(\overline{Y_n} > 1\right) \tag{17.10.8}$$

Consider

$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\overline{Y_n} - \mu}{\sigma}\right) \tag{17.10.9}$$

Since $\overline{Y_n} > 1$, we have $Z > 0$.
By central limit theorem, we have Z to be a standard normal distribution.

$$Z \sim \mathcal{N}(0,1) \tag{17.10.10}$$
$$\lim_{n\to\infty} \Pr\left(\overline{Y_n} > 1\right) = \Pr(Z > 0) \tag{17.10.11}$$
$$= \frac{1}{2} \tag{17.10.12}$$

$$\therefore \lim_{n\to\infty} \Pr\left(-\ln(1-X_1) - \cdots - \ln(1-X_n) > n\right) = 0.5 \tag{17.10.13}$$