GEOMETRY

Through Algebra

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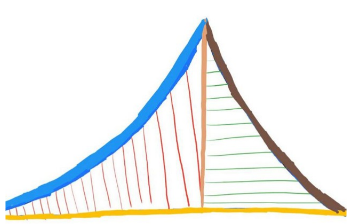

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Introduction

This book shows how to solve problems in geometry using trigonometry.

Chapter 1

Trigonometry

1.1. Ratios

A right angled triangle looks like Fig. 1.1. with angles $\angle A, \angle B$ and $\angle C$ and

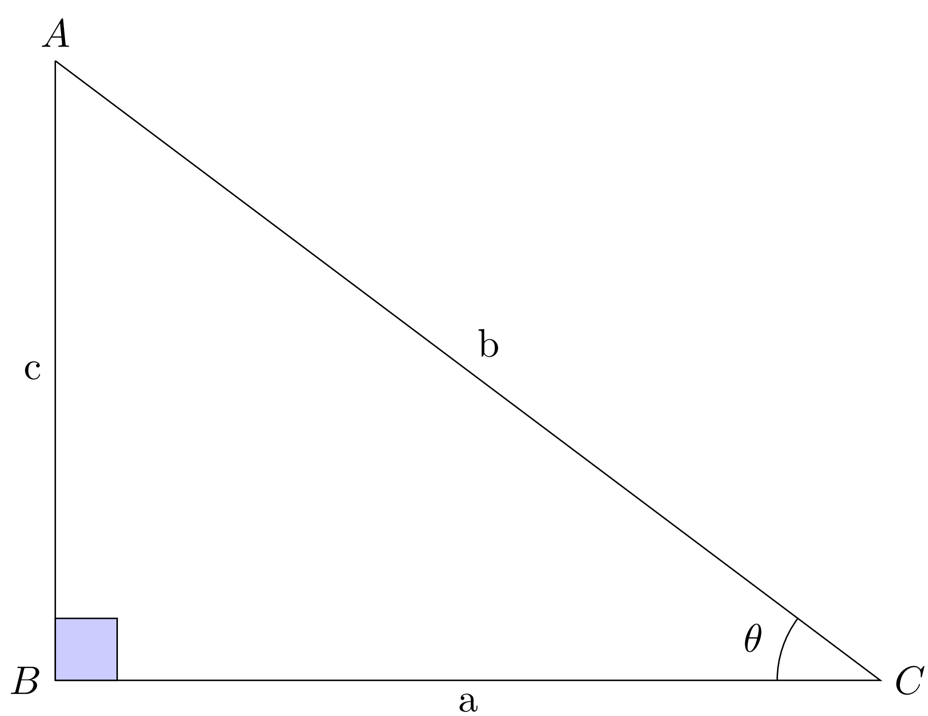

Figure 1.1: Right Angled Triangle

sides a, b and c. The unique feature of this triangle is $\angle B$ which is defined to be $90°$.

1.1.1. For simplicity, let the greek letter $\theta = \angle C$. We have the following definitions.

$$\begin{aligned} \sin\theta &= \tfrac{c}{b} & \cos\theta &= \tfrac{a}{b} \\ \tan\theta &= \tfrac{c}{a} & \cot\theta &= \tfrac{1}{\tan\theta} \\ \csc\theta &= \tfrac{1}{\sin\theta} & \sec\theta &= \tfrac{1}{\cos\theta} \end{aligned} \tag{1.1.1.1}$$

1.1.2. Show that

$$\cos\theta = \sin\left(90° - \theta\right) \tag{1.1.2.1}$$

Solution: From (1.1.1.1),

$$\cos\angle BAC = \cos\alpha = \cos\left(90° - \theta\right) = \frac{c}{b} = \sin\angle ABC = \sin\theta \tag{1.1.2.2}$$

1.2. The Baudhayana Theorem

Use Fig. 1.2 for all problems in this section.

1.2.1. Show that

$$b = a\cos\theta + c\sin\theta \tag{1.2.1.1}$$

Solution: We observe that

$$BD = a\cos\theta \tag{1.2.1.2}$$

$$AD = c\cos\alpha = c\sin\theta \quad \text{(From} \quad (1.1.2.2)) \tag{1.2.1.3}$$

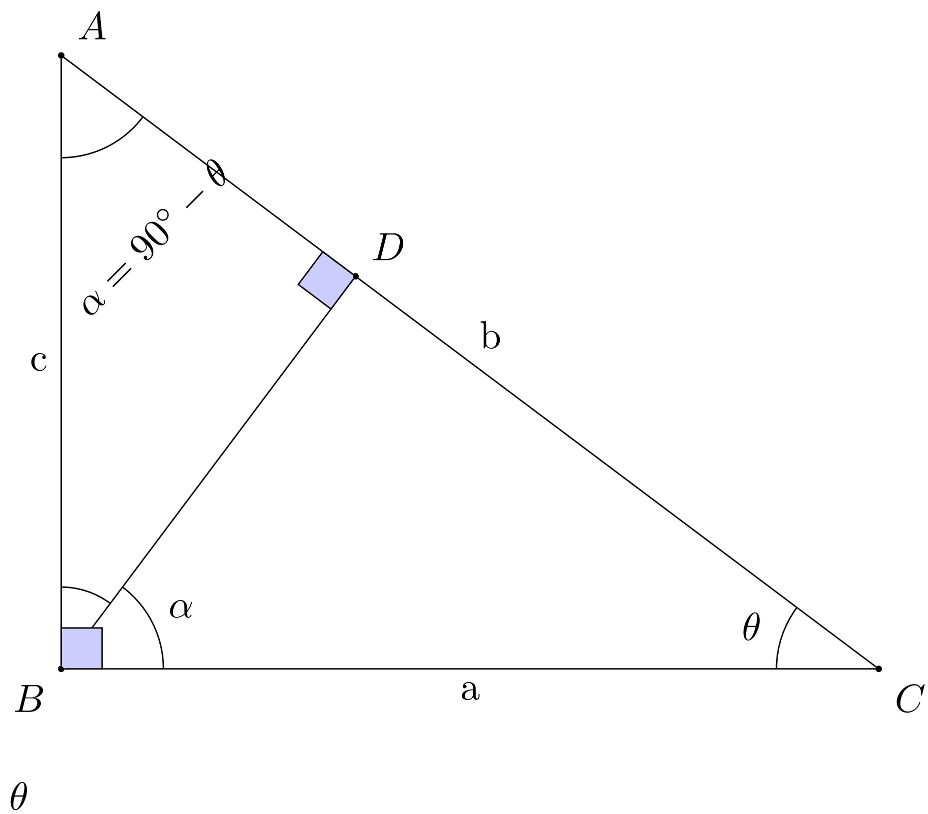

Figure 1.2: Baudhayana Theorem

Thus,

$$BD + AD = b = a\cos\theta + c\sin\theta \tag{1.2.1.4}$$

1.2.2. From (1.2.1.1), show that

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2.2.1}$$

Solution: Dividing both sides of (1.2.1.1) by b,

$$1 = \frac{a}{b}\cos\theta + \frac{c}{b}\sin\theta \tag{1.2.2.2}$$

$$\Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{(from} \quad (1.1.1.1)) \tag{1.2.2.3}$$

1.2.3. In a right angled triangle, the hypotenuse is the longest side.

Solution: From (1.2.2.1),

$$0 \le \sin\theta, \cos\theta \le 1 \tag{1.2.3.1}$$

Hence,

$$b\sin\theta \le b \implies c \le b \tag{1.2.3.2}$$

Similalry,

$$a \le b \tag{1.2.3.3}$$

1.2.4. Using (1.2.1.1), show that

$$b^2 = a^2 + c^2 \tag{1.2.4.1}$$

(1.2.4.1) is known as the Baudhayana theorem. It is also known as the Pythagoras theorem.

Solution: From (1.2.1.1),

$$b = a\frac{a}{b} + c\frac{c}{b} \quad \text{(from} \quad (1.1.1.1)) \tag{1.2.4.2}$$

$$\implies b^2 = a^2 + c^2 \tag{1.2.4.3}$$

1.3. Area of a Triangle

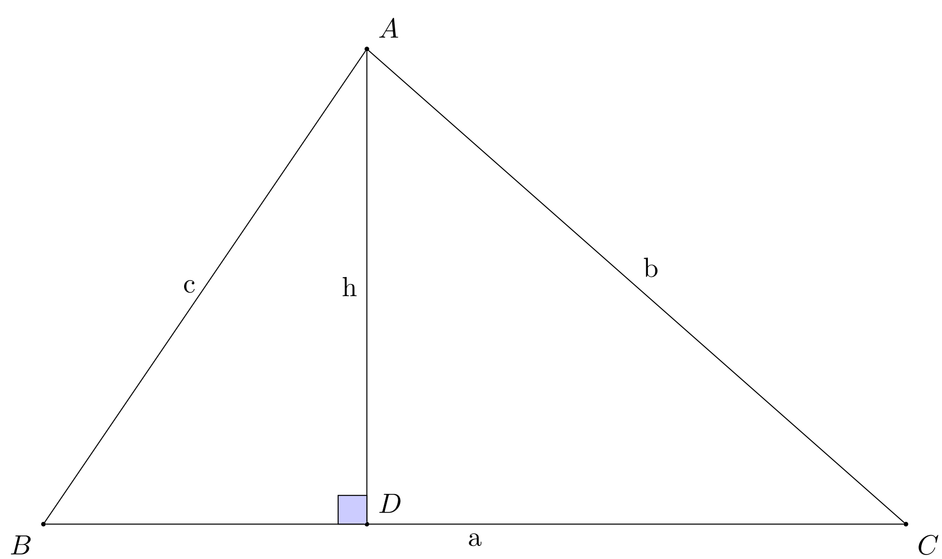

Figure 1.3: Area of a Triangle

1.3.1. Show that the area of ΔABC in Fig. 1.3 is $\frac{1}{2}ab\sin C$.

Solution: We have

$$ar\left(\Delta ABC\right) = \frac{1}{2}ah = \frac{1}{2}ab\sin C \quad \left(\because \quad h = b\sin C\right). \tag{1.3.1.1}$$

1.3.2. Show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.3.2.1}$$

Solution: Fig. 1.3 can be suitably modified to obtain

$$ar\left(\Delta ABC\right) = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B \tag{1.3.2.2}$$

Dividing the above by abc, we obtain

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1.3.2.3}$$

This is known as the sine formula.

1.3.3. Show that

$$\alpha > \beta \implies \sin\alpha > \sin\beta \tag{1.3.3.1}$$

Solution: In Fig. 1.4,

$$ar\left(\triangle ABD\right) < ar\left(\triangle ABC\right) \tag{1.3.3.2}$$
$$\implies \frac{1}{2}lc\sin\theta_1 < \frac{1}{2}ac\sin\left(\theta_1 + \theta_2\right) \tag{1.3.3.3}$$
$$\implies \frac{l}{a} < \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} \tag{1.3.3.4}$$
$$\text{or, } 1 < \frac{l}{a} < \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} \tag{1.3.3.5}$$
$$\implies \frac{\sin\left(\theta_1 + \theta_2\right)}{\sin\theta_1} > 1 \tag{1.3.3.6}$$

from Theorem 1.2.3. This proves (1.3.3.1).

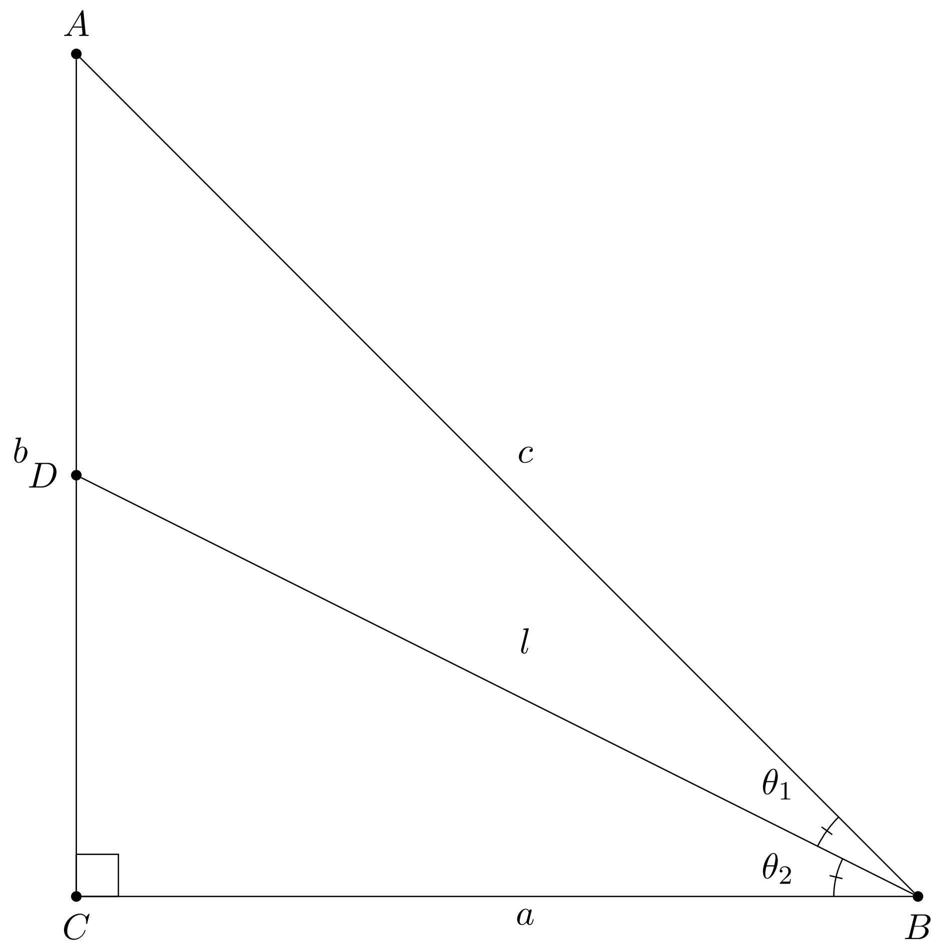

Figure 1.4:

1.3.4. Using Fig. 1.4, show that

$$\sin\theta_1 = \sin(\theta_1+\theta_2)\cos\theta_2 - \cos(\theta_1+\theta_2)\sin\theta_2 \tag{1.3.4.1}$$

Solution: The following equations can be obtained from the figure using the forumula for the area of a triangle

$$ar\left(\Delta ABC\right) = \frac{1}{2}ac\sin\left(\theta_1 + \theta_2\right) \tag{1.3.4.2}$$

$$= ar\left(\Delta BDC\right) + ar\left(\Delta ADB\right) \tag{1.3.4.3}$$

$$= \frac{1}{2}cl\sin\theta_1 + \frac{1}{2}al\sin\theta_2 \tag{1.3.4.4}$$

$$= \frac{1}{2}ac\sin\theta_1\sec\theta_2 + \frac{1}{2}a^2\tan\theta_2 \tag{1.3.4.5}$$

($\because l = a\sec\theta_2$). From the above,

$$\sin\left(\theta_1 + \theta_2\right) = \sin\theta_1\sec\theta_2 + \frac{a}{c}\tan\theta_2 \tag{1.3.4.6}$$

$$= \sin\theta_1\sec\theta_2 + \cos\left(\theta_1 + \theta_2\right)\tan\theta_2 \tag{1.3.4.7}$$

Multiplying both sides by $\cos\theta_2$,

$$\sin\left(\theta_1 + \theta_2\right)\cos\theta_2 = \sin\theta_1 + \cos\left(\theta_1 + \theta_2\right)\sin\theta_2 \tag{1.3.4.8}$$

resulting in (1.3.4.1).

1.3.5. Find Hero's formula for the area of a triangle.

Solution: From (1.3.1), the area of $\triangle ABC$ is

$$\frac{1}{2}ab\sin C = \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \text{(from (1.2.2.1))} \tag{1.3.5.1}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \text{(from (2.3.4.1))} \tag{1.3.5.2}$$

$$= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)} \tag{1.3.5.3}$$

$$= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \tag{1.3.5.4}$$

$$= \frac{1}{4}\sqrt{\left\{(a+b)^2-c^2\right\}\left\{c^2-(a-b)^2\right\}} \tag{1.3.5.5}$$

$$= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{1.3.5.6}$$

Substituting

$$s = \frac{a+b+c}{2} \tag{1.3.5.7}$$

in (1.3.5.6), the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} \tag{1.3.5.8}$$

This is known as Hero's formula.

1.4. Angle Bisectors

1.4.1. In Fig. 1.4.1.1, the bisectors of $\angle B$ and $\angle C$ meet at **I**. Show that IA bisects $\angle A$.

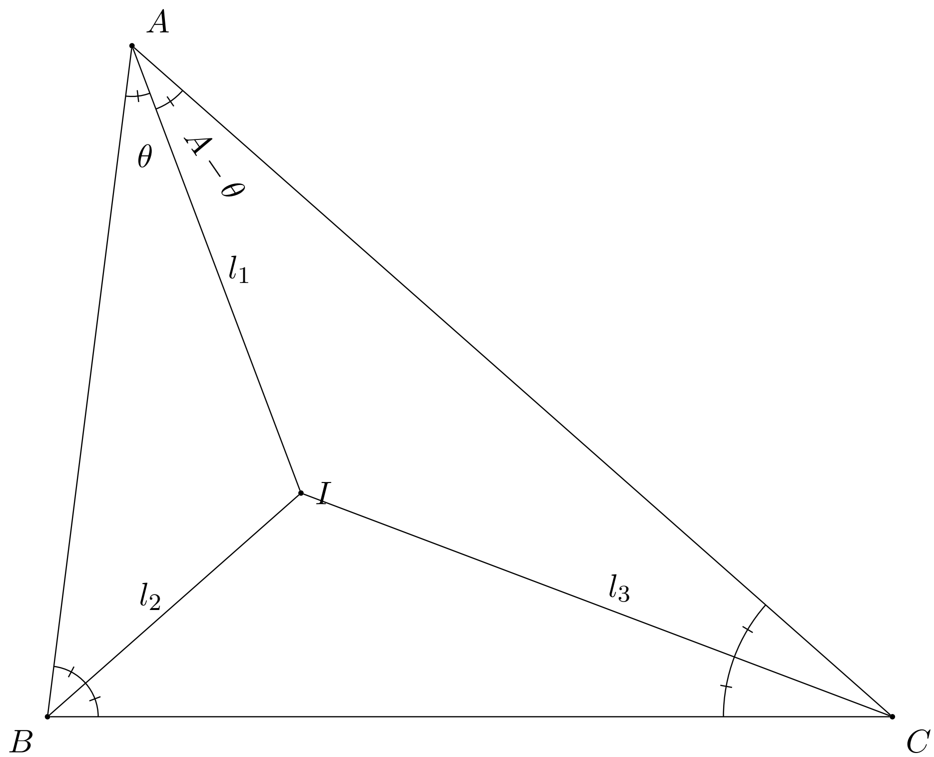

Figure 1.4.1.1: Incentre I of $\triangle ABC$

Solution: Using sine formula

$$\frac{l_1}{\sin\frac{C}{2}} = \frac{l_3}{\sin(A-\theta)} \tag{1.4.1.1}$$

$$\frac{l_3}{\sin\frac{B}{2}} = \frac{l_2}{\sin\frac{C}{2}} \tag{1.4.1.2}$$

$$\frac{l_1}{\sin\frac{B}{2}} = \frac{l_2}{\sin\theta} \tag{1.4.1.3}$$

Multiplying the above equations,

$$\sin\theta = \sin(A-\theta) \implies \theta = \frac{A}{2} \tag{1.4.1.4}$$

1.4.2. In Fig. 1.4.2.1,

$$ID \perp BC,\ IE \perp AC,\ IF \perp AB. \tag{1.4.2.1}$$

Show that

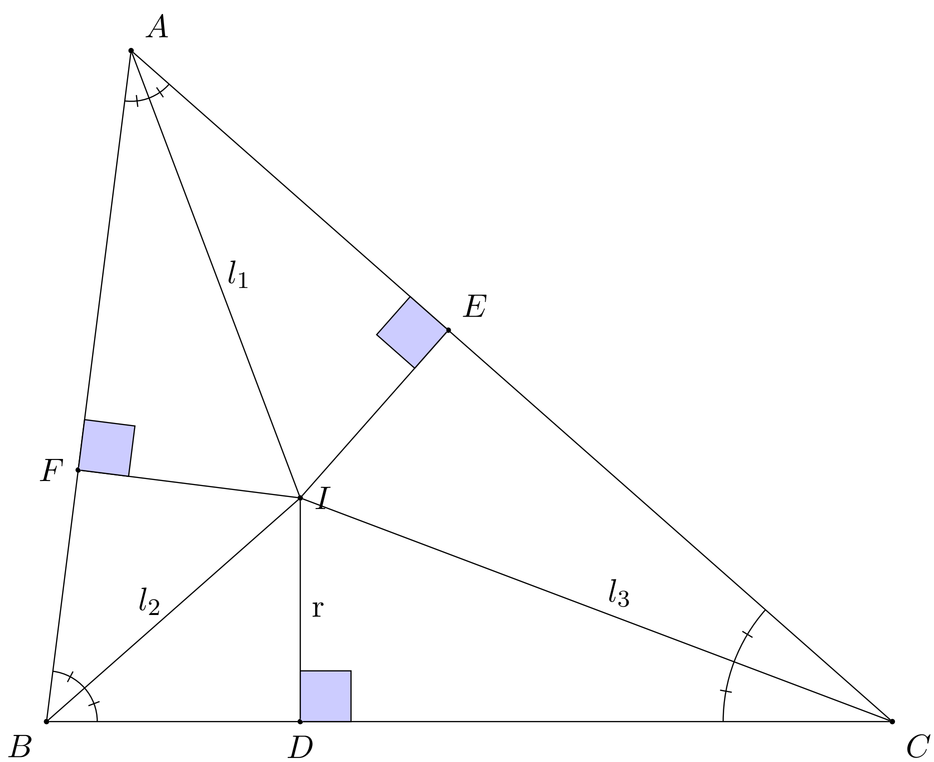

Figure 1.4.2.1: Inradius r of $\triangle ABC$

$$ID = IE = IF = r \tag{1.4.2.2}$$

Solution: In $\triangle$s IDC and IEC,

$$ID = IE = \frac{l_3}{\sin \frac{C}{2}} \tag{1.4.2.3}$$

Similarly, in $\triangle$s IEA and IFA,

$$IF = IE = \frac{l_1}{\sin \frac{A}{2}} \tag{1.4.2.4}$$

yielding (1.4.2.2)

1.4.3. In Fig. 1.4.2.1, show that

$$BD = BF,\ AE = AF,\ CD = CE \tag{1.4.3.1}$$

Solution: From Fig. 1.4.2.1, in $\triangle$s IBD and IBF,

$$x = BD = BF = r \cot \frac{B}{2} \tag{1.4.3.2}$$

Similarly, other results can be obtained.

1.4.4. The circle with centre **I** and radius r in Fig. 1.4.4.1 is known as the <u>incircle</u>. Find the radius r.

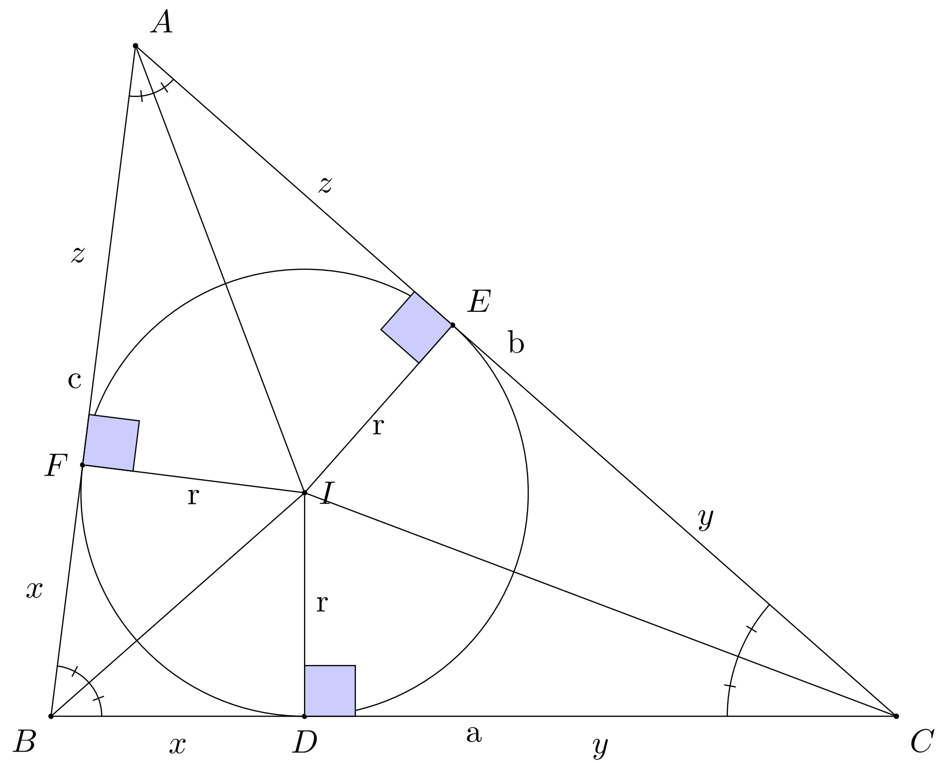

Figure 1.4.4.1: Circumcircle of $\triangle ABC$

Solution: In $\triangle IBC$,

$$a = x + y = r\cot\frac{B}{2} + r\cot\frac{C}{2} \tag{1.4.4.1}$$

$$\implies r = \frac{a}{\cot\frac{B}{2} + \cot\frac{C}{2}} \tag{1.4.4.2}$$

1.5. Identities

1.1. Show that

$$\sin 90^\circ = 1 \tag{1.1.1}$$

Solution: In Fig. **??**, using (1.3.1.1) and (**??**)

$$ar\left(\triangle ABC\right) = \frac{1}{2}ac\sin B = \frac{1}{2}ac \tag{1.1.2}$$

$$\implies \sin B = \sin 90^\circ = 1 \tag{1.1.3}$$

1.2. Show that

$$\cos 90^\circ = 1 \tag{1.2.1}$$

Solution: Trivial using (1.2.2.1).

1.3. Prove the following identities

(a)

$$\sin\left(\alpha - \beta\right) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{1.3.1}$$

(b)

$$\cos\left(\alpha + \beta\right) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{1.3.2}$$

Solution: In (1.3.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{1.3.3}$$

This gives (B.3.1). In (B.3.1), replace α by $90° - \alpha$. This results in

$$\begin{aligned} &\sin(90° - \alpha - \beta) \\ &\quad = \sin(90° - \alpha)\cos\beta - \cos(90° - \alpha)\sin\beta \\ &\qquad \implies \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned} \tag{1.3.4}$$

1.4. Using (1.3.4.1) and (B.3.2), show that

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{1.4.1}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{1.4.2}$$

Solution: From (1.3.4.1),

$$\sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1 + \theta_2)\sin\theta_2 \tag{1.4.3}$$

Using (B.3.2) in the above,

$$\begin{aligned} \sin(\theta_1 + \theta_2)\cos\theta_2 = \sin\theta_1 + (&\cos\theta_1\cos\theta_2 \\ &- \sin\theta_1\sin\theta_2)\sin\theta_2 \end{aligned} \tag{1.4.4}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 \\ + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \quad (1.4.5)$$

Since

$$\sin^2\theta_2 = 1-\cos^2\theta_2, \quad (1.4.6)$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 \\ + \sin\theta_1\cos^2\theta_2 \quad (1.4.7)$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \quad (1.4.8)$$

after factoring out $\cos\theta_2$. Using a similar approach, (B.4.2) can also be proved.

1.5. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{1.5.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{1.5.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \tag{1.5.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \tag{1.5.4}$$

Solution: Let

$$\begin{aligned}\theta_1 &= \alpha+\beta\\ \theta_2 &= \alpha-\beta\end{aligned} \tag{1.5.5}$$

From (B.4.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{1.5.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1.5.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{1.5.8}$$

$$= 2\sin\alpha\cos\beta \tag{1.5.9}$$

resulting in (B.5.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2} \tag{1.5.10}$$

$$\beta = \frac{\theta_1-\theta_2}{2} \tag{1.5.11}$$

from (B.5.5). Other identities may be proved similarly.

Chapter 2

Coordinate Geometry

2.1. Vectors

2.1.1. A matrix of the form

$$\mathbf{A} \triangleq \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{2.1.1.1}$$

is defined be column vector, or simply, vector. In Fig. 1.1 the point vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ can be defined as

$$\mathbf{A} = \begin{pmatrix} 0 \\ c \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.1.1.2}$$

2.1.2.

$$\lambda\mathbf{A} \triangleq \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \tag{2.1.2.1}$$

2.1.3. For

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2.1.3.1}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \tag{2.1.3.2}$$

2.1.4. The transpose of $\mathbf{A}$ is the <u>row vector</u> defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{2.1.4.1}$$

2.1.5. The <u>inner product</u> or <u>dot product</u> is defined as

$$\mathbf{A}^\top\mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{2.1.5.1}$$

In Fig. 1.1,

$$\mathbf{A}^\top\mathbf{C} = 0 \tag{2.1.5.2}$$

2.1.6. The <u>norm</u> of $\mathbf{A}$ is defined as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}^\top\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{2.1.6.1}$$

2.1.7. In Fig. 1.1, it is easy to verify that

$$\|\mathbf{A}-\mathbf{C}\|^2 = \begin{pmatrix} -c & a \end{pmatrix} \begin{pmatrix} -c \\ a \end{pmatrix} = a^2 + c^2 = b^2 \tag{2.1.7.1}$$

from (1.2.4.1). Thus, the distance betwen any two points **A** and **B** is given by

$$\|\mathbf{A}-\mathbf{B}\| \tag{2.1.7.2}$$

2.1.8. Show that

$$\|\lambda\mathbf{A}\| = |\lambda| \, \|\mathbf{A}\| \tag{2.1.8.1}$$

2.2. Collinear Points

2.2.1. The direction vector of the line AB is

$$\mathbf{A}-\mathbf{B} \equiv \mathbf{B}-\mathbf{A} \equiv \kappa \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{2.2.1.1}$$

where m is defined to be the slope of AB. In Fig. 1.1,

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} -c \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -\frac{a}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix} \tag{2.2.1.2}$$

the slope of AC is $-\tan\theta$

2.2.2. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are on a line if they have the same direction vector, i.e.

$$p(\mathbf{B}-\mathbf{A})+q(\mathbf{C}-\mathbf{B})=0 \implies p, q \neq 0. \tag{2.2.2.1}$$

$(\mathbf{A}-\mathbf{B}), (\mathbf{C}-\mathbf{B})$ are then said to be <u>linearly dependent</u>.

2.2.3. If points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are collinear,

$$\mathbf{B}=\frac{k\mathbf{A}+\mathbf{C}}{k+1} \tag{2.2.3.1}$$

Solution: From (2.2.2.1),

$$p(\mathbf{A}-\mathbf{B})+q(\mathbf{A}-\mathbf{C})=0 \implies \mathbf{B}=\frac{p\mathbf{A}+q\mathbf{C}}{p+q} \tag{2.2.3.2}$$

yielding (2.2.3.1) upon substituting

$$k=\frac{p}{q}. \tag{2.2.3.3}$$

This is known as <u>section formula</u>.

2.2.4. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A}-\mathbf{B})+q(\mathbf{C}-\mathbf{B}) \tag{2.2.4.1}$$

$$=(p+q)\mathbf{B}-p\mathbf{A}-q\mathbf{C}=0 \tag{2.2.4.2}$$

$$\implies p=0, q=0 \tag{2.2.4.3}$$

2.2.5. In Fig. 2.2.5.1,

$$AF = BF,\ AE = BE, \tag{2.2.5.1}$$

and the medians BE and CF meet at **G**. Show that

$$\frac{GB}{GE} = \frac{GC}{GF} = 2 \tag{2.2.5.2}$$

Solution: From (2.2.3.1),

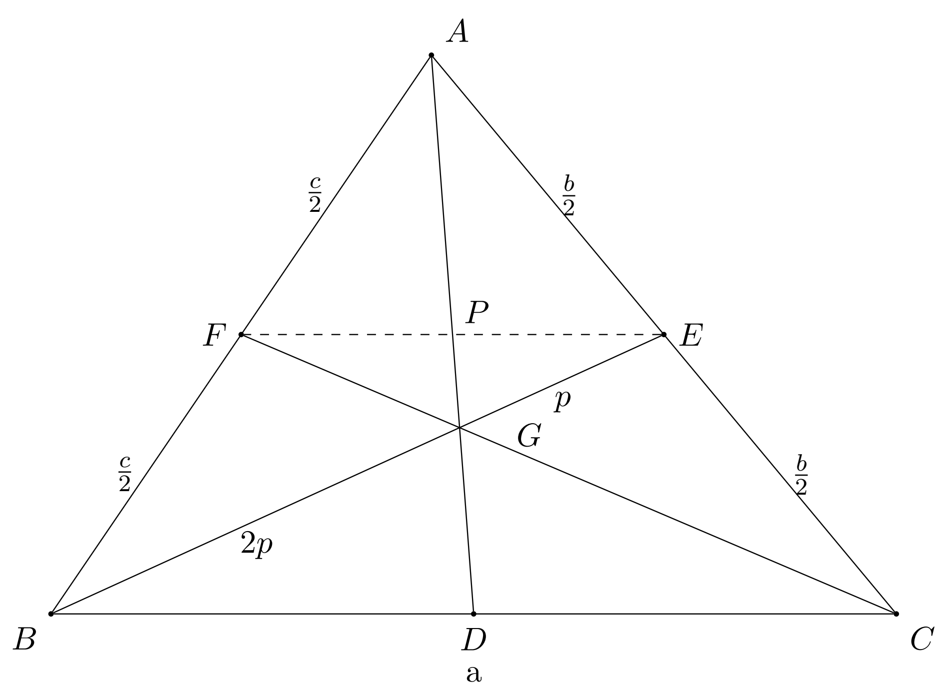

Figure 2.2.5.1: $\frac{GA}{GD} = 2$

$$\mathbf{G} = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \tag{2.2.5.3}$$

$$\implies \frac{k_1\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) + \mathbf{B}}{k_1 + 1} = \frac{k_2\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) + \mathbf{C}}{k_2 + 1} \tag{2.2.5.4}$$

$$\implies (k_2 + 1)\left\{k_1(\mathbf{A} + \mathbf{C}) + 2\mathbf{B}\right\} = (k_1 + 1)\left\{k_2(\mathbf{A} + \mathbf{B}) + 2\mathbf{C}\right\} \tag{2.2.5.5}$$

which can be expressed as

$$\left\{2 + k_2 - k_1k_2\right\}\mathbf{B} - (k_2 - k_1)\mathbf{A} - \left\{k_1 + 2 - k_1k_2\right\}\mathbf{C} = 0 \tag{2.2.5.6}$$

and is of the form (2.2.4.3) with

$$p = k_2 - k_1, q = k_1 + 2 - k_1k_2. \tag{2.2.5.7}$$

Thus, from (2.2.4.3)

$$k_2 - k_1 = 0, \tag{2.2.5.8}$$

$$k_1 + 2 - k_1k_2 = 0 \tag{2.2.5.9}$$

Thus, from (2.2.5.9)

$$k_1 = k_2 \tag{2.2.5.10}$$

and substituting the above in (2.2.5.9) results in the quadratic

$$k_1^2 - k_1 - 2 = 0 \tag{2.2.5.11}$$

$$\implies (k_1 - 2)(k_1 + 1) = 0 \tag{2.2.5.12}$$

admitting $k_1 = k_2 = 2$ as the only possible solution.

2.2.6. Substituting $k_1 = 2$ in (2.2.5.3)

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{2.2.6.1}$$

2.2.7. AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

Solution: Considering the ratios in Fig. **??**,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \tag{2.2.7.1}$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{2.2.7.2}$$

Substituting from (2.2.6.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{2.2.7.3}$$

$$\implies (k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\left\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\right\} \tag{2.2.7.4}$$

which can be expressed as

$$
\begin{aligned}
(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A}& \\
&- (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B} \\
&\qquad - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0} \qquad (2.2.7.5)
\end{aligned}
$$

Comparing the above with (2.2.4.3),

$$
p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \qquad (2.2.7.6)
$$

yielding

$$
-k_3k_4 - k_4 + 2k_3 - 1 = 0 \qquad (2.2.7.7)
$$

$$
-k_3 - k_4 - 1 + 2k_3k_4 = 0 \qquad (2.2.7.8)
$$

Subtracting (2.2.7.7) from (2.2.7.8),

$$
3k_3\,(k_4 - 1) = 0 \qquad (2.2.7.9)
$$

$$
\implies k_4 = 1 \qquad (2.2.7.10)
$$

which upon substituting in (2.2.7.7) yields

$$
k_3 = 2 \qquad (2.2.7.11)
$$

2.3. Matrices: Cosine Formula

2.3.1. The determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{2.3.1.1}$$

is defined as

$$\begin{vmatrix} \mathbf{M} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{B} \end{vmatrix} \tag{2.3.1.2}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{2.3.1.3}$$

2.3.2. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{2.3.2.1}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \mathbf{C}_{ij} = \begin{pmatrix} c_i \\ c_j \end{pmatrix}. \tag{2.3.2.2}$$

Then,

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} = a_1 \begin{vmatrix} \mathbf{B}_{23} & \mathbf{C}_2 3 \end{vmatrix} + a_2 \begin{vmatrix} \mathbf{C}_{23} & \mathbf{A}_{23} \end{vmatrix} + a_3 \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \tag{2.3.2.3}$$

2.3.3. In Fig. 2.3.3.1, show that

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos B \\ \cos C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{2.3.3.1}$$

Solution: From Fig. 2.3.3.1,

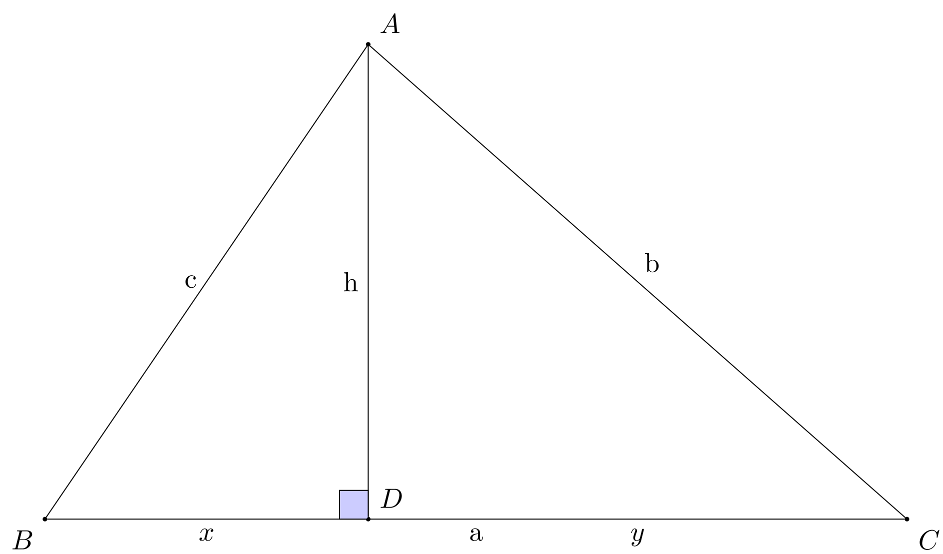

Figure 2.3.3.1: The cosine formula

$$a = x + y = b\cos C + c\cos B = \begin{pmatrix} \cos C & \cos B \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \tag{2.3.3.2}$$

$$= \begin{pmatrix} 0 & b & c \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{2.3.3.3}$$

Similarly,

$$b = c\cos A + a\cos C = \begin{pmatrix} c & 0 & a \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{2.3.3.4}$$

$$c = b\cos A + a\cos B = \begin{pmatrix} b & a & 0 \end{pmatrix} \begin{pmatrix} \cos A \\ \cos C \\ \cos B \end{pmatrix} \tag{2.3.3.5}$$

The above equations can be expressed in matrix form as (2.3.3.1).

2.3.4. Show that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{2.3.4.1}$$

Solution: Using the properties of determinants,

$$\cos A = \frac{\begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix}}{\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}} = \frac{ab^2 + ac^2 - a^3}{abc + abc} = \frac{b^2 + c^2 - a^2}{2abc} \tag{2.3.4.2}$$

2.4. Area of a Triangle: Cross Product

2.4.1. The <u>cross product</u> or <u>vector product</u> defined as $\mathbf{A} \times \mathbf{B}$ is given by (2.3.1.2) for 2×1 vectors.

2.4.2. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{C})\right\| = \frac{1}{2}\left\|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\right\| \quad (2.4.2.1)$$

2.4.3. If

$$\left\|\mathbf{A} \times \mathbf{B}\right\| = \left\|\mathbf{C} \times \mathbf{D}\right\|, \quad \text{then} \quad (2.4.3.1)$$

$$\mathbf{A} \times \mathbf{B} = \pm\left(\mathbf{C} \times \mathbf{D}\right) \quad (2.4.3.2)$$

where the sign depends on the orientation of the vectors.

2.4.4. The median divides the triangle into two triangles of equal area.

2.4.5. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$\left(\mathbf{A}-\mathbf{B}\right)^{\top} \mathbf{x} = \frac{\left\|\mathbf{A}\right\|^2 - \left\|\mathbf{B}\right\|^2}{2} \quad (2.4.5.1)$$

Solution:

$$\left\|\mathbf{x}-\mathbf{A}\right\| = \left\|\mathbf{A}-\mathbf{B}\right\| \quad (2.4.5.2)$$

$$\implies \left\|\mathbf{x}-\mathbf{A}\right\|^2 = \left\|\mathbf{x}-\mathbf{B}\right\|^2 \quad (2.4.5.3)$$

which can be expressed as

$$\begin{aligned}\left(\mathbf{x}-\mathbf{A}\right)^{\top}\left(\mathbf{x}-\mathbf{A}\right) &= \left(\mathbf{x}-\mathbf{B}\right)^{\top}\left(\mathbf{x}-\mathbf{B}\right) \\ \implies \left\|\mathbf{x}\right\|^2 - 2\mathbf{x}^{\top}\mathbf{A} + \left\|\mathbf{A}\right\|^2 & \\ &= \left\|\mathbf{x}\right\|^2 - 2\mathbf{x}^{\top}\mathbf{B} + \left\|\mathbf{B}\right\|^2 \quad (2.4.5.4)\end{aligned}$$

which can be simplified to obtain (2.4.5.1).

2.4.6. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{2.4.6.1}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1} \tag{2.4.6.2}$$

Solution: From (2.4.5.1).

$$x\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{2.4.6.3}$$

yielding (2.4.6.2).

2.4.7. The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^\top\mathbf{B}}{\|A\|\,\|B\|} \tag{2.4.7.1}$$

2.4.8. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top\mathbf{B} = 0 \tag{2.4.8.1}$$

2.4.9. For an isoceles triangle ABC ith $AB = AC$, the median $AD \perp BC$.

2.4.10. The <u>direction vector</u> of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{2.4.10.1}$$

2.4.11. The points $\mathbf{AAA}$

2.4.12. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{2.4.12.1}$$

2.4.13. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{2.4.13.1}$$

the m is defined to be the slope of the line.

2.4.14. $AB \parallel CD$ if

$$\mathbf{A} - \mathbf{B} = k\left(\mathbf{C} - \mathbf{D}\right) \tag{2.4.14.1}$$

2.4.15. The <u>normal vector</u> to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{2.4.15.1}$$

2.4.16. If

$$\mathbf{m}^\top \mathbf{n}_1 = 0 \tag{2.4.16.1}$$

$$\mathbf{m}^\top \mathbf{n}_2 = 0, \tag{2.4.16.2}$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \tag{2.4.16.3}$$

2.4.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.4.17.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.4.17.2}$$

2.5. Parallelogram

2.5.1. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{2.5.1.1}$$

2.5.2. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \tag{2.5.2.1}$$

2.6. Altitudes of a Triangle:Line Equation

2.6.1. Find the equation of the line BC.

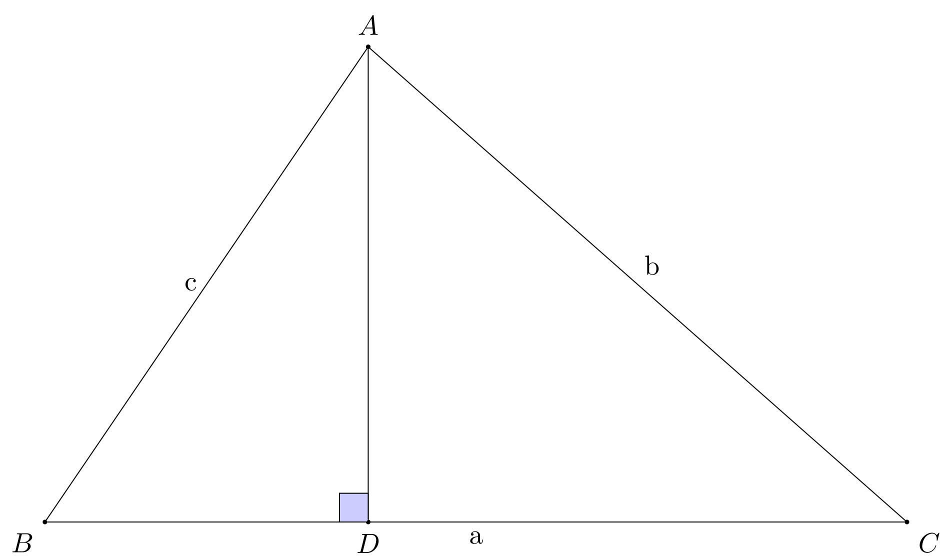

Figure 2.6.1.1: Drawing the altitude

Solution: Let $\mathbf{x}$ be any point on BC. Using section formula, for some k,

$$\mathbf{x} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} = \frac{(k+1)\,\mathbf{C} + (\mathbf{B} - \mathbf{C})}{k+1} \tag{2.6.1.1}$$

$$\implies \mathbf{x} = \mathbf{C} + \lambda\mathbf{m} \tag{2.6.1.2}$$

where

$$\mathbf{m} = \frac{\mathbf{B} - \mathbf{C}}{k+1} \equiv \mathbf{B} - \mathbf{C} \tag{2.6.1.3}$$

2.6.2. The <u>normal vector</u> to $\mathbf{m}$ is defined as

$$\mathbf{n}^\top \mathbf{m} = 0 \tag{2.6.2.1}$$

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{m} \tag{2.6.2.2}$$

2.6.3. From (2.6.2.1) and (2.6.1.2), it can be verified that

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} + \lambda \mathbf{n}^\top \mathbf{m} \tag{2.6.3.1}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{2.6.3.2}$$

(2.6.3.2) is defined to be the <u>normal form</u> of the line BC.

2.6.4. In Fig. 2.6.5.1, $AD \perp BC$ and $BE \perp AC$ are defined to be the altitudes of $\triangle ABC$.

2.6.5. Let $\mathbf{H}$ be the intersection of the altitudes AD and BE as shown in Fig. 2.6.5.1. CH is extended to meet AB at $\mathbf{F}$. Show that $CF \perp AB$.

Solution: From (2.6.1.3) (2.6.2.1) and (2.6.3.2), the equations of AD

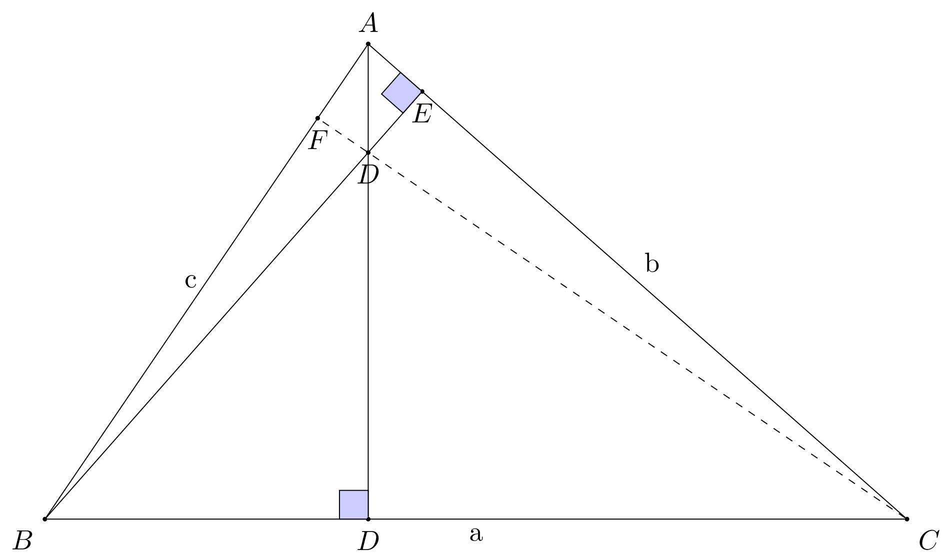

Figure 2.6.5.1: Altitudes of a triangle meet at the orthocentre H

and BE are

$$(\mathbf{B}-\mathbf{C})^\top(\mathbf{x}-\mathbf{A}) = 0 \tag{2.6.5.1}$$

$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{x}-\mathbf{B}) = 0 \tag{2.6.5.2}$$

$\because$ $\mathbf{H}$ lies on both AD and BE, it satisfies the above equations, and

$$(\mathbf{B}-\mathbf{C})^\top(\mathbf{H}-\mathbf{A}) = 0 \tag{2.6.5.3}$$

$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{H}-\mathbf{B}) = 0 \tag{2.6.5.4}$$

Adding both the above and simplifying,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{H}-\mathbf{C}) = 0 \tag{2.6.5.5}$$

$\implies CH \perp AB$ from Theorem **??**, or $CF \perp AB$. The python code for Fig. 2.6.5.1 is

codes/triangle/tri_alt_h.py

and the equivalent latex-tikz code is

figs/triangle/tri_alt_h.tex

2.6.6. Altitudes of a $\triangle$ meet at the orthocentre H.

2.6.7. Find $\mathbf{H}$.

2.7. Properties

Refer to Fig. 2.7.1.1.

2.7.1. In $\triangle OBC$, $OB = OC = R$. Such a triangle is known as an isoceles triangle.

2.7.2. Show that $\angle OBC = \angle OCB$. In an isoceles triangle, opposite sides and corresponding opposite angles are equal.

Solution: Using the sine formula in (1.3.2.3),

$$\frac{\sin \angle OBC}{R} = \frac{\sin \angle OCB}{R} \tag{2.7.2.1}$$

$$\implies \sin \angle OBC = \sin \angle OCB \tag{2.7.2.2}$$

2.7.3. Show that $\angle BOC = 2\angle A$.

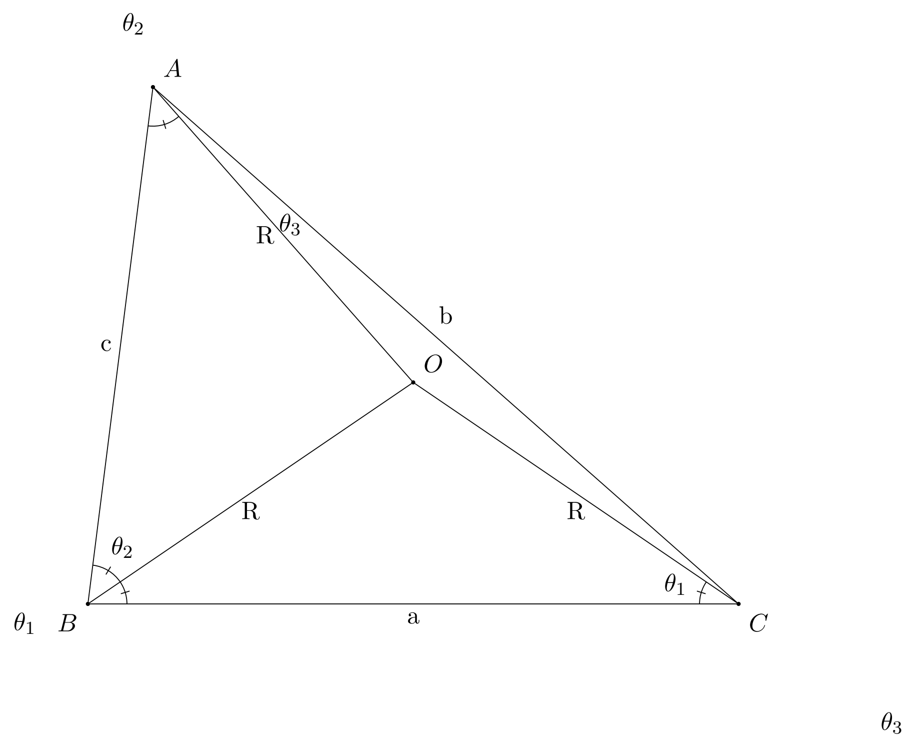

Figure 2.7.1.1: Circumcentre O of $\triangle ABC$

Solution: In Fig. 2.7.1.1,

$$A = \theta_2 + \theta_3 \tag{2.7.3.1}$$

$$B = \theta_1 + \theta_2 \tag{2.7.3.2}$$

$$C = \theta_3 + \theta_1 \tag{2.7.3.3}$$

$$\implies 2\left(\theta_1 + \theta_2 + \theta_3\right) = A + B + C = 180^\circ \tag{2.7.3.4}$$

$$\implies \theta_1 + \theta_2 + \theta_3 = 90^\circ \tag{2.7.3.5}$$

From (2.7.3.1) and (2.7.3.5),

$$A = 90^\circ - \theta_1 \tag{2.7.3.6}$$

Also, in $\triangle OBC$, all angles add up to 180°. Hence,

$$\angle BOC + 2\theta_1 = 180^\circ \tag{2.7.3.7}$$

$$\implies \angle BOC = 180^\circ - 2\theta_1 = 2\left(90^\circ - \theta_1\right) = 2\angle A \tag{2.7.3.8}$$

upon substituting from (2.7.3.6).

2.7.4. Let $\mathbf{D}$ be the mid point of BC. Show that $OD \perp BC$.

Solution: From (2.7.1),

$$\left\|\mathbf{O}-\mathbf{C}\right\| = \left\|\mathbf{O}-\mathbf{B}\right\| = R \tag{2.7.4.1}$$

$$\implies \left\|\mathbf{O}-\mathbf{C}\right\|^2 = \left\|\mathbf{O}-\mathbf{B}\right\|^2 \left(\mathbf{B}-\mathbf{C}\right)^T \mathbf{O} = \frac{\left\|\mathbf{B}\right\|^2 - \left\|\mathbf{C}\right\|^2}{2} \tag{2.7.4.2}$$

$$\implies \left(\mathbf{B}-\mathbf{C}\right)^T \mathbf{O} = \frac{1}{2}\left(\mathbf{B}-\mathbf{C}\right)^T\left(\mathbf{B}+\mathbf{C}\right) \tag{2.7.4.3}$$

$$\implies \left(\mathbf{B}-\mathbf{C}\right)^T \left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right) = 0 \tag{2.7.4.4}$$

$$\text{or, } \left(\mathbf{B}-\mathbf{C}\right)^T \left(\mathbf{O}-\mathbf{D}\right) = 0 \tag{2.7.4.5}$$

$\because \mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2}$ is the mid point of BC. From (**??**) we then conclude that $OD \perp BC$.

2.7.5. Perpendicular bisectors of a triangle meet at the circumcentre.

2.7.6. In the isosceles $\triangle OBC$, if $BD = DC$, $OD \perp BC$.

2.8. Circumradius

2.8.1. Show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{2.8.1.1}$$

Solution: In $\triangle OBC$, (see Fig. 2.7.1.1) using the cosine formula,

$$\cos 2A = \frac{R^2 + R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \tag{2.8.1.2}$$

Using the sine formula,

$$\frac{\sin 2A}{a} = \frac{\sin \theta_1}{R} = \frac{\sin (90^\circ - A)}{R} \tag{2.8.1.3}$$

$$\implies \sin 2A = \frac{a \cos A}{R} \tag{2.8.1.4}$$

from (2.7.3.6) and (1.1.2.1). Using (1.2.2.1),

$$\cos^2 2A + \sin^2 2A = 1 \tag{2.8.1.5}$$

$$\implies \left(1 - \frac{a^2}{2R^2}\right)^2 + \left(\frac{a \cos A}{R}\right)^2 = 1 \tag{2.8.1.6}$$

upon substituting from (2.8.1.2) and (2.8.1.4). Letting

$$x = \left(\frac{a}{R}\right)^2, \tag{2.8.1.7}$$

in the previous equation yields

$$\left(1-\frac{x}{2}\right)^2 + x\cos^2 A = 1 \tag{2.8.1.8}$$

$$\implies 1-\frac{x^2}{4} - x + x\cos^2 A = 1 \tag{2.8.1.9}$$

$$\implies x\left(1-\cos^2 A\right) - \frac{x^2}{4} = 0 \tag{2.8.1.10}$$

From (1.2.2.1), the above equation can be expressed as

$$x\sin^2 A - \frac{x^2}{4} = 0 \tag{2.8.1.11}$$

$$\implies x\left(\sin^2 A - \frac{x}{4}\right) = 0 \tag{2.8.1.12}$$

$$\text{or, } \frac{x}{4} - \sin^2 A = 0 \tag{2.8.1.13}$$

$\because x \neq 0$. Thus, substituting from (2.8.1.7),

$$x = \left(\frac{a}{R}\right)^2 = 4\sin^2 A \tag{2.8.1.14}$$

$$\implies \frac{a}{R} = 2\sin A, \tag{2.8.1.15}$$

$$\text{or,} \quad \frac{a}{\sin A} = 2R \tag{2.8.1.16}$$

2.8.2. Show that

$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 \tag{2.8.2.1}$$

$$= \cos^2 A - \sin^2 A \quad \text{and} \tag{2.8.2.2}$$

$$\sin 2A = 2\sin A\cos A \tag{2.8.2.3}$$

2.8.3. Find R.

Solution: From (1.3.1.1),

$$ar\left(\triangle ABC\right) = \frac{1}{2}bc\sin A = \frac{abc}{4R} \tag{2.8.3.1}$$

$$\implies R = \frac{abc}{4s\sqrt{\left(s-a\right)\left(s-b\right)\left(s-c\right)}} \tag{2.8.3.2}$$

upon substituting from (2.8.1.1) and using Hero's formula.

2.8.4. Show that

$$ar\left(\triangle OBC\right) = \frac{1}{2}R^2\sin 2A \tag{2.8.4.1}$$

2.9. Circumcircle

2.9.1. In Fig. 2.7.1.1, points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are at a distance R from $\mathbf{O}$. Trace all such points. The locus of such points is defined as a circle.

2.9.2. Line segements joining any two points on the circle are defined to be chords. The sides of $\triangle ABC$ are chords of the circle in (2.9.1.1)

2.9.3. From (2.7.4), it is established that the line segment joining the centre of a circle to the mid point of a chord bisects the chord.

2.9.4. From (2.7.3), it is clear that the angle subtended by a chord at the centre of the circle is twice the angle subtended at any point on the circle.

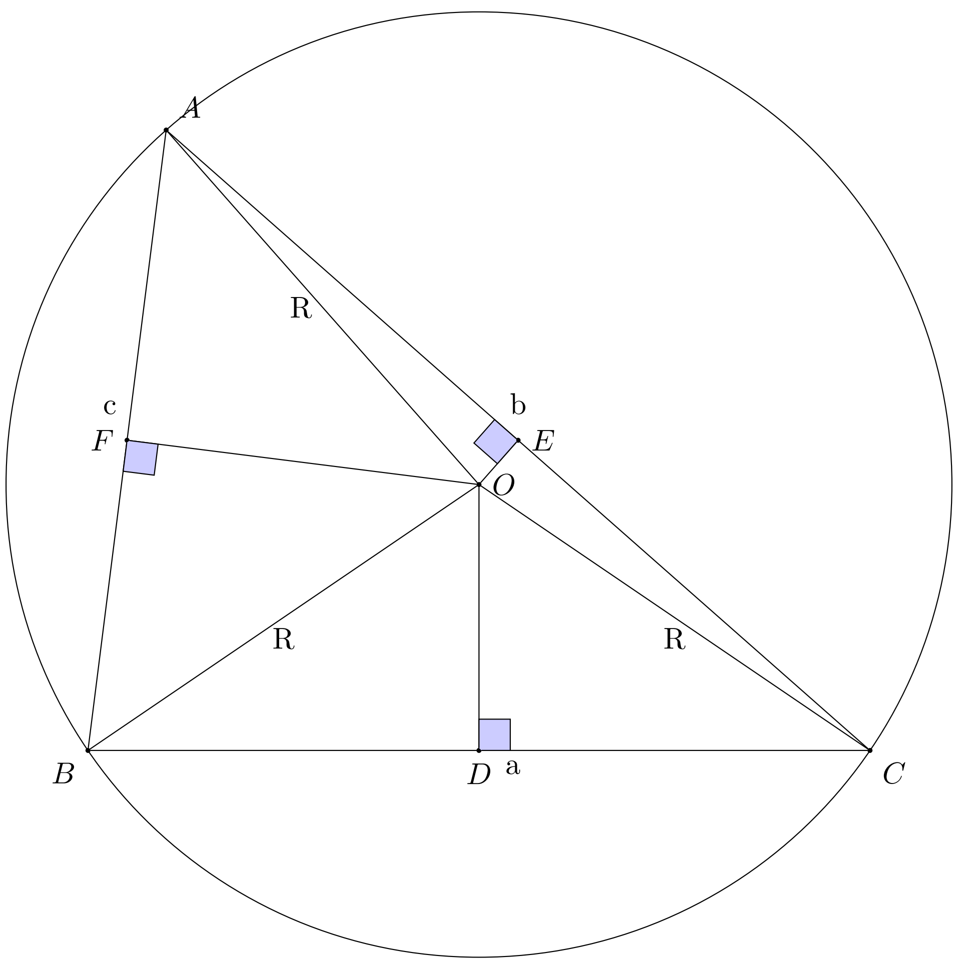

Figure 2.9.1.1: Circumcircle of $\triangle ABC$

Chapter 3

Incentre

3.1. Locating the Incentre

1. Find a point **O** that is equidistant from the sides of $\triangle ABC$ for $a = 5, b = 6, c = 4$. Here, distance means the perpendicular distanace.
Solution: Let **I** be the desired point and $\mathbf{D}, \mathbf{E}, \mathbf{F}$ are on BC, CA, AB such that $ID \perp BC, IE \perp CA, IF \perp AB$ and $ID = IE = IF = r$, then, applying (1.2.4.1) in $\triangle s$ IDB and IEB,

$$\begin{aligned} IB^2 &= ID^2 + BD^2 = r^2 + BD^2 \\ IB^2 &= IF^2 + BF^2 = r^2 + BF^2 \end{aligned} \tag{3.1.0.1.1}$$

From the above, it is obvious that $BD = BF$. Similarly, $AE = AF, CD = CF$. Denoting these lengths as x, y and z, as shown in Fig. 1.4.1.1,

$$x + y = ay + z = bx + z = c \tag{3.1.0.1.2}$$

which can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{3.1.0.1.3}$$

Section formula can be used to compute

$$\mathbf{D} = \frac{x\mathbf{C} + y\mathbf{B}}{x+y} \tag{3.1.0.1.4}$$

$$\mathbf{E} = \frac{y\mathbf{A} + z\mathbf{C}}{y+z} \tag{3.1.0.1.5}$$

$$\mathbf{F} = \frac{z\mathbf{B} + x\mathbf{A}}{z+x} \tag{3.1.0.1.6}$$

Note that **I** is the circumcentre of $\triangle DEF$. Thus, (**??**) can be used to compute **I**. This is done by the python code below

codes/circle/tri_icentre.py

and the equivalent latex-tikz code to draw Fig. 1.4.1.1 is

figs/circle/tri_icentre.tex

2. r is known as the <u>inradius</u> of $\triangle ABC$. Find r for the given values of a, b, c.

Solution: From Fig. 1.4.1.1

$$\because ar(ABC) = ar(IBC) + ar(ICA) + ar(IAB) \tag{3.1.0.2.1}$$

$$= \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{a+b+c}{2}r, \tag{3.1.0.2.2}$$

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \tag{3.1.0.2.3}$$

using Hero's formula. The following python code computes the inradius

```
codes/circle/tri_iradius.py
```

3.2. Drawing the Incircle

1. In Fig. 1.4.1.1, points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ are at a distance r from $\mathbf{I}$. The circle with centre $\mathbf{I}$ through these points is known as the incircle. Draw the incircle of $\triangle ABC$.

   Solution: This is done by the following python code

   ```
   codes/circle/tri_icircle.py
   ```

   and the equivalent latex-tikz code to draw Fig. 1.4.4.1 is

   ```
   figs/triangle/tri_icircle.tex
   ```

2. Sides AB, BC and CA touch the circle at exactly one point. Such lines are known as tangents to the circle.

3. Tangents to the circle are perpendicular to the radius at the point of contact.

4. From (3.1.0.1.1), it is obvious that tangents to the circle from a given point are equal.

3.3. Congruent Triangles

1. RHS: For two right angled triangles, if the hypotenuse and one of the sides are equal, show that the triangles are congruent.

   Solution: In $\triangle$s IDB and IFB in Fig. 1.4.1.1, $ID \perp BC, IF \perp AB, IB$ is a common side and $ID = IF$, i.e. both triangles are right angled, have the same hypotenuse and one equal leg. This information was sufficient to show that $BD = BF$. Similarly, it can be shown that all angles of both triangles are equal. Such triangles are known as congruent triangles and denoted by $\triangle IDB \cong \triangle IEB$.

2. Show that IA, IB, IC bisect the angles A, B and C respectively.

3. Angle bisectors of $\triangle ABC$ meet at the incentre **I**.

4. To show that two triangles are congruent, it is sufficient to show that some corresponding angles and sides are equal. Sine and cosine formulae are sufficient to show that two triangles are congruent.

5. SSS: Show that if the corresponding sides of three triangles are equal, the triangles are congruent.

6. ASA: Show that if two angles and any one side are equal in corresponding triangles, the triangles are congruent.

7. SAS: Show that if two sides and the angle between them are equal in corresponding triangles, the triangles are congruent.

Chapter 4

Medians of a Triangle

4.1. Basic Proportionality Theorem

1. Draw Fig. 4.1.0.2.1 with $a = 5$, $b = 6$ and $c = 4$. **E** and **F** are the mid points of AC and AB respectively. **Solution:** Using (**??**),

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{4.1.0.1.1}$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{4.1.0.1.2}$$

   The latex-tikz code is

figs/triangle/tri_med_sim.tex

2. In Fig. 4.1.0.2.1, BE and CF are the medians. show that $EF = \frac{a}{2}$.

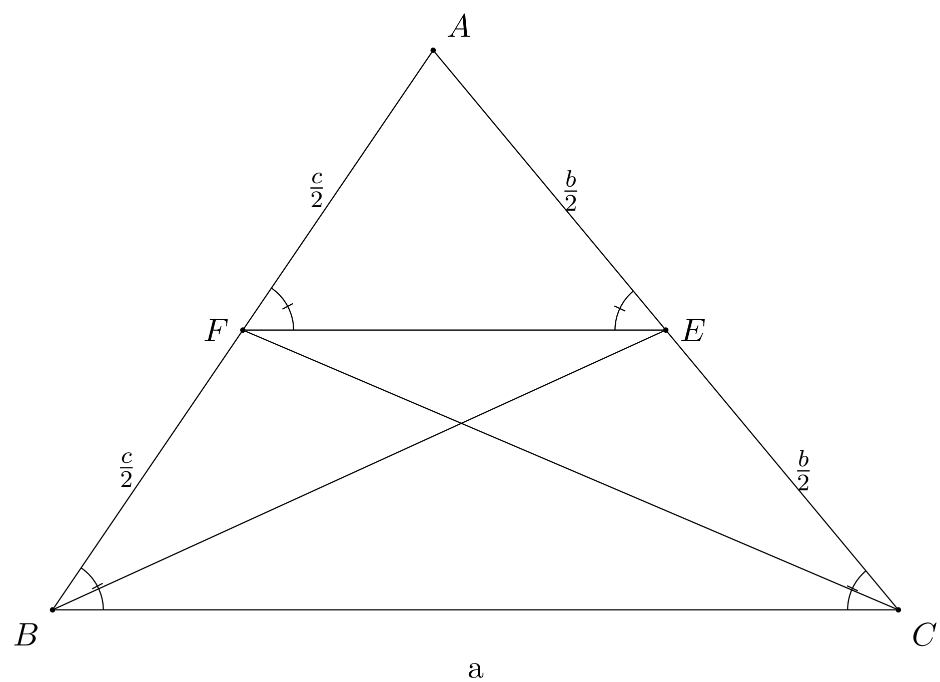

Figure 4.1.0.2.1: Similar Triangles

Solution: Using the cosine formula for $\triangle AEF$,

$$EF^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2 - 2\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)\cos A \tag{4.1.0.2.1}$$

$$= \frac{b^2 + c^2 - 2bc\cos A}{4} \tag{4.1.0.2.2}$$

$$= \frac{a^2}{4} \tag{4.1.0.2.3}$$

$$\Rightarrow EF = \frac{a}{2} \tag{4.1.0.2.4}$$

3. The ratio of sides of triangles AEF and ABC is the same. Such triangles are known as similar triangles.

4. Show that similar triangles have the same angles.

   Solution: Use cosine formula and the proof is trivial.

5. Show that in Fig. 4.1.0.2.1, $EF||BC$.

   Solution: Since $\triangle AEF \sim \triangle ABC$, $\angle AEF = \angle ACB$. Hence the line $EF||BC$

6. The line joining the mid points of two sides of a triangle is parallel to the third side. This is known as the basic proportionality theorem.

4.2. Similar Triangles

Use the sine and cosine formulae to show that triangles are similar if

1. AAA: all angles are equal

2. SAS: two corresponding sides are in the same ratio and the common angle is equal.

3. SSS: all sides have the same ratio.

4.3. Centroid

1. In Fig. 4.3.0.1.1, BE and CF are the medians. Show that $\triangle GEF \sim \triangle GBC$.

   Solution: $\because EF \parallel BC$, alternate interior angles are equal. Hence,

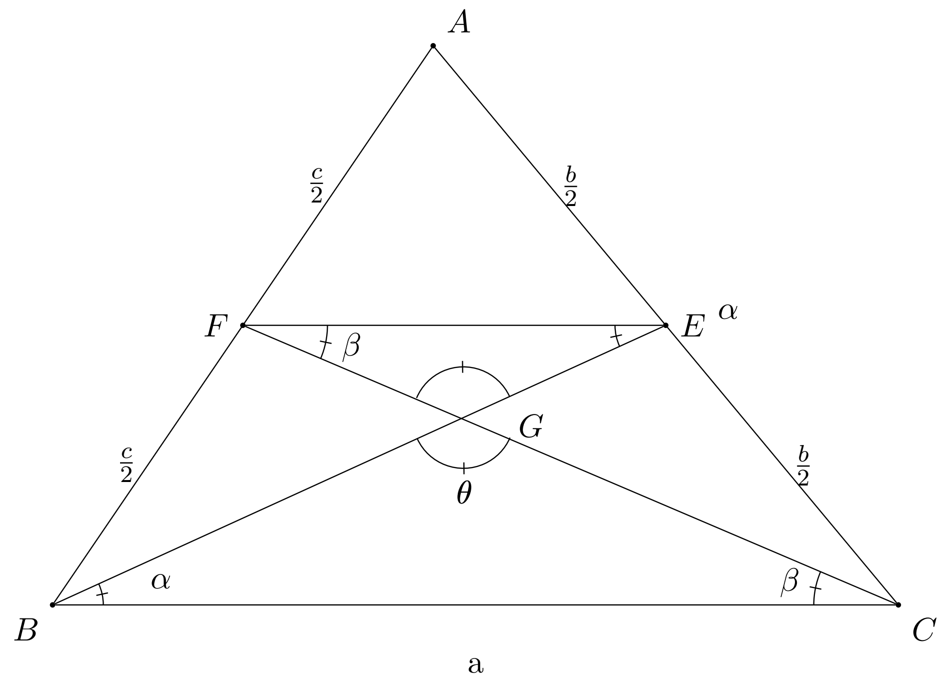

Figure 4.3.0.1.1: $\frac{GB}{GE} = \frac{GC}{GF} = 2$

corresponding angles of both triangles are equal and the triangles are similar.

2. In Fig. 2.2.5.1, AG is extended to meet BC at D. Show that

$$\frac{GA}{GD} = 2 \tag{4.3.0.2.1}$$

Solution: From Theorem 4.1.0.6, $EF||BC$. Hence,

$$\triangle APE \sim \triangle ADC \quad (AAA) \tag{4.3.0.2.2}$$
$$\implies AP = PD = \frac{AD}{2} \tag{4.3.0.2.3}$$
$$\implies AG - GP = GP + GD \tag{4.3.0.2.4}$$
$$\text{or, } GP = \frac{AG - GD}{2} \tag{4.3.0.2.5}$$

Similarly,

$$\triangle PGE \sim \triangle BGD \quad (AAA) \tag{4.3.0.2.6}$$
$$\implies \frac{GP}{GD} = \frac{GE}{GB} = \frac{1}{2} \tag{4.3.0.2.7}$$
$$\text{or, } GP = \frac{GD}{2} \tag{4.3.0.2.8}$$

using (2.2.5.2). From (4.3.0.2.5) and (4.3.0.2.8),

$$GP = \frac{GA - GD}{2} = \frac{GD}{2} \tag{4.3.0.2.9}$$
$$\implies \frac{GA}{GD} = 2 \tag{4.3.0.2.10}$$

3. In Fig. 4.3.0.3.1, BE is a median and

$$\frac{GA}{GD} = 2 \tag{4.3.0.3.1}$$

Show that

$$BD = DC \tag{4.3.0.3.2}$$

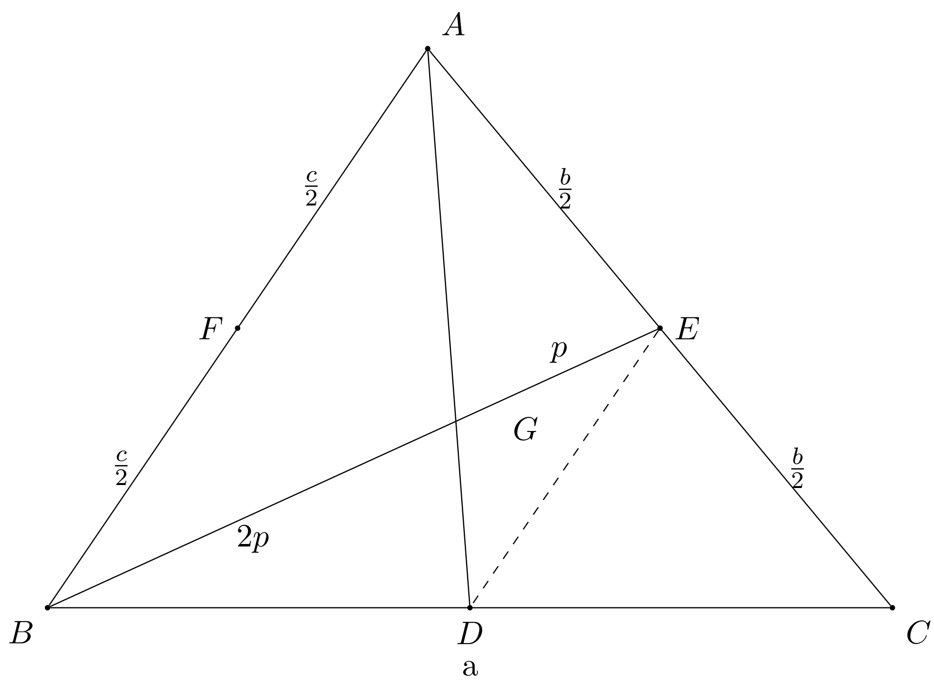

Figure 4.3.0.3.1: AD is also a median.

Solution: In Fig. 4.3.0.3.1,

$$\triangle GDE \sim \triangle GAB \quad (SAS) \tag{4.3.0.3.3}$$

Hence, all angles of both the above triangles are equal resulting in $DE \parallel AB$. As a consequence,

$$\triangle DDE \sim \triangle CAB \quad (AAA) \tag{4.3.0.3.4}$$

resulting in (4.3.0.3.2)

4. Medians of a triangle meet at a point.

5. The centroid divides the median in the ratio $2:1$.

6. Show that

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{4.3.0.6.1}$$

4.4. Median and Area

1. In Fig. 4.3.0.1.1, show that

$$ar\left(\triangle ABE\right) = ar\left(\triangle EBC\right)$$
$$ar\left(\triangle AFC\right) = ar\left(\triangle BFC\right) \tag{4.4.0.1.1}$$
$$ar\left(\triangle BCF\right) = ar\left(\triangle EBC\right)$$

Solution: We have

$$ar\left(\triangle EBC\right) = \frac{1}{2}\left(\frac{b}{2}\right)a\sin C = \frac{1}{4}ab\sin C \tag{4.4.0.1.2}$$
$$= \frac{1}{2}ar\left(\triangle ABC\right) \tag{4.4.0.1.3}$$
$$ar\left(\triangle BCF\right) = \frac{1}{2}\left(\frac{c}{2}\right)a\sin B = \frac{1}{4}ac\sin B \tag{4.4.0.1.4}$$
$$= \frac{1}{2}ar\left(\triangle ABC\right) \tag{4.4.0.1.5}$$

Using the sine formula, $b\sin C = c\sin B$ we obtain 4.4.0.1.1

2. The median divides a triangle into two triangles that have equal area.

3. Triangles on the same base between two parallel lines have the same area.

4.5. Parallelogram

1. In Fig. 4.5.0.1.1, $BDEF$ is defined as a parallelogram. Based on the properties of medians and similar triangles, we know that

$$BD \parallel EF, BD = EF \tag{4.5.0.1.1}$$

$$BF \parallel DE, BF = DE \tag{4.5.0.1.2}$$

$$OD = OF, OB = OE \tag{4.5.0.1.3}$$

Hence,

   (a) opposite sides are equal

   (b) opposite angles are equal

   (c) diagonals bisect each other

   (d) Adjacent angles of a paralleogram are supplementary.

2. In Fig. 4.5.0.1.1, $BFEC$ is a quadrilateral with $EF \parallel BC$ and is known as a trapezium.

3. Sum of the angles of a quadrilateral is $360°$.

4. Construct parallelogram $ABCD$ in Fig. 4.5.0.4.1 given that $BC = 5, AB = 6, \angle C = 85°$.

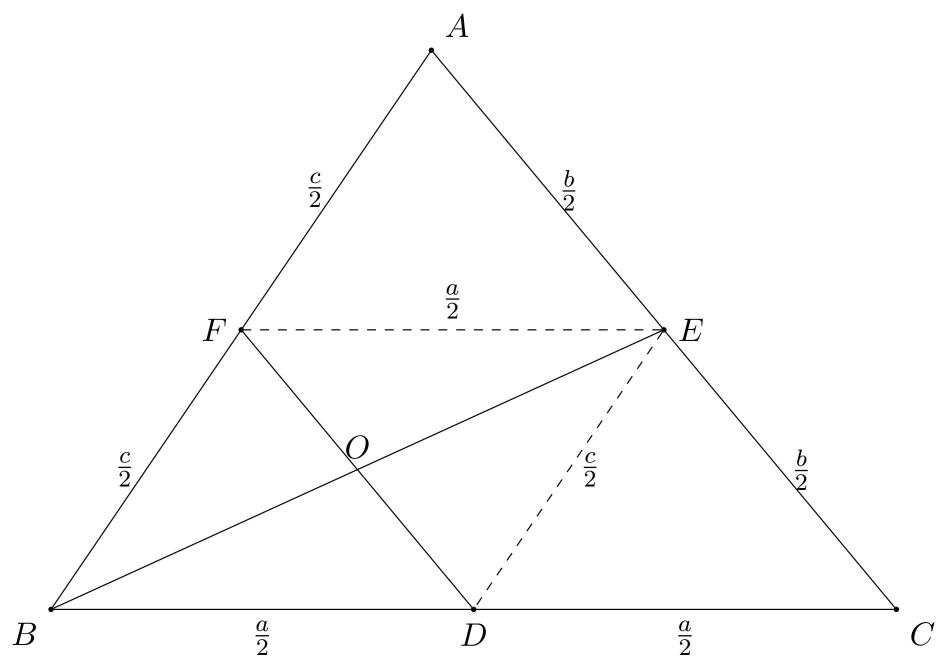

Figure 4.5.0.1.1: Parallelogram

Solution: BD is found using the cosine formula and $\triangle BDC$ is drawn using the approach in Construction **??** with

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \tag{4.5.0.4.1}$$

Since the diagonals bisect each other,

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{D}}{2} \tag{4.5.0.4.2}$$

$$\mathbf{A} = 2\mathbf{O} - \mathbf{C}. \tag{4.5.0.4.3}$$

AB and AD are then joined to complete the $\|$gm. The python code

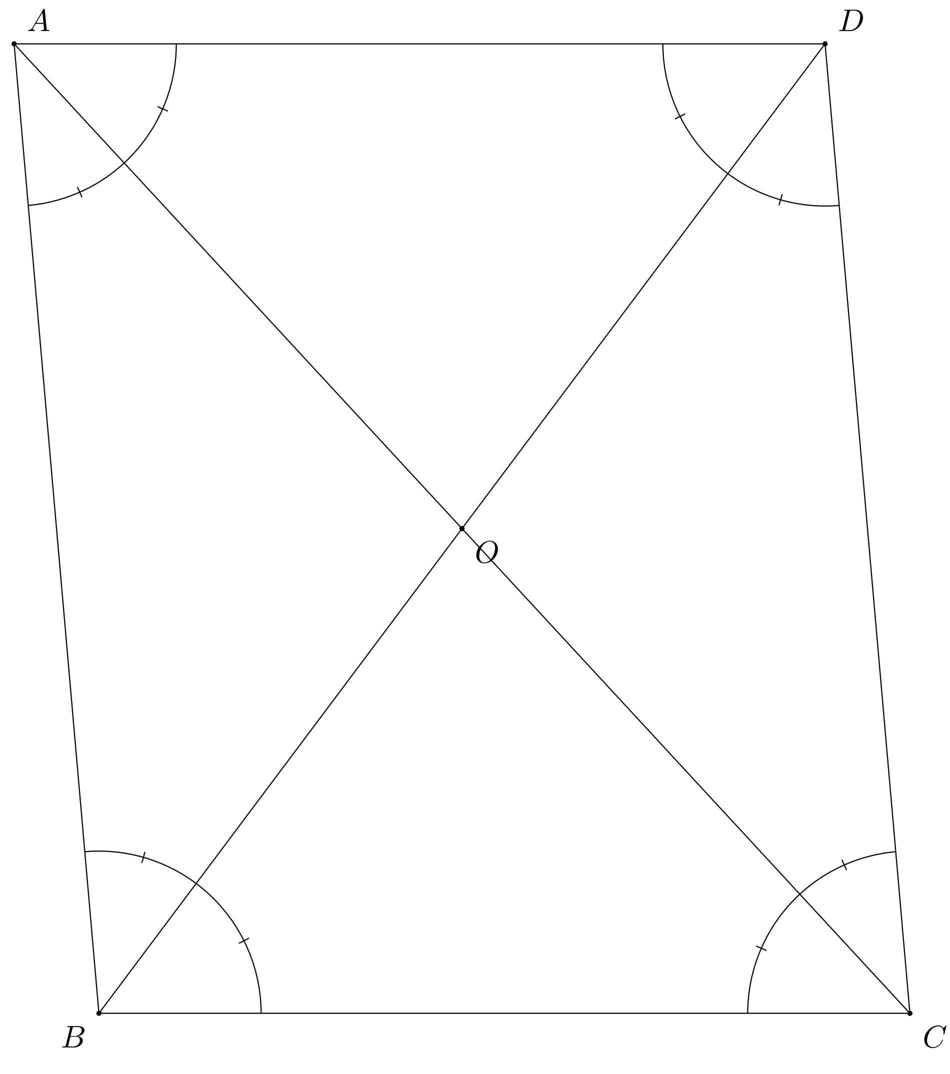

Figure 4.5.0.4.1: Parallelogram Properties

for Fig. 4.5.0.4.1 is

```
codes/quad/pgm_sas.py
```

and The equivalent latex-tikz code is

figs/quad/pgm_sas.tex

5. A rectangle is a parallelogram with one right angle.

6. Draw the $\|$gm $ABCD$ in Fig. 4.5.0.6.1 with $BC = 6, CD = 4.5$ and $BD = 7.5$. Show that it is a rectangle.

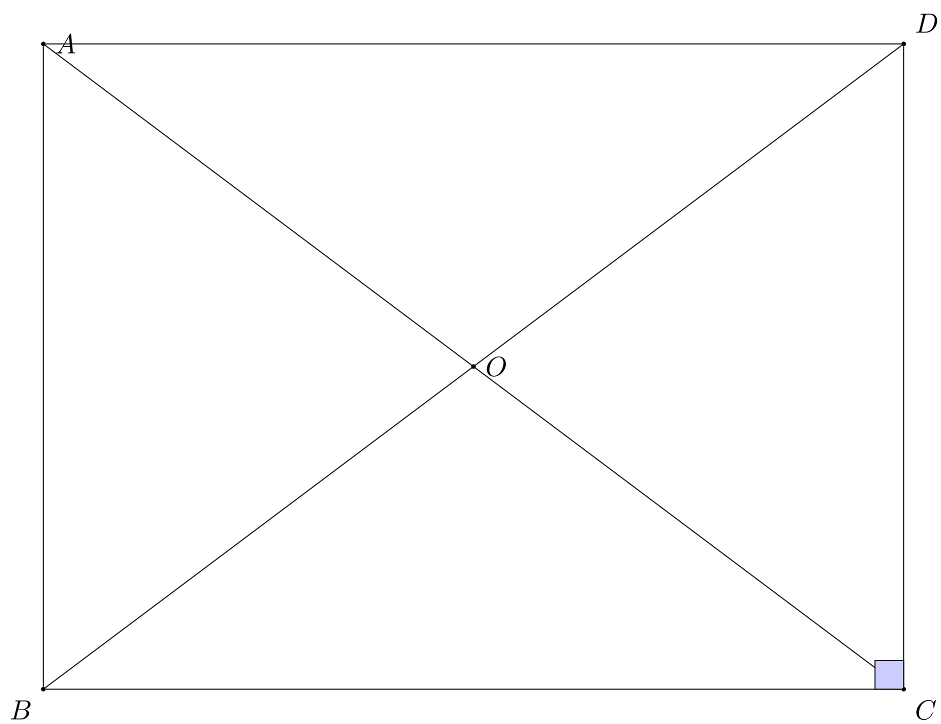

Figure 4.5.0.6.1: Rectangle

Solution: It is easy to verify that

$$BD^2 = BC^2 + C^2 \tag{4.5.0.6.1}$$

Hence, using Baudhayana theorem,

$$\angle BCD = 90^\circ \tag{4.5.0.6.2}$$

and $ABCD$ is a rectangle.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \tag{4.5.0.6.3}$$

The python code for Fig. 4.5.0.6.1 is

```
codes/quad/pgm_sss.py
```

and the equivalent latex-tikz code is

```
figs/quad/pgm_sss.tex
```

7. Diagonals of a rectangle are equal and vice-versa.

   Solution: Follows from the fact that $\triangle BCD \cong \triangle ABC$.

8. A rhombus, shown in Fig. 4.5.0.8.1 is a parallelogram with equal sides.

9. Diagonals of a rhombus bisect each other at right angles.

   Solution: In Fig. 4.5.0.8.1, from Theorem 4.5.0.1, $OB = OS$. From Theorem 2.7.6, $OE \perp BS$.

10. Draw the rhombus $BEST$ with $BE = 4.5$ and $ET = 6$.

    Solution: The coordinates of the various points in Fig. 4.5.0.8.1 are obtained as

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4.5 \end{pmatrix} \tag{4.5.0.10.1}$$

$$\mathbf{E} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{4.5.0.10.2}$$

11. A square is a rectangle whose sides are equal. Draw a square of side 4.5.

    Solution: The coordinates of the various points in Fig. 4.5.0.11.1 are obtained as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4.5 \end{pmatrix} \tag{4.5.0.11.1}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4.5 \\ 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix} \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{4.5.0.11.2}$$

12. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.

    Solution: A square has the properties of a rectangle as well as a rhombus.

13. Area of a parallelogram is the product of its base and the corresponding altitude.

4.6. Triangle

1. Each angle of an equilateral triangle is of $60°$.

2. Triangles on the same base (or equal bases) and between the same parallels are equal in area.

3. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

4. In $\triangle ABC, D, E$ and F are respectively the mid-points of sides AB, BC and CA. Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F.

5. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

6. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

7. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that (i) D is the mid-point of AC (ii) $MD \perp AC$ (iii) $CM = MA = \frac{1}{2}AB$

8. Sides opposite to equal angles of a triangle are equal.

9. Each angle of an equilateral triangle is of $60°$.

10. Using cosine formula in an equilateral $\triangle$, show that $\cos 60° = \frac{1}{2}$.

11. Using (1.2.2.1), show that $\sin 60° = \frac{\sqrt{3}}{2}$.

12. Find $\sin 30^\circ$ and $\sin 30^\circ$ using (1.1.2.2).

13. Triangles on the same base (or equal bases) and between the same parallels are equal in area.

14. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

15. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.

16. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.

17. In an isosceles $\triangle ABC$ with $AB =$ AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.

18. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.

19. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.

20. D is a point on side BC of $\triangle ABC$ such that $AD = AC$. Show that $AB > AD$

21. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

22. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that

    (a) $\triangle AOB \cong \triangle DOC$

    (b) O is also the mid-point of BC.

23. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?

24. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that

    (a) $\triangle ABD \cong \triangle BAC$

    (b) $BD = AC$

    (c) $\angle ABD = \angle BAC$.

25. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.

26. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$. Show that:

    (a) $\triangle APB \cong \triangle AQB$

    (b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.

27. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.

28. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:

    (a) $\triangle AMC \cong \triangle BMD$

    (b) $\angle DBC$ is a right angle.

    (c) $\triangle DBC \cong \triangle ACB$

    (d) $CM = \frac{1}{2}AB$

29. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :

    (a) $OB = OC$

    (b) AO bisects $\angle A$

30. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.

31. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are equal.

32. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

    (a) $\triangle ABE \cong \triangle ACF$

    (b) $AB = AC$, i.e., ABC is an isosceles triangle.

33. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$.

34. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that

    (a) $\triangle ABD \cong \triangle ACD$

    (b) $\triangle ABP \cong \triangle ACP$

    (c) AP bisects $\angle A$ as well as $\angle D$.

    (d) AP is the perpendicular bisector of BC.

35. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that

    (a) AD bisects BC

    (b) AD bisects $\angle A$.

36. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:

    (a) $\triangle ABM \cong \triangle PQN$

    (b) $\triangle ABC \cong \triangle PQR$

37. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

38. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.

39. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.

40. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.

41. Show that in a right angled triangle, the hypotenuse is the longest side.

42. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

43. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

44. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.

45. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

46. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.

47. $ABCD$ is a trapezium with $AB \parallel DC$. E and F are points on non-parallel sides AD and BC respectively such that EF is parallel to AB . Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

48. ST is a line joining two points on PQ and PR in $\triangle PQR$. If $\frac{PS}{SQ} = \frac{PT}{TR}$ and $\angle PST = \angle PRQ$, prove that PQR is an isosceles triangle.

49. If $LM \parallel CB$ and $LN \parallel CD$, prove that $\frac{AM}{AB} = \frac{AN}{AD}$.

50. D is a point on AB and E, F are points on BC such that $DE \parallel AC$ and $DF \parallel AE$. Prove that $\frac{BF}{FE} = \frac{BE}{EC}$.

51. O is a point in the interior of $\triangle ABC$. D is a point on OA. If $DE \parallel OB$ and $DF \parallel OC$. Show that $EF \parallel BC$.

52. O is a point in the interior of $\triangle PQR$. $A, BandC$ are points on OP, OQ and OR respectively such that $AB \parallel PQ$ and $AC \parallel PR$. Show that $BC \parallel QR$.

53. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$

54. The diagonals of a quadrilateral $ABCD$ intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that $ABCD$ is a trapezium.

55. $PQ \parallel RS$ and PS intersects QR at O. Show that $\triangle OPQ \sim \triangle ORS$.

56. CM and RN are respectively the medians of $\triangle ABC$ and $\triangle PQR$. If $\triangle ABC \sim \triangle PQR$, prove that

    (a) $\triangle AMC \sim \triangle PNR$

    (b) $\frac{CM}{RN} = \frac{AB}{PQ}$

    (c) $\triangle CMB \sim \triangle RNQ$

57. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. Using a similarity criterion for two triangles, show that $\frac{OA}{OC} = \frac{OB}{OD}$

58. In $\triangle PQR$, QP is extended to T and S is a point on QR such that $\frac{QR}{QS} = \frac{QT}{PR}$. If $\angle PRQ = \angle PQS$, show that that $\triangle PQS \sim \triangle TQR$.

59. S and T are points on sides PR and QR of $\triangle PQR$ such that $\angle P = \angle RTS$. Show that $\triangle RPQ \sim \triangle RTS$.

60. In $\triangle ABC$, D and E are points on the sides AB and AC respectively. If $\triangle ABE \cong \triangle ACD$, show that $\triangle ADE \sim \triangle ABC$.

61. Altitudes AD and CE of $\triangle ABC$ intersect each other at the point P. Show that:

    (a) $\triangle AEP \sim \triangle CDP$

    (b) $\triangle ABD \sim \triangle CBE$

    (c) $\triangle AEP \sim \triangle ADB$

    (d) $\triangle PDC \sim \triangle BEC$

62. E is a point on the side AD produced of a parallelogram $ABCD$ and BE intersects CD at F. Show that $\triangle ABE \sim \triangle CFB$.

63. ABC and AMP are two right triangles, right angled at B and M respectively. M lies on AC and AB is extended to meet P. Prove that:

    (a) $\triangle ABC \sim \triangle AMP$

(b) $\frac{CA}{PA} = \frac{BC}{MP}$

64. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of $\triangle ABC$ and $\triangle EFG$ respectively. If $\triangle ABC \sim \triangle FEG$, show that:

65. $\frac{CD}{GH} = \frac{AC}{FG}$

66. $\triangle DCB \sim \triangle HGE$

67. $\triangle DCA \sim \triangle HGF$

68. E is a point on side CB produced of an isosceles $\triangle ABC$ with $AB = AC$. If $AD \perp BC$ and $EF \perp AC$, prove that $\triangle ABD \sim \triangle ECF$.

69. Sides AB and BC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and QR and median PM of $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.

70. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that $CA^2 = CB.CD$.

71. Sides AB and AC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and PR and median PM of another $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.

72. If AD and PM are medians of $\triangle sABC$ and PQR, respectively where $\triangle ABC \sim \triangle PQR$, prove that $\frac{AB}{PQ} = \frac{AD}{PM}$

73. The line segment XY is parallel to side AC of $\triangle ABC$ and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$

74. Diagonals of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. If $AB = 2CD$, find the ratio of the areas of $\triangle sAOB$ and COD.

75. ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, show that $\frac{ar(ABC)}{ar(DBC)} = \frac{AO}{DO}$.

76. If the areas of two similar triangles are equal, prove that they are congruent.

77. D, E and F are respectively the mid-points of sides AB, BC and CA of $\triangle ABC$. Find the ratio of the areas of $\triangle DEF$ and $\triangle ABC$.

78. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.

79. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.

80. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. Find the ratio of the areas of triangles ABC and BDE.

81. The sides of two similar triangles are in the ratio $4 : 9$. Find the ratio the area of these triangles are in the ratio

82. In $\triangle ABC, \angle ACB = 90°$ and $CD \perp AB$. Prove that $\frac{BC^2}{AC^2} = \frac{BD}{AD}$.

83. In $\triangle ABC$, if $AD \perp BC$, prove that $AB^2 + CD^2 = BD^2 + AC^2$.

84. BL and CM are medians of a $\triangle ABC$ right angled at A. Prove that $4(BL^2 + CM^2) = 5BC^2$.

85. O is any point inside a rectangle $ABCD$. Prove that $OB^2 + OD^2 = OA^2 + OC^2$.

86. PQR is a triangle right angled at P and M is a point on QR such that $PM \perp QR$. Show that $PM^2 = QM.MR$.

87. ABD is a triangle right angled at A and $AC \perp BD$. Show that

    (a) $AB^2 = BC.BD$

    (b) $AC^2 = BC.DC$

    (c) $AD^2 = BD.CD$

88. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.

89. ABC is an isosceles triangle with $AC = BC$. If $AB^2 = 2AC^2$, prove that ABC is a right triangle.

90. ABC is an equilateral triangle of side $2a$. Find each of its altitudes.

91. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

92. O is a point in the interior of a $\triangle ABC, OD \perp BC, OE \perp ACandOF \perp AB$. Show that

    (a) $OA^2 + OB^2 + BD^2–OD2–OE2–OF2 = AF^2 + BD^2 + CE^2$.

(b) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

93. D and E are points on the sides CA and CB respectively of a $\triangle ABC$ right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.

94. The perpendicular from A on side BC of a $\triangle ABC$ intersects BC at D such that $DB = 3CD$. Prove that $2AB^2 = 2AC^2 + BC^2$.

95. In an equilateral $\triangle ABC$, D is a point on side BC such that $BD = \frac{1}{3}BC$. Prove that $9AD^2 = 7AB^2$.

96. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.

97. PS is the bisector of $\angle QPR$ of $\triangle PQR$. Prove that $\frac{QS}{SR} = \frac{PQ}{PR}$

98. D is a point on hypotenuse AC of $\triangle ABC$, such that $BD \perp AC, DM \perp BC$ and $DN \perp AB$. Prove that :

(a) DM2 = DN . MC

(b) DN2 = DM . AN

99. ABC is a triangle in which $\angle ABC > 90°$ and $AD \perp CB$ produced. Prove that $AC^2 = AB^2 + BC^2 + 2BC.BD$.

100. ABC is a triangle in which $\angle ABC < 90°$ and $AD \perp BC$. Prove that $AC^2 = AB^2 + BC^2–2BC.BD$.

101. AD is a median of a $\triangle ABC$ and $AM \perp BC$. Prove that :

(a) $AC^2 = AD^2 + BC.DM + \left(\frac{BC}{2}\right)^2$

(b) $AB^2 = AD^2 – BC.DM + \left(\frac{BC}{2}\right)^2$

(c) $AC^2 + AB^2 = 2AD^2 + \frac{1}{2}BC^2$

102. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.

103. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD}\frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

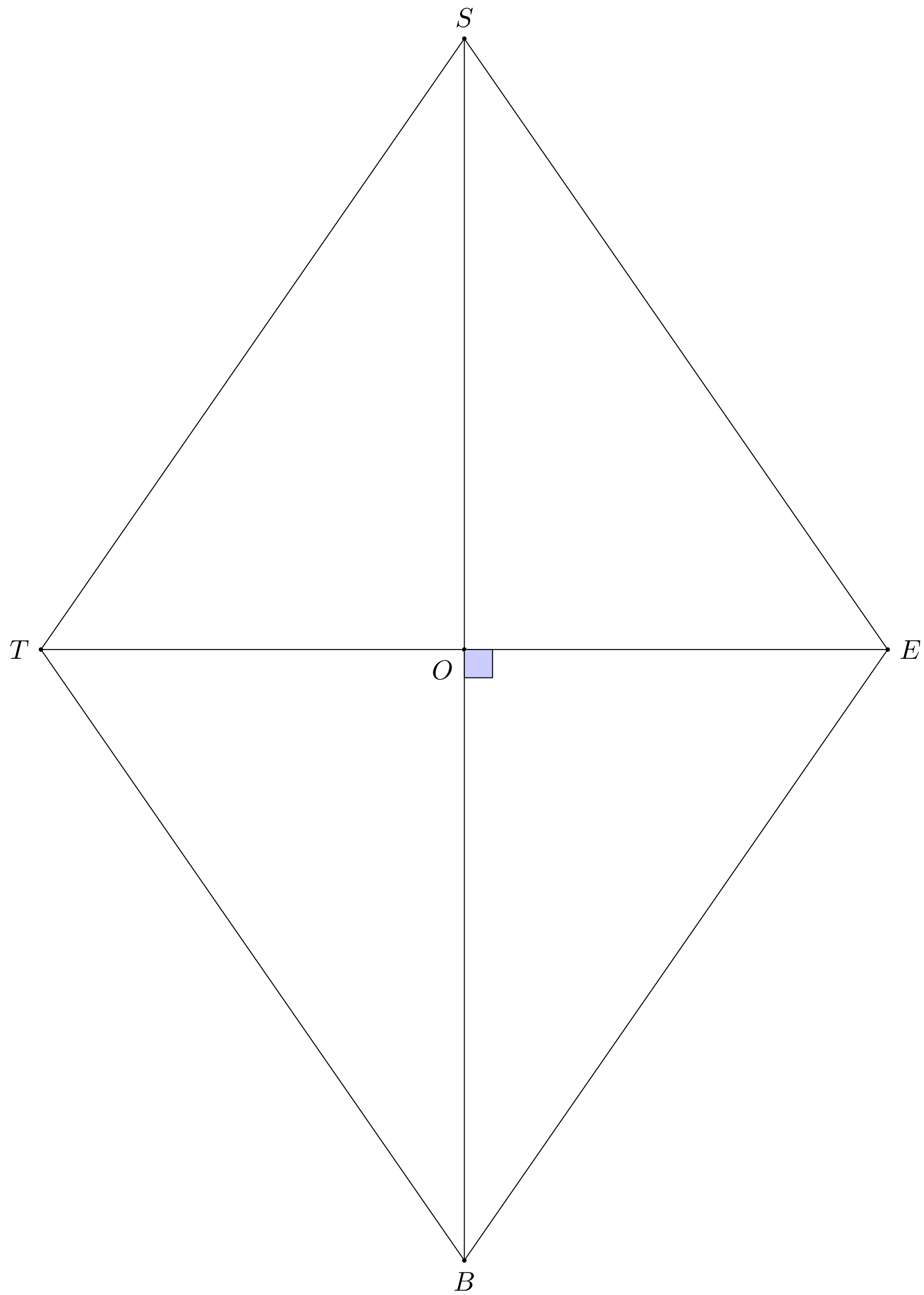

Figure 4.5.0.8.1: Rhombus

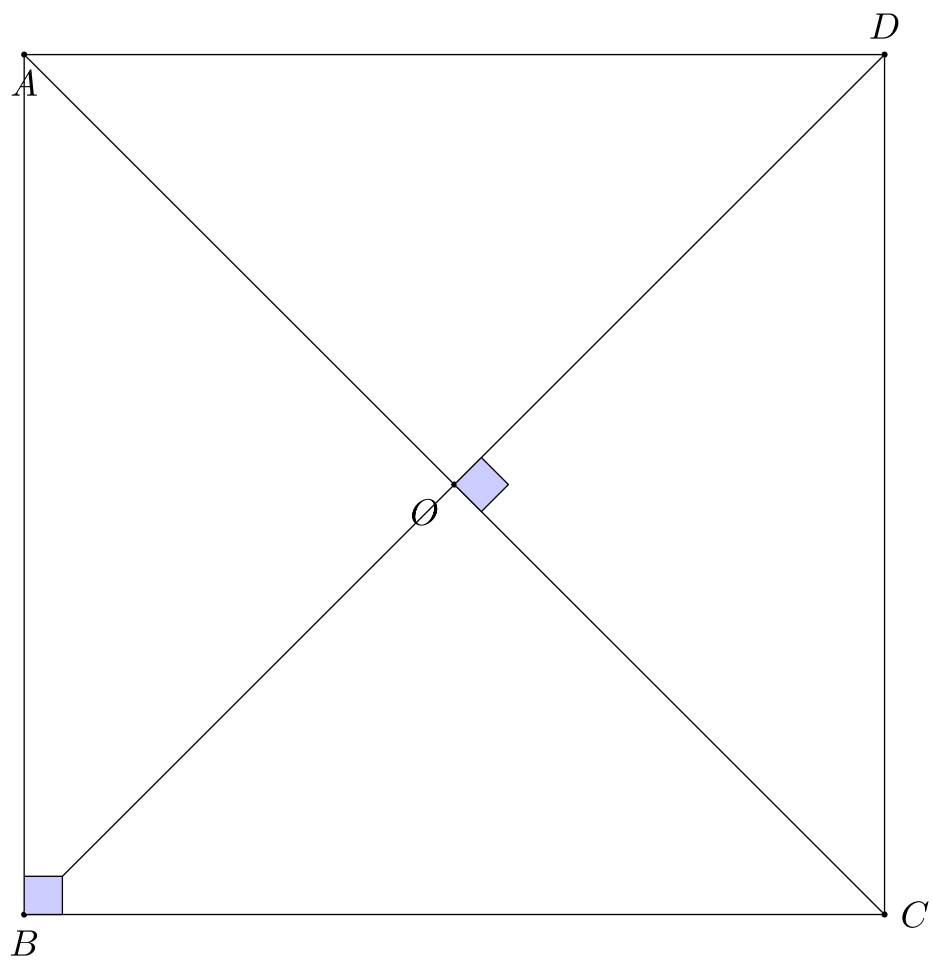

Figure 4.5.0.11.1: Square

Chapter 5

Quadrilateral

1. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

2. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

3. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

4. Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

5. Show that the bisectors of angles of a parallelogram form a rectangle.

6. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:

   (a) $APCQ$ is a parallelogram.

(b) $DPBQ$ is a parallelogram.

(c) $PSQR$ is a parallelogram.

7. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.

8. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

9. Area of a parallelogram is the product of its base and the corresponding altitude.

10. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.

11. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

12. In parallelogram $ABCD$, two points P and Q are taken on diagonal BD such that $DP = BQ$. show that

    (a) $\triangle APD \cong \triangle CQB$

    (b) $AP = CQ$

    (c) $\triangle AQB \cong \triangle CPD$

    (d) $AQ = CP$

    (e) $APCQ$ is a parallelogram

13. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD . Show that

(a) $\triangle APB \cong \triangle CQD$

(b) $AP = CQ$

14. In $\triangle ABC$ and $\triangle DEF, AB = DE, AB \parallel DE, BC = EF$ and $BC \parallel EF$. Vertices A, B and C are joined to vertices D, E and F respectively. Show that

(a) quadrilateral $ABED$ is a parallelogram

(b) quadrilateral $BEFC$ is a parallelogram

(c) $AD \parallel CF$ and $AD = CF$

(d) quadrilateral $ACFD$ is a parallelogram

(e) $AC = DF$

(f) $\triangle ABC \cong \triangle DEF$.

15. $ABCD$ is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that

(a) $\angle A = \angle B$

(b) $\angle C = \angle D$

(c) $\triangle ABC \cong \triangle BAD$

(d) diagonal $AC =$ diagonal BD

16. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA AC is a diagonal. Show that

(a) $SR \parallel AC$ and $SR = \frac{1}{2}AC$

(b) $PQ = SR$

(c) $PQRS$ is a parallelogram.

17. $ABCD$ is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rectangle.

18. $ABCD$ is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.

19. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and E is the mid-point of AD. A line is drawn through $E \parallel AB$ intersecting BC at F. Show that F is the mid-point of BC.

20. In a parallelogram $ABCD$, E and F are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.

21. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

22. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:

    (a) $APCQ$ is a parallelogram.

    (b) $DPBQ$ is a parallelogram.

    (c) $PSQR$ is a parallelogram.

23. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.

24. Diagonal AC of a parallelogram $ABCD$ bisects $\angle A$. show that

    (a) it bisects $\angle C$ also,

    (b) $ABCD$ is a rhombus.

25. $ABCD$ is a rhombus. Show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.

26. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that

    (a) $ABCD$ is a square

    (b) diagonal BD bisects $\angle B$ as well as $\angle D$.

27. If E, F, G and H are respectively the mid-points of the sides of a parallelogram $ABCD$, show that

$$ar\left(EFGH\right) = \frac{1}{2} ar\left(ABCD\right). \tag{5.0.0.27.1}$$

28. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$. Show that $ar(APB) = ar(BQC)$.

29. P is a point in the interior of a parallelogram $ABCD$. Show that

    (a) $ar(APB) + ar(PCD) = \frac{1}{2} ar(ABCD)$

(b) $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

30. $PQRS$ and $ABRS$ are parallelograms and X is any point on side BR. show that

    (a) $ar(PQRS) = ar(ABRS)$

    (b) $ar(AXS) = \frac{1}{2}ar(PQRS)$

31. A farmer was having a field in the form of a parallelogram $PQRS$. She took any point A on RS and joined it to points P and Q. In how many parts the fields is divided? What are the shapes of these parts? The farmer wants to sow wheat and pulses in equal portions of the field separately. How should she do it?

32. $ABCD$ is a quadrilateral and $BE \parallel AC$ and also BE meets DC produced at E. Show that area of $\triangle ADE$ is equal to the area of the quadrilateral $ABCD$.

33. E is any point on median AD of a $\triangle ABC$. Show that $ar(ABE) = ar(ACE)$.

34. In a $\triangle ABC, E$ is the mid-point of median AD. Show that $ar(BED) = \frac{1}{4}ar(ABC)$.

35. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

36. ABC and ABD are two triangles on the same base AB. If line- segment CD is bisected by AB at O, show that $ar(ABC) = ar(ABD)$.

37. D, E and F are respectively the mid-points of the sides BC, CA and AB of a $\triangle ABC$. show that

    (a) $BDEF$ is a parallelogram.

    (b) $ar(BDEF) = \frac{1}{2}ar(ABC)$

38. Diagonals AC and BD of quadrilateral $ABCD$ intersect at O such that $OB = OD$. If $AB = CD$, then show that

    (a) $ar(DOC) = ar(AOB)$

    (b) $ar(DCB) = ar(ACB)$

    (c) $ar(DEF) = \frac{1}{4}ar(ABC)$

39. D and E are points on sides AB and AC respectively of $\triangle ABC$ such that $ar(DBC) = ar(EBC)$. Prove that $DE \parallel BC$.

40. XY is a line parallel to side BC of a $\triangle ABC$. If $BE \parallel AC$ and $CF \parallel AB$ meet XY at E and F respectively, show that $ar(ABE) = ar(ACF)$.

41. The side AB of a parallelogram $ABCD$ is produced to any point P. A line through A and parallel to CP meets CB produced at Q and then parallelogram $PBQR$ is completed. Show that $ar(\text{ABCD}) = ar(PBQR)$.

42. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at O. Prove that $ar(AOD) = ar(BOC)$.

43. $ABCDE$ is a pentagon. A line through B parallel to AC meets DC produced at F. Show that

(a) $ar(ACB) = ar(ACF)$

(b) $ar(AEDF) = ar(ABCDE)$.

44. A villager Itwaari has a plot of land of the shape of a quadrilateral. The Gram Panchayat of the village decided to take over some portion of his plot from one of the corners to construct a Health Centre. Itwaari agrees to the above proposal with the condition that he should be given equal amount of land in lieu of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented.

45. $ABCD$ is a trapezium with $AB \parallel DC$. A line parallel to AC intersects AB at X and BC at Y. Prove that $ar(ADX) = ar(ACY)$.

46. $AP \parallel BQ \parallel CR$. Prove that $ar(AQC) = ar(PBR)$.

47. Diagonals AC and BD of a quadrilateral $ABCD$ intersect at O in such a way that $ar(AOD) = ar(BOC)$. Prove that $ABCD$ is a trapezium.

48. $AB \parallel DC \parallel RP$. $ar(DRC) = ar(DPC)$ and $ar(BDP) = ar(ARC)$. Show that both the quadrilaterals $ABCD$ and $DCPR$ are trapeziums.

49. Parallelogram $ABCD$ and rectangle $ABEF$ are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

50. In $\triangle ABC$, D and E are two points on BC such that $BD = DE = EC$. Show that $ar(ABD) = ar(ADE) = ar(AEC)$.

51. $ABCD, DCFE$ and $ABFE$ are parallelograms. Show that ar$(ADE) = ar(BCF)$.

52. $ABCD$ is a parallelogram and BC is produced to a point Q such that $AD = CQ$. If AQ intersect DC at P, show that $ar(BPC) = ar(DPQ)$. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. If AE intersects BC at$F$, show that

   (a) $ar(BDE) = \frac{1}{4}ar(ABC)$

   (b) $ar(BDE) = \frac{1}{2}ar(BAE)$

   (c) $ar(ABC) = 2ar(BEC)$

   (d) $ar(BFE) = ar(AFD)$

   (e) $ar(BFE) = 2ar(FED)$

   (f) $ar(FED) = \frac{1}{8}ar(AFC)$

53. Diagonals AC and BD of a quadrilateral $ABCD$ intersect each other at P. Show that $ar(APB) \times ar(CPD) = ar(APD) \times ar(BPC)$.

54. P and Q are respectively the mid-points of sides AB and BC of a $\triangle ABC$ and R is the mid-point of AP, show that

   (a) $ar(PRQ) = \frac{1}{2}ar(ARC)$

   (b) $ar(PBQ) = ar(ARC)$

   (c) $ar(RQC) = \frac{3}{8}ar(ABC)$

55. ABC is a right triangle right angled at A. $BCED$, $ACFG$ and $ABMN$ are squares on the sides BC, CA and AB respectively. Line segment $AX \perp DE$ meets BC at Y. Show that

   (a) $\triangle MBC \cong \triangle ABD$

(b) $ar(BYXD) = ar(ABMN)$

(c) $ar(CYXE) = 2ar(FCB)$

(d) $ar(BYXD) = 2ar(MBC)$

(e) $\triangle FCB \cong \triangle ACE$

(f) $ar(CYXE) = ar(ACFG)$

(g) $ar(BCED) = ar(ABMN) + ar(ACFG)$

56. L is a point on the diagonal AC of quadrilateral $ABCD$. If LM —— CB and LN —— CD, prove that $\frac{AM}{AB} = \frac{AN}{AD}$

57. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.

Solution: Let the measure of angles $\underline{/A}, \underline{/B}, \underline{/C}, \underline{/D}$ of a quadrilateral are 3x, 5x, 9x and 13x respectively, where x is a real number.
Using angle sum property, the sum of interior angles of a quadrilateral is 360 degree.

$$3x + 5x + 9x + 13x = 360^\circ \tag{5.0.0.57.1}$$

$$30x = 360^\circ \tag{5.0.0.57.2}$$

$$x = 12^\circ \tag{5.0.0.57.3}$$

From the above calculations,

$$\angle A = 3x = 3(12) = 36^\circ \tag{5.0.0.57.4}$$

$$\angle B = 5x = 5(12) = 60^\circ \tag{5.0.0.57.5}$$

$$\angle C = 9x = 9(12) = 108^\circ \tag{5.0.0.57.6}$$

$$\angle D = 13x = 13(12) = 156^\circ \tag{5.0.0.57.7}$$

Chapter 6

Circles

6.1. Miscellaneous Properties

1. In Fig. 6.1.0.1.1, AB is the diameter and passes through the centre O. show that $\angle APB = 90^{\circ}$.

   Solution: From Theorem 2.9.4,

   $$\angle APB = \frac{1}{2}\angle AOB = 90^{\circ} \tag{6.1.0.1.1}$$

2. In right $\triangle APB$, right angled at $\mathbf{P}$, the median

   $$PO = AO = OB \tag{6.1.0.2.1}$$

   Solution: See Fig. 6.1.0.1.1. The median PO is a radius of the circle.

3. In Fig. 6.1.0.3.1, show that

   $$PA.PB = PC.PD \tag{6.1.0.3.1}$$

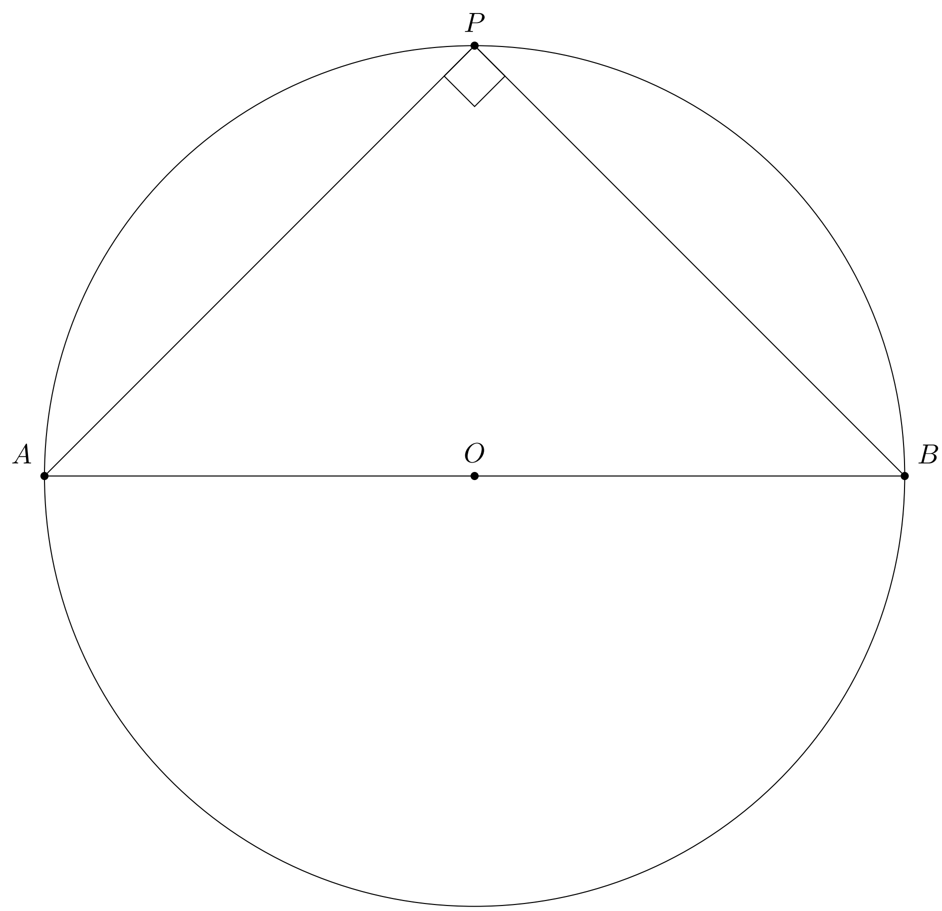

Figure 6.1.0.1.1: Diameter of a circle.

Solution: From Theorem 2.9.4,

$$\begin{aligned} \angle ABD &= \angle ACD \\ \angle CAB &= \angle CDB \end{aligned} \tag{6.1.0.3.2}$$

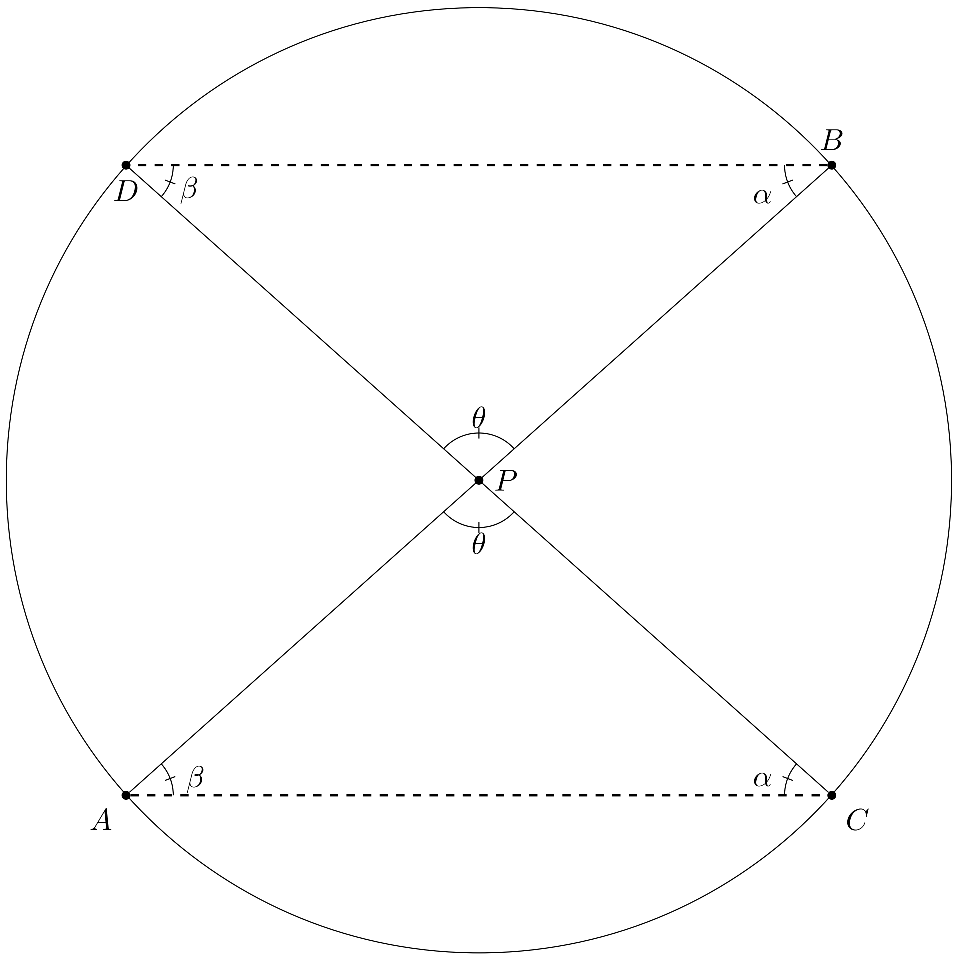

Figure 6.1.0.3.1: $PA.PB = PC.PD$

Hence,

$$\triangle PAC \sim \triangle PBD \quad (AAA) \tag{6.1.0.3.3}$$

and

$$\frac{PA}{PD} = \frac{PC}{PB} \tag{6.1.0.3.4}$$

$$\implies PA.PB = PC.PD \tag{6.1.0.3.5}$$

4. In Fig. 6.1.0.4.1 show that

$$\angle PCA = \angle PBC \tag{6.1.0.4.1}$$

O is the centre of the circle and PC is the tangent.

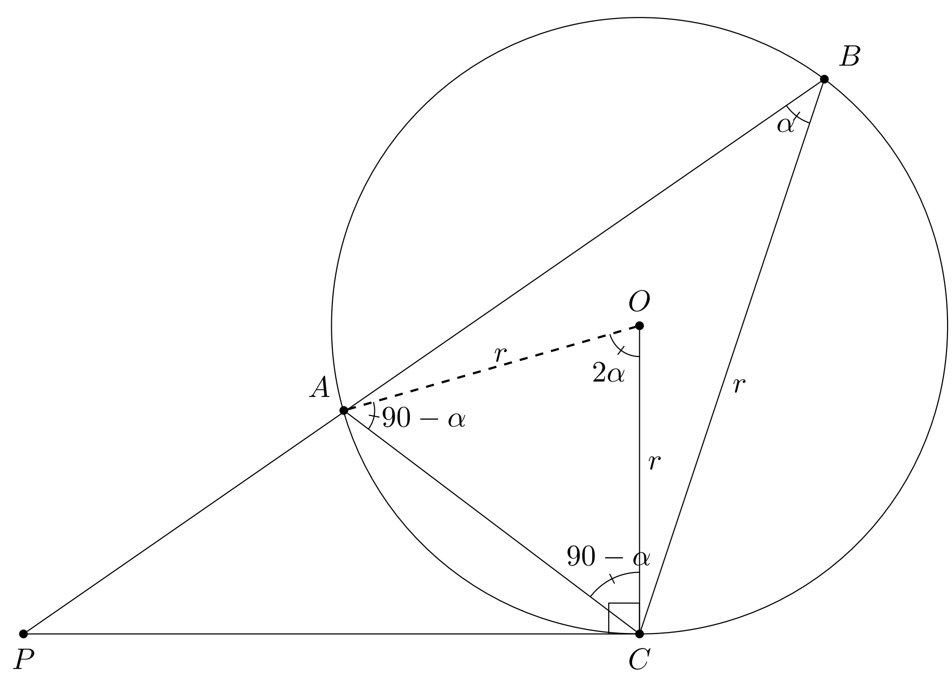

Figure 6.1.0.4.1: $PA.PB = PC^2$.

Solution: Obvious from the figure once we observe that $\triangle OAC$ is isosceles.

5. In Fig. 6.1.0.4.1, show that $PA.PB = PC^2$. **Solution:** Using Theorem 6.1.0.4,

$$\triangle PAC \sim \triangle PBC \quad (AAA) \tag{6.1.0.5.1}$$

Hence,

$$\frac{PA}{PC} = \frac{PC}{PB} \tag{6.1.0.5.2}$$

$$\implies PA.PB = PC^2 \tag{6.1.0.5.3}$$

6. In Fig. 6.1.0.6.1, show that

$$PA.PB = PC.PD \tag{6.1.0.6.1}$$

Solution: From Theorem 6.1.0.5, if PT be a tangent to the circle,

$$PA.PB = PT^2 = PC.PD \tag{6.1.0.6.2}$$

6.2. Examples

1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

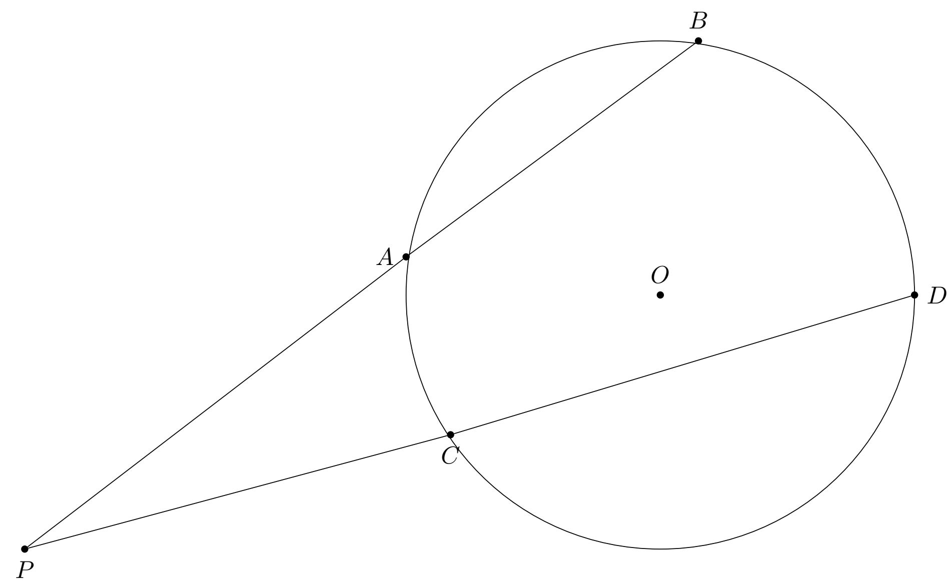

Figure 6.1.0.6.1: $PA.PB = PC^2$.

3. The perpendicular from the centre of a circle to a chord bisects the chord.

4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

5. There is one and only one circle passing through three non-collinear points.

6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).

7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.

8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.

9. Congruent arcs of a circle subtend equal angles at the centre.

10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

11. Angles in the same segment of a circle are equal.

12. Angle in a semicircle is a right angle.

13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.

14. The sum of either pair of opposite angles of a cyclic quadrilateral is $180°$.

15. If sum of a pair of opposite angles of a quadrilateral is $180°$, the quadrilateral is cyclic.

16. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.

17. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$

18. Two circles intersect at two points $AandB$. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.

19. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

20. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

21. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

22. The perpendicular from the centre of a circle to a chord bisects the chord.

23. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

24. There is one and only one circle passing through three non-collinear points.

25. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).

26. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.

27. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.

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29. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

30. Angles in the same segment of a circle are equal.

31. Angle in a semicircle is a right angle.

32. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.

33. The sum of either pair of opposite angles of a cyclic quadrilateral is $180°$.

34. If sum of a pair of opposite angles of a quadrilateral is $180°$, the quadrilateral is cyclic.

35. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.

36. Two circles intersect at two points A and B. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.

37. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

38. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the

other chord.

39. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.

40. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.

41. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

42. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.

43. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

44. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.

45. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

46. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.

47. Prove that a cyclic parallelogram is a rectangle.

48. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

49. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.

50. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.

51. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.

52. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.

53. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90° - \frac{A}{2}, 90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.

54. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.

55. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.

56. The lengths of tangents drawn from an external point to a circle are equal.

57. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

58. Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2\angle OPQ$.

59. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.

60. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

61. A quadrilateral $ABCD$ is drawn to circumscribe a circle. Prove that $AB + CD = AD + BC$.

62. XY and $X'Y'$ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and $X'Y'$ at B. Prove that $\angle AOB = 90°$

63. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

64. Prove that the parallelogram circumscribing a circle is a rhombus.

65. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

66. Find the area of a sector of angle p (in degrees) of a circle with radius R.

67. Two chords AB and CD intersect each other at the point P. Prove that :

    (a) $\triangle APC \sim \triangle DPB$

    (b) $AP.PB = CP.DP$

68. Two chords AB and CD of a circle intersect each other at the point P (when produced) outside the circle. Prove that

    (a) $\triangle PAC \sim \triangle PDB$

    (b) $PA.PB = PC.PD$

Chapter 7

Miscellaneous

1. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55^\circ$ and $\angle BAC = 45^\circ$, find $\angle BCD$

2. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.

3. A,B and C are three points on a circle with centre O such that $\angle BOC = 30^\circ$ and $\angle AOB = 60^\circ$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.

4. $\angle PQR = 100^\circ$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$.

5. A, B, C, D are points on a circle such that $\angle ABC = 69^\circ, \angle ACB = 31^\circ$, find $\angle BDC$.

6. A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130^\circ$ and $\angle ECD = 20^\circ$. Find $\angle BAC$.

7. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at a point

E. If $\angle DBC = 70°, \angle BAC$ is $30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.

8. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.

9. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?

10. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that $OQ = 12$ cm. Find the length of PQ.

11. PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.

12. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. Find the radius of the circle is

13. If TP and TQ are the two tangents to a circle with centre O so that $\angle POQ = 110°$, then find $\angle PTQ$

14. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80 ° , then find $\angle POA$

15. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

16. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

17. A $\triangle ABC$ is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively. Find the sides AB and AC.

18. The radii of two circles are 19 cm and 9 cm respectively. Find the radius of the circle which has circumference equal to the sum of the circumferences of the two circles.

19. The radii of two circles are 8 cm and 6 cm respectively. Find the radius of the circle having area equal to the sum of the areas of the two circles.

20. A circular archery target is marked with its five scoring regions from the centre outwards as Gold, Red, Blue, Black and White. The diameter of the region representing Gold score is 21 cm and each of the other bands is 10.5 cm wide. Find the area of each of the five scoring regions.

21. The wheels of a car are of diameter 80 cm each. How many complete revolutions does each wheel make in 10 minutes when the car is travelling at a speed of 66 km per hour?

22. Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector.

23. Find the area of the segment AYB, if radius of the circle is 21 cm and $\angle AOB = 120^\circ$.

24. Find the area of a sector of a circle with radius 6 cm if angle of the sector is 60 $^\circ$.

25. Find the area of a quadrant of a circle whose circumference is 22 cm. 3. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes.

26. A chord of a circle of radius 10 cm subtends a right angle at the centre. Find the area of the corresponding :

    (a) minor segment

    (b) major sector.

27. In a circle of radius 21 cm, an arc subtends an angle of 60 $^\circ$ at the centre. Find:

    (a) the length of the arc

    (b) area of the sector formed by the arc

    (c) area of the segment formed by the corresponding chord

28. A chord of a circle of radius 15 cm subtends an angle of 60 $^\circ$ at the centre. Find the areas of the corresponding minor and major segments of the circle.

29. A chord of a circle of radius 12 cm subtends an angle of 120 $^\circ$ at the centre. Find the area of the corresponding segment of the circle.

30. A horse is tied to a peg at one corner of a square shaped grass field of side 15 m by means of a 5 m long rope. Find

    (a) the area of that part of the field in which the horse can graze.

    (b) the increase in the grazing area if the rope were 10 m long instead of 5 m.

31. A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire is also used in making 5 diameters which divide the circle into 10 equal sectors. Find :

    (a) the total length of the silver wire required.

    (b) the area of each sector of the brooch

32. An umbrella has 8 ribs which are equally spaced. Assuming umbrella to be a flat circle of radius 45 cm, find the area between the two consecutive ribs of the umbrella.

33. A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115 ° . Find the total area cleaned at each sweep of the blades.

34. To warn ships for underwater rocks, a lighthouse spreads a red coloured light over a sector of angle 80 ° to a distance of 16.5 km. Find the area of the sea over which the ships are warned.

35. Two circular flower beds are located on opposite sides of a square lawn $ABCD$ of side 56 m. If the centre Of each circular flower bed is the

point of intersection O of the diagonals of the square lawn, find the sum of the areas of the lawn and the flower beds.

36. Four circles are inscribed inside a square $ABCD$ of side 14 cm such that each one touches exernally two adjacent sides of the square and two other circles. Find the region between the circles and the square.

37. $ABCD$ is a square of side 10 cm and semicircles are drawn with each side of the square as diameter. Find the area enclosed by the circular arcs.

38. P is a point on the semi-circle formed with diameter QR. Find the area between the semi-circle and $\triangle PQR$ if $PQ = 24$ cm, PR = 7 cm and O is the centre Of the circle.

39. AC and BD are two arcs on concentric circles with radii 14 cm and 7 cm respectively, such that $\angle AOC = 40°$. Find the area of the region $ABDC$.

40. Find the area between a square $ABCD$ of side 14cm and the semi circles APD and BPC.

41. Find the area of the region enclosed by a circular arc of radius 6 cm drawn with vertex O of an equilateral triangle OAB of side 12 cm as centre.

42. From each corner of a square of side 4 cm a quadrant of a circle of radius 1 cm is cut and also a circle of diameter 2 cm is cut. Find the area of the remaining portion of the square.

43. In a circular table cover of radius 32 cm, a design is formed leaving an equilateral $\triangle ABC$ in the middle. Find the area of the design.

44. $ABCD$ is a square of side 14 cm. With centres A, B, C and D, four circles are drawn such that each circle touches externally two of the remaining three circles. Find the area within the square that lies outside the circles.

45. The left and right ends of a racing track are semicircular. The distance between the two inner parallel line segments is 60 m and they are each 106 m long. If the track is 10 m wide, find :

    (a) the distance around the track along its inner edge

    (b) the area of the track.

46. AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OD is the diameter of a smaller circle inside. If OA = 7 cm, find the area of the smaller circle.

47. The area of an equilateral $\triangle ABC$ is 17320.5 cm^2 . With each vertex of the triangle as centre, a circle is drawn with radius equal to half the length of the side of the triangle. Find the area of region within the triangle but outside the circles.

48. On a square handkerchief, nine circular designs are inscribed touching each other, each of radius 7 cm. Find the area of the remaining portion of the handkerchief.

49. $OACB$ is a quadrant of a circle with centre O and radius 3.5 cm. D is a point on OA. If OD = 2 cm, find the area of the

   (a) quadrant $OACB$,

   (b) the region between the quadrant and $\triangle OBD$.

50. A square $OABC$ is inscribed in a quadrant $OPBQ$. If $OA = 20$ cm, find the area between the square and the quadrant.

51. AB and CD are respectively arcs of two concentric circles of radii 21 cm and 7 cm and centre O. If $\angle AOB = 30^\circ$, find the area of the region $ABCD$.

52. ABC is a quadrant of a circle of radius 14 cm and a semicircle is drawn with BC as diameter. Find the area of the crescent formed.

53. Find the area common between the two quadrants of circles of radius 8 cm each if the centres of the circles lie on opposite sides of a square.

54. Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector.

55. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

56. Draw a triangle whose sides are 8cm and 11cm and the perimeter is 32 cm and find its area.

57. The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Draw the plot and find its area.

58. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be $60°$. Find the height of the tower.

59. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of $60°$ to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

60. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is $45°$. What is the height of the chimney?

61. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is $30°$. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is $45°$. Find the length of the flagstaff and the distance of the building from the point **P**.

62. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is $30°$ than when it is $60°$. Find the height of the tower.

63. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

64. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?

65. There is a slide in a park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN". If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.

66. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.

67. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.

68. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.

69. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

70. A circus artist is climbing a 20m long rope, which is tightly stretched

and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is $30°$.

71. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of $30°$ with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.

72. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of $30°$ to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of $60°$ to the ground. What should be the length of the slide in each case?

73. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is $30°$. Find the height of the tower.

74. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is $60°$. Find the length of the string, assuming that there is no slack in the string.

75. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from $30°$ to $60°$ as he walks towards the building. Find the distance he walked towards the building.

76. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are $45°$ and $60°$ respectively. Find the height of the tower.

77. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is $60°$ and from the same point the angle of elevation of the top of the pedestal is $45°$. Find the height of the pedestal.

78. The angle of elevation of the top of a building from the foot of the tower is $30°$ and the angle of elevation of the top of the tower from the foot of the building is $60°$. If the tower is 50 m high, find the height of the building.

79. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are $60°$ and $30°$, respectively. Find the height of the poles and the distances of the point from the poles.

80. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is $60°$. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is $30°$. Find the height of the tower and the width of the canal.

81. From the top of a 7 m high building, the angle of elevation of the top

of a cable tower is $60°$ and the angle of depression of its foot is $45°$. Determine the height of the tower.

82. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are $30°$ and $45°$. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

83. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is $60°$. After some time, the angle of elevation reduces to $30°$. Find the distance travelled by the balloon during the interval.

84. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of $30°$, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be $60°$. Find the time taken by the car to reach the foot of the tower from this point.

85. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

86. E and F are points on the sides PQ and PR respectively of a $\triangle PQR$. For each of the following cases, state whether $EF \parallel QR$

    (a) $PE = 3.9cm, EQ = 3cm, PF = 3.6cm$ and $FR = 2.4cm$

(b) $PE = 4cm, QE = 4.5cm, PF = 8cm$ and $RF = 9cm$

(c) $PQ = 1.28cm, PR = 2.56cm, PE = 0.18cm$ and $PF = 0.36cm$

87. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

88. $\triangle ODC \sim \triangle OBA, \angle BOC = 125°$ and $\angle CDO = 70°$. Find $\angle DOC, \angle DCO$ and $\angle OAB$.

89. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

90. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

91. Let $\triangle ABC \sim \triangle DEF$ and their areas be, respectively, 64 cm^2 and 121 cm^2. If $EF = 15.4cm$, find BC.

92. A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

93. Sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse.

   (a) 7 cm, 24 cm, 25 cm

   (b) 3 cm, 8 cm, 6 cm

   (c) 50 cm, 80 cm, 100 cm

   (d) 13 cm, 12 cm, 5 cm

94. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

95. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

96. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

97. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

98. In $\triangle ABC, AB = 6\sqrt{3}cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.

99. A park, in the shape of a quadrilateral $ABCD$, has $\angle C = 90°, AB = 9m, BC = 12m, CD = 5m$ and $AD = 8m$. How much area does it

occupy? 2. Find the area of a quadrilateral $ABCD$ in which $AB = 3cm, BC = 4cm, CD = 4cm, DA = 5cm$ and $AC = 5cm$.

100. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 26 cm, 28 cm and 30 cm, and the parallelogram stands on the base 28 cm, find the height of the parallelogram.

101. A rhombus shaped field has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48 m, how much area of grass field will each cow be getting?

102. A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The non-parallel sides are 14 m and 13 m. Find the area of the field.

103. $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8$ cm and $CF = 10$ cm, find AD.

104. Kamla has a triangular field with sides 240 m, 200 m, 360 m, where she grew wheat. In another triangular field with sides 240 m, 320 m, 400 m adjacent to the previous field, she wanted to grow potatoes and onions. She divided the field in two parts by joining the mid-point of the longest side to the opposite vertex and grew patatoes in one part and onions in the other part. Draw the figure for this problem. How much area (in hectares) has been used for wheat, potatoes and onions? (1 hectare $= 10000\ m^2$).

105. Students of a school staged a rally for cleanliness campaign. They walked through the lanes in two groups. One group walked through

the lanes AB, BC and CA; while the other through AC, CD and DA. Then they cleaned the area enclosed within their lanes. If AB = 9 m, BC = 40 m, CD = 15 m, DA = 28 m and $\angle B = 90^\circ$, which group cleaned more area and by how much? Draw the corresponding figure. Find the total area cleaned by the students (neglecting the width of the lanes).

106. Sanya has a piece of land which is in the shape of a rhombus. She wants her one daughter and one son to work on the land and produce different crops. She divided the land in two equal parts. If the perimeter of the land is 400 m and one of the diagonals is 160 m, how much area each of them will get for their crops? Draw the rhombus.

107. Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?

108. A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

109. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest

side.

110. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m. Find its length and breadth.

111. The area of a rectangular plot is 528 m^2 . The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

112. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

113. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

114. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2 ? If so, find its length and breadth.

115. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2 If so, find its length and breadth.

116. On an open ground, a motorist follows a track that turns to his left by an angle of 600 after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn.

Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

117. A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is

(a) the average speed of the taxi,

(b) the magnitude of average velocity ? Are the two equal ?

118. An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s apart is $30°$, what is the speed of the aircraft ?

Appendix A

Area of a Circle

1. The diameter of a circle is the chord that divides the circle into two equal parts. The diameter is equal to twice the radius

2. The ratio of the perimeter of a circle to its diameter is π.

3. Radian is a another unit of the angle defined by

$$\pi \text{ radians} = 180^\circ \tag{A.0.0.3.1}$$

4. In Fig. A.0.0.4.1, 6 congruent triangles are arranged in a circular fashion. Such a figure is known as a regular hexagon. In general, n number of traingles can be arranged to form a regular polygon.

5. The angle formed by each of the congruent triangles at the centre of a regular polygon of n sides is $\frac{2\pi}{n} = \frac{2\pi}{n}$ rad.

6. The triangle that forms a polygon of n sides is given in Fig. A.0.0.6.1. Show that

$$BC = 2r \sin \frac{\pi}{n} \tag{A.0.0.6.1}$$

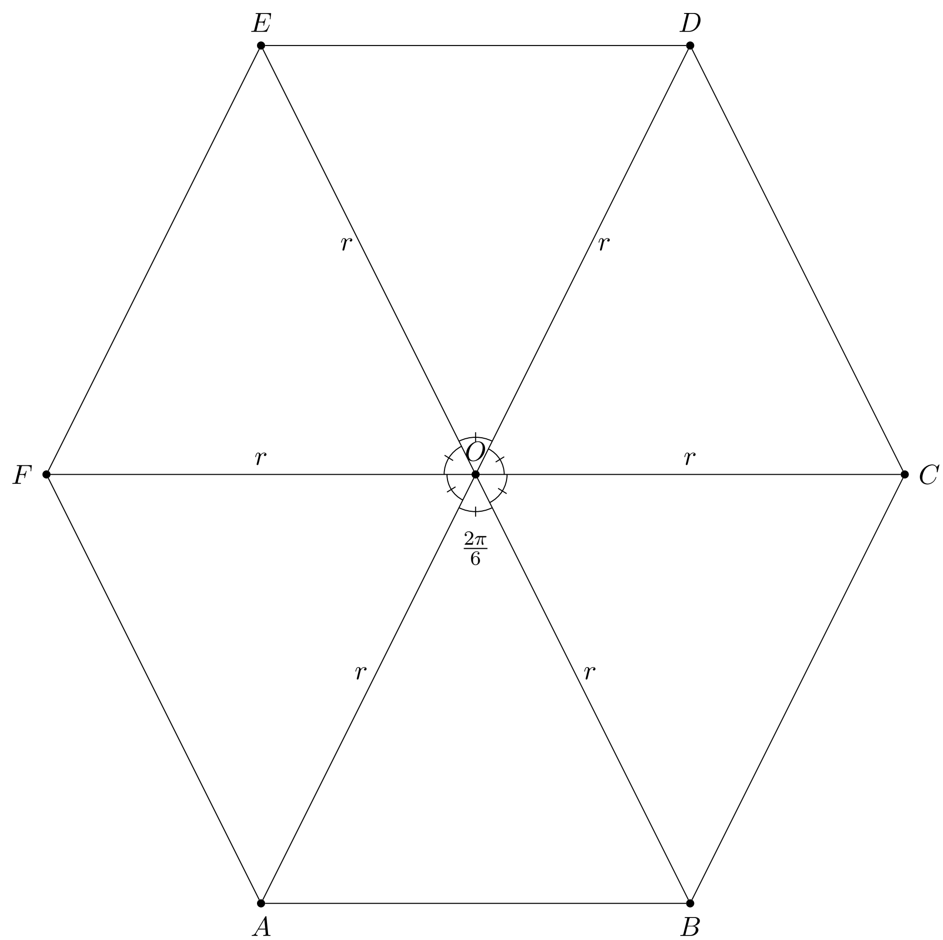

Figure A.0.0.4.1: Polygon Definition

Solution: From (2.8.1.1).

$$BC = 2r \sin \frac{A}{2} = 2r \sin \frac{\pi}{n} \tag{A.0.0.6.2}$$

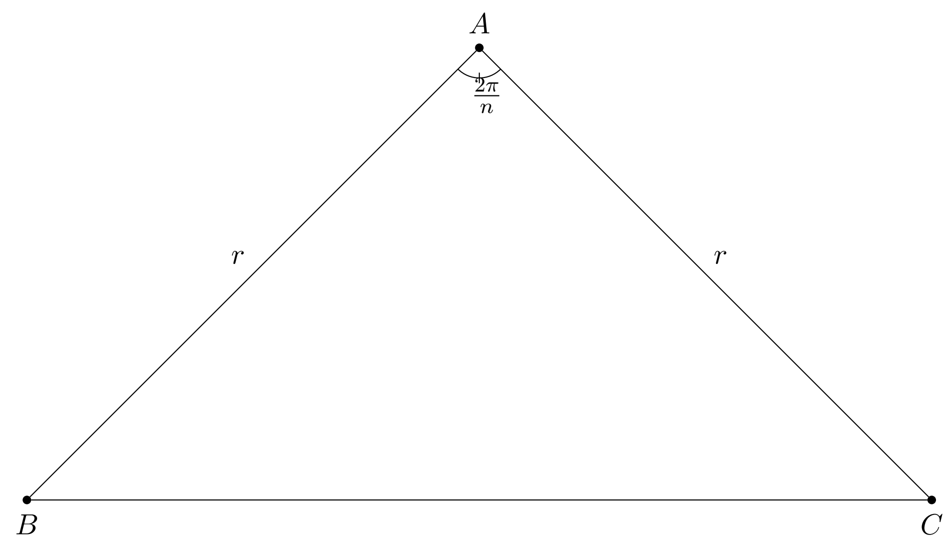

Figure A.0.0.6.1: Triangle that forms a polygon

7. Show that the perimeter of a regular polygon is given by

$$2rn\sin\frac{\pi}{n} \tag{A.0.0.7.1}$$

8. Show that the area of a regular polygon is given by

$$\frac{n}{2}r^2\sin\frac{2\pi}{n} \tag{A.0.0.8.1}$$

Solution: From Fig. A.0.0.6.1

$$\begin{aligned} ar\left(polygon\right) &= n \times ar\left(\Delta ABC\right) \\ &= \frac{n}{2}r^2\sin\frac{2\pi}{n} \end{aligned} \tag{A.0.0.8.2}$$

using (2.8.4.1)

9. Using Fig. A.0.0.9.1, show that

$$\frac{n}{2}r^2 \sin\frac{2\pi}{n} < \text{ area of circle } < nr^2 \tan\frac{\pi}{n} \tag{A.0.0.9.1}$$

The portion of the circle visible in Fig. A.0.0.9.1 is defined to be a sector of the circle.

Solution: Note that the circle is squeezed between the inner and outer regular polygons. As we can see from Fig. A.0.0.9.1, the area of the circle should be in between the areas of the inner and outer polygons. Since

$$ar\left(\Delta OAB\right) = \frac{1}{2}r^2 \sin\frac{2\pi}{n} \tag{A.0.0.9.2}$$

$$ar\left(\Delta OPQ\right) = 2 \times \frac{1}{2} \times r \tan\frac{2\pi/n}{2} \times r \tag{A.0.0.9.3}$$

$$= r^2 \tan\frac{\pi}{n}, \tag{A.0.0.9.4}$$

we obtain (A.0.0.9.1).

10. Show that

$$\cos^2\frac{\pi}{n} < \frac{\text{area of circle}}{nr^2 \tan\frac{\pi}{n}} < 1 \tag{A.0.0.10.1}$$

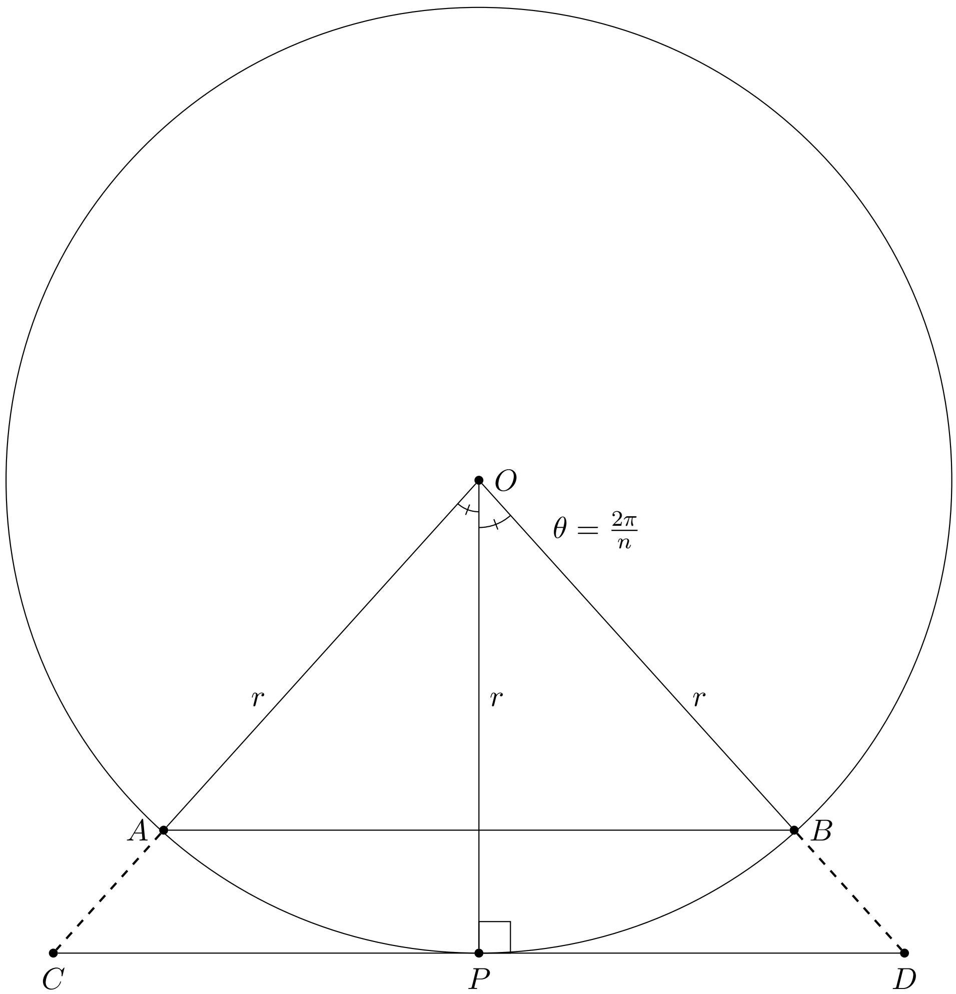

Figure A.0.0.9.1: Circle Area in between Area of Two Polygons

Solution: From (A.0.0.9.1) and (2.8.2.3),

$$\frac{n}{2}r^2 \sin\frac{2\pi}{n} < \text{ area of circle } < nr^2 \tan\frac{\pi}{n} \tag{A.0.0.10.2}$$

$$\Rightarrow nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} < \text{ area of circle } < nr^2 \tan\frac{\pi}{n} \tag{A.0.0.10.3}$$

which yields (A.0.0.10.1) upon making use of the fact that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \tag{A.0.0.10.4}$$

11. Show that

$$\cos 0^{\circ} = 1 \tag{A.0.0.11.1}$$

Solution: Follows from the fact that $\cos 0° = \sin(90° - 0°) = \sin(90°) = 1$ using (B.1.1) and (1.1.2.2).

12. Show that

$$\sin 0^{\circ} = 0 \tag{A.0.0.12.1}$$

13. Show that for large values of n

$$\cos^2\frac{\pi}{n} = 1 \tag{A.0.0.13.1}$$

Solution: As $n \to \infty$, $\frac{\pi}{n} \to 0$. From (A.0.0.11.1), this yields (A.0.0.13.1).

14. (A.0.0.13.1) is a limit and expressed as

$$\lim_{n\to\infty}\cos^2\frac{\pi}{n} = 1 \tag{A.0.0.14.1}$$

15. Show that

$$\text{area of circle } = r^2 \lim_{n\to\infty} n\tan\frac{\pi}{n} \tag{A.0.0.15.1}$$

Solution: From (A.0.0.10.1) and (A.0.0.14.1),

$$\lim_{n\to\infty} \cos^2 \frac{\pi}{n} < \lim_{n\to\infty} \frac{\text{area of circle}}{nr^2 \tan \frac{\pi}{n}} < 1 \tag{A.0.0.15.2}$$

$$1 = \lim_{n\to\infty} \frac{\text{area of circle}}{nr^2 \tan \frac{\pi}{n}} < 1 \tag{A.0.0.15.3}$$

resulting in (A.0.0.15.1).

16. Show that

$$\lim_{n\to\infty} n \sin \frac{\pi}{n} = \pi \tag{A.0.0.16.1}$$

Solution: From (A.0.0.2) and (A.0.0.7.1), the perimeter of the circle is

$$\lim_{n\to\infty} 2rn \sin \frac{\pi}{n} = 2\pi r \implies \lim_{n\to\infty} n \sin \frac{\pi}{n} \qquad = \pi \tag{A.0.0.16.2}$$

17. Show that

$$\pi = \lim_{n\to\infty} n \tan \frac{\pi}{n} \tag{A.0.0.17.1}$$

Solution: From Fig. (A.0.0.9.1), using the fact that the inner and outer polygons converge into a circle for large n,

$$\lim_{n\to\infty} nCD - nAB = 0 \tag{A.0.0.17.2}$$

$$\implies \lim_{n\to\infty} 2rn \tan \frac{\pi}{n} - 2rn \sin \frac{\pi}{n} = 0 \tag{A.0.0.17.3}$$

from which, we obtain (A.0.0.17.1) by substituting from (A.0.0.16.1).

18. Show that the area of a circle is πr^2.

    Solution: Use (A.0.0.17.1) in (A.0.0.15.1).

19. Show that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1 \tag{A.0.0.19.1}$$

20. Show that the area of a sector with angle θ in radians is $\frac{1}{2} r^2 \theta$.

Appendix B

Trigonometric Identities

B.1. Show that

$$\sin 90^\circ = 1 \tag{B.1.1}$$

Solution: In Fig. **??**, using (1.3.1.1) and (**??**)

$$ar\,(\triangle ABC) = \frac{1}{2}ac\sin B = \frac{1}{2}ac \tag{B.1.2}$$

$$\implies \sin B = \sin 90^\circ = 1 \tag{B.1.3}$$

B.2. Show that

$$\cos 90^\circ = 1 \tag{B.2.1}$$

Solution: Trivial using (1.2.2.1).

B.3. Prove the following identities

(a)

$$\sin\,(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{B.3.1}$$

(b)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{B.3.2}$$

Solution: In (1.3.4.1), let

$$\begin{aligned} \theta_1 + \theta_2 &= \alpha \\ \theta_2 &= \beta \end{aligned} \tag{B.3.3}$$

This gives (B.3.1). In (B.3.1), replace α by $90^\circ - \alpha$. This results in

$$\begin{aligned} &\sin(90^\circ - \alpha - \beta) \\ &\qquad = \sin(90^\circ - \alpha)\cos\beta - \cos(90^\circ - \alpha)\sin\beta \\ &\qquad\qquad \implies \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned} \tag{B.3.4}$$

B.4. Using (1.3.4.1) and (B.3.2), show that

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{B.4.1}$$

$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2 \tag{B.4.2}$$

Solution: From (1.3.4.1),

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos(\theta_1+\theta_2)\sin\theta_2 \tag{B.4.3}$$

Using (B.3.2) in the above,

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\sin\theta_2 \quad \text{(B.4.4)}$$

which can be expressed as

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \sin\theta_1 + \cos\theta_1\cos\theta_2\sin\theta_2 - \sin\theta_1\sin^2\theta_2 \quad \text{(B.4.5)}$$

Since

$$\sin^2\theta_2 = 1-\cos^2\theta_2, \quad \text{(B.4.6)}$$

we obtain

$$\sin(\theta_1+\theta_2)\cos\theta_2 = \cos\theta_1\cos\theta_2\sin\theta_2 + \sin\theta_1\cos^2\theta_2 \quad \text{(B.4.7)}$$

resulting in

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \quad \text{(B.4.8)}$$

after factoring out $\cos\theta_2$. Using a similar approach, (B.4.2) can also be proved.

B.5. Show that

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{B.5.1}$$

$$\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{B.5.2}$$

$$\sin\theta_1 - \sin\theta_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right) \tag{B.5.3}$$

$$\cos\theta_1 - \cos\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_2-\theta_1}{2}\right) \tag{B.5.4}$$

Solution: Let

$$\begin{aligned}\theta_1 &= \alpha+\beta \\ \theta_2 &= \alpha-\beta\end{aligned} \tag{B.5.5}$$

From (B.4.1),

$$\sin\theta_1 + \sin\theta_2 = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{B.5.6}$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{B.5.7}$$

$$+ \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{B.5.8}$$

$$= 2\sin\alpha\cos\beta \tag{B.5.9}$$

resulting in (B.5.1)

$$\because \alpha = \frac{\theta_1+\theta_2}{2} \tag{B.5.10}$$

$$\beta = \frac{\theta_1-\theta_2}{2} \tag{B.5.11}$$

from (B.5.5). Other identities may be proved similarly.