Portfolio Replication using The Binomial Tree Method to Accurately Price Options

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Introduction

In the domain of financial derivatives, the accurate pricing of options is a cornerstone for investment decision making and risk management. Conventional methodologies often rely on complex mathematical modes, which require sophisticated quantitative tools and expertise. However, as the financial market landscape evolves, a growing demand for transparent, robust strategies to price options with practical implementation strategies emerges.

Portfolio replication, which is defined as a portfolio of assets sharing the same properties as the reference asset, Aemerged as a compelling strategy to satisfy this issue, grounded in the principle of the no-arbitrage argument, and leveraging the dynamics of underlying assets and risk-free instruments. This technique presents a strong framework for option pricing that enhances accuracy and provides insights into the underlying factors that drive market dynamics.

Motivation

The motivation for exploring portfolio replication techniques as a method for option pricing stems from several key considerations natural in modern financial markets.

Primarily the adoption of portfolio replication addresses the need for a methodical approach to option pricing that can navigate the complexities of market dynamics, due to the inherent transparency in the portfolio replication model, it not only facilitates accuracy in pricing options, but enables a stronger understanding of the relationship between option values and underlying asset movements for the average trader. Furthermore, this approach allows investors to hedge against market volatility and mitigate the risks associated with mispriced options, which is a crucial capability for optimizing investments and return.

Data

The study incorporates a comprehensive dataset comprising various financial instruments obtained from diverse sources. This includes treasury yields for different maturities ranging from 30-year to 1-month treasury notes and bills, sourced from the Federal Reserve. Additionally, data on a SPY ETF call option with a strike price of 650 and an expiration date of 2025/19/12 was obtained to construct the following data.

Table 1. DATA APPENDIX

Variable	Measured	Years	Source
DGS10	Yield on 10 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS20	Yield on 20 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS30	Yield on 5 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS5	Yield on 7 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS3	Yield on 3 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS2	Yield on 2 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS1	Yield on 1 year treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS6MO	Yield on 6 month treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS3MO	Yield on 3 month treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
DGS1MO	Yield on 1 month treasury notes	2023-01-09 - 2024-04-08	Federal Reserve
Call Price	Call price of SPY	2023-01-09 - 2024-04-08	Yahoo Finance
SPYClose	Closing option price of SPY	2023-01-09 - 2024-04–08	Polygon.io
IV	Spy corresponding Implied volatility	2023-01-09 - 2024-04-08	Calculated

Methods

Using the Binomial Pricing Tree Model, our portfolio replication technique aims to replicate the payoff of the option regardless of the underlying assets movement. The replicated portfolio comprises two positions: a position in shares (S) and a position in a risk-free bond (B).

$(i)\ \ A\ Position\ in\ \Delta\ Shares,\ S$

$(ii)\ A\ position\ in\ risk\ free\ bond,\ B$

The pricing rules that would apply to a two step tree would be as follows:

$$if\ S_T = S_U\ then\ \Delta S_U +\ Be^{rT} = P_U$$

$$if\ S_T = S_d\ then\ \Delta S_d +\ Be^{rT} = P_d$$

Using these rules equations we can solve for Δ getting: $\Delta\ = P_U - P_d\ /\ S_U - S_d$.

Plugging back into one of the equations we can solve for B getting:

$$B\ = e^{-rT}(P_U - S_U(P_U - P_d\ /\ S_U - S_d)).$$

From this the Price of the option should be: $P\ \ =\ \ S_0\Delta\ +\ B$

To implement this theory, we integrated quantitative techniques using R(cite), and Excel(cite). Leveraging functions designed to simulate binomial trees for call option and stock prices, we iteratively computed option and stock prices at different nodes of the tree. These computations considered parameters such as the underlying price, the strike price, the risk free rate, implied volatility, time to expiration, and the number of steps.

The majority of the data was sourced from various data sources described above, with an expectation to the implied volatility which was calculated by solving for the sigma parameter of the Black and Scholes Model using a numerical solver.

For each observation in our dataset, we extracted relevant data points and iteratively applied our binomial tree function. This function generated matrices representing call option prices and stock prices for each scenario. From these matrices, we derived additional metrics such as Delta values to gauge option price sensitivity to stock price changes. Moreover, using delta values, we computed the bonds required to replicate option price movements, forming a comprehensive portfolio replication strategy. This synthesized approach facilitated a detailed analysis of market dynamics and helped provide insights for informed decision making and portfolio management strategies.

Results

Accuracy: defined as the magnitude of the difference between the replicated price and the actual price divided by the actual price. This metric will be used to evaluate the effectiveness of various approaches to the replication.

Our initial approach involves replicating the options value at each time step using the 2 year treasury yield as the risk free rate, a volatility rate that minimized the error in the replication that was held constant throughout the simulation, the price of the SPY ETF, and the strike and call of a SPY ETF option with a strike price of 650 and an expiration date of 2025/19/12. Note that at the start of the simulation the option was far out of the money and had an expiration over 2 years out. Our initial replication yielded an accuracy of 72.77% and revealed that across time the accuracy fluctuated on average by 21.04% between each time step. This high level of fluctuation is apparent in Figure 3 and 4 below.

Figure 1. Delta stock over time

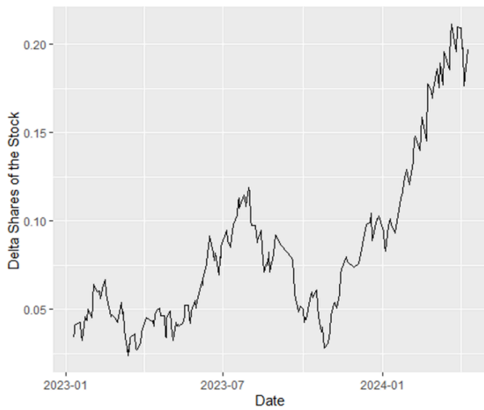


Figure 2. Units of the bond bought/sold over time

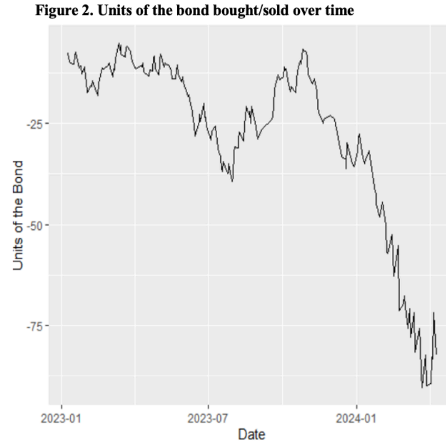


Figure 3.
Red = actual option price Black = replicated portfolio price

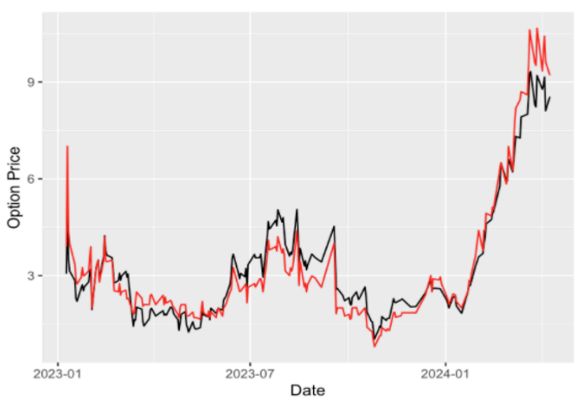
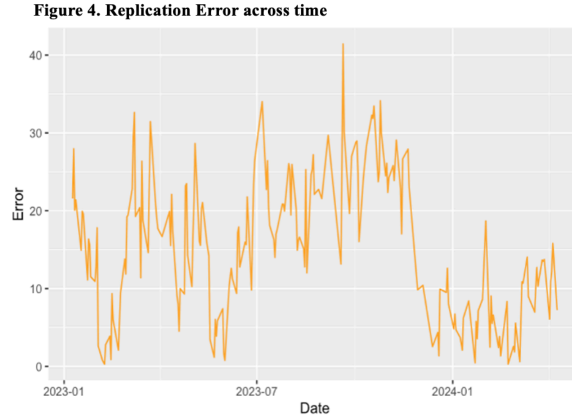


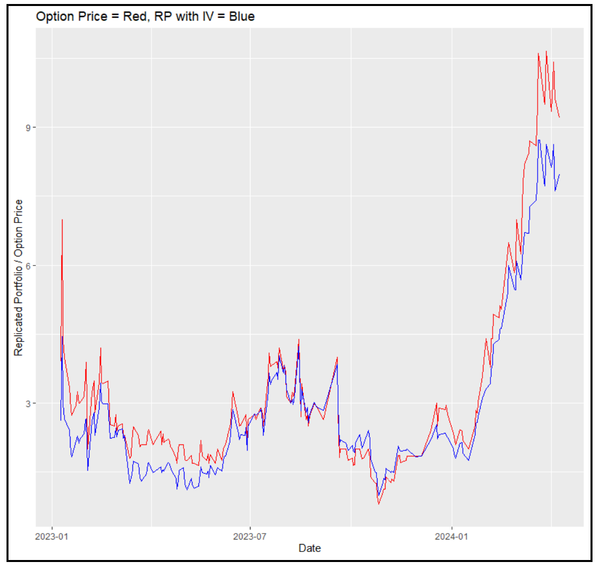
Figure 4. Replication Error across time

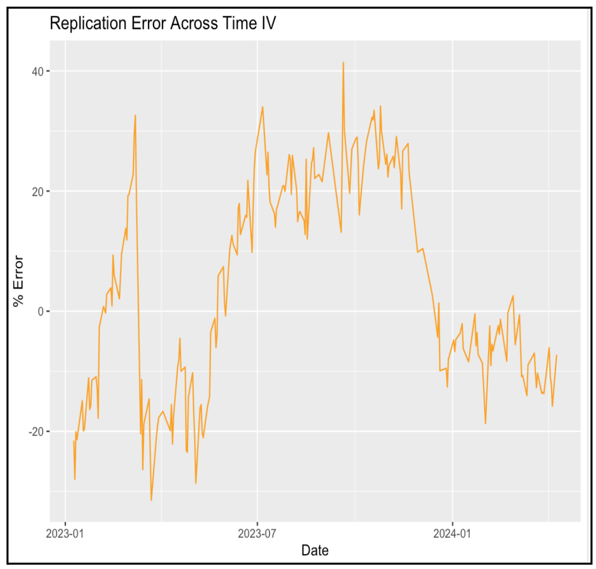


The Delta of the underlying option across time increases as the SPY ETF price increases (more in the money) and decreases as the SPY ETF price decreases. This relationship is evident in Figures 1 and 4 above. The units of shorted bonds required to replicate the option can be seen in Figure 2 and appears to increase as the option price increases as well. This is confirmed by regressing units of shorted bonds on option price which yielded a p-value far below 0.05 as seen in figure

Dynamically Updating Volatility

One approach to improve the accuracy of the replication is to update the sigma used to derive the quantity of stocks and bonds to match the value of the option on a daily basis instead of keeping it set for the duration of the replication. This approach replicates the options price throughout time with 84.28% accuracy and fluctuated on average by 8.97% across each time step. The correlation between sudden changes in volatility and changes in replication accuracy appears to be non-existent. When regressing the first differenced accuracy by the first differenced volatility the p-values for all of the coefficients were far above 0.05.



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Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.06246    0.42456  -0.147    0.883
diffIV      63.59842   79.22571   0.803    0.423
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Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -0.8755     0.6302  -1.389    0.166
OrigP        -9.9728     0.1626 -61.320   <2e-16 ***
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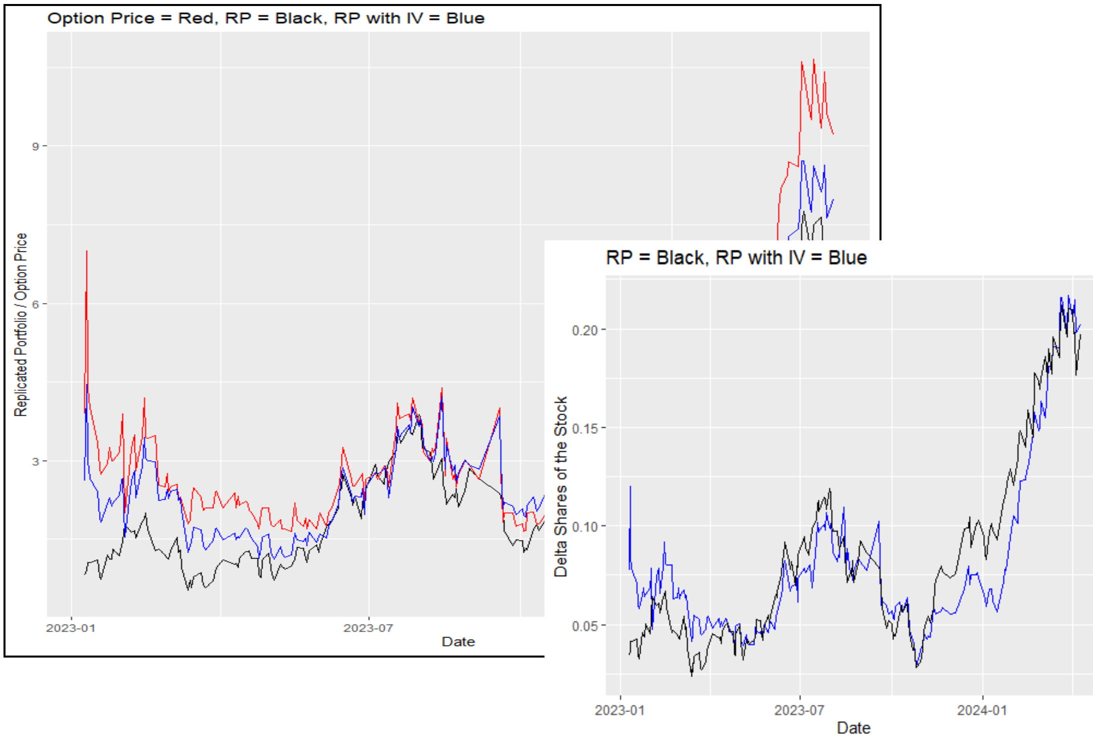

Key Findings

- Updating implied volatility on a daily basis increased replication accuracy over using a fixed initial volatility at t = 0 from ~73% to ~84% using a binomial pricing model with 1000 steps.
- There is a statistically significant association between the number of bonds required to replicate and option price.
- The accuracy of the binomial pricing model with the daily updated implied volatility appears to not be sensitive to changes in volatility from one step to another.
- Changing the risk free rate in our model appears to systematically cause the replication to be more or less valuable than the actual option.

Applications

- Entities that are legally unable to own an option can imitate their values using a combination of bonds and the underlying stock.

- The binomial pricing model with implied volatility can be used to price options with reasonable accuracy.
- The replicated portfolio can be a good Risk Managment tool used for hedging or constructing a broader portfolio with less risk dependency.

Limitations

One aspect that could limit the accuracy of the model is determining a fair risk free rate. For example, changing the risk free rate proxy from the 2-year to the 10-year treasury severely impacts the accuracy of the replication. Another assumption involved in using a binomial pricing model is that stocks either follow an up or a down move. In reality stocks value could increase or decrease by a continuous price distribution which could cause slight differences in value from the binomial pricing model. Our model also doesn’t account for illiquidity in the options market and how it could impact an options value. Another downside to our model is that it appears to take 20-80 times the amount of capital to replicate an option compared to the options price. Given more time these may be additional topics worth looking into to improve our models performance.