Forecasting day-ahead expected shortfall on the EUR/USD exchange rate

Report for the Computational Project

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Abstract

In this project, we replicate and extend the findings of a recent paper on the best models for forecasting Expected Shortfall (ES) for EUR/USD foreign exchange returns (Lyócsa et. al). We check whether incorporating implied volatility information from options markets improves these risk forecasts.

Our empirical strategy is based on a comparison of the performances of two groups of models: one that relies only on historical realized price measures, such as Exponential GARCH and quantile regression models, and another that supplements those with implied volatilities from options with daily, weekly, and monthly maturities. We adhere to the methodology of the original paper using strict ES regression-based backtesting and joint ES evaluation frameworks to validate our models. Our findings attempt to reproduce the original while contributing to knowledge regarding the added value of implied volatility in enhancing the accuracy of risk forecasting.

Keywords: Value at Risk, Expected Shortfall, EGARCH, Quantile Regression, Implied Volatility, Risk Forecasting, EUR/USD, Replication Study

LITERATURE REVIEW

General Previous Studies

Risk management and forecasting methodologies for financial markets have been a major focus of research, specifically in terms of predicting risk metrics such as Value at Risk (VaR) and Expected Shortfall (ES). VaR and ES are used to quantify potential losses in assets or portfolios and are critical tools in both regulatory frameworks and risk management practices. Most of the early research on this topic relied on historical price data and volatility measures, such as GARCH models, to forecast risk. The development of the Exponential Generalized

Autoregressive Conditional Heteroskedasticity (EGARCH) model by Nelson (1991) was a crucial advancement, allowing for asymmetric volatility to be modeled in financial time series.

The various applications of EGARCH and its supplementary features in VaR and ES forecasting in financial markets indeed prove their efficiency in volatility clustering and asymmetry of the distribution of returns. Gradually, researchers have incorporated more forward-looking information into their risk forecasting approaches, particularly the implied volatility obtained from options prices. Implied volatility reflects the market's expectations of future volatility and can provide early warnings of market stress. Many studies have examined the effectiveness of implied volatility in improving risk prediction. For instance, Bakshi et al. (2003) and Doran et al. (2013) have shown that implied volatility can be used to predict future stock returns and to enhance the accuracy of risk measures. Besides, implied volatility is usually perceived as a leading indicator of market uncertainty. In this line, Christensen and Prabhala (1998) find that implied volatility tends to increase with high market stress, thereby making it an exceptionally useful predictor of the risk metric of ES.

Specific Previous Studies

The study by Lyócsa et al. (2024) investigates the role of implied volatility (IV) in forecasting Expected Shortfall (ES) for the EUR/USD exchange rate. By integrating multiple econometric models—including EGARCH, quantile regression, and joint VaR and ES frameworks—with implied volatility data from options markets, Lyócsa et al. address a critical gap in the literature. Their findings demonstrate that incorporating IV significantly improves the accuracy of ES forecasts, specifically in periods where the market has higher uncertainty. This is achieved through the use of IV measures from options with varying maturities (daily, weekly, and monthly), providing insights into the dynamics of risk forecasting in foreign exchange markets. Lyócsa et al. Investigate the conventional reliance on realized variance by highlighting the forward-looking nature of IV and its ability to capture market expectations of future risks. Their methodology explores the robustness of these findings across different time periods and market conditions, offering a comprehensive framework for evaluating IV's role in financial risk management.

Early contributions to volatility modeling laid the groundwork for the approaches used in ES forecasting today. Bollerslev (1986) introduced the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, which enabled the modeling of time-varying volatility. This model captured volatility clustering, a common feature in financial markets, where periods of high volatility are followed by similar periods. Nelson (1991) extended this work by developing the Exponential GARCH (EGARCH) model, which accounts for asymmetric effects in volatility, such as the leverage effect—where negative returns lead to higher volatility increases than positive returns. These models became essential tools for analyzing financial time series, but they rely heavily on historical data and often fail to anticipate risks during periods of market stress.

Implied volatility addresses the limitations of historical volatility measures by incorporating market expectations of future risk. Christensen and Prabhala (1998) were among the first to establish the predictive superiority of IV over realized volatility. Their findings showed that IV provides more accurate forecasts of future price variation, particularly during volatile periods. Building on this, Busch et al. (2011) examined the role of IV across numerous asset classes, proving its ability to improve risk forecasts, especially during periods of high stress. These studies note the importance of IV as a forward-looking measure, as it reflects market sentiment and tail risks that are missed by backward-looking metrics like realized variance.

Quantile regression has become a powerful tool for ES forecasting due to its flexibility in modeling relationships at specific quantiles of a distribution. Developed by Koenker and Bassett (1978), this method enables the estimation of ES at extreme quantiles, such as 1% or 99%, which is notably valuable for extreme risk analysis. Dimitriadis and Bayer (2019) advanced this methodology by proposing a joint regression framework for VaR and ES, ensuring continuity between these risk measures. Bayer and Dimitriadis (2022) further refined this approach with strict regression-based back testing techniques, yielding a rigorous standard for validating ES forecasts.

These developments highlight the value of quantile regression in evaluating the accuracy of risk models and ensuring their reliability.

The foreign exchange market, particularly the EUR/USD exchange rate, presents unique challenges for risk assessment. As one of the most liquid and actively traded currency pairs, the EUR/USD market is subject to a wide range of influences, including macroeconomic policies, geopolitical events, and shifts in investor interests. Implied volatility derived from options on the EUR/USD exchange rate captures these dynamics, making it a robust predictor of future risks. Studies like Christensen and Prabhala (1998) and Busch et al. (2011) have demonstrated the effectiveness of IV in anticipating extreme risks in financial markets. This forward-looking measure is particularly valuable in foreign exchange markets, where volatility can shift rapidly in response to global events.

PROBLEM DESCRIPTION

Real World Problem

The global financial markets are inherently uncertain and volatile, and effective risk management is crucial for both investors and financial institutions. The ability to predict the risk of extreme losses, especially in the form of Expected Shortfall (ES), is of critical importance for decision-makers. Expected Shortfall, also known as Conditional Value at Risk (CVaR), quantifies the average loss in the worst-case scenarios beyond a specified quantile (usually between 1% and 5%) of the loss distribution. Accurate forecasting of ES allows market participants to hedge against extreme market moves, make informed decisions, and comply with regulatory requirements. However, traditional methods of estimating financial risk, such as using historical realized volatility, often fail to capture the full scope of market uncertainty, especially during periods of market distress. The EUR/USD exchange rate market is one of the most liquid and widely traded financial markets in the world, representing the value of the Euro against the U.S. Dollar. As with other financial assets, the EUR/USD exchange rate is subject to fluctuations driven by factors such as economic data releases, geopolitical events, interest rate changes, and market sentiment. These fluctuations, particularly during times of high market uncertainty, can result in significant financial losses for traders and investors. Therefore, accurately forecasting the risk of these extreme price movements is crucial for organizations' risk management in the foreign exchange market. Traditional models for forecasting risk, such as the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models and its extensions (like EGARCH), rely on historical price data to estimate future volatility. While these models are effective in capturing certain features of financial time series, such as volatility clustering, they often fail to provide accurate forecasts during periods of high market uncertainty or extreme events. Realized volatility, which is derived from past price movements, does not necessarily capture future volatility or market expectations. In the case of the EUR/USD exchange rate, volatility can change rapidly, and historical data may not reflect the market's forward-looking expectations about potential risks. Implied volatility, on the other hand, is derived from the prices of options in the market and reflects the market's expectation of future volatility. Thus, implied volatility can offer valuable insights into the risk of extreme price movements, especially during uncertain periods. For instance, when investors anticipate increased risk or volatility, the implied volatility of options tends to rise, signaling a higher likelihood of extreme price movements in the underlying asset.

Figure 1 below illustrates the concept of Expected Shortfall (ES), a critical metric for understanding extreme market risks. The figure shows the Value at Risk (VaR), which defines the threshold of extreme losses, and the Expected Shortfall (ES), calculated as the average loss beyond this threshold. The shaded region highlights the ES zone, allowing a comparison between ES estimates derived using historical volatility and those incorporating implied volatility. This visual demonstrates how forward-looking implied volatility can capture market expectations about extreme risks more effectively than traditional methods relying solely on historical data.

Figure 1: *Approximation of Expected Shortfall (ES) for EUR/USD daily returns, demonstrating the risk threshold*

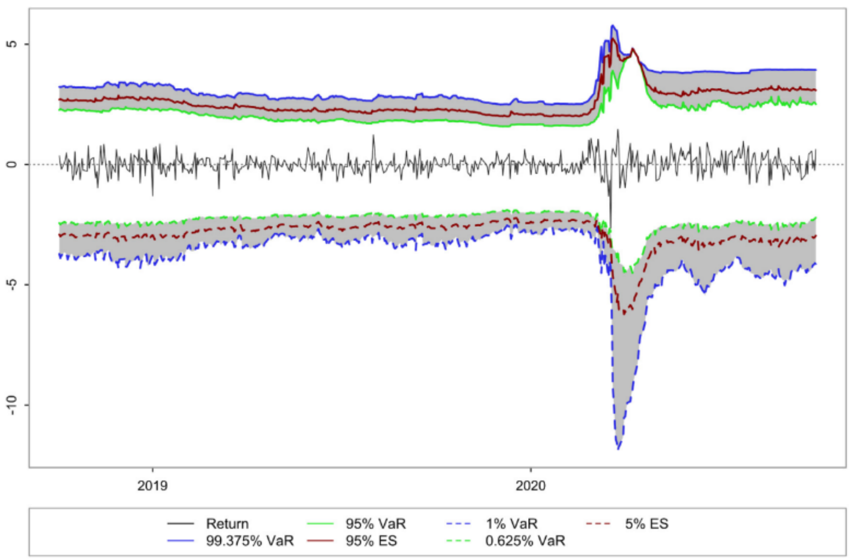

Fig. 1. Example of the approximation of the expected shortfall.

(VaR) and the average loss beyond this threshold (ES). The shaded region highlights the ES zone, with separate estimations using historical volatility and implied volatility for comparison.

Inaccurate risk forecasts can have significant financial consequences, particularly for institutions with large exposure to foreign exchange markets. For example, a financial institution might hold a large portfolio of EUR/USD positions, and inaccurate risk assessments could result in inadequate hedging strategies, potentially leading to large losses. Regulatory bodies such as the Basel Committee on Banking Supervision also require financial institutions to accurately forecast and manage risk using metrics such as VaR and ES (BIS, 2011). Failure to meet these requirements can result in fines, increased capital charges, and reputational damage.

Our Approach

The problem we aim to address in this study is how to improve the accuracy of ES forecasts for EUR/USD returns by incorporating implied volatility from options markets. Our work builds on the premise that implied volatility, which reflects forward-looking market expectations, can enhance the accuracy of risk forecasts compared to traditional volatility measures based on historical data. By using a combination of econometric models (such as EGARCH, quantile regression-based HAR, and joint VaR/ES models) and augmenting them with implied volatility data, we aim to assess whether implied volatility can better predict the extreme losses (ES) in the EUR/USD exchange rate.

MATHEMATICAL MODELING

Modeling Approach

To address the problem of accurately forecasting Expected Shortfall (ES) for EUR/USD returns, we use a combination of econometric models and incorporate implied volatility as a forward-looking measure of market expectations. Below, we describe the mathematical foundations of our approach.

Expected Shortfall and Value at Risk

To address the problem of accurately forecasting Expected Shortfall (ES) for EUR/USD returns, we use a combination of econometric models augmented with implied volatility data as a forward-looking measure of market expectations.

The Expected Shortfall (ES) at confidence level α is a critical risk measure that quantifies the expected loss in the tail of the loss distribution beyond a specified quantile. ES is calculated as:

$$(1)\quad \mathrm{ES}_t^\alpha = \mathbb{E}[L_t \mid L_t > \mathrm{VaR}_t^\alpha]$$

Where Lt represents financial loss, and VaR (Value at Risk) is the loss threshold such that the probability of exceeding it is α.

$$(2)\quad \mathrm{VaR}_t^\alpha = \inf\{l \in \mathbb{R} : P(L_t > l) \leq \alpha\}$$

To validate the accuracy of ES forecasts, we employ a regression-based back testing framework where (·)+ denotes the positive part (x+=max(x,0)) and ϵt is the regression error. This framework allows us to assess the performance of the models by evaluating whether the forecasts satisfy statistical properties consistent with ES definitions.

$$(3)\quad \log(\sigma_t^2) = \omega + \beta \log(\sigma_{t-1}^2) + \gamma \frac{|L_{t-1}|}{\sigma_{t-1}} + \lambda \frac{L_{t-1}}{\sigma_{t-1}}$$

The forecasting procedure utilized employed a rolling window approach, with a window size of 1500 days (roughly 6 years). For each model, the parameters were re-evaluated for each subsequent day. We did not filter for outliers in our models, as any extreme observations represent real fluctuations in the EUR/USD exchange rate. Including these extreme values did not result in decreased performance by the models.

EGARCH Models

To model the conditional volatility of returns, we use the Exponential Generalized Autoregressive Conditional Heteroskedasticity (EGARCH) model:

This model captures the volatility clustering and asymmetries in returns, which are characteristic of financial time series. In addition, the Heterogeneous Autoregressive (HAR) model is employed to account for volatility persistence across different time horizons. The HAR model is expressed as:

Both models are augmented with implied volatility (IV) from options markets, which serves as a forward-looking explanatory variable:

$$(4)\quad \sigma_t = \beta_0 + \beta_s \sigma_{t-1} + \beta_m \frac{1}{5} \sum_{i=1}^{5} \sigma_{t-i} + \beta_l \frac{1}{22} \sum_{i=1}^{22} \sigma_{t-i}$$

$$(5)\quad \sigma_t = f(\text{Historical Data}) + \delta IV_{t,m}$$

where δ measures the contribution of implied volatility, and mmm refers to the option maturity (e.g., daily, weekly, or monthly). By integrating implied volatility, the models account for market expectations, enhancing the accuracy of risk forecasts.

A key relationship between VaR and ES under normality assumptions is utilized for joint estimation:

$$(6)\quad \mathrm{ES}_t^\alpha = \mathrm{VaR}_t^\alpha \left(1 + \frac{1}{\alpha} \phi\left(-\Phi^{-1}(\alpha)\right)\right)$$

where ϕ is the standard normal probability density function, and Φ the inverse of the standard normal cumulative distribution function.

$$(7) \quad L_t = \beta_0 + \beta_1 \mathrm{VaR}_t^{\alpha} + \beta_2(\mathrm{VaR}_t^{\alpha} - L_t)_+ + \epsilon_t$$

Quantile Regression Models

The next set of models utilized were estimated via quantile regression. The conditional quantiles used in the regression correspond to the level of τ used in the tails of the "returns" of the EUR/USD exchange rate. The u_j'th quantile from these returns is calculated as:

$$(8) \quad \boldsymbol{z}'_{t-1}\boldsymbol{\beta}(\boldsymbol{u_j}), P(R_t \leq \boldsymbol{z}'_{t-1}\boldsymbol{\beta}(u_j)|\boldsymbol{x}_{t-1}) = u_j.$$

Where R_t is a linear function of a K set of variables, $\beta(u_j)$ is the vector of coefficients, and x_{t-1} is defined as:

$$(9) \quad \boldsymbol{x}_{t-1} = (x_{1,t-1}, \ldots, \quad x_{K,t-1})', \boldsymbol{z}_{t-1} = (1, \boldsymbol{x}'_{t-1})$$

The benefits of using this estimation method is that the $\beta(u_j)$ values can vary across the different quantiles, which allows the model to capture potential nonlinear relationships between the predicted and predictor variables. The standard asymptomatic loss function was used in these models' estimation:

$$(10) \quad \rho(u_j, v) = v(u_j - I(v < 0)) = 2^{-1}\left[|v| + (2u_j - 1)v\right]$$

The model coefficients were then estimated as:

$$(11) \quad \hat{\boldsymbol{\beta}}(u_j) = arg \min_{\boldsymbol{\beta}(u_j)} \sum_{t=1}^{T} \rho(R_t - \boldsymbol{z}'_{t-1}\boldsymbol{\beta}(u_j))$$

These models were estimated using the "quantreg" package in R Studio, which implicitly uses the methods described above. For each rolling window in the data, the model was fitted to the window data, and used to predict the next day's return for the FX rate, for each level of τ. There were 10 QR models used in this application, with five being realized variance (RV) models, and the other five being implied volatility (IV) models. Beginning with the RV models, the first one estimated was a simple QR-RV model that used only the weekly and monthly realized variance as predictors (the original paper employed daily realized variance as well, but our data did not have adequate observations to use this predictor):

$$(12) \quad VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{2,u_j} RV^{W}_{t-1} + \beta_{3,u_j} RV^{M}_{t-1}$$

Given that extreme returns will occur when volatility is high, the realized variance should be well suited to predict these returns. The next model was the QR-RV-CJ model, which is defined as:

$$(13)\quad VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j}CC_{t-1} + \beta_{2,u_j}JC_{t-1} + \beta_{3,u_j}RV^{W}_{t-1} + \beta_{4,u_j}RV^{M}_{t-1}$$

Where CC_{t-1} is the continuous component and JC_{t-1} is the jump component. The continuous component represents the price variation not driven by discontinuous price movements, which should be indicative of future return variation given its continual nature. This component was defined as:

$$(14)\quad CC_t = \frac{\pi}{6-4\sqrt{3}+\pi}\left(\frac{N}{N-2}\right)\sum_{j=2}^{N-1} med\{|R_{t,j-1}|, |R_{t,j}|, |R_{t,j+1}|\}^2$$

The CC_t value was estimated for each rolling window, which resulted in much longer computation times. The values (R_t realized variances) were summed over the rolling window, so the length of N was equal to 1500, The jump component equation was defined as RV_t - CC_t if this difference was positive, and 0 otherwise.

The next model considered was the QR-RV-CQ model, which utilized the continuous component along with a volatility uncertainty measurement, CQ, defined as:

$$(15)\quad CQ_t = CC_t \times \sqrt{MedRQ_t}$$

With the $MedRQ_t$ (median realized quarticity) component being calculated as:

$$(16)\quad MedRQ_t = \frac{3\pi N}{9\pi + 72 - 52\sqrt{3}}\left(\frac{N}{N-2}\right) \times \sum_{j=2}^{N-1}\left[med|r_{t,j-1}|, |r_{t,j}|, |r_{t,j+1}|\right]^4$$

With the summation again being done over the length of the window. Since uncertainty regarding the volatility estimate will be higher during periods of higher volatility, the volatility persistence should be lower when the uncertainty surrounding $MedRQ_t$ is higher. Thus, extreme returns should be well predicted, assuming that the value for $MedRQ_t$ should be lower during times of high uncertainty.

The next model is the QR-RV-SV model, which utilizes negative and positive semi variances. Splitting up the realized variance into these two components was deemed useful, since *Patton and Sheppard* (2015) found that negative price variation is more persistent and better predicts the one-day-ahead variation in prices on US equity markets.These values are defined as:

$$(17)\quad NV_t = \sum_{j=2}^{N} r_{t,j}^2 \times I\left(r_{t,j} < 0\right) \quad PV_t = \sum_{j=2}^{N} r_{t,j}^2 \times I\left(r_{t,j} \geq 0\right)$$

The function I(.) was a simple indicator that took on a value of 1 if the next-day return was negative and 0 otherwise for the NV_t. For PV_t it was the opposite, with I having a value of 1 if the return was positive, and 0 otherwise. Using both these components as predictors allowed us to exploit both sides of the return distribution, as the negative variance component should be more predictive of negative losses, and the positive variance component should better estimate positive losses. The regression equation was thus set up as:

(18)

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j}PV_{t-1} + \beta_{2,u_j}NV_{t-1} + \beta_{3,u_j}RV_{t-1}^{W} + \beta_{4,u_j}RV_{t-1}^{M}$$

The final model is the QR-RV-SJ model, which uses the difference between the positive and negative semi variances in an SJ_t term, defined as:

$$SJ_t = \left(\sum_{j=2}^{N} r_{t,j}^2 \times I\left(r_{t,j} \geq 0\right) - \sum_{j=2}^{N} r_{t,j}^2 \times I\left(r_{t,j} < 0\right)\right) \tag{19}$$

The SJ_t term is positive when the positive jumps have a stronger presence, and is negative when negative price variation dominates. Including this term along with the continuous term again creates a good balance between volatility uncertainty and price variation. The model equation can be written as:

$$\begin{aligned} VaR_{R_t}(u_j) = {} & \beta_{0,u_j} + \beta_{1,u_j}CC_{t-1} + \beta_{2,u_j}SJ_{t-1} \\ & + \beta_{3,u_j}RV_{t-1}^{W} + \beta_{4,u_j}RV_{t-1}^{M} \end{aligned} \tag{20}$$

The remaining 5 models make up the group of IV models. The only difference between these models and their RV versions, is that these models have two added predictors of the implied volatilities: the at-the-money implied volatility of EUR/USD options with one week and one month maturities. Given that these models use the same components as their RV counterparts, they will simply be listed below:

the QR-IV model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{2,u_j}IV_{t-1}^{W} + \beta_{3,u_j}IV_{t-1}^{M} \tag{21}$$

the QR-IV-RV model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j} RV_{t-1}^{D} + \beta_{2,u_j} RV_{t-1}^{W} + \beta_{3,u_j} RV_{t-1}^{M} + \beta_{5,u_j} IV_{t-1}^{W} + \beta_{6,u_j} IV_{t-1}^{M} \tag{22}$$

the QR-IV-CJ model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j} CC_{t-1} + \beta_{2,u_j} JC_{t-1} + \beta_{3,u_j} RV_{t-1}^{W} + \beta_{4,u_j} RV_{t-1}^{M} + \beta_{6,u_j} IV_{t-1}^{W} + \beta_{7,u_j} IV_{t-1}^{M} \tag{23}$$

the QR-IV-CQ model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j} CC_{t-1} + \beta_{2,u_j} CQ_{t-1} + \beta_{3,u_j} RV_{t-1}^{W} + \beta_{4,u_j} RV_{t-1}^{M} + \beta_{6,u_j} IV_{t-1}^{W} + \beta_{7,u_j} IV_{t-1}^{M} \tag{24}$$

the QR-IV-SV model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j} PV_{t-1} + \beta_{2,u_j} NV_{t-1} + \beta_{3,u_j} RV_{t-1}^{W} + \beta_{4,u_j} RV_{t-1}^{M} + \beta_{6,u_j} IV_{t-1}^{W} + \beta_{7,u_j} IV_{t-1}^{M} \tag{25}$$

and the QR-IV-SJ model:

$$VaR_{R_t}(u_j) = \beta_{0,u_j} + \beta_{1,u_j} CC_{t-1} + \beta_{2,u_j} SJ_{t-1} + \beta_{3,u_j} RV_{t-1}^{W} + \beta_{4,u_j} RV_{t-1}^{M} + \beta_{6,u_j} IV_{t-1}^{W} + \beta_{7,u_j} IV_{t-1}^{M} \tag{26}$$

Since these models predict the VaR at the quantile specified, to get the expected shortfall value for these estimates, the mean of the returns within the rolling window that were at least as extreme as the VaR value (greater than or equal to) was calculated.

Dimitriadis and Bayer Models

To further enhance our forecast of Expected Shortfall (ES) for EUR/USD returns, we integrate the Dimitriadis and Bayer (DB) models, which focus on incorporating volatility measures (both implied volatility and realized variance) as key drivers of market risk. These models are particularly useful for capturing the dynamic nature of volatility and its relationship with extreme risk events, such as those seen in the tails of the return distribution.

The DB-IV model is specified as a linear regression where ES is a function of implied volatility (IV), which represents the market's expectations of future volatility. This relationship can be expressed as:

$$\mathrm{ES}_{\tau} = \alpha + \beta_{\mathrm{IV}} \cdot \mathrm{IV}_t + \epsilon_t \tag{27}$$

The implied volatility is extracted from EUR/USD options data, and this model reflects how the market's forward-looking expectations of volatility influence the forecast of ES.

The DB-RV model, on the other hand, uses realized variance (RV) as an explanatory variable. Realized variance is a measure of actual volatility observed from past returns. The model is expressed as:

$$(28) \quad \mathrm{ES}_\tau = \alpha + \beta_{\mathrm{RV}} \cdot \mathrm{RV}_t + \epsilon_t$$

Realized variance is computed over a defined window, typically 1 week or 1 month, based on historical EUR/USD return data.

To model the conditional volatility of returns, we use the Exponential Generalized Autoregressive Conditional Heteroskedasticity (EGARCH) model:

$$(29) \quad \log(\sigma_t^2) = \omega + \sum_{i=1}^{p} \beta_i \log(\sigma_{t-i}^2) + \sum_{i=1}^{q} \alpha_i \frac{\epsilon_{t-i}}{\sigma_{t-i}} + \gamma \left(\frac{\epsilon_{t-i}}{\sigma_{t-i}} \right)^2$$

This model captures the volatility clustering and asymmetries in returns, which are characteristic of financial time series. In addition, the Heterogeneous Autoregressive (HAR) model is employed to account for volatility persistence across different time horizons. The HAR model is expressed as:

$$(30) \quad \sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-5}^2 + \beta_3 \sigma_{t-22}^2$$

Both models are augmented with implied volatility (IV) from options markets, which serves as a forward-looking explanatory variable:

$$(31) \quad \sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \beta_2 IV_t + \epsilon_t$$

Where δ measures the contribution of implied volatility. By integrating implied volatility, the models account for market expectations, enhancing the accuracy of risk forecasts.

A key relationship between VaR and ES under normality assumptions is utilized for joint estimation:

$$(32) \quad \mathrm{VaR}_\alpha = -\sigma_t \Phi^{-1}(\alpha)$$

where ϕ is the standard normal probability density function, and Φ\ the inverse of the standard normal cumulative distribution function.

Finally, to validate the accuracy of ES forecasts, we employ a regression-based backtesting framework:

$$(33) \quad \mathrm{ES}_\alpha = \sum_{t=1}^{T} (L_t - \mathrm{VaR}_\alpha)^+$$

Where (x)+ and ϵt is the regression error. This framework allows us to assess the performance of the models by evaluating whether the forecasts satisfy statistical properties consistent with ES definitions.

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SOLUTION METHOD

General Solution Methodology

To evaluate the effectiveness of various econometric models for forecasting Expected Shortfall (ES) on the EUR/USD exchange rate, a robust methodology combining a rolling window framework, multiple model specifications, and strict backtesting procedures was employed. The key advantage of this approach lies in its ability to dynamically adapt to evolving market conditions and rigorously validate model performance.

Specific Solution Methodology

A 1500-day rolling window approach was used to ensure that model parameters reflect the most recent market dynamics. For each day, models were trained on the preceding 1500 days of data and used to forecast ES at two quantiles: ES 0.01 (1% tail) and ES 0.99 (99% tail). This rolling framework allows for consistent evaluation across different volatility regimes and market environments, ensuring robustness in model predictions.

The statistical programming language **R** was selected for its advanced libraries and tools tailored for econometric modeling, time series analysis, and financial risk assessment. Additionally, the availability of R packages developed by Dimitriadis and Bayer in their prior works (2019, 2022) facilitated seamless implementation and backtesting of ES models. Key packages used in this study include:

- **rugarch**: For dynamic latent volatility models like eGARCH.
- **quantreg**: For quantile regression modeling.
- **esreg**: For Expected Shortfall regression models.
- **esback**: For strict regression-based ES backtesting.

These tools provided a robust framework for constructing, evaluating, and validating the models under study.

The eGARCH models were specified with two underlying distributions to account for the skewness and heavy tails in financial returns:

- Student's skewed t (SSK) distribution.
- Johnson's (JSU) distribution.

QR models were directly used to estimate tail quantiles and ES with all explained inputs. DB models ensured joint estimation of VaR and ES under consistent regression frameworks. In total, 16 models were constructed, combining variations of the eGARCH, QR, and DB frameworks with external regressors like RV and IV.

Figure 2 illustrates the implementation of the models by plotting the 1% ES (red line) and 99% ES (blue line) for the EUR/USD exchange rate. This visualization highlights the models' ability to capture extreme market movements.

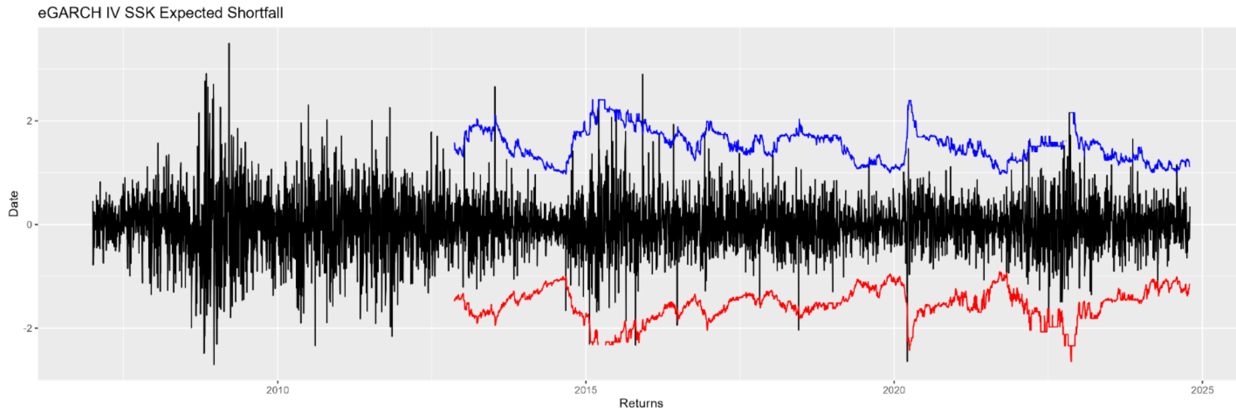

Figure 2

The econometric models used in this study are built on foundations and capture key aspects of financial return distributions. The eGARCH model is designed to model time-varying volatility and account for asymmetric responses to positive and negative shocks, often referred to as the leverage effect. This model estimates conditional volatility as a function of past volatility, lagged residuals, and additional explanatory variables, such as realized variance or implied volatility. These extensions allow the eGARCH framework to better capture the complex dynamics of financial returns, particularly during periods of heightened market stress.

Quantile regression models are used to estimate conditional quantiles of the return distribution, making them especially useful for tail risk analysis. Unlike traditional regression models that focus on mean relationships, quantile regression directly estimates specific points in the distribution, such as the 1st or 99th percentiles. This enables precise modeling of extreme events and tail behavior, which is essential for calculating measures like Expected Shortfall (ES). The explanatory variables in the quantile regression framework include realized variance and implied volatility, allowing for a comparison of backward-looking and forward-looking predictors.

Expected Shortfall, as a risk metric, measures the average loss beyond a specified Value at Risk (VaR) threshold. It provides a more comprehensive view of tail risk by considering the magnitude of losses in the extreme tail of the distribution. For this study, ES forecasts are generated for both the lower (1%) and upper (99%) quantiles. The accuracy of these forecasts depends on the ability of the models to align their predictions with observed extreme losses.

Finally, strict regression-based backtesting is employed to evaluate the alignment between forecasted and observed ES values. This method involves regressing observed losses on forecasted ES values to assess whether the model is well specified. If the forecasts are unbiased and proportional to the observed losses, the model is considered accurate. This backtesting framework allows for a rigorous comparison of model performance, particularly in evaluating the contribution of external variables such as realized variance and implied volatility.

RESULTS AND CONCLUSION

In figure 2 we plot the EUR/USD exchange rate returns, implied volatility, and realized variance for the days that we calculated the expected shortfalls for. We can see large spikes in the realized variance and the implied volatility during certain times with large fluctuations in the returns during certain periods. Most notably in 2015 through the beginning of 2017 where multiple factors including the European Central Bank Quantitative Easing Program began and the U.S. Presidential elections were also taking place. We can also see the large spike in the beginning of 2020 as a result of Covid-19.

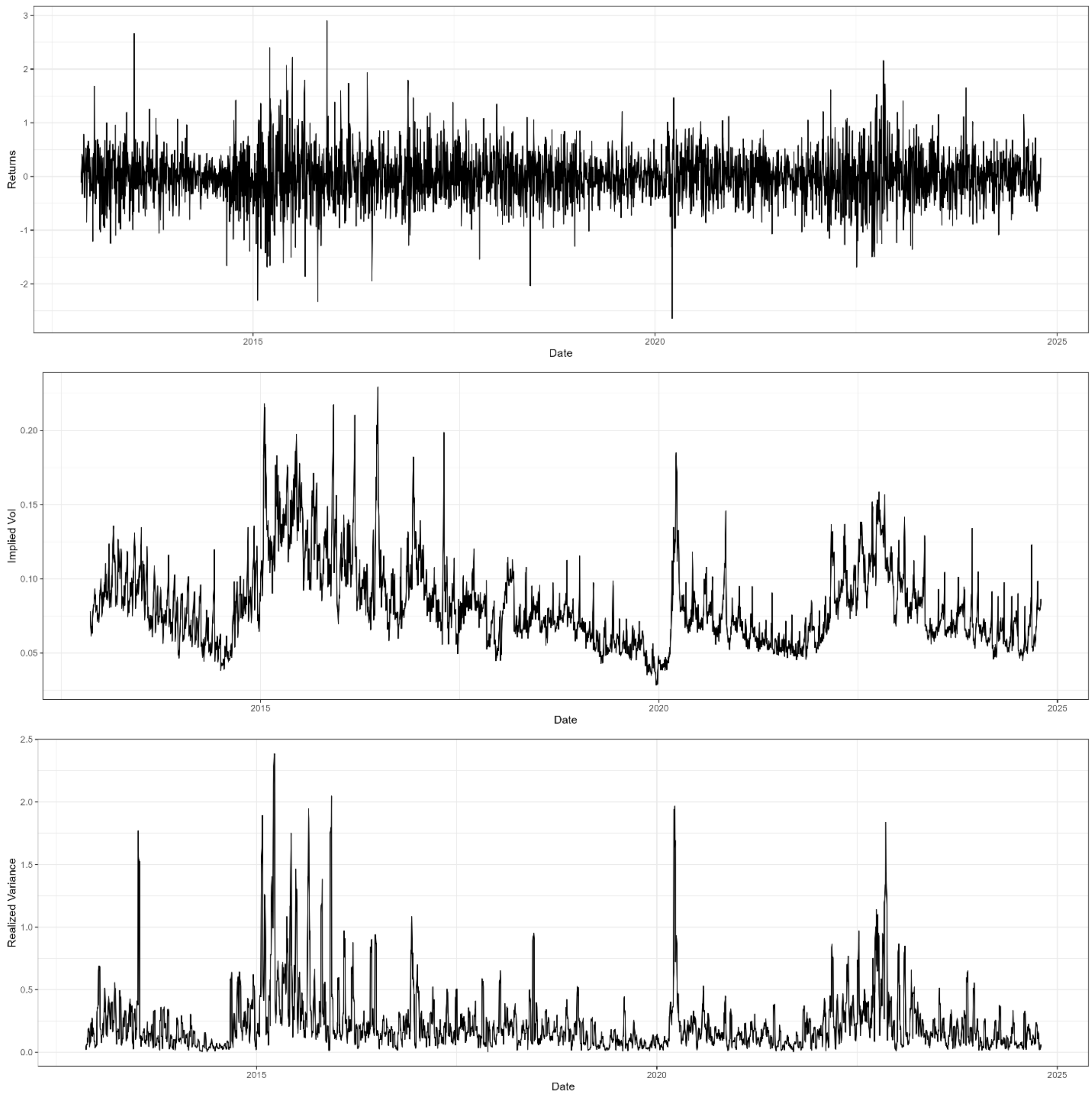

Figure 2: EUR/USD returns, realized and implied volatilities

The correlation between our realized variance and implied volatility is 0.535, which is less than the original authors had at 0.79 correlation between the two, which could give us different results than their findings. But it does suggest an overlap of information content between the two market uncertainty measures.

For model comparison, each expected shortfall was tested in a strict expected shortfall regression-based backtest hypothesis (Bayer and Dimitriadis 2022), where under the null, the intercept and the slope coefficients are 0 and 1 respectively. A non-rejection of the null-hypothesis implies a well-specified forecast of the expected shortfall. We will be using this metric to determine if our replication of the paper and our new observed IV measures work since the higher the p-value the more well-suited it is to forecasting the ES. The p-values from this test are shown in table 1.

Table 1
Strict regression-based expected shortfall backtesting.

Model	ES 0.01	ES 0.99
Panel A: Dynamic Latent Volatility Expected Shortfall Forecast Models		
eGARCH-SSK	0.554	0.382
eGARCH-JSU	0.562	0.351
Panel B: Dynamic Latent Volatility Expected Shortfall Forecast Models with Realized Volatility		
eGARCH-RV-SSK	0.522	0.493
eGARCH-RV-JSU	0.524	0.521
Panel C: Dynamic Latent Volatility Expected Shortfall Forecast Models with Implied Volatility		
eGARCH-IV-SSK	0.495	0.538
eGARCH-IV-JSU	0.522	0.55
Panel D: Dimitriadis and Bayer's (2019) Expected Shortfall Forecast Models		
DB-IV	0.411	0.521
DB-RV	0.304	0.263
Panel E: Quantile Regression Approximate Expected Shortfall Forecasting Models with Implied Volatility		
QR-IV	0.856	0.903
QR-IV-SV	0.229	0.173
QR-IV-CJ	0.016	0.262
QR-IV-CQ	0.007	0.117
Panel F: Quantile Regression Approximate Expected Shortfall Forecasting Models with Realized Volatility		
QR-RV	0.517	0.649
QR-RV-SV	0.252	0.033
QR-RV-CJ	0.668	0.444
QR-RV-CQ	0.056	0.001

Dynamic Latent Volatility Models

In Panel A, the eGARCH models without realized or implied volatility components (eGARCH-SSK and eGARCH-JSU) exhibited moderate p-values for both ES 0.01 and ES 0.99. These results suggest that while these baseline models provide reasonable forecasts, they do not fully capture the dynamics of extreme tail risks. Incorporating additional information appears essential for improving ES forecast accuracy.

Realized Variance Models

Adding realized variance (Panel B) marginally improved the performance of the eGARCH models. For example, the eGARCH-RV-JSU model achieved a p-value of 0.521 for ES 0.99, a notable improvement over its baseline counterpart. This finding aligns with the broader literature, which emphasizes the value of realized measures in enhancing volatility and risk predictions.

Implied Volatility Models

Panel C demonstrates that implied volatility provides a clear enhancement in forecast performance. The eGARCH-IV-JSU model achieved a p-value of 0.55 for ES 0.99, surpassing both the baseline and realized variance-augmented models. These results reinforce the notion that forward-looking information embedded in implied volatility captures additional market dynamics, especially during periods of heightened uncertainty.

Quantile Regression Models

Quantile regression models that incorporated implied volatility (Panel E) significantly outperformed all other model classes. For instance, the QR-IV model achieved the highest p-values across both extreme quantiles, with 0.856 for ES 0.01 and 0.903 for ES 0.99. These results emphasize the superior ability of QR-IV models to capture extreme market movements compared to models relying solely on realized variance.

Performance Gaps Between Realized and Implied Volatility

The correlation between realized variance and implied volatility in our dataset (0.535) was notably lower than the 0.79 correlation reported in the original study. This reduced overlap may explain why realized variance models performed less robustly than implied volatility models in our replication. Despite this, the inclusion of implied volatility consistently improved forecasting accuracy across all models.

Implications for Risk Forecasting

Overall, our findings validate the use of implied volatility as a critical component for ES forecasting. The QR-IV and eGARCH-IV models emerged as the most reliable approaches, offering well-specified forecasts even under extreme market conditions. These results underscore the importance of forward-looking measures for effective risk management, particularly in volatile markets like EUR/USD.

Conclusion

This study sought to replicate and extend the findings of Lyócsa et al. (2024) by investigating the role of implied volatility (IV) in forecasting Expected Shortfall (ES) for the EUR/USD exchange rate. Utilizing a range of econometric models—including EGARCH and quantile regression (QR) frameworks—we compared traditional risk models based on historical realized variance (RV) to models augmented with forward-looking implied volatility measures.

Our findings confirm that implied volatility provides significant improvements in forecasting accuracy, particularly for extreme quantiles. Specifically, QR models incorporating IV, such as the QR-IV framework, achieved the highest p-values, indicating their robustness in predicting ES. This result underscores the forward-looking nature of IV, which captures market expectations of risk better than backward-looking realized measures, particularly during periods of elevated market uncertainty.

The correlation between realized variance and implied volatility in our dataset was 0.535, lower than the 0.79 correlation reported by the original authors. Despite this discrepancy, both measures exhibited a meaningful overlap in capturing market uncertainty. However, the lower correlation may account for some divergence in model performance between our findings and the original study.

Dynamic latent volatility models, such as EGARCH, also benefited from the inclusion of implied volatility, particularly for higher quantile ES forecasts. For example, the eGARCH-IV-JSU model achieved a p-value of 0.55 for ES 0.99, highlighting the complementary nature of IV when combined with traditional volatility measures. Models without additional explanatory variables, such as eGARCH-SSK and eGARCH-JSU, performed moderately but left room for improvement.

Overall, our results corroborate the value of incorporating implied volatility into risk forecasting models. These findings have important implications for risk managers and policymakers, particularly in foreign exchange markets, where effective prediction of extreme price movements is crucial. The robustness of IV-enhanced models suggests their suitability for improving financial risk assessments, especially during volatile market periods.

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