ECE 595: Homework 1

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(Spring 2019)

Exercise 2

(a) For a guassian distribution:

$$
\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} (x+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= 0 + \mu \frac{1}{\sqrt{2\pi\sigma^2}} \times \sigma\sqrt{2\pi} \\
&= \mu
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
Var[x] &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx \\
& let\ y = \frac{x}{\sigma},\ then\ dy = \frac{1}{\sigma} dx \\
&= \frac{\sigma^3}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy \\
&= \sigma^2
\end{aligned} \tag{2}
$$

(b) Data generated and plotted as follows.

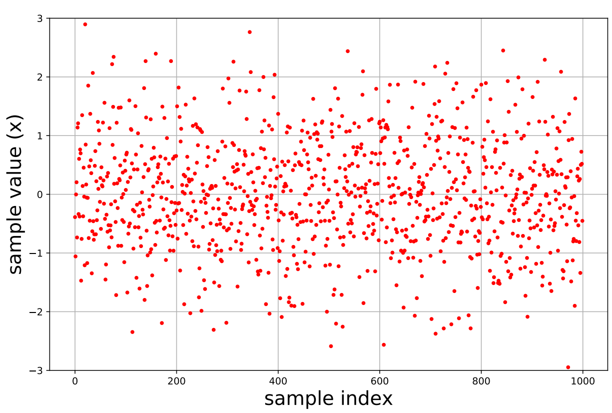
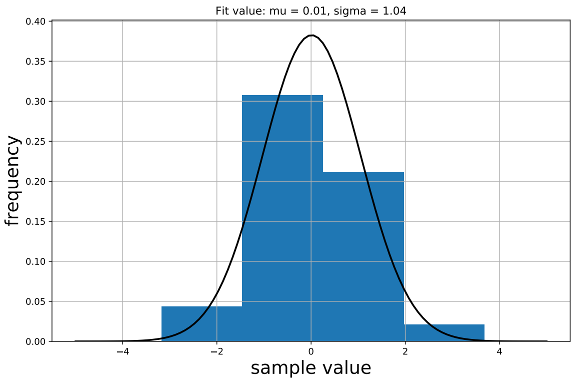


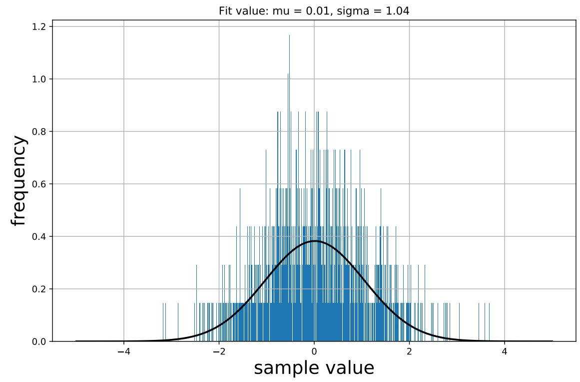
Figure 1: Gussian random data.

(c)

(i)..(iv) plots shown below



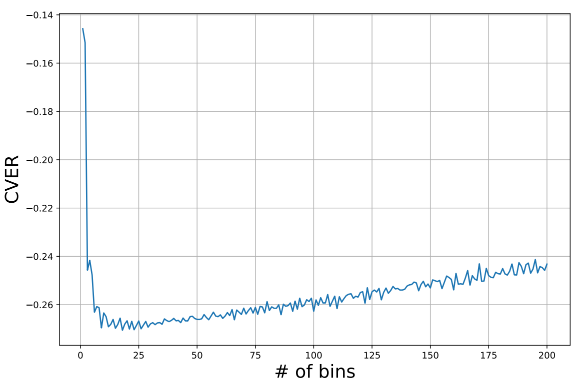
(a) 4 bins



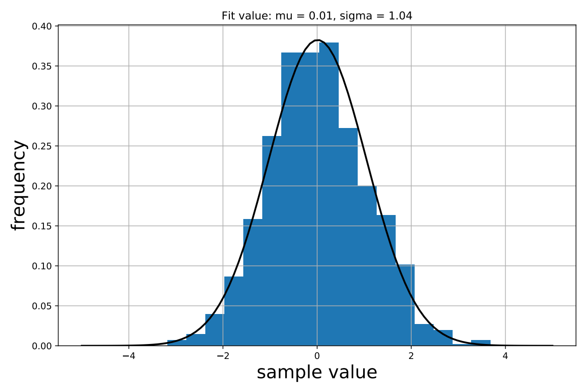
(b) 1000 bins

(v) TODO: fill this

(d) compare to part (c), the histogram fits a lot better with the PDF plots shown below



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 3

(a)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi^2|\mathbf{\Sigma}|}} exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$$

(i) plug in

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{and } \mathbf{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we get

$$\begin{aligned} f_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \frac{1}{\sqrt{2\pi^2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}} exp\left\{-\frac{1}{2}\begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}\right\} \\ &= \frac{1}{\pi\sqrt{6}} exp\left\{-\frac{1}{3}\left((x_1-2)^2-(x_1-2)(x_2-6)+(x_2-6)^2\right)\right\} \end{aligned} \quad (3)$$

(ii) plot shown below

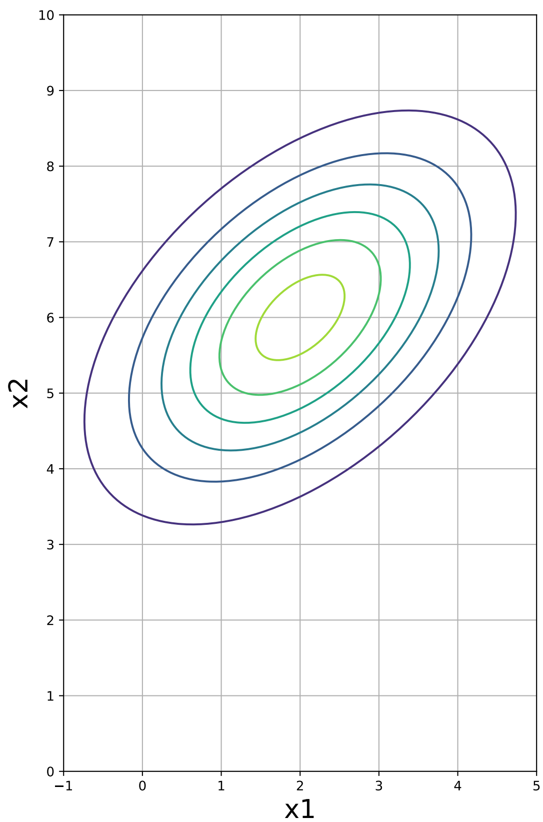


Figure 2: Gussian random data

(b)

(i)

To prove: $\boldsymbol{\mu}_{\mathbf{Y}} = b$

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}} &= \mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{AX} + \mathbf{b}] = \mathbb{E}[\mathbf{AX}] + \mathbf{b} \\ &= \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b} \\ & since\ \mathbf{X} \in \mathcal{N}(\mathbf{0}, \mathbf{I}),\ \mathbb{E}[\mathbf{X}] = \mathbf{0} \\ &= \mathbf{A0} + \mathbf{b} \\ &= \mathbf{b} \end{aligned} \tag{4}$$

To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}} &= \mathbb{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})^T] \\ &= \mathbb{E}[\mathbf{AXX}^T\mathbf{A}^T] \\ &= \mathbf{A}\mathbb{E}[\mathbf{XX}^T]\mathbf{A}^T \\ &= \mathbf{A}(\mathbf{I} + \mathbf{0})\mathbf{A}^T \\ &= \mathbf{AA}^T \end{aligned} \tag{5}$$

(ii) To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}}$ is symmetric and positive semi-definite

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}ij} &= \Sigma_{x=1}^{n} \mathbf{A}_{ix}\mathbf{A}_{xj}^T \\ &= \Sigma_{x=1}^{n} \mathbf{A}_{jx}\mathbf{A}_{xi}^T \\ &= \boldsymbol{\Sigma}_{\mathbf{Y}ji} \\ & thus\ symmetric \\ \mathbf{x}^T\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{x} &= \mathbf{x}^T\mathbf{AA}^T\mathbf{x} \\ let\ \mathbf{y} = \mathbf{x}^T\mathbf{A} \\ &= \mathbf{yy}^T \\ &= \|\mathbf{y}\|^2 \geq 0 \\ & thus\ positive\ semi-definite \end{aligned} \tag{6}$$

(iii)

$$Null(\mathbf{A}) = \mathbf{0}$$

(iv) By inspection,

$$b = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

By Cholesky decomposition,

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$$

$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

(c)

(i) Data points drawn from 2D standard guassian distribution are shown below.

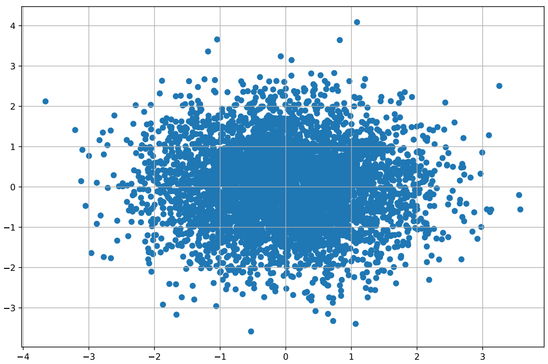
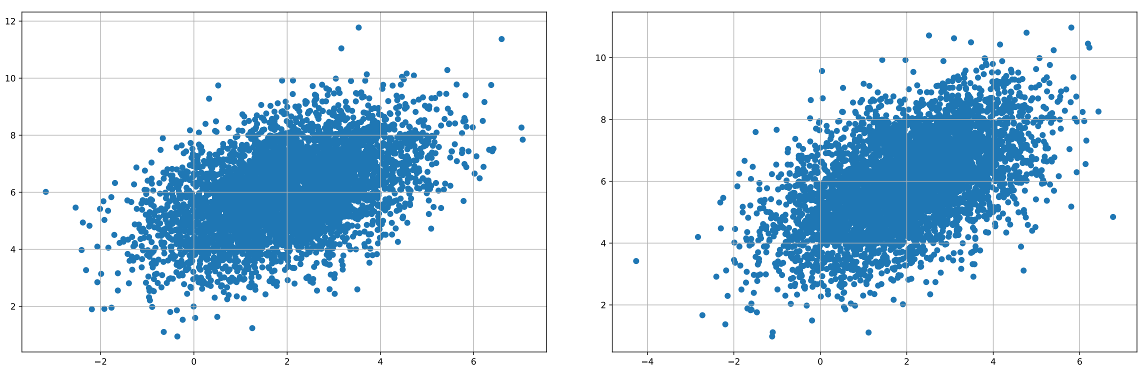


Figure 3: 2D standard Gussian random data

(ii) After applying the affine transformation, the data plot shown below.



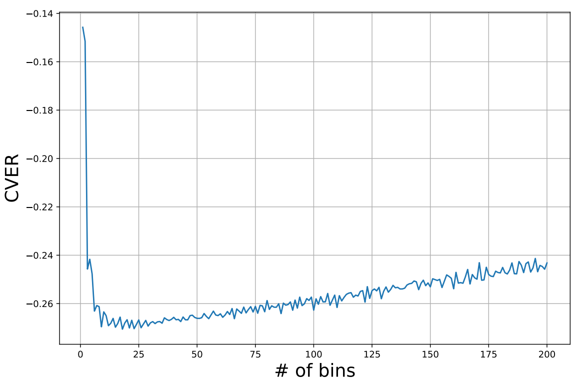
(a) After applying affine transformation

(b) After applying transform get from eigen

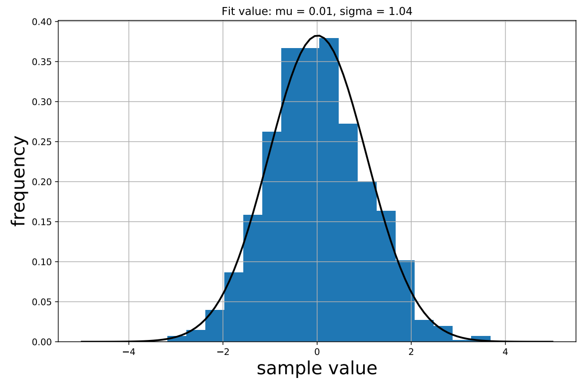
(iii) From my favorite python and numpy, for the data above:

$$\boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2.003 \\ 6.004 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} 2.017 & 1.050 \\ 1.050 & 2.074 \end{bmatrix}$$



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 4

(a)

Proof

$$\begin{aligned}
\left|\mathbf{x}^T\mathbf{A}\mathbf{y}\right| &= \left|\sum_i\sum_j a_{ij}x_iy_j\right| \\
&\leq \sum_i\sum_j |a_{ij}|\,|x_i|\,|y_j| \\
&= \sum_i\sum_j (|a_{ij}|^{\frac{1}{2}})^2\,|x_i|\,|y_j| \\
&\mathit{by\ Cauchy\ Schwarz} \\
&\leq \sqrt{\sum_i\sum_j |a_{ij}|\,|x_i|^2}\sqrt{\sum_i\sum_j |a_{ij}|\,|y_j|^2} \\
&= \sqrt{\sum_i |x_i|^2\sum_j |a_{ij}|}\sqrt{\sum_j |y_j|^2\sum_i |a_{ij}|} \\
&\leq \sqrt{\sum_i |x_i|^2\,C}\sqrt{\sum_j |y_j|^2\,R} \\
&= \sqrt{RC}||x||_2||y||_2
\end{aligned} \tag{7}$$

(b)

(i) For a invertible $n \times n$ matrix $\mathbf{A}$

$$\begin{aligned}
\mathbf{A} &= \mathbf{PDP} \\
\mathbf{A}^{-1} &= (\mathbf{PDP}^{-1})^{-1} \\
&= \mathbf{P}^{-1}\mathbf{D}^{-1}\mathbf{P}
\end{aligned} \tag{8}$$

Since A is positive definite, $\lambda_i > 0\ for\ 1 \leq i \leq n$, $\mathbf{D}$ is invertible.

Thus $\mathbf{A}$ is also invertible.

(ii)

$$f\left(\begin{bmatrix} x1 \\ x2 \end{bmatrix}\right) = x_1^2 - x_2^2$$

the Hessian of this function is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

which is invertible but not positive definite

(iii) If a matrix is invertible while positive semi-definite, the matrix is positive definite.
By definition of positive semi-definite, $\mathbf{x}\mathbf{A}\mathbf{x}^T \geq 0$, and for a eigenvalue of $\mathbf{A}$, λ_i, $\lambda_i \mathbf{u}_i = \mathbf{A}\mathbf{u}_i$
So,

$$\begin{aligned} \mathbf{u}_i^T \mathbf{A}\mathbf{u}_i &= \lambda_i \\ \lambda_i &\geq 0 \end{aligned} \tag{9}$$

If the matrix is invertible, it cannot have $\lambda_i = 0$.
So, $\lambda_i > 0$, thus $\mathbf{x}\mathbf{A}\mathbf{x}^T > 0$.
The matrix $\mathbf{A}$ is positive definite.

(c) To prove: $(\exists \mathbf{A}^\dagger)\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ &simplify \\ \mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\boldsymbol{\Lambda} &= \boldsymbol{\Lambda}\boldsymbol{\Lambda}^- \\ \mathbf{A}^\dagger &= \mathbf{U}\boldsymbol{\Lambda}^-\mathbf{U}^T \end{aligned} \tag{10}$$

Since $\mathbf{A}$ is symmetric, it is guaranteed that can be decompose, thus $\mathbf{A}^\dagger$ exist.

Code