ECE 595: Homework 1

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Exercise 2

(a) For a guassian distribution:

$$\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} (x+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= 0 + \mu \frac{1}{\sqrt{2\pi\sigma^2}} \times \sigma\sqrt{2\pi} \\
&= \mu
\end{aligned} \tag{1}$$

$$\begin{aligned}
Var[x] &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx \\
&let\ y = \frac{x}{\sigma},\ then\ dy = \frac{1}{\sigma} dx \\
&= \frac{\sigma^3}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy \\
&= \sigma^2
\end{aligned} \tag{2}$$

(b) Data generated and plotted as follows.

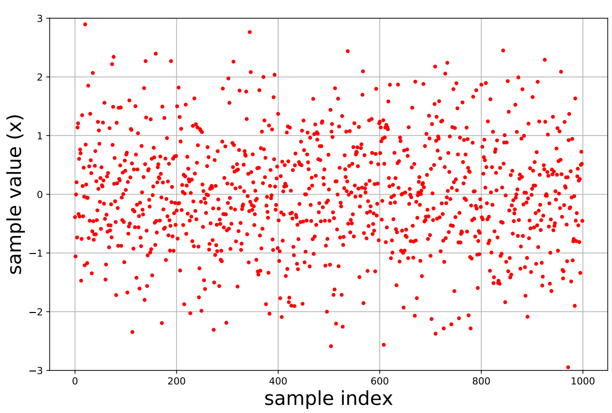
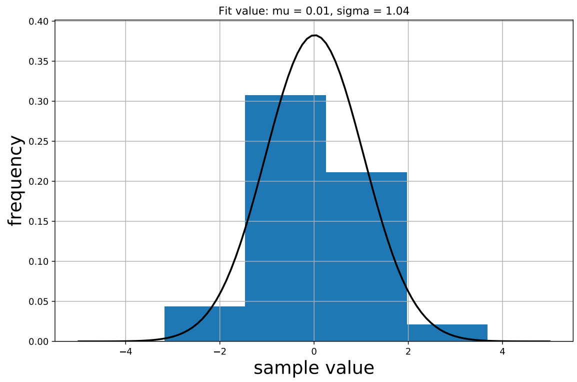


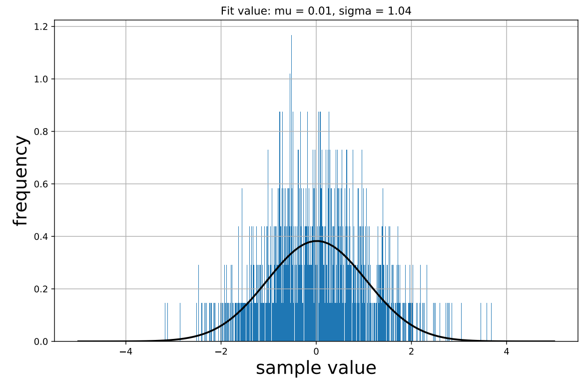
Figure 1: Gussian random data.

(c)

(i)..(iv) plots shown below



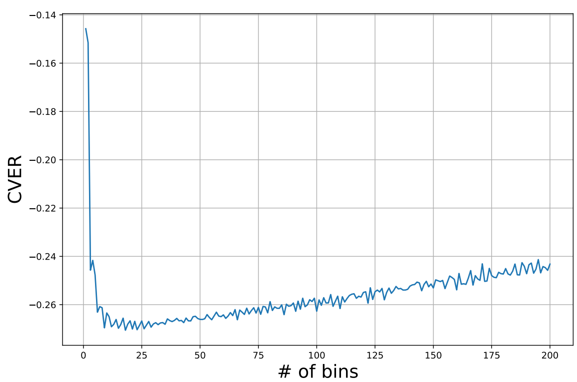
(a) 4 bins



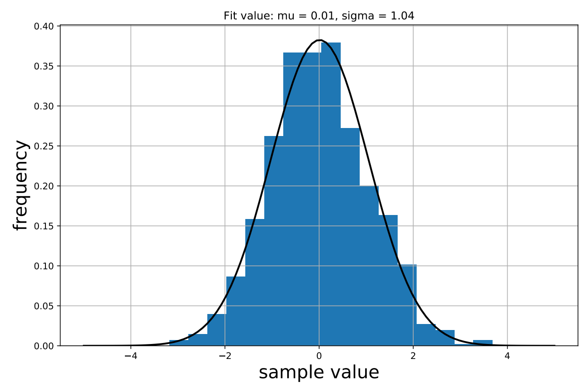
(b) 1000 bins

(v) TODO: fill this

(d) compare to part (c), the histogram fits a lot better with the PDF plots shown below



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 3

(a)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi^2|\mathbf{\Sigma}|}} exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$$

(i) plug in

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{ and } \mathbf{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we get

$$\begin{aligned} f_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \frac{1}{\sqrt{2\pi^2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}} exp\left\{-\frac{1}{2}\begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}\right\} \\ &= \frac{1}{\pi\sqrt{6}} exp\left\{-\frac{1}{3}\left((x_1-2)^2-(x_1-2)(x_2-6)+(x_2-6)^2\right)\right\} \end{aligned} \tag{3}$$

(ii) plot shown below

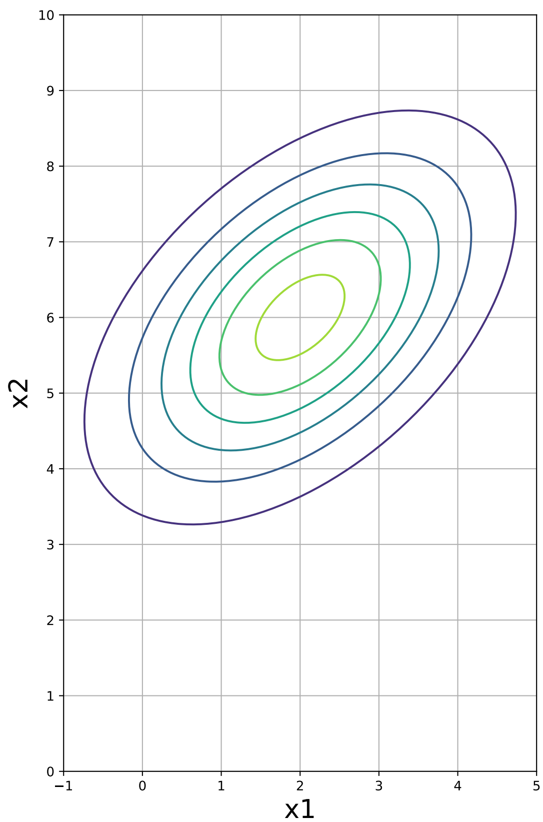


Figure 2: Gussian random data

(b)

(i)

To prove: $\boldsymbol{\mu}_{\mathbf{Y}} = b$

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}} &= \mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{AX} + \mathbf{b}] = \mathbb{E}[\mathbf{AX}] + \mathbf{b} \\ &= \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b} \\ & since\ \mathbf{X} \in \mathcal{N}(\mathbf{0}, \mathbf{I}),\ \mathbb{E}[\mathbf{X}] = \mathbf{0} \\ &= \mathbf{A0} + \mathbf{b} \\ &= \mathbf{b} \end{aligned} \tag{4}$$

To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}} &= \mathbb{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})^T] \\ &= \mathbb{E}[\mathbf{AXX}^T\mathbf{A}^T] \\ &= \mathbf{A}\mathbb{E}[\mathbf{XX}^T]\mathbf{A}^T \\ &= \mathbf{A}(\mathbf{I} + \mathbf{0})\mathbf{A}^T \\ &= \mathbf{AA}^T \end{aligned} \tag{5}$$

(ii) To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}}$ is symmetric and positive semi-definite

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}ij} &= \Sigma_{x=1}^{n} \mathbf{A}_{ix}\mathbf{A}_{xj}^T \\ &= \Sigma_{x=1}^{n} \mathbf{A}_{jx}\mathbf{A}_{xi}^T \\ &= \boldsymbol{\Sigma}_{\mathbf{Y}ji} \\ & thus\ symmetric \\ \mathbf{x}^T\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{x} &= \mathbf{x}^T\mathbf{AA}^T\mathbf{x} \\ let\ \mathbf{y} = \mathbf{x}^T\mathbf{A} & \\ &= \mathbf{yy}^T \\ &= \|\mathbf{y}\|^2 \geq 0 \\ & thus\ positive\ semi-definite \end{aligned} \tag{6}$$

(iii)

$$Null(\mathbf{A}) = \mathbf{0}$$

(iv) By inspection,

$$b = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

By Cholesky decomposition,

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$$

$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

(c)

(i) Data points drawn from 2D standard guassian distribution are shown below.

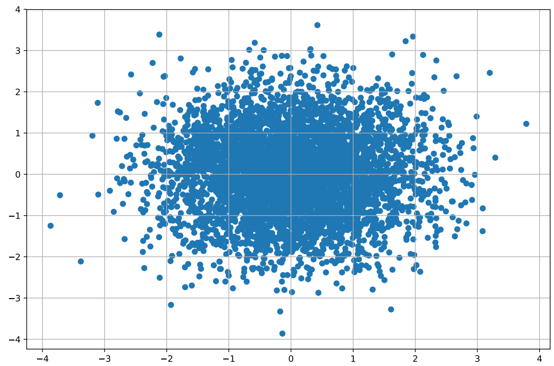


Figure 3: 2D standard Gussian random data

(ii) After applying the affine transformation, the data plot shown below.

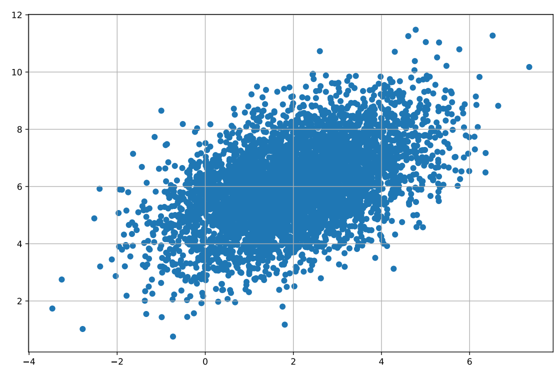


Figure 4: After applying affine transformation

(iii) From my favorite python and numpy, for the data above:

$$\boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2.003 \\ 6.004 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} 2.017 & 1.050 \\ 1.050 & 2.074 \end{bmatrix}$$

Exercise 4

(a)

Proof

$$\begin{aligned}
\left|\mathbf{x}^T\mathbf{A}\mathbf{y}\right| &= \left|\sum_i\sum_j a_{ij}x_iy_j\right| \\
&\leq \sum_i\sum_j |a_{ij}|\,|x_i|\,|y_j| \\
&= \sum_i\sum_j (|a_{ij}|^{\frac{1}{2}})^2\,|x_i|\,|y_j| \\
&by\ Cauchy\ Schwarz \\
&\leq \sqrt{\sum_i\sum_j |a_{ij}|\,|x_i|^2}\sqrt{\sum_i\sum_j |a_{ij}|\,|y_j|^2} \\
&= \sqrt{\sum_i |x_i|^2\sum_j |a_{ij}|}\sqrt{\sum_j |y_j|^2\sum_i |a_{ij}|} \\
&\leq \sqrt{\sum_i |x_i|^2\,C}\sqrt{\sum_j |y_j|^2\,R} \\
&= \sqrt{RC}||x||_2||y||_2
\end{aligned} \tag{7}$$

(b)

(i) For a invertible $n \times n$ matrix $\mathbf{A}$

$$\begin{aligned}
\mathbf{A} &= \mathbf{PDP} \\
\mathbf{A}^{-1} &= (\mathbf{PDP}^{-1})^{-1} \\
&= \mathbf{P}^{-1}\mathbf{D}^{-1}\mathbf{P}
\end{aligned} \tag{8}$$

Since A is positive definite, $\lambda_i > 0\ for\ 1 \leq i \leq n$, $\mathbf{D}$ is invertible.

Thus $\mathbf{A}$ is also invertible.

(ii)

$$f\left(\begin{bmatrix} x1 \\ x2 \end{bmatrix}\right) = x_1^2 - x_2^2$$

the Hessian of this function is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

which is invertible but not positive definite

(iii) If a matrix is invertible while positive semi-definite, the matrix is positive definite.
By definition of positive semi-definite, $\mathbf{x}\mathbf{A}\mathbf{x}^T \geq 0$, and for a eigenvalue of $\mathbf{A}$, λ_i, $\lambda_i \mathbf{u}_i = \mathbf{A}\mathbf{u}_i$
So,

$$\begin{aligned} \mathbf{u}_i^T \mathbf{A}\mathbf{u}_i &= \lambda_i \\ \lambda_i &\geq 0 \end{aligned} \tag{9}$$

If the matrix is invertible, it cannot have $\lambda_i = 0$.
So, $\lambda_i > 0$, thus $\mathbf{x}\mathbf{A}\mathbf{x}^T > 0$.
The matrix $\mathbf{A}$ is positive definite.

(c) To prove: $(\exists \mathbf{A}^\dagger)\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T \\ & simplify \\ \mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\boldsymbol{\Lambda} &= \boldsymbol{\Lambda}\boldsymbol{\Lambda}^- \\ \mathbf{A}^\dagger &= \mathbf{U}\boldsymbol{\Lambda}^-\mathbf{U}^T \end{aligned} \tag{10}$$

Since $\mathbf{A}$ is symmetric, it is guaranteed that can be decompose, thus $\mathbf{A}^\dagger$ exist.