ECE 595: Homework 1

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(Spring 2019)

Exercise 2

(a) For a guassian distribution:

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} (x+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 0 + \mu \frac{1}{\sqrt{2\pi\sigma^2}} \times \sigma\sqrt{2\pi} \\ &= \mu \end{aligned} \tag{1}$$

$$\begin{aligned} Var[x] &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx \\ &\quad let\ y = \frac{x}{\sigma},\ then\ dy = \frac{1}{\sigma} dx \\ &= \frac{\sigma^3}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy \\ &= \sigma^2 \end{aligned} \tag{2}$$

(b) Data generated and plotted as follows.

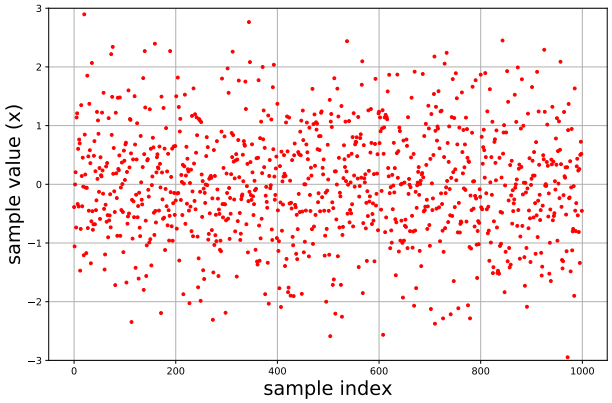
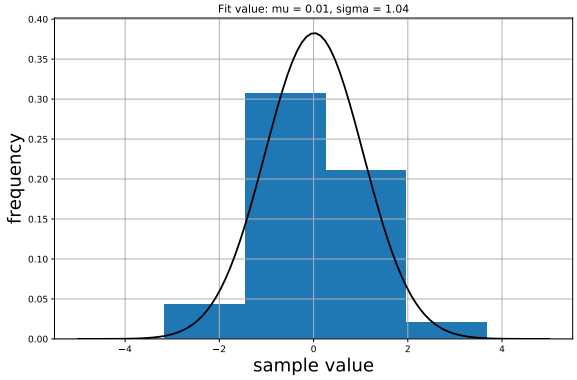


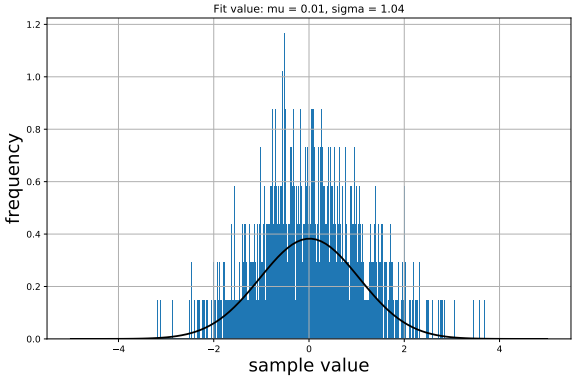
Figure 1: Gussian random data.

(c)

(i)..(iv) plots shown below



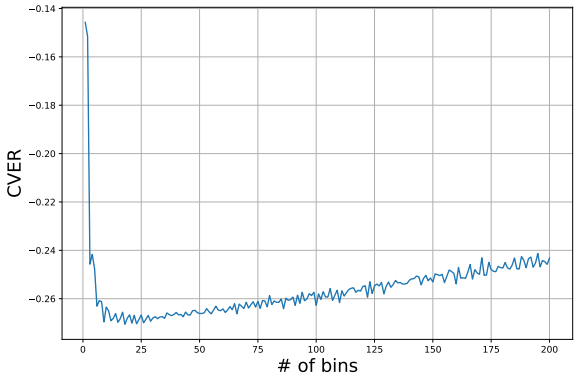
(a) 4 bins



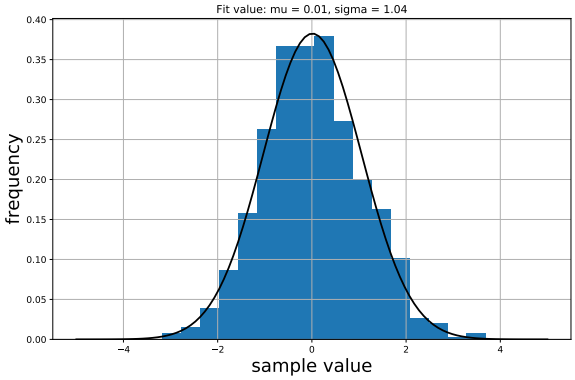
(b) 1000 bins

(v) Though they kind of show the distribution of the data, the resolution of first one with 4 bins is too low, while the second one with 1000 bins is too sparse and cannot show the overall picture.

(d) compare to part (c), the histogram fits a lot better with the PDF plots shown below



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 3

(a)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi^2|\mathbf{\Sigma}|}} exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$$

(i) plug in

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{ and } \mathbf{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we get

$$\begin{aligned} f_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \frac{1}{\sqrt{2\pi^2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}} exp\left\{-\frac{1}{2}\begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}\right\} \\ &= \frac{1}{\pi\sqrt{6}} exp\left\{-\frac{1}{3}\left((x_1-2)^2-(x_1-2)(x_2-6)+(x_2-6)^2\right)\right\} \end{aligned} \tag{3}$$

(ii) plot shown below

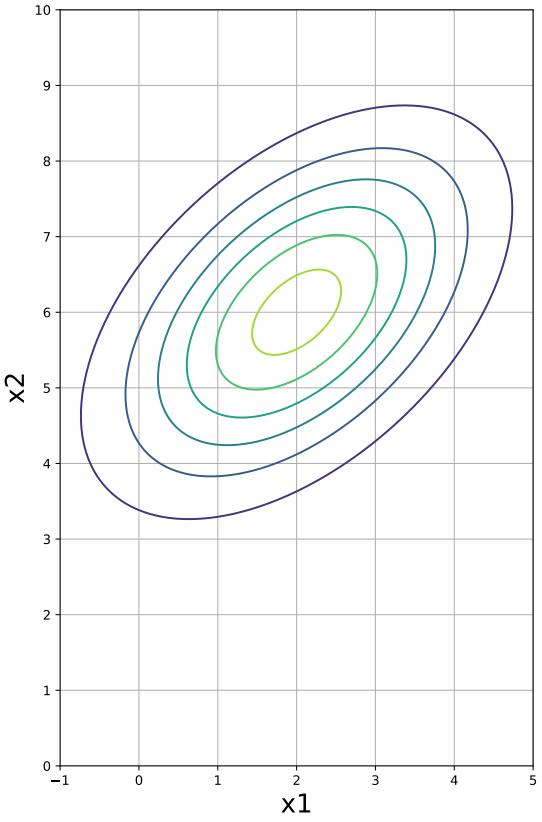


Figure 2: Gussian random data

(b)

(i)

To prove: $\boldsymbol{\mu}_{\mathbf{Y}} = b$

$$\begin{aligned}\boldsymbol{\mu}_{\mathbf{Y}} &= \mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{AX} + \mathbf{b}] = \mathbb{E}[\mathbf{AX}] + \mathbf{b} \\ &= \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b} \\ &since\ \mathbf{X} \in \mathcal{N}(\mathbf{0}, \mathbf{I}),\ \mathbb{E}[\mathbf{X}] = \mathbf{0} \\ &= \mathbf{A0} + \mathbf{b} \\ &= \mathbf{b}\end{aligned} \tag{4}$$

To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{Y}} &= \mathbb{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})^T] \\ &= \mathbb{E}[\mathbf{AXX}^T\mathbf{A}^T] \\ &= \mathbf{A}\mathbb{E}[\mathbf{XX}^T]\mathbf{A}^T \\ &= \mathbf{A}(\mathbf{I} + \mathbf{0})\mathbf{A}^T \\ &= \mathbf{AA}^T\end{aligned} \tag{5}$$

(ii) To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}}$ is symmetric and positive semi-definite

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{Y}ij} &= \Sigma_{x=1}^{n} \mathbf{A}_{ix}\mathbf{A}_{xj}^T \\ &= \Sigma_{x=1}^{n} \mathbf{A}_{jx}\mathbf{A}_{xi}^T \\ &= \boldsymbol{\Sigma}_{\mathbf{Y}ji} \\ &thus\ symmetric \\ \mathbf{x}^T\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{x} &= \mathbf{x}^T\mathbf{AA}^T\mathbf{x} \\ let\ \mathbf{y} = \mathbf{x}^T\mathbf{A} & \\ &= \mathbf{yy}^T \\ &= \|\mathbf{y}\|^2 \geq 0 \\ &thus\ positive\ semi - definite\end{aligned} \tag{6}$$

(iii)

$$Null(\mathbf{A}) = \mathbf{0}$$

(iv) By inspection,

$$b = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

By Cholesky decomposition,

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$$

$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

(c)

(i) Data points drawn from 2D standard guassian distribution are shown below.

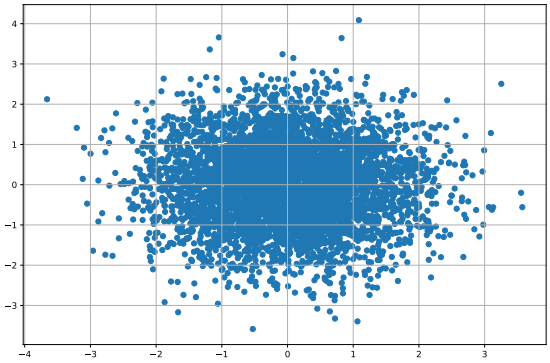
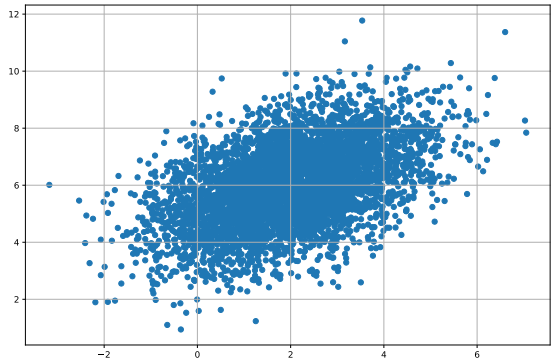
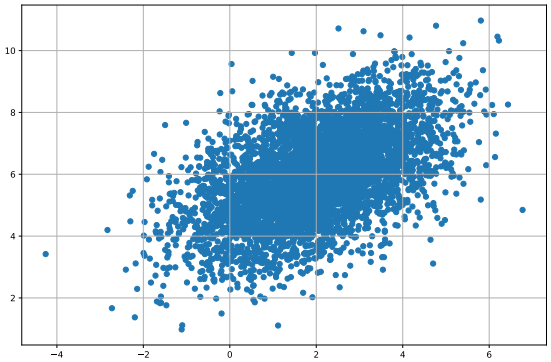


Figure 3: 2D standard Gussian random data

(ii) After applying the affine transformation, the data plot shown below.



(a) After applying affine transformation

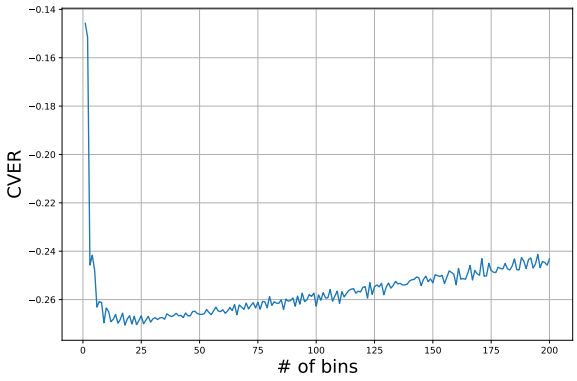


(b) After applying transform get from eigen

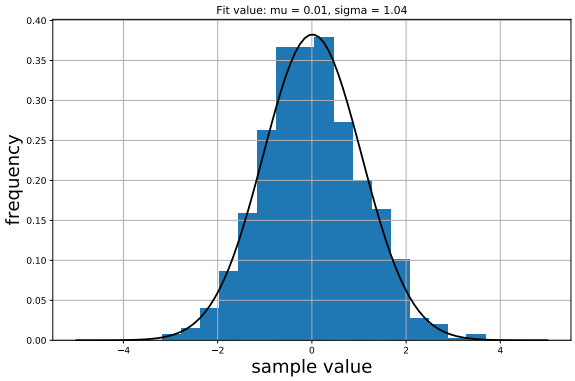
(iii) From my favorite python and numpy, for the data above:

$$\boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2.003 \\ 6.004 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} 2.017 & 1.050 \\ 1.050 & 2.074 \end{bmatrix}$$



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 4

(a)
Proof

$$
\begin{aligned}
|\mathbf{x}^T\mathbf{A}\mathbf{y}| &= \left|\sum_i \sum_j a_{ij} x_i y_j\right| \\
&\leq \sum_i \sum_j |a_{ij}|\,|x_i|\,|y_j| \\
&= \sum_i \sum_j (|a_{ij}|^{\frac{1}{2}})^2\,|x_i|\,|y_j| \\
&\textit{by Cauchy Schwarz} \\
&\leq \sqrt{\sum_i \sum_j |a_{ij}|\,|x_i|^2}\sqrt{\sum_i \sum_j |a_{ij}|\,|y_j|^2} \\
&= \sqrt{\sum_i |x_i|^2 \sum_j |a_{ij}|}\sqrt{\sum_j |y_j|^2 \sum_i |a_{ij}|} \\
&\leq \sqrt{\sum_i |x_i|^2 C}\sqrt{\sum_j |y_j|^2 R} \\
&= \sqrt{RC}||x||_2||y||_2
\end{aligned}
\tag{7}
$$

(b)

(i) For a invertible $n \times n$ matrix $\mathbf{A}$

$$
\begin{aligned}
\mathbf{A} &= \mathbf{P}\mathbf{D}\mathbf{P} \\
\mathbf{A}^{-1} &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^{-1} \\
&= \mathbf{P}^{-1}\mathbf{D}^{-1}\mathbf{P}
\end{aligned}
\tag{8}
$$

Since A is positive definite, $\lambda_i > 0\ for\ 1 \leq i \leq n$, $\mathbf{D}$ is invertible.

Thus **A** is also invertible.

(ii)

$$f\left(\begin{bmatrix} x1 \\ x2 \end{bmatrix}\right) = x_1^2 - x_2^2$$

the Hessian of this function is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

which is invertible but not positive definite

(iii) If a matrix is invertible while positive semi-definite, the matrix is positive definite.
By definition of positive semi-definite, $\mathbf{x}\mathbf{A}\mathbf{x}^T \geq 0$, and for a eigenvalue of $\mathbf{A}$, λ_i, $\lambda_i \mathbf{u}_i = \mathbf{A}\mathbf{u}_i$
So,

$$\begin{aligned} \mathbf{u}_i^T \mathbf{A} \mathbf{u}_i &= \lambda_i \\ \lambda_i &\geq 0 \end{aligned} \tag{9}$$

If the matrix is invertible, it cannot have $\lambda_i = 0$.
So, $\lambda_i > 0$, thus $\mathbf{x}\mathbf{A}\mathbf{x}^T > 0$.
The matrix $\mathbf{A}$ is positive definite.

(c) To prove: $(\exists \mathbf{A}^\dagger)\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} &= \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \\ &= \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \\ & \mathit{simplify} \\ \mathbf{U}^T\mathbf{A}^\dagger\mathbf{U}\mathbf{\Lambda} &= \mathbf{\Lambda}\mathbf{\Lambda}^- \\ \mathbf{A}^\dagger &= \mathbf{U}\mathbf{\Lambda}^-\mathbf{U}^T \end{aligned} \tag{10}$$

Since $\mathbf{A}$ is symmetric, it is guaranteed that can be decompose, thus $\mathbf{A}^\dagger$ exist.

Code

Exercise2

```
import matplotlib.pyplot as plt
from scipy.stats import norm
import numpy as np

def newfig():
    fig = plt.figure(figsize=(9,6), dpi=300)
    ax = fig.add_subplot(111)
    return fig, ax

def final_adjust(fn):
    plt.tight_layout()
    plt.savefig(fn, bbox='tight')

if __name__ == '__main__':

    # (b)
    fig, ax = newfig()
    mu, sigma = 0, 1
    X = np.random.normal(mu, sigma, 1000)
    ax.set_ylim([-3,3])
    ax.set_xlabel('sample index', fontsize = 20)
    ax.set_ylabel('sample value (x)', fontsize = 20)
    ax.grid(True)
    ax.plot(X,'r.')
    final_adjust('../pix/exercise2_b.pdf')

    # (c)
    X = np.random.normal(mu, sigma, 1000)

    # 4 bins
    fig, ax = newfig()
    ax.hist(X, 4, density=True)
    mu, sigma = norm.fit(X)
    x_ticks = np.linspace(-5, 5, 100)
    pdf = norm.pdf(x_ticks, mu, sigma)
    ax.grid(True)
    ax.plot(x_ticks, pdf, 'k', linewidth=2)
    ax.set_title("Fit value: mu = %.2f, sigma = %.2f"%(mu, sigma))
    ax.set_xlabel('sample value', fontsize = 20)
    ax.set_ylabel('frequency', fontsize = 20)
    final_adjust('../pix/exercise2_c1.pdf')

    # 1000 bins
    fig, ax = newfig()
```

```
ax.hist(X, 1000, density=True)
mu, sigma = norm.fit(X)
x_ticks = np.linspace(-5, 5, 100)
pdf = norm.pdf(x_ticks, mu, sigma)
ax.grid(True)
ax.plot(x_ticks, pdf, 'k', linewidth=2)
ax.set_title("Fit value: mu = %.2f, sigma = %.2f"%(mu, sigma))
ax.set_xlabel('sample value', fontsize = 20)
ax.set_ylabel('frequency', fontsize = 20)
final_adjust('../pix/exercise2_c2.pdf')

# (d)
n = 1000
J_h = np.zeros(200)
for m in range(1,201):
    h = (max(X) - min(X))/m
    p = np.array([((X >= min(X) + h*x) & (X < min(X) + h*(x+1))).sum()/n for x in range(m)]
    J_h[m-1] = 2 / h / (n-1) - (n+1) / h /(n-1) * sum(p**2)

fig, ax = newfig()
ax.grid(True)
ax.plot([x for x in range(1,201)], J_h)
ax.set_xlabel('# of bins', fontsize = 20)
ax.set_ylabel('CVER', fontsize = 20)
final_adjust('../pix/exercise2_d1.pdf')

m_star = J_h.argmin()

fig, ax = newfig()
ax.hist(X, m_star, density=True)
mu, sigma = norm.fit(X)
x_ticks = np.linspace(-5, 5, 100)
pdf = norm.pdf(x_ticks, mu, sigma)
ax.grid(True)
ax.plot(x_ticks, pdf, 'k', linewidth=2)
ax.set_title("Fit value: mu = %.2f, sigma = %.2f"%(mu, sigma))
ax.set_xlabel('sample value', fontsize = 20)
ax.set_ylabel('frequency', fontsize = 20)
final_adjust('../pix/exercise2_d2.pdf')
```

Exercise3 (a)

```
import numpy as np
import matplotlib.pyplot as plt

def newfig():
    fig = plt.figure(figsize=(6,9), dpi=300)
    ax = fig.add_subplot(111)
    return fig, ax

def final_adjust(fn):
    plt.tight_layout()
    plt.savefig(fn, bbox='tight')


if __name__ == '__main__':
    x1 = np.arange(-1,5,0.01)
    x2 = np.arange(0,10,0.01)
    Z = np.zeros((1000,600))
    for i in range(len(x1)):
        for j in range(len(x2)):
            Z[j][i] = 1/(np.pi*6**(1/2))*np.exp(-1/3*((x1[i]-2)**2
                    -(x1[i]-2)*(x2[j]-6)+(x2[j]-6)**2))

    fig, ax = newfig()
    ax.contour(x1,x2,Z)
    ax.grid(True)
    ax.set_yticks([i for i in range(11)])
    ax.set_xticks([i for i in range(-1,6)])
    ax.set_xlabel('x1', fontsize = 20)
    ax.set_ylabel('x2', fontsize = 20)

    final_adjust('../pix/exercise3_a.pdf')
```

Exercise3 (c)

```
import numpy as np
import matplotlib.pyplot as plt

def newfig():
    fig = plt.figure(figsize=(6,9), dpi=300)
    ax = fig.add_subplot(111)
    return fig, ax

def final_adjust(fn):
    plt.tight_layout()
    plt.savefig(fn, bbox='tight')


if __name__ == '__main__':
    x1 = np.arange(-1,5,0.01)
    x2 = np.arange(0,10,0.01)
    Z = np.zeros((1000,600))
    for i in range(len(x1)):
        for j in range(len(x2)):
            Z[j][i] = 1/(np.pi*6**(1/2))*np.exp(-1/3*((x1[i]-2)**2
                     -(x1[i]-2)*(x2[j]-6)+(x2[j]-6)**2))

    fig, ax = newfig()
    ax.contour(x1,x2,Z)
    ax.grid(True)
    ax.set_yticks([i for i in range(11)])
    ax.set_xticks([i for i in range(-1,6)])
    ax.set_xlabel('x1', fontsize = 20)
    ax.set_ylabel('x2', fontsize = 20)

    final_adjust('../pix/exercise3_a.pdf')
```