ECE 595: Homework 1

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Exercise 2

(a) For a guassian distribution:

$$\begin{aligned} E[x] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} (x+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 0 + \mu \frac{1}{\sqrt{2\pi\sigma^2}} \times \sigma\sqrt{2\pi} \\ &= \mu \end{aligned} \tag{1}$$

$$\begin{aligned} Var[x] &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx \\ let\ y &= \frac{x}{\sigma},\ then\ dy = \frac{1}{\sigma} dx \\ &= \frac{\sigma^3}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2}} dy \\ &= \sigma^2 \end{aligned} \tag{2}$$

(b) Data generated and plotted as follows.

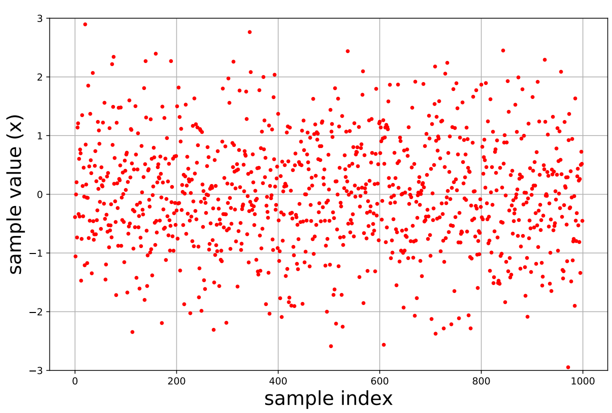
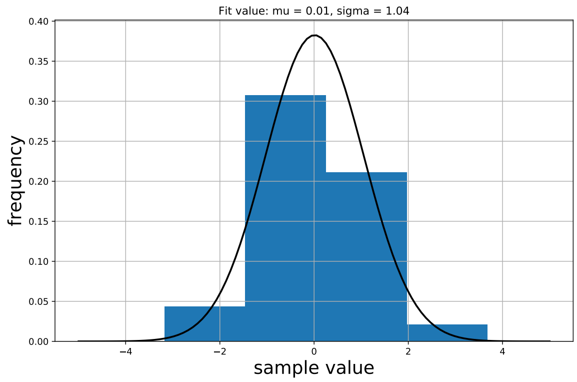


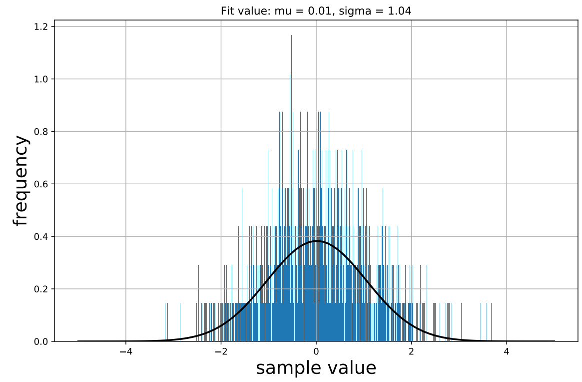
Figure 1: Gussian random data.

(c)

(i)..(iv) plots shown below



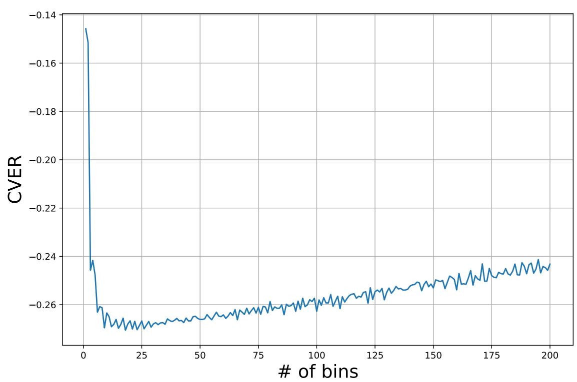
(a) 4 bins



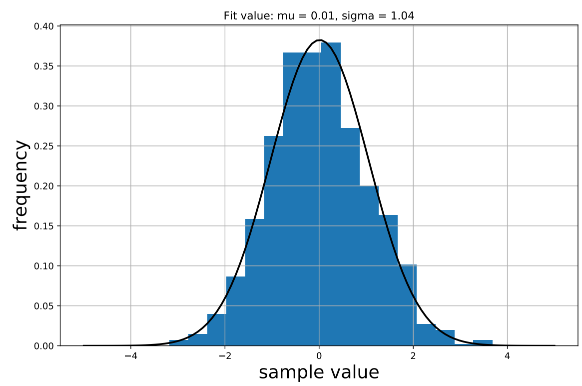
(b) 1000 bins

(v) TODO: fill this

(d) compare to part (c), the histogram fits a lot better with the PDF plots shown below



(c) Cross validation estimator of risk vs. # of bins



(d) Histogram and PDF overlayed with optimized number of bins

Exercise 3

(a)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi^2|\mathbf{\Sigma}|}} exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$$

(i) plug in

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{\mu} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{and } \mathbf{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

we get

$$\begin{aligned} f_{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \frac{1}{\sqrt{2\pi^2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}} exp\left\{-\frac{1}{2}\begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-2 \\ x_2-6 \end{bmatrix}\right\} \\ &= \frac{1}{\pi\sqrt{6}} exp\left\{-\frac{1}{3}\left((x_1-2)^2-(x_1-2)(x_2-6)+(x_2-6)^2\right)\right\} \end{aligned} \tag{3}$$

(ii) plot shown below

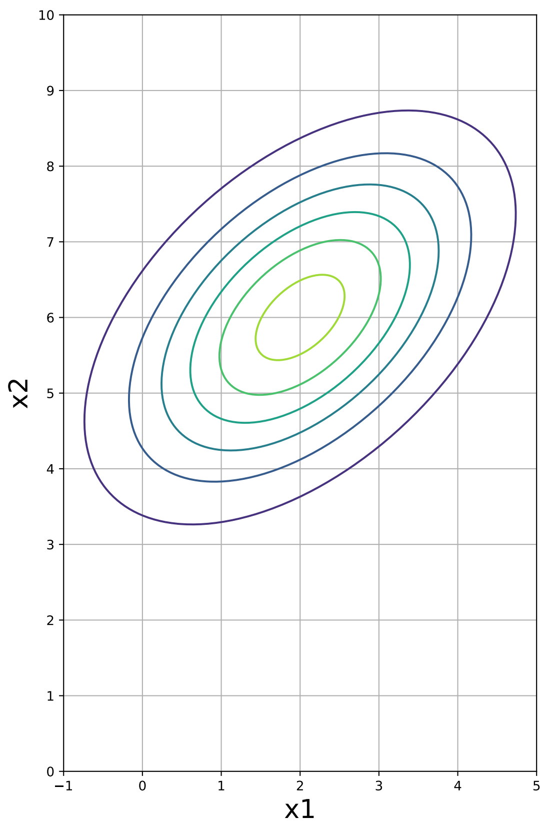


Figure 2: Gussian random data

(b)

(i)

To prove: $\boldsymbol{\mu}_{\mathbf{Y}} = b$

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}} &= \mathbb{E}[\mathbf{Y}] = \mathbb{E}[\mathbf{AX} + \mathbf{b}] = \mathbb{E}[\mathbf{AX}] + \mathbf{b} \\ &= \mathbf{A}\mathbb{E}[\mathbf{X}] + \mathbf{b} \\ & since\ \mathbf{X} \in \mathcal{N}(\mathbf{0}, \mathbf{I}),\ \mathbb{E}[\mathbf{X}] = \mathbf{0} \\ &= \mathbf{A0} + \mathbf{b} \\ &= \mathbf{b} \end{aligned} \tag{4}$$

To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}} &= \mathbb{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})^T] \\ &= \mathbb{E}[\mathbf{AX} \times \mathbf{X}^T\mathbf{A}^T] \\ &= \mathbf{A}\mathbb{E}[\mathbf{XX}^T]\mathbf{A}^T \\ &= \mathbf{A}(\mathbf{I} + \mathbf{0})\mathbf{A}^T \\ &= \mathbf{AA}^T \end{aligned} \tag{5}$$

(ii) To prove: $\boldsymbol{\Sigma}_{\mathbf{Y}}$ is symmetric and positive semi-definite

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathbf{Y}ij} &= \Sigma_{x=1}^{n}\mathbf{A}_{ix}\mathbf{A}_{xj}^T \\ &= \Sigma_{x=1}^{n}\mathbf{A}_{jx}\mathbf{A}_{xi}^T \\ &= \boldsymbol{\Sigma}_{\mathbf{Y}ji} \\ & thus\ symmetric \\ \mathbf{x}^T\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{x} &= \mathbf{x}^T\mathbf{AA}^T\mathbf{x} \\ let\ \mathbf{y} = \mathbf{x}^T\mathbf{A} \\ &= \mathbf{yy}^T \\ &= \|\mathbf{y}\|^2 \geq 0 \\ & thus\ positive\ semi-definite \end{aligned} \tag{6}$$

(iii)

$$Null(\mathbf{A}) = \mathbf{0}$$

(iv) By inspection,

$$b = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

By Cholesky decomposition,

$$\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{AA}^T$$

$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

(c)

TODO: write this

Exercise 4

(a)

(b)

(i) For a invertible $n \times n$ matrix $\mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{PDP} \\ \mathbf{A}^{-1} &= (\mathbf{PDP}^{-1})^{-1} \\ &= \mathbf{P}^{-1}\mathbf{D}^{-1}\mathbf{P} \end{aligned} \tag{7}$$

Since A is positive definite, $\lambda_i > 0 \ for \ 1 \leq i \leq n$, $\mathbf{D}$ is invertible.

Thus $\mathbf{A}$ is also invertible.

(ii) TODO: write this

(iii) If a matrix is invertible while positive semi-definite, the matrix is positive definite.
By definition of positive semi-definite, $\mathbf{xAx}^T \geq 0$, and for a eigenvalue of $\mathbf{A}$, λ_i, $\lambda_i \mathbf{u}_i = \mathbf{Au}_i$
So,

$$\begin{aligned} \mathbf{u}_i^T \mathbf{Au}_i &= \lambda_i \\ \lambda_i &\geq 0 \end{aligned} \tag{8}$$

If the matrix is invertible, it cannot have $\lambda_i = 0$.
So, $\lambda_i > 0$, thus $\mathbf{xAx}^T > 0$.
The matrix $\mathbf{A}$ is positive definite.

(c) To prove: $(\exists \mathbf{A}^\dagger)\ \mathbf{AA}^\dagger\mathbf{A} = \mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{U\Lambda U}^T \\ \mathbf{AA}^\dagger\mathbf{A} &= \mathbf{U\Lambda U}^T\mathbf{A}^\dagger\mathbf{U\Lambda U}^T \\ &= \mathbf{U\Lambda U}^T \\ & simplify \\ \mathbf{U}^T\mathbf{A}^\dagger\mathbf{U\Lambda} &= \mathbf{\Lambda\Lambda}^- \\ \mathbf{A}^\dagger &= \mathbf{U\Lambda}^-\mathbf{U}^T \end{aligned} \tag{9}$$

Since $\mathbf{A}$ is symmetric, it is guaranteed that can be decompose, thus $\mathbf{A}^\dagger$ exist.