两角差、两角和的余弦公式

$\cos(\alpha+\pi) = -\cos\alpha$ （诱导公式）

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{cases}$$

公式证明后续在向量讲中

例：求 $\cos 15^\circ$

解：$\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

例：求 $\cos 75^\circ$

解：$\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

注：$\cos 75^\circ = \sin 15^\circ$ $\quad \cos\theta = \sin(\frac{\pi}{2} - \theta)$

$\cos 15^\circ = \sin 75^\circ$ $\quad \cos\theta = \sin(\frac{\pi}{2} - \theta)$

例：$\cos 555^\circ$

解：$\cos 555^\circ = \cos(555^\circ - 360^\circ)$

$= \cos 195^\circ$

$= \cos(180^\circ + 15^\circ)$

$= -\cos 15^\circ$

$= -\cos(45^\circ - 30^\circ)$

$= -(\cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ)$

$= -(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4})$

$= -\frac{\sqrt{6}+\sqrt{2}}{4}$

例：$\sin(a+\frac{\pi}{4}) = \frac{4}{5}$，且 $\frac{\pi}{4} < a < \frac{3\pi}{4}$，求 $\cos a$

解：思路：构造角思想

$\cos a = \cos(a+\frac{\pi}{4}-\frac{\pi}{4})$

$= \cos(a+\frac{\pi}{4})\cos\frac{\pi}{4} + \sin(a+\frac{\pi}{4})\sin\frac{\pi}{4}$

$= \frac{\sqrt{2}}{2}\cos(a+\frac{\pi}{4}) + \frac{4}{5}\cdot\frac{\sqrt{2}}{2}$ ①

$\because \sin(a+\frac{\pi}{4}) = \frac{4}{5}$

$\therefore \cos(a+\frac{\pi}{4}) = \pm\frac{3}{5}$

又 $\frac{\pi}{4} < a < \frac{3\pi}{4}$

$\therefore \frac{\pi}{2} < a+\frac{\pi}{4} < \pi$

$\therefore \cos(a+\frac{\pi}{4}) = -\frac{3}{5}$

$\therefore$ ①式 $= \frac{\sqrt{2}}{2}\cdot(-\frac{3}{5}) + \frac{4}{5}\cdot\frac{\sqrt{2}}{2}$

$= \frac{1}{5}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{10}$

例：$\sin\alpha+\sin\beta=\frac{4}{5}$

$\cos\alpha+\cos\beta=\frac{3}{5}$

求 $\cos(\alpha-\beta)=$ ______

解：$\sin\alpha+\sin\beta=\frac{4}{5} \Rightarrow \sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta=\frac{16}{25}$ ①

$\cos\alpha+\cos\beta=\frac{3}{5} \Rightarrow \cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta=\frac{9}{25}$ ②

① + ② $\Rightarrow 1+1+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=\frac{25}{25}=1$

$\Rightarrow 2\cos(\alpha-\beta)=-1$

$\Rightarrow \cos(\alpha-\beta)=-\frac{1}{2}$

例：$\sin\alpha\cdot\sin\beta=1$. 则 $\cos(\alpha-\beta)=$ ______

解：$\because \sin\alpha\cdot\sin\beta=1$

$$\therefore \begin{cases} \sin\alpha=\sin\beta=1 \text{ ①} \Rightarrow \cos\alpha=\cos\beta=0 \\ \text{或} \sin\alpha=\sin\beta=-1 \text{ ②} \Rightarrow \cos\alpha=\cos\beta=0 \end{cases}$$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=0+1$

$=1$

例：$\sin(\pi+\theta)=-\frac{3}{5}$. θ 第二象限

$\sin(\frac{\pi}{2}+\theta)=-\frac{2\sqrt{5}}{5}$, φ 第三象限. 求 $\cos(\theta-\varphi)$

解：$\sin(\pi+\theta)=-\sin\theta=-\frac{3}{5} \Rightarrow \sin\theta=\frac{3}{5} \Rightarrow \cos\theta=-\frac{4}{5}$ (二象限)

$\sin(\frac{\pi}{2}+\theta)=\cos\theta=-\frac{2\sqrt{5}}{5} \Rightarrow \sin\theta=-\frac{\sqrt{5}}{5}$

$\therefore \cos(\theta-\varphi)=\cos\theta\cos\varphi+\sin\theta\sin\varphi=\cdots=\frac{\sqrt{5}}{5}$

例：求 $\dfrac{2\cos 10^\circ - \sin 20^\circ}{\cos 20^\circ}$

解：$\dfrac{2\cos 10^\circ - \sin 20^\circ}{\cos 20^\circ}$

$= \dfrac{2\cos(30^\circ - 20^\circ) - \sin 20^\circ}{\cos 20^\circ}$

$= \dfrac{2(\cos 30^\circ \cdot \cos 20^\circ + \sin 30^\circ \cdot \sin 20^\circ) - \sin 20^\circ}{\cos 20^\circ}$

$= \dfrac{2(\frac{\sqrt{3}}{2} \cdot \cos 20^\circ + \frac{1}{2} \cdot \sin 20^\circ) - \sin 20^\circ}{\cos 20^\circ}$

$= \dfrac{\sqrt{3} \cdot \cos 20^\circ + \sin 20^\circ - \sin 20^\circ}{\cos 20^\circ}$

$= \sqrt{3}$

例：$\cos(\alpha - \frac{\pi}{3}) = \cos\alpha$. 求 $\tan\alpha$ ______

解：$\cos\alpha \cdot \cos\frac{\pi}{3} + \sin\alpha \cdot \sin\frac{\pi}{3} = \cos\alpha$

$\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha = \cos\alpha$

两边同除以 $\cos\alpha$. 则：

$\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\alpha = 1 \Rightarrow \frac{\sqrt{3}}{2}\tan\alpha = \frac{1}{2}$

$\therefore \tan\alpha = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

例：$\alpha, \beta \in (\frac{3\pi}{4}, \pi)$ $\sin(\alpha+\beta) = -\frac{3}{5}$

$\sin(\beta - \frac{\pi}{4}) = \frac{12}{13}$，求 $\cos(\alpha + \frac{\pi}{4})$

解：$\sin(\alpha+\beta) = -\frac{3}{5} \Rightarrow \cos(\alpha+\beta) = \pm\frac{4}{5}$

$\because \begin{cases} \frac{3\pi}{4} < \alpha < \pi \\ \frac{3\pi}{4} < \beta < \pi \end{cases} \Rightarrow \frac{3\pi}{2} < \alpha+\beta < 2\pi$ 第四象限

$\therefore \cos(\alpha+\beta) = \frac{4}{5}$

$\because \sin(\beta - \frac{\pi}{4}) = \frac{12}{13}$

$\therefore \cos(\beta - \frac{\pi}{4}) = \pm\frac{5}{13}$

又 $\frac{\pi}{2} < \beta - \frac{\pi}{4} < \frac{3\pi}{4}$ 第二象限

$\therefore \cos(\beta - \frac{\pi}{4}) = -\frac{5}{13}$

由上可知：$\cos(\alpha+\beta) = \frac{4}{5}$

$\cos(\beta - \frac{\pi}{4}) = -\frac{5}{13}$

$\sin(\alpha+\beta) = -\frac{3}{5}$

$\sin(\beta - \frac{\pi}{4}) = \frac{12}{13}$

$\therefore \cos(\alpha + \frac{\pi}{4}) = \cos[(\alpha+\beta) - (\beta - \frac{\pi}{4})]$

$= \cos(\alpha+\beta)\cdot\cos(\beta - \frac{\pi}{4}) + \sin(\alpha+\beta)\cdot\sin(\beta - \frac{\pi}{4})$

$= \frac{4}{5}\cdot(-\frac{5}{13}) + (-\frac{3}{5})\cdot\frac{12}{13}$

$= -\frac{20}{65} + (-\frac{36}{65})$

$= -\frac{56}{65}$

两角差、和正弦公式

$$\boxed{\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\alpha\end{aligned}}$$

例：$\sin 43^\circ\cos 13^\circ-\cos 43^\circ\sin 13^\circ$

$=\sin(43^\circ-13^\circ)=\sin 30^\circ=\frac{1}{2}$

例：$\cos 44^\circ\sin 14^\circ-\sin 44^\circ\cos 14^\circ$

$=\sin 14^\circ\cos 44^\circ-\cos 14^\circ\sin 44^\circ$

$=\sin(14^\circ-44^\circ)$

$=\sin(-30^\circ)$

$=-\sin 30^\circ$

$=-\frac{1}{2}$

例：$\triangle ABC$中，$\sin(A-B)\cos B+\cos(A-B)\sin B\geq 1$

则$\triangle ABC$为___$\triangle$

解：$\sin(A-B)\cos B+\cos(A-B)\sin B$

$=\sin[(A-B)+B]=\sin A\geq 1$

又 $\sin A\leq 1$

$\therefore A=\frac{\pi}{2}$

$\therefore\triangle ABC$为直角三角形

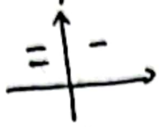
例：$\sin\alpha=\frac{2}{3}$，$\cos\beta=-\frac{1}{4}$，α、β 相邻象限，

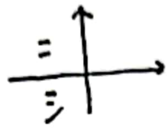
求 $\sin(\alpha+\beta)$

解：$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\cdot\sin\beta$

$=\frac{2}{3}\cdot(-\frac{1}{4})+\cos\alpha\cdot\sin\beta$ ①

即需求出 $\cos\alpha$ 和 $\sin\beta$

$\because \sin\alpha=\frac{2}{3} \Rightarrow$  α 在一、二象限

$\because \cos\beta=-\frac{1}{4} \Rightarrow$  β 在二、三象限

又$\because$ α、β 相邻象限

$\therefore$ $\begin{cases} \alpha\text{第一象限、}\beta\text{第二象限} & \langle 1\rangle \\ \text{或} & \\ \alpha\text{第二象限、}\beta\text{第三象限} & \langle 2\rangle \end{cases}$

第<1>种情况、$\sin\alpha=\frac{2}{3} \Rightarrow \cos\alpha=\frac{\sqrt{5}}{3}$

$\cos\beta=-\frac{1}{4} \Rightarrow \sin\beta=\frac{\sqrt{15}}{4}$

~~①式 $=-\frac{2}{3}+\cos\alpha\cdot\sin\beta=-\frac{2}{3}+\frac{\sqrt{5}}{3}\cdot\frac{\sqrt{15}}{4}=-\frac{2}{3}+\frac{5\sqrt{3}}{12}=$~~

①式 $=\frac{2}{3}\cdot(-\frac{1}{4})+\cos\alpha\cdot\sin\beta=-\frac{2}{12}+\frac{5\sqrt{3}}{12}=\frac{5\sqrt{3}-2}{12}$

第2种情况、$\sin\alpha=\frac{2}{3} \Rightarrow \cos\alpha=-\frac{\sqrt{5}}{3}$

$\cos\beta=-\frac{1}{4} \Rightarrow \sin\beta=-\frac{\sqrt{15}}{4}$

①式 $=-\frac{2}{12}+\cos\alpha\cdot\sin\beta+(-\frac{\sqrt{5}}{3})\cdot(-\frac{\sqrt{15}}{4})=\frac{5\sqrt{3}-2}{12}$

$\therefore \sin(\alpha+\beta)=\frac{5\sqrt{3}-2}{12}$

例：α、β为锐角，$\sin\alpha=\frac{2\sqrt{5}}{5}$，$\sin(\alpha+\beta)=\frac{3}{5}$

则 $\cos\beta=$ ____

解：构造角思想

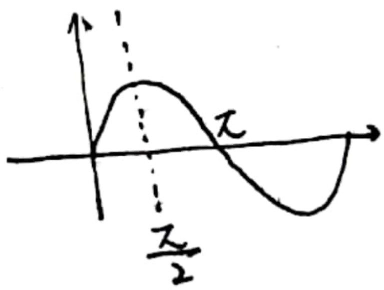
$$\cos\beta=\cos[(\alpha+\beta)-\alpha]$$

$$=\cos(\alpha+\beta)\cdot\cos\alpha+\underbrace{\sin(\alpha+\beta)}_{\frac{3}{5}}\cdot\underbrace{\sin\alpha}_{\frac{2\sqrt{5}}{5}}$$

$\because \sin(\alpha+\beta)=\frac{3}{5}$

$\therefore \cos(\alpha+\beta)=\pm\frac{4}{5}$

又 $\sin(\alpha+\beta)=\frac{3}{5}<\sin\alpha=\frac{2\sqrt{5}}{5}$



$\therefore \alpha+\beta>\frac{\pi}{2}$

$\therefore \cos(\alpha+\beta)=-\frac{4}{5}$

又 $\cos\alpha=\sqrt{1-\sin^2\alpha}=\frac{\sqrt{5}}{5}$

$\therefore \cos(\alpha+\beta)\cos\alpha+\sin(\alpha+\beta)\cdot\sin\alpha$

$=\cdots$

$=\frac{2\sqrt{5}}{25}$