SOLUTIONS / GRADING GUIDE

Note: Do not distribute solutions to contestants upfront. These are for graders.

Section A Suggested Solutions:

1. **Random Walk Probability:**
   After 10 flips, to end at position 0, you need exactly 5 heads and 5 tails. Probability = C(10,5)/2^{10} = 252/1024 = 63/256.

2. **Painted Cube Probability:**
   In a 5x5x5 cube, the cubes with exactly one face painted are those not on edges or corners of a face. Each face's interior layer has (5-2)×(5-2)=3×3=9 such cubes per face. There are 6 faces, so 54 cubes with exactly one face painted. Total small cubes = 125. Probability = 54/125.

3. **Bond Prices and Maturity:**
   A bond with the longest maturity is discounted the most. Prices: $96.15, $92.45, $88.90. Highest price = shortest maturity, lowest price = longest maturity. Order of maturity from shortest to longest: $96.15 (1-year), $92.45 (2-year), $88.90 (3-year).

4. **Market Indicator Expectation:**
   After two days: The outcomes are up then up: 1.1 × 1.1 = 1.21; up then down or down then up: 1.1 × 0.9 = 0.99; down then down: 0.9 × 0.9 = 0.81. Equal probabilities (1/4 each): Expected value = (1.21 + 0.99 + 0.99 + 0.81)/4 = (1.21 + 1.98 + 0.81)/4 = (4.00)/4 = 1.00. So the expected value is exactly 1.00. Thus equal to the original value.

5. **Counting Common Factors: (adapted from [MathCounts](#))**
   396,040 integers.

6. **Counting Potential Sum of Powers: (adapted from [MathCounts](#))**
   52 integers, The relevant powers of 3 are: $3^0 = 1$; $3^1 = 3$; $3^2 = 9$; $3^3 = 27$; $3^4 = 81$; $3^5 = 243$; $3^6 = 729$. The terms being separated in value by a factor of 2 or more means that distinct contents in the sums yield distinct sum values, so we need not be concerned with overlapping sum values. There are 7 terms from which to choose potentially 2, 4 or 6 to form the sum. The total count of distinct combinations permitted is 7C2 + 7C4 + 7C6 = 7!/2! 5! + 7!/4! 3! + 7!/6! 1! = 21 + 35 + 7 = 63. However, some of these violate the criterion of the sum not exceeding 1000. Exceeding 1000 requires at least 4 terms, with two of them being 729 and 243, with these latter two summing to 972, leaving 28 to spare. The only way to exceed 1000 is if at least one of 81 and 27 must be included. If 81 is included, we are definitely over: 81 must be paired with either one other term, with 4 to choose from, or 81 must be combined with three other terms, thus leaving out any one lower term, of which there are 4 to chose from; this means we have so far rejected 8

of the 63 to leave 55 to consider. The last case to consider is 81 is excluded and 27 is included, with one (any one of 9, 3 or 1) or three other terms (all of 9, 3 and 1) also included—a total of 4 possibilities. The only way for 27 to work is to be paired with 1, so the other 3 possibilities fail, leaving 52 that work.

7. **Zeros and Factorials:**
   The number of trailing zeros in 100! is determined by counting the factors of 5 (since factors of 2 are plentiful). Compute $\lfloor 100/5 \rfloor = 20$ and $\lfloor 100/25 \rfloor = 4$. Thus, total zeros = 20 + 4 = 24.

8. **Dice Rolls:**
   With two dice, there are 6×6 = 36 outcomes. For the first number to be greater than the second, if the first die shows 2 there's 1 possibility (only 1 is lower), for 3 there are 2, and so on, giving 1+2+3+4+5 = 15 favorable outcomes. Hence, the probability = 15/36 = 5/12.

9. **Rope-Triangle:**
   Cutting a rope at two random points yields three segments. A triangle can be formed if no segment is longer than half the rope's length. It can be shown that the probability of satisfying this triangle inequality is 1/4.

10. **Uphill Numbers:**
    A three-digit number with strictly increasing digits is formed by choosing 3 distinct digits from {1, 2, …, 9} (since the hundreds digit must be nonzero) and writing them in increasing order. The count is C(9, 3) = 84.

11. **Non-Crossing Handshakes:**
    This problem is equivalent to finding the 5th Catalan number (since 10 people form 5 handshake pairs). The formula for the nth Catalan number is: $C_n = (1/(n+1)) \times C(2n, n)$
    For n = 5: $C_5 = (1/6) \times C(10, 5) = (1/6) \times 252 = 42$.

12. **Expected Coin Flips for Two Consecutive Heads:**
    Let E be the expected number of flips starting from scratch, and E_1 be the expected additional flips after getting one head. We set up the following equations:

    a. Starting with no heads: $E = 1 + (½)\cdot E_1 + (½)\cdot E$ (The first flip takes one toss; with probability ½ you get a head and move to state E_1, and with probability ½ you get a tail and remain in state E.)

    b. After one head: $E_1 = 1 + (½)\cdot 0 + (½)\cdot E$ (After one head, the next flip either gives a head (and you're done, requiring 0 extra flips) with probability ½, or a tail, which resets you to the starting state, adding E flips, with probability ½.)

    Solving these equations yields $E_1 = 4$ and $E = 6$, so the expected number of flips is 6.

13. **Noodles Slurp:**

Given a bowl with 100 noodles, each with two ends hanging outside, we randomly tie pairs of ends until none remain, forming loops. The goal is to determine a 0.5-wide interval containing the expected number of loops. Each noodle starts with two ends, yielding 200 ends total, and pairing them randomly connects the noodles into cycles. The expected number of such cycles is given by the sum of the reciprocals of the first 100 odd integers: $E = \sum_{k=1}^{100} 1/(2k-1)$. This sum approximates to 3.284 through precise calculation or asymptotic approximation (e.g., $(1/2) \ln 100 + \ln 2 + \gamma/2$, where $\gamma \approx 0.57721$). A 0.5-wide interval must span from a to a + 0.5 and include 3.284. The interval [3.0, 3.5] has width 3.5 - 3.0 = 0.5 and contains 3.284, satisfying the requirement. Thus, the solution is the interval [3.0, 3.5].

14. **Which is Greater?**
Raise both to power 30, and we get 2^{10} vs 10^3. This makes it apparent that $2^{(1/3)}$ is greater. Note in order for this reasoning to be valid, we need both numbers at hand to be greater than 1, which is the case thankfully!

15. **Future Minimizer :)**
2345 06 17

(1) 2

(2) $\frac{41}{2}$

(3) 190

(4) 4017

(5) 1156

(6) 396

(7) $-\frac{17}{56}$

(8) 487

(9) 8062

*(10) $1173-1295$

(11) 3.2 or $3\frac{1}{5}$

(12) 38

(13) 1584

(14) 5

(15) 303303

(16) 570

(17) 0

(18) 210

(19) 4

*(20) $140-154$

(21) 4

(22) 72

(23) 2.56

(24) 2

(25) 41

(26) -2

(27) 1.25, $\frac{5}{4}$, $1\frac{1}{4}$

(28) 1

(29) $22\frac{2}{9}$

*(30) $444-489$

(31) 38295

(32) -4

(33) 7

(34) 5

(35) 280

(36) $19\frac{8}{25}$

(37) 48

(38) $-\frac{16}{231}$

(39) 4545

*(40) $2541-2808$

(41) 700

(42) 196

(43) $\frac{25}{2}$, $12\frac{1}{2}$, 12.5

(44) 8

(45) 12

(46) 3

(47) $-\frac{80}{27}$, $-2\frac{26}{27}$

(48) 13

(49) 16

*(50) $980-1082$

(51) 6

(52) 3249

(53) $\frac{1}{2}$

(54) 36

(55) $\frac{2}{3}$

(56) 10

(57) -3

(58) 14

(59) 6

*(60) $122-134$

(61) 3.5, $\frac{7}{2}$, $3\frac{1}{2}$

(62) .5, $\frac{1}{2}$

(63) 343

(64) .78, $\frac{39}{50}$

(65) 60

(66) 5

(67) 4

(68) $\frac{35}{12}$, $2\frac{9}{13}$

(69) 1221

*(70) $2849998 - 3149996$

(71) 72

(72) 2

(73) 0

(74) 7

(75) $-\frac{16}{3}$, $-5\frac{1}{3}$

(76) 3

(77) 2

(78) $\frac{8}{3}$, $2\frac{2}{3}$

(79) 288

*(80) $102-112$