More Brownian Motion

Math 365

April 26, 2022

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In this lab we will do some further exploration of Brownian motion.

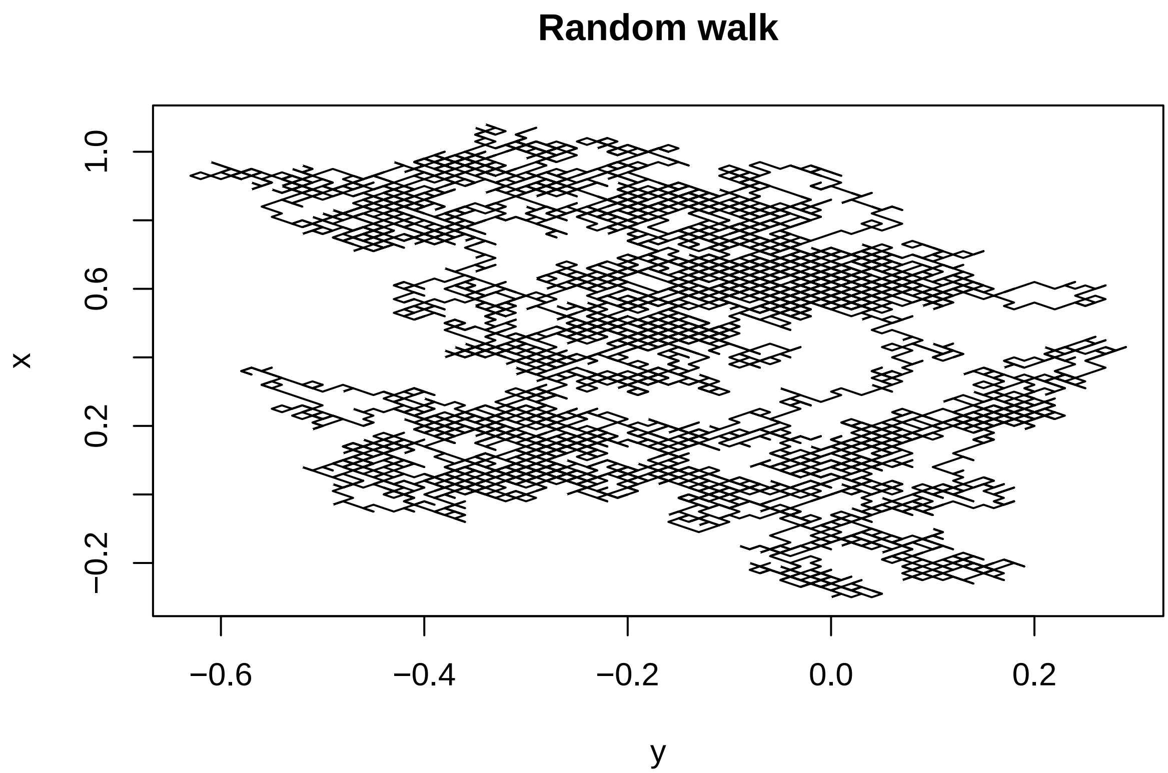
Multi-Dimensional Brownian Motion

Multi-dimensional Brownian motion can be constructed using independent one-dimensional Brownian motions for the coordinates of a vector-valued function $B_t = (B_t^{(1)}, \ldots, B_t^{(n)})$.

Exercise 1

Update the R code from last week's lab to run simulations of 2-dimensional Brownian motion by generating two standard Brownian motions X_t and Y_t. Plot the resulting path (X_t, Y_t) in the xy-plane.

```
duration <- 1  # total time
numsteps <- 10000 # number of steps
steps <- seq(0,1,duration/numsteps)
deltax <- sample(c(-1,1),size=numsteps,replace=TRUE)/sqrt(numsteps)
deltay <- sample(c(-1,1),size=numsteps,replace=TRUE)/sqrt(numsteps)
x <- c(0, cumsum(deltax)) # compute cumulative sum
y <- c(0, cumsum(deltay)) # compute cumulative sum
plot(y, x, type = "l", main = "Random walk")
```



Exercise 2

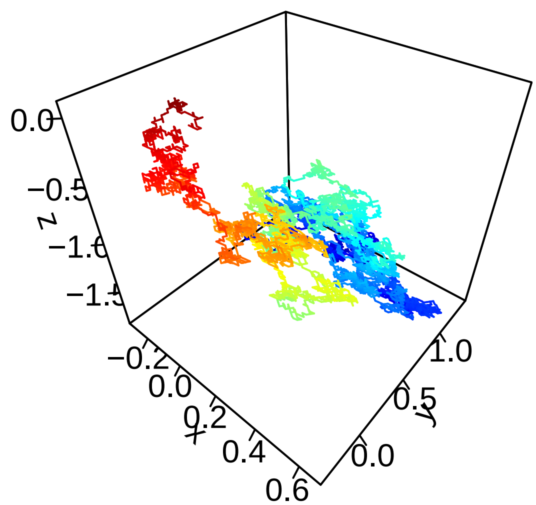
Add a third standard Brownian motion Z_t to run simulations of 3-dimensional Brownian motion. Plot the resulting path using plot3D, which requires loading the plot3D package: in the RStudio window, click on Install in the Packages tab, then choose plot3D. It should then download and install the package; you may need to click on the box next to plot3D in the list of packages to actually load it.

```
library("plot3D")

duration <- 1  # total time
numsteps <- 10000 # number of steps
steps <- seq(0,1,duration/numsteps)
deltax <- sample(c(-1,1),size=numsteps,replace=TRUE)/sqrt(numsteps)
deltay <- sample(c(-1,1),size=numsteps,replace=TRUE)/sqrt(numsteps)
deltaz <- sample(c(-1,1),size=numsteps,replace=TRUE)/sqrt(numsteps)
x <- c(0, cumsum(deltax)) # compute cumulative sum
y <- c(0, cumsum(deltay)) # compute cumulative sum
z <- c(0, cumsum(deltaz)) # compute cumulative sum

lines3D(x,y,z,ticktype="detailed",main="3D Brownian motion simulation",colkey=FALSE)
```

3D Brownian motion simulation



Martingales and Brownian Motion

For continuous-time stochastic processes, we say that $(Y_t)_{t\geq 0}$ is a martingale with respect to stochastic process $(X_t)_{t\geq 0}$ if for all $t \geq 0$ we have $E(Y_t \,|\, Y_r, 0 \leq r \leq s) = E(Y_s)$ for all $0 \leq s \leq t$ and $E(|Y_t|) < \infty$.

Geometric Brownian motion can be defined as

$$G_t = e^{\mu t + \sigma B_t},$$

where B_t is a standard Brownian motion.

Exercise 3

Show that $e^{-(\mu+\sigma^2/2)t} G_t$ is a martingale with respect to $(B_t)_{t\geq 0}$

Q3

Martingale

$(Y_t)_{t\geq 0}$ is a <u>martingale</u> w.r.t stochastic process X_t

if $E(Y_t \mid X_r,\ 0\leq r\leq s) = Y_s \quad \forall\ 0\leq s\leq t.$

$\boxed{\text{AND}}$ $\quad E(|Y_t|) < \infty.$

Geom. Brownian Motion

$$G_t = e^{\mu t + \sigma B_t}$$

$\Rightarrow$ show that $\underbrace{e^{-(\mu + \frac{\sigma^2}{2})t} G_t}$ is martingale wrt $(B_t)_{t\geq 0}$

$$Y_t = e^{-(\mu+\frac{\sigma^2}{2})t} e^{\mu t + \sigma B_t} = \boxed{e^{(\sigma B_t - \frac{\sigma^2}{2}t)}}$$

we are breaking into product of Y_s because we have to break B_s in the sum on the exponent

$$E(Y_t \mid X_r,\ 0\leq r\leq s) = E\left((Y_t - Y_s)\cdot Y_s \mid X_r,\ 0\leq r\leq s\right)$$

$$= E(Y_t - Y_s)\cdot \underbrace{E(Y_s \mid X_r,\ 0\leq r\leq s)}$$

$$= E(Y_{t-s}) \cdot Y_s$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{(\sigma x - \frac{\sigma^2(t-s)}{2})} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{-x^2/2(t-s)}\, dx\right]}_{\text{comes out to be } 1} \cdot Y_s = \boxed{Y_s}$$

$$E(|Y_t|) = \int_{-\infty}^{\infty} e^{(\sigma x - \frac{\sigma^2 t}{2})} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx = \boxed{1} < \infty$$

Hence Proved $\blacksquare$

Figure 1: Problem 3