

# MATH-365 Week 3 HW - Stochastic Processes

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**3.3** Determine which of the following matrices are regular.

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \\ 1 & 0 & 0 \end{pmatrix}.$$

Sol [Using R studio, every element of  $P^6$  &  $R^3$  is  $> 0$ ]

Ans

$P$  &  $R$  are regular.  
 $Q$  is regular iff  $p \in [0, 1]$

Regular Transition matrix  
is a matrix  $A$  such that for  
some  $n \geq 1$ ,  $(A^n > 0)$ .

→ we can notice that each of the matrices  $P$ ,  $Q$  and  $R$   
have sum of rows = 1 and thus are transition  
matrices representing Markov chains.

→ Regularity of a transition matrix representing Markov chain  
corresponds directly to the ability to possess a limiting  
distribution, which is unique stationary distribution of chain.

→ If for some power  $n$ , all zeros in  $A^n$  appears at the same  
place as  $A^{n+1}$ , then they will appear at same place for  
all higher powers (textbook Pg84)

→ Trying to raise matrices  $P$  and  $R$  to different powers  
we can say that they are regular as  $(P^n > 0) \& (R^n > 0)$  ] used  
for some  $n \geq 0$ . ] used this !!

$$Q^2 = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix} = \begin{pmatrix} p & 1-p \\ p(1-p) & p+(1-p)^2 \end{pmatrix} = \begin{pmatrix} p & 1-p \\ p-p^2 & p^2-p+1 \end{pmatrix}$$

for  $Q$  to be regular ( $p > 0$ ) & ( $1-p > 0$ ) & ( $p-p^2 > 0$ ) & ( $p^2-p+1 > 0$ )

⇒ as  $p^2-p+1$  is always positive, we can say that  
matrix  $Q$  is regular as long as  $p \in (0, 1)$

exploration  
from  
textbook

### 3.5 A Markov chain has transition matrix

$$P = \begin{pmatrix} 0 & 1/4 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix}.$$

- (a) Describe the set of stationary distributions for the chain.
- (b) Use technology to find  $\lim_{n \rightarrow \infty} P^n$ . Explain the long-term behavior of the chain.
- (c) Explain why the chain does not have a limiting distribution, and why this does not contradict the existence of a limiting matrix as shown in (b).

S8

(a) Stationary distribution ( $\pi$ ) must satisfy:  $\pi = \pi P$ , where  $\pi = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$

$$\pi = \pi P = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) \begin{pmatrix} 0 & 1/4 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} \pi_1 = \frac{3}{4}\pi_2 + \frac{1}{4}\pi_5 \quad \text{--- (1)} \\ \pi_2 = \frac{1}{4}\pi_1 + \frac{3}{4}\pi_5 \quad \text{--- (2)} \\ \pi_3 = \pi_3, \pi_4 = \pi_4 \\ \pi_5 = \frac{3}{4}\pi_1 + \frac{1}{4}\pi_2 \quad \text{--- (3)} \end{array} \begin{array}{l} \xrightarrow{3 \times (1) - (2)} \\ \Rightarrow 3\pi_1 - \pi_2 = \frac{9}{4}\pi_5 - \frac{1}{4}\pi_1 \\ \Rightarrow \frac{13}{4}\pi_1 = \frac{13}{4}\pi_2 \Rightarrow \boxed{\pi_1 = \pi_2} \end{array}$$

$$\Rightarrow \pi = [\pi_1 \pi_2 \pi_3 \pi_4 \pi_5] = [\pi_1 \pi_1 \pi_3 \pi_4 \pi_1], \text{ as } \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \Rightarrow \boxed{\pi_1 + \pi_3 + \pi_4 = 1}$$

All the non-negative vectors of the form  $(\pi_1, \pi_1, \pi_3, \pi_4, \pi_1)$  which satisfy the equation:  $\boxed{3\pi_1 + \pi_3 + \pi_4 = 1}$ . Ans

$$(b) \text{ By raising } P \text{ to high power } (n), \text{ using R, } \lim_{n \rightarrow \infty} P^n = \begin{pmatrix} "A" & 1/3 & 1/3 & 0 & 0 & 1/3 \\ "B" & 1/3 & 1/3 & 0 & 0 & 1/3 \\ "C" & 0 & 0 & 1 & 0 & 0 \\ "D" & 0 & 0 & 0 & 1 & 0 \\ "E" & 1/3 & 1/3 & 0 & 0 & 1/3 \end{pmatrix}$$

- In the long-term, we can say by looking at  $\lim_{n \rightarrow \infty} P^n$  that if chain starts at C or D, then it stays there forever.

- Moreover, if our chain starts from A, B or E then we have an equal chance to land at any one of A, B or E in long term.

(c) As the long-term behaviour of chain depends on the initial state (as per part (b)), therefore it does not have a limiting distribution although it has a set of possible stationary distributions. As we can see that the matrix  $(\lim_{n \rightarrow \infty} P^n)$  doesn't have equal rows, we can say that this doesn't contradict the fact that limiting matrix exists.

**3.10** A Markov chain has transition matrix  $P$  and limiting distribution  $\pi$ . Further assume that  $\pi$  is the initial distribution of the chain. That is, the chain is in stationarity. Find the following:

- $\lim_{n \rightarrow \infty} P(X_n = j | X_{n-1} = i)$
- $\lim_{n \rightarrow \infty} P(X_n = j | X_0 = i)$
- $\lim_{n \rightarrow \infty} P(X_{n+1} = k, X_n = j | X_0 = i)$
- $\lim_{n \rightarrow \infty} P(X_0 = j | X_n = i)$

Sol (a)  $\lim_{n \rightarrow \infty} P(X_n = j | X_{n-1} = i) = \lim_{n \rightarrow \infty} P(X_1 = j | X_0 = i) = [P_{ij}] \text{ Ans}$

(b)  $\lim_{n \rightarrow \infty} P(X_n = j | X_0 = i) = (P^n)_{jj} = [\pi_j] \text{ Ans}$

(c)  $\lim_{n \rightarrow \infty} P(X_{n+1} = k, X_n = j | X_0 = i)$

$$= \lim_{n \rightarrow \infty} P(X_{n+1} = k | X_n = j) P(X_n = j | X_0 = i)$$

$$= \lim_{n \rightarrow \infty} P(X_1 = k | X_0 = j) P(X_n = j | X_0 = i) = [P_{jk} \pi_j] \text{ Ans}$$

(d)  $\lim_{n \rightarrow \infty} P(X_0 = j | X_n = i) = \lim_{n \rightarrow \infty} \frac{P(X_0 = j, X_n = i)}{P(X_n = i)}$

$$= \lim_{n \rightarrow \infty} \frac{\overbrace{P(X_n = i | X_0 = j)}^{\pi_i} P(X_0 = j)}{\underbrace{P(X_n = i)}_{\pi_i}} = \frac{\pi_j \pi_i}{\pi_i} = [\pi_j] \text{ Ans}$$

[here  $P(X_0 = j)$  is  $\pi_j$  because we are told to assume that the initial distribution and limiting distribution are both equal to  $\pi$ ]

- 3.14** The California Air Resources Board warns the public when smog levels are above certain thresholds. Days when the board issues warnings are called *episode* days. Lin (1981) models the daily sequence of episode and nonepisode days as a Markov chain with transition matrix

$$P = \begin{pmatrix} \text{Nonepisode} & \text{Episode} \\ \text{Nonepisode} & \begin{pmatrix} 0.77 & 0.23 \\ 0.24 & 0.76 \end{pmatrix} \\ \text{Episode} & \end{pmatrix}.$$

- (a) What is the long-term probability that a given day will be an episode day?
- (b) Over a year's time about how many days are expected to be episode days?
- (c) In the long-term, what is the average number of days that will transpire between episode days?

Sol

(a) Longterm probability of an episode day =  $\lim_{n \rightarrow \infty} P^n$  12.02.22

$$= \boxed{0.489\dots} \xrightarrow{\text{Ans}} [\text{from calculator}]$$

(b) The expected number of episode days over a year

$$= 365 \times \text{longterm probability of an episode day.}$$

$$= 365 \times 0.489362\dots \approx \boxed{178.6 \text{ days}} \xrightarrow{\text{Ans}}$$

(c) In the long-term, average number of days that will transpire between episode days =  $\boxed{1 / (\text{longterm prob. of ep. day})}$

$$= (1 / 0.489362\dots) \approx \boxed{2} \xrightarrow{\text{Ans}}$$

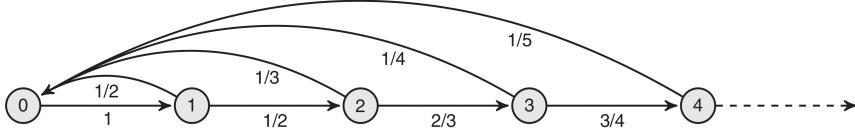
3.27 Sinclair (2005). Consider the infinite Markov chain on the non-negative integers described by Figure 3.16.

- Show that the chain is irreducible and aperiodic.
- Show that the chain is recurrent by computing the first return time to 0 for the chain started at 0.
- Show that the chain is null recurrent.



Figure 3.16

Sol



- (a) observations: — state 0 can be reached from any other non-zero state.  
— we can go from 0 to whichever state we want with a non-zero probability (if we observe  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \dots$ )

$\Rightarrow$  chain is **[IRREDUCIBLE]**

observations: — the loops like  $0 \rightarrow 1 \rightarrow 0$ ,  $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ , ... exist for  $\forall x \in (0 \rightarrow 1 \rightarrow 2 \dots x \rightarrow 0)$  where  $x \geq 1$

$\Rightarrow$  state 0 is accessible in more than or equal to 2 steps.

$\Rightarrow$  0 is **[APERIODIC]**  $\Rightarrow$  chain is **[APERIODIC]**

- (b) The probability of returning at zero after  $n$  steps is by moving straight  $(n-1)$  steps & then looping round to 0 state

$$\Rightarrow \left( \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{n-3}{n-2} \times \frac{n-2}{n-1} \right) \times \frac{1}{n} = \frac{1}{n(n-1)} \quad (\forall n \geq 2)$$

$$\Rightarrow P_{\text{total of returning to 0}} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots - \infty = 1$$

$\Rightarrow$  state 0 is **[RECURRENT]**  $\Rightarrow$  chain is **[RECURRENT]**

(c) on next page...

3.27(c)

Let Expected time to return to k state be  $E_{\text{time}}(k)$ ..

$$E_{\text{time}}(0) = \sum_i (\text{time}_i)(p_{0i}) = \sum_{n=2}^{\infty} \text{gauged by number of steps} \left( \frac{1}{\pi(n-1)} \right)$$

$$= \sum_{n=2}^{\infty} \left( \frac{1}{n-1} \right) = \infty \cdot (\text{doesn't converge}) \\ \Rightarrow \text{diverges}$$

$\Rightarrow$  Hence, 0 is **NULL RECURRENT**  $\Rightarrow$  chain is **NULL RECURRENT**

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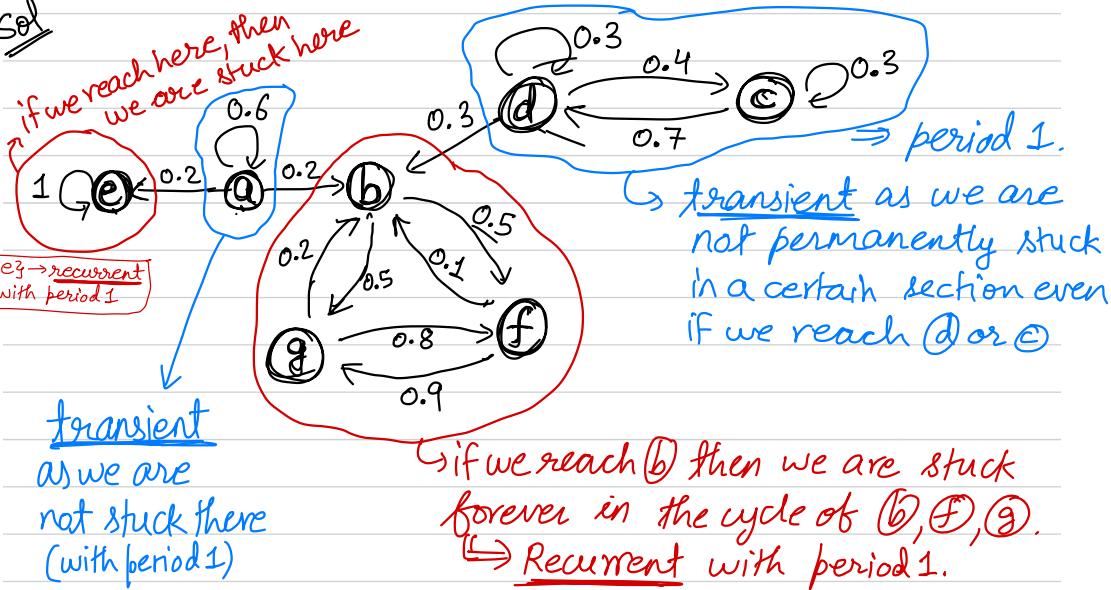
Next Problem Starts on next page...

3.29 Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} a & b & c & d & e & f & g \\ a & 0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ c & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ d & 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ f & 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\ g & 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \end{pmatrix}.$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

Sol



- $\Rightarrow \{e\}$  is recurrent with period 1
- $\Rightarrow \{a\}$  is transient with period 1
- $\Rightarrow \{b, f, g\}$  is recurrent with period 1
- $\Rightarrow \{c, d\}$  is transient with period 1.

Ans

$\{e\}$ : period =  $\text{gcd}(1) = 1$  (aperiodic)

$\{a\}$ : period =  $\text{gcd}(1) = 1$  (aperiodic) → transient is always aperiodic

$\{b, f, g\}$ : period =  $\text{gcd}(2, 3, 4, 5, \dots) = 1$  (aperiodic) ⇒ all diagonal elements of  $P^2, P^3$  are > 0.

$\{c, d\}$ : period =  $\text{gcd}(1, 2, 3, \dots) = 1$  (aperiodic) → transient is always aperiodic