Class 18: Independence & conditioning

Thursday, November 14, 2019 11:37 AM

Definition: Random variables $X_1, X_2, \ldots, X_n$ are <u>independent</u> if
$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n)$$
$$= \mathbb{P}(X_1 \in A_1)\mathbb{P}(X_2 \in A_2)\cdots\mathbb{P}(X_n \in A_n)$$
for all subsets $A_1, \ldots, A_n \subseteq \mathbb{R}$.

↳ If r.v.'s are continuous, the <u>joint pdf</u> factorizes:
$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_n}(x_n),$$

Similar for discrete case, using pmf instead of pdf.

Def: We say $X_1, \ldots, X_n$ are <u>iid</u> (independent & identically distributed) if they're independent RVs and each has the same distribution.

↳ $$\mathbb{P}(X_1 \in A) = \mathbb{P}(X_2 \in A) = \ldots = \mathbb{P}(X_n \in A)$$
for every $A \subseteq \mathbb{R}$.

↳ If continuous, this means each X_i has same PDF.

Ex 1: Maximum of iid:

Let $X_1, X_2, \ldots, X_n$ be iid continuous RVs with common PDF $f(x)$.

Let $Y = \max(X_1, \dots, X_n)$.

Goal: Find CDF/PDF of Y.

Start with CDF. Let $y \in \mathbb{R}$,

$$\mathbb{P}(Y \le y) = \mathbb{P}(\max(X_1, \dots, X_n) \le y)$$

same event!

$$= \mathbb{P}(X_1 \le y, X_2 \le y, \dots, X_n \le y)$$

$$= \mathbb{P}(X_1 \le y)\mathbb{P}(X_2 \le y) \cdots \mathbb{P}(X_n \le y)$$

by independence

$$= F_{X_1}(y) F_{X_2}(y) \cdots F_{X_n}(y).$$

where F_{X_i} is the CDF of X_i. But these CDFs are all identical. Let $F = F_{X_1} = F_{X_2} = \dots = F_{X_n}$.

So

$$F_Y(y) = \mathbb{P}(Y \le y) = (F(y))^n$$ is the CDF of Y.

The PDF is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F(y)^n \right]$$

$$= n F(y)^{n-1} \frac{d}{dy} F(y)$$

$$= n F(y)^{n-1} f(y),$$ where f was the PDF of the X_i's.

Ex 2: Minimum of iid

Let $X_1, \dots, X_n$ are iid continuous RVs with CDF F and PDF f. Let $Z = \min(X_1, \dots, X_n)$.

Goal: Find CDF & PDF of Z.

Let $r \in \mathbb{R}$. Start with CDF:

$$\mathbb{P}(Z \le r) = \mathbb{P}(\min(X_1, \dots, X_n) \le r)$$
$$= \mathbb{P}(\text{at least one of the } X_i\text{'s is } \le r)$$
$$= \dots \text{ hard to use independence here} \dots$$

Instead note $\mathbb{P}(Z \le r) = 1 - \mathbb{P}(Z > r)$.

Calculate

$$\mathbb{P}(Z > r) = \mathbb{P}(\underline{\min(X_1, \dots, X_n)} > r)$$
$$= \mathbb{P}(\underline{X_1 > r, X_2 > r, \dots,} X_n > r) \quad \text{same event!}$$
$$= \mathbb{P}(X_1 > r)\mathbb{P}(X_2 > r) \cdots \mathbb{P}(X_n > r) \quad \text{by independence}$$
$$= (1 - \mathbb{P}(X_1 \le r))(1 - \mathbb{P}(X_2 \le r)) \cdots (1 - \mathbb{P}(X_n \le r))$$
$$= (1 - F(r))(1 - F(r)) \cdots (1 - F(r))$$
$$= (1 - F(r))^n.$$

So CDF of Z is

$$F_Z(r) = \mathbb{P}(Z \le r) = 1 - \mathbb{P}(Z > r)$$
$$= 1 - (1 - F(r))^n.$$

The PDF is then

$$f_Z(r) = \frac{d}{dr} F_Z(r) = \frac{d}{dr}\left[1 - (1 - F(r))^n\right]$$
$$= -n(1 - F(r))^{n-1} \frac{d}{dr}\left[1 - F(r)\right]$$
$$= n f(r) (1 - F(r))^{n-1}$$

$F' = f$

where f and F are the PDF / CDF of X_i.

Ex 3: Let $X_1, \dots, X_n$ be iid $\mathrm{Exp}(\lambda)$ where $\lambda > 0$.

The PDF is
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Let $Z = \min(X_1, \dots, X_n)$.

Think of X_i as time until bus #i arrives.

Then Z models the time until any bus arrives.

We know CDF of X_i is $F(x) = 1 - e^{-\lambda x}$ from before.

The PDF of Z is
$$f_Z(r) = n f(r) \left(1 - F(r)\right)^{n-1} = n \lambda e^{-\lambda r} \left(e^{-\lambda r}\right)^{n-1}$$
$$= n \lambda e^{-n\lambda r} \quad \text{for } r \geq 0$$

and $f_Z(r) = 0$ for $r < 0$.

Thus $Z \sim \mathrm{Exp}(n\lambda)$.

Conditioning with Continuous RVs

Recap: If X and Y are discrete RVs with joint PMF $p_{X,Y}$ and marginals p_X, p_Y, then the conditional PMF of X given Y is the function

$$p_{X|Y}(x|y) = \mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$
$$= \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad \text{when } p_Y(y) > 0.$$

$$= \frac{p_{X,Y}(x,y)}{p_Y(y)}, \quad \text{when} \quad p_Y(y) > 0.$$

Tricky to define conditional distribution for continuous RVs because $P(Y=y)=0$.

Def:
Let X and Y be continuous RVs with joint PDF $f_{X,Y}$ and marginals f_X and f_Y. The conditional PDF of X given Y is defined as the function

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{for } x, y \text{ such that } f_Y(y) > 0.$$

Note PDFs are not probabilities! This is not equal to

$$\frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

like in discrete case.

How to use this:

1. $$P(a \le X \le b \mid Y=y) = \int_a^b f_{X|Y}(x|y)\,dx \quad \text{for } a < b.$$

Think of y as fixed.

The function $f_{X|Y}(x|y)$ is a PDF, for fixed y.

2. $$\mathbb{E}[X \mid Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$$

and similarly

and similarly

$$\mathbb{E}[g(X) \mid Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\, dx.$$

Same rules as in unconditional case apply, just with conditioning on $Y=y$ throughout.

③ $f_{X|Y}(x|y)$ is a PDF, for fixed y:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

ⓐ $f_{X|Y} \geq 0$ ✓

ⓑ Check if $\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = 1$?

Recall the definition of marginal: $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$.

Thus

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$$

$$= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \frac{1}{f_Y(y)} f_Y(y)$$

$$= 1.$$ ✓

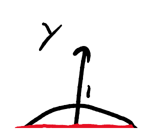
④ If X and Y are independent, then $f_{X|Y}(x|y) = f_X(x)$.

Proof:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)}$$ by independence

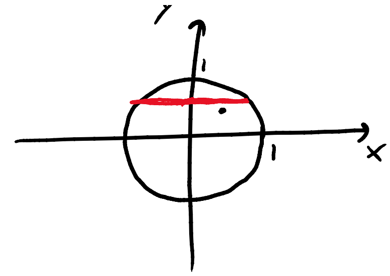
$$= f_X(x).$$

Recall: A indep of B if $\mathbb{P}(A|B) = \mathbb{P}(A)$



Ex 4: Let (X,Y) is a

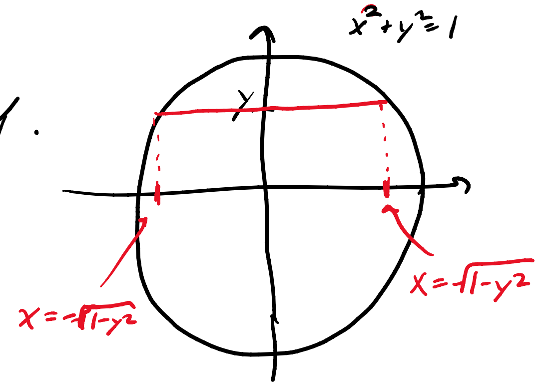
Ex 4: Let (X,Y) is a uniformly random point in the circle of radius 1 around origin. The joint PDF is



$$f_{X,Y}(x,y) = \begin{cases} \text{const.} & \text{if } (x,y) \text{ lies in circle} \\ 0 & \text{otherwise} \end{cases}$$

Here const. should be 1/area of circle. So

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



Goal 1: Find conditional PDF of X given Y.

First need marginal of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx$$

$$= \frac{2}{\pi}\sqrt{1-y^2} \quad \text{for } -1 \le y \le 1$$

and $f_Y(y) = 0$ for $|y| > 1$.

Conditional PDF:

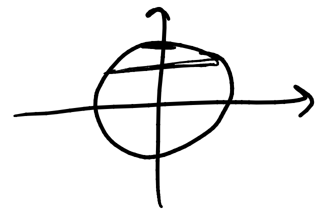
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-y^2}}$$

$$= \frac{1}{2\sqrt{1-y^2}} \quad \text{for } (x,y) \text{ in the circle.}$$

So
$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{2\sqrt{1-y^2}} & \text{for } x^2+y^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{1-y^2}} & \text{for } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \\ 0 & \text{otherwise} \end{cases}$$

Since y is fixed, as a function of x, $f_{X|Y}(x|y)$ is the PDF of a $\text{Unif}\left[-\sqrt{1-y^2}, \sqrt{1-y^2}\right]$.



Goal 2: Find $\mathbb{E}[X|Y=y]$ for $-1<y<1$.

This is 0, midpoint of $[-\sqrt{1-y^2}, \sqrt{1-y^2}]$.

Or directly:

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \cdot \frac{1}{2\sqrt{1-y^2}}\,dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \left[\tfrac{1}{2}x^2\right]_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 0.$$

Similarly:

$$\mathbb{E}[X^2|Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \frac{1}{2\sqrt{1-y^2}}\,dx$$

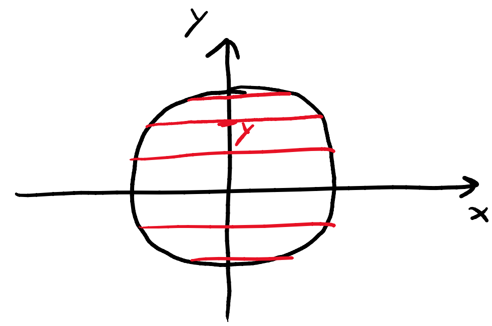
$$= \ldots = \frac{1-y^2}{3}.$$

Def: The conditional variance of X given $Y=y$ is $\text{Var}(X|Y=y) = \mathbb{E}\left[(X-\mathbb{E}[X|Y=y])^2 \mid Y=y\right]$

$$\mathrm{Var}(X \mid Y=y) = \mathbb{E}[(X - \mathbb{E}[X|Y=y])^2 \mid Y=y]$$
$$= \mathbb{E}[X^2 \mid Y=y] - (\mathbb{E}[X \mid Y=y])^2 .$$

In Ex 4 above:

$$\mathrm{Var}(X|Y=y) = \mathbb{E}[X^2|Y=y] - (\mathbb{E}[X|Y=y])^2$$
$$= \frac{1-y^2}{3} - 0$$
$$= \frac{1-y^2}{3} .$$



Note this approaches 0 as $y \to \pm 1$.