Week 9 HW Stochastic

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7.21 For an M/M/∞ queue with $\lambda = \mu = 1$, find the mean time until state 4 is hit for the process started in state 1.

Sol

Making state 4 an absorbing state, we get...

$$(-\bar{V})^{-1} = \left(- \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 10 & 9 & 4 & 1 \\ 9 & 9 & 4 & 1 \\ 8 & 8 & 4 & 1 \\ 6 & 6 & 3 & 1 \end{pmatrix}$$

The row sums are: $(24, 23, 21, 16)$

$\Rightarrow$ desired meantime $= \boxed{23}$ Ans

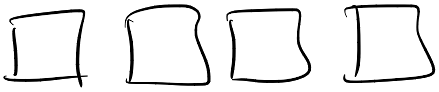
7.23 A facility has four machines, with two repair workers to maintain them. Individual machines fail on average every 10 hours. It takes an individual repair worker on average 4 hours to fix a machine. Repair and failure times are independent and exponentially distributed.

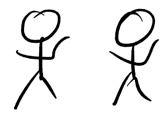
(a) Find the generator matrix.

(b) In the long term, how many machines are typically operational?

(c) If all four machines are initially working, find the probability that only two machines are working after 5 hours.

Sol

 4 machines

 2 workers

avg. failing rate (λ) = $0.1\ hr^{-1}$
avg. repair rate (μ) = $0.25\ hr^{-1}$
$\hookrightarrow \Sigma\mu = 0.5 hr^{-1}$

(a)

no. of working machines

$$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & -1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 1/10 & -3/5 & 1/2 & 0 & 0 \\ 2 & 0 & 1/5 & -7/10 & 1/2 & 0 \\ 3 & 0 & 0 & 3/10 & -11/20 & 1/4 \\ 4 & 0 & 0 & 0 & 2/5 & -2/5 \end{array}$$

(b) $\pi Q = \bar{0} \Rightarrow$ stationary distribution:

π: left eigenvector

$\pi = (0.0191, 0.0955, 0.2388, 0.2487)$

The long-term expected # working machines

$= 0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 = \boxed{2.76}$ Ans

(c) Desired probability $= P_{42}(5) = (e^{5Q})_{42}$

[using R]

$= \boxed{0.188}$ Ans

7.33 Consider an M/M/1 queue. Assume that the arrival and service time rates are both increased by a factor of k. What effect does this have on

(a) the long-term expected number of customers in the system?

(b) the long-term expected time that a customer is in the system?

Sol (a) In M/M/1 queue, 1 person gets served at a time. $\begin{pmatrix} \text{Arrival rate} : \lambda \\ \text{Serving rate} : \mu \end{pmatrix}$

As per the lab, the long-term expected number of customers in queue is the mean of a geometric distribution on 0, 1, 2, ... with parameter of $\left(1-\frac{\lambda}{\mu}\right)$.

$\Rightarrow$ if $\begin{pmatrix} \lambda \rightarrow \lambda k \\ \mu \rightarrow \mu k \end{pmatrix}$ we can say that the parameter $= \left(1-\frac{\lambda k}{\mu k}\right) = \left(1-\frac{\lambda}{\mu}\right)$

$\Rightarrow$ Hence, the value of long-term expected number of customers in queue stay same

(b) We know from the lab that expected waiting time $(W) = \left(\frac{L}{\lambda}\right)$ where L is long-term expected number of people in queue

When $\lambda \rightarrow k\lambda$,

$\Rightarrow$ The new waiting time $(W') = \frac{L}{\lambda k} = \boxed{\frac{W}{K}}$

4. Project Topic

Black Scholes Equation for Options Pricing: An application of Exponential Brownian motion and Ito's Calculus.

How can we use exponential brownian motion and Ito's calculus (specifically Ito's Lemma) to get to the Black Scholes equation which is being used to estimate/model the fair price or market price of options as a function of time and other parameters in real world

Used references: Black-Scholes original paper, Ito's original paper, and two other sources on "Financial derivatives & partial diff. equations" and "Ito's calculus in financial decision making"