MATH-365 Week 2 HW - Stochastic Processes

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2.2 Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (1/2, 0, 1/2)$. Find the following:

(a) $P(X_2 = 1 | X_1 = 3)$

(b) $P(X_1 = 3, X_2 = 1)$

(c) $P(X_1 = 3 | X_2 = 1)$

(d) $P(X_9 = 1 | X_1 = 3, X_4 = 1, X_7 = 2)$

Sol

$$P^n_{ij} = P(X_n = j \mid X_0 = i) \quad \forall\, i,j$$

(a) $P_{31} = P(X_2 = 1 \mid X_1 = 3) = \boxed{\frac{1}{3}}$ Ans

(b) $P(X_1=3, X_2=1) = P(X_2=1 \mid X_1=3)\, P(X_1=3)$ [By conditional probability]

$$P(X_1=3, X_2=1) = P_{31} \cdot P(X_1=3) = P_{31} \cdot (\alpha P)_3$$

$$\alpha P = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \left(\tfrac{1}{6}, \tfrac{5}{12}, \tfrac{5}{12}\right) \Rightarrow (\alpha P)_3 = \frac{5}{12}$$

$$\Rightarrow P(X_1=3, X_2=1) = \frac{1}{3} \times \frac{5}{12} = \boxed{\frac{5}{36}} \text{ Ans}$$

(c) $$P(X_1 = 3 \mid X_2 = 1) = \frac{P(X_1=3, X_2=1)}{P(X_2=1)} = \left(\frac{(5/36)}{(\alpha P^2)_1}\right)$$

$$\alpha P^2 = (\alpha P)P = \left(\tfrac{1}{6}, \tfrac{5}{12}, \tfrac{5}{12}\right)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \left(\tfrac{5}{9}, \tfrac{2}{9}, \tfrac{2}{9}\right) \Rightarrow (\alpha P^2)_1 = \frac{5}{9}$$

$$\Rightarrow P(X_1=3 \mid X_2=1) = \frac{(5/36)}{(5/9)} = \boxed{\frac{1}{4}} \text{ Ans}$$

(d) $P(X_9=1 \mid X_1=3, X_4=1, X_7=2) = P(X_9=1 \mid X_7=2)$

$$= P(X_3 = 1 \mid X_0 = 2) = (P^2)_{21} = \boxed{0}$$

Ans

2.4 For the general two-state chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, find the following:

(a) the two-step transition matrix

(b) the distribution of X_1

Sol

(a) $P^2 = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = \begin{pmatrix} (1-p)^2+pq & (1-p)p+p(1-q) \\ q(1-p)+q(1-q) & pq+(1-q)^2 \end{pmatrix}$

$P^2 = \boxed{\begin{pmatrix} (1+p^2+pq-2p) & (2p-p^2-pq) \\ (2q-q^2-pq) & (1+q^2+pq-2q) \end{pmatrix}}$ Ans

(b) Distribution of $X_1 = \alpha P = (\alpha_1 \; \alpha_2)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$

$= \boxed{\big((\alpha_1(1-p)+\alpha_2 q), (\alpha_2(1-q)+\alpha_1 p)\big)}$ Ans

2.5 Consider a random walk on $\{0, \ldots, k\}$, which moves left and right with respective probabilities q and p. If the walk is at 0 it transitions to 1 on the next step. If the walk is at k it transitions to $k-1$ on the next step. This is called *random walk with reflecting boundaries*. Assume that $k = 3$, $q = 1/4$, $p = 3/4$, and the initial distribution is uniform. For the following, use technology if needed.

(a) Exhibit the transition matrix.

(b) Find $P(X_7 = 1 | X_0 = 3, X_2 = 2, X_4 = 2)$.

(c) Find $P(X_3 = 1, X_5 = 3)$.

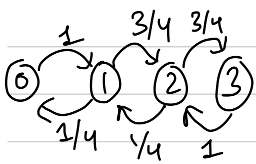
Sol

(a)

Transition Matrix

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Ans



(b) $P(X_7=1 | X_0=3, X_2=2, X_4=2) = P(X_7=1 | X_4=2) = P(X_3=1|X_0=2)$

$= P^3_{21} = \boxed{0.296875}$ Ans

$$P^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0.4375 & 0 & 0.5625 \\ 0.109375 & 0 & 0.890625 & 0 \\ 0 & 0.296875 & 0 & 0.703125 \\ 0.0625 & 0 & 0.9375 & 0 \end{pmatrix}}$$

(c) $P(X_3=1, X_5=3) = P(X_5=3 | X_3=1)\, P(X_3=1)$

$\alpha = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

as it says uniform dist^r for initial dist^r case.

$= P(X_2=3|X_0=1)\, P(X_3=1)$

$= P^2_{13} (\alpha P^3)_1 = \left(\frac{3}{4}\right)^2 (\alpha P^3)_1$

$= \frac{9}{16} (\alpha P^3)_1 = \frac{9}{16}\left(\frac{1}{4}(0.4375) + \frac{1}{4}(0.296875)\right)$

$= 0.10327148\ldots \approx \boxed{0.1033}$ Ans

2.16 Assume that $\boldsymbol{P}$ is a stochastic matrix with equal rows. Show that $\boldsymbol{P}^n = \boldsymbol{P}$, for all $n \geq 1$.

Sol As per the question, P is a stochastic matrix with equal rows. $\Rightarrow \boxed{P_{ij} = p_j}$

We will prove this through mathematical induction,

$\mathbb{P}_1 : P^1 = P$ ✓ $\boxed{\text{TRUE}}$

Let's assume that $\mathbb{P}_k$ is true: $\mathbb{P}_k : P^k = P$

We have to prove that $\mathbb{P}_{k+1}$ is true...

$\mathbb{P}_{k+1} : P^{k+1} = P^k \cdot P = P^2 \Rightarrow$ now, To prove $P^2 = P$

Since, P is a stochastic matrix with all rows same (let p_j)

P^2 is matrix with all rows $= \left(\sum_k p_k\right) p_j = p_j$ since sum of $\sum_k p_k$ is sum of all elements in a row of a stochastic matrix, which is 1.

$\Rightarrow P^2 = P$ ✓ $\boxed{\text{TRUE}}$, Hence Proved

$\Rightarrow \mathbb{P}_k$ is true, hence $\Rightarrow (\mathbb{P}_k \Rightarrow \mathbb{P}_{k+1})$ and

Hence, $\forall\, k \in \mathbb{N}$ $\mathbb{P}_k$ holds true by mathematical induction

In general, thus $\boxed{P^n = P}$, Hence Proved

Q2.27 in the rmd file continued from next page.....

q2.27

dm

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Please run the code chunk below to load the gamble and matrix power functions first! The problem and solution starts on next page.

```r
# gamblersruin.R
# Example 1.11

# gamble(k, n, p)
  #   k: Gambler's initial state
  #   n: Gambler plays until either $n or Ruin
  #   p: Probability of winning $1 at each play
  #   Function returns 1 if gambler is eventually ruined
  #                    returns 0 if gambler eventually wins $n

gamble <- function(k,n,p) {
    stake <- k
    while (stake > 0 & stake < n) {
        bet <- sample(c(-1,1),1,prob=c(1-p,p))
        stake <- stake + bet
    }
    if (stake == 0) return(1) else return(0)
    }

#k <- 10
#n <-  40
#p <- 1/2
#trials <- 1000
#simlist <- replicate(trials, gamble(k, n, p))
#mean(simlist) # Estimate of probability that gambler is ruined
# For p = 0.5, exact probability is (n-k)/n

##### Matrix powers ##############################
# matrixpower(mat,k) mat^k
#
matrixpower <- function(mat,k) {
    if (k == 0) return (diag(dim(mat)[1]))
    if (k == 1) return(mat)
    if (k > 1) return( mat %*% matrixpower(mat, k-1))
 }
```

R Markdown

2.27 R : See gamblersruin.R. Simulate gambler's ruin for a gambler with initial stake \$2, playing a fair game. (a) Estimate the probability that the gambler is ruined before he wins \$5. (b) Construct the transition matrix for the associated Markov chain. Estimate the desired probability in (a) by taking high matrix powers. (c) Compare your results with the exact probability.

ANS:

```
######## part (a) #########

n_trials <- 100000
# initial stake = 2, gambler ruined before he wins 5,
# fair game => prob = 0.5
simulation_list <- replicate(n_trials,gamble(2,5,0.5))
# finding mean to "estimate"
mean(simulation_list)
```

```
## [1] 0.60006
```

```
######## part (b) ########
p_matrix <- matrix(c(1,1/2,0,0,0,0,
                     0,0,1/2,0,0,0,
                     0,1/2,0,1/2,0,0,
                     0,0,1/2,0,1/2,0,
                     0,0,0,1/2,0,0,
                     0,0,0,0,1/2,1),nrow=6)
rownames(p_matrix) <- 0:5
colnames(p_matrix) <- 0:5
matrixpower(p_matrix,100)
```

```
##     0            1            2            3            4   5
## 0 1.0 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.0
## 1 0.8 1.726998e-10 0.000000e+00 2.794341e-10 0.000000e+00 0.2
## 2 0.6 0.000000e+00 4.521339e-10 0.000000e+00 2.794341e-10 0.4
## 3 0.4 2.794341e-10 0.000000e+00 4.521339e-10 0.000000e+00 0.6
## 4 0.2 0.000000e+00 2.794341e-10 0.000000e+00 1.726998e-10 0.8
## 5 0.0 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 1.0
```

```
print("As we can see in the resultant matrix, the requested probability is 0.6")
```

```
## [1] "As we can see in the resultant matrix, the requested probability is 0.6"
```

```
########## part (c) #########
# desired probability can be given as (n-k)/n
print((5-2)/5)
```

```
## [1] 0.6
```