Class 14: Continuous random variables

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Recap on PMFs:

(1) random variables are not events

↳ If X is a random var, and A an event,

"$X \cap A$" does not make sense. "$\cap$" for events, not r.v.'s

↳ "$\mathbb{P}(X)$" or "$\mathbb{P}(X-4)$" do not make sense

but $\mathbb{P}(X=1)$ makes sense because $\{X=1\}$ is an event.

(2) The PMF of a r.v. X is the function which maps a number k to the number $P(X=k)$. (Usually $p_X(k)$ for short.)

Every discrete r.v. has a PMF.

E.g. $3X^2-4$ is a r.v., so its PMF is $\mathbb{P}(3X^2-4=k)$ for $k \in \mathbb{R}$.

Recap 2: Covariance/correlation

$$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$$

$$\rho(X,Y) = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{Cov(X,Y)}{sd(X)\,sd(Y)}.$$

$$-1 \le \rho(X,Y) \le 1$$

Ex: Flip n coins. Let $X = \#$ heads, $Y = \#$ tails.

Here $\rho(X,Y) = -1$. Perfectly negatively correlated.

Notice $\boxed{X+Y=n}$. Both $X \sim Bin(n,\frac{1}{2})$, $Y \sim Bin(n,\frac{1}{2})$.

Recall variance of a $Bin(n,p)$ is $np(1-p)$.

$\Rightarrow$ $Var(X) = \frac{1}{4}n = Var(Y)$.

Also $Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$ since $Y=n-X$

$$= \mathbb{E}[(X-\mathbb{E}[X])(n-X-\underbrace{\mathbb{E}[n-X]}_{n-\mathbb{E}[X]})]$$

$$= -\mathbb{E}[(X-\mathbb{E}[X])^2]$$

$n - \mathbb{E}[X]$

$$= -\mathbb{E}[(X-\mathbb{E}[X])^2]$$
$$= -Var(X).$$

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-Var(X)}{\sqrt{Var(X)\cdot Var(X)}}$$ since $Var(Y) = Var(X)$

$$= -1.$$

__More generally:__ If $Y = aX + b$ for constants $a, b \in \mathbb{R}$,

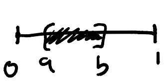
Then $\rho(X,Y) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a < 0. \end{cases}$

__Continuous Random Variables__

- Back to basics.
- Suppose our sample space Ω is uncountably infinite.
  ↳ continuum of possible outcomes
- Recall our __first__ definition of a probability measure.

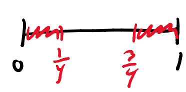
  $\mathbb{P}$ assigns a probability to each __event__ $E \subseteq \Omega$,
  with $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(E) \geq 0$ for all E,
  and $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ for __disjoint__ E and F.

__Ex 1:__ $\Omega = [0,1]$ interval of real numbers from 0 to 1.



Define $\mathbb{P}$ by requiring, for $0 \leq a \leq b \leq 1$
$$\mathbb{P}([a,b]) = b - a = \text{length of interval}.$$

Using union property:



$$\mathbb{P}\left([0,\tfrac{1}{4}] \cup [\tfrac{3}{4},1]\right) = \mathbb{P}\left([0,\tfrac{1}{4}]\right) + \mathbb{P}\left([\tfrac{3}{4},1]\right)$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Also $\mathbb{P}(\{\tfrac{1}{2}\}) = \mathbb{P}([\tfrac{1}{2},\tfrac{1}{2}]) = \frac{1}{2} - \frac{1}{2} = 0.$

The probability of any __single point__ is zero.

The probability of any single point is zero.

A consequence of this:

$$\mathbb{P}([a,b]) = \mathbb{P}((a,b)) = \mathbb{P}((a,b]) = \mathbb{P}([a,b)).$$

Recall:

$x \in [a,b]$ means $a \le x \le b$

$x \in (a,b)$ means $a < x < b$

$x \in (a,b]$ mean $a < x \le b$

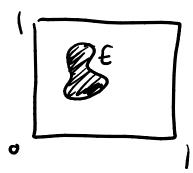
Careful derivation:

$$[a,b] = \{a\} \cup \{b\} \cup (a,b) \qquad \text{disjoint}$$

$$\hookrightarrow \mathbb{P}([a,b]) = \underbrace{\mathbb{P}(\{a\})}_{0} + \underbrace{\mathbb{P}(\{b\})}_{0} + \mathbb{P}((a,b)) = \mathbb{P}((a,b)).$$

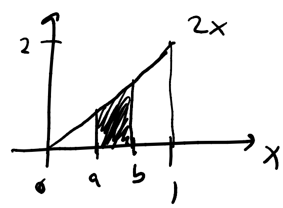
Ex 2: $\Omega = [0,1]^2$

Picking a random point in the box.



$$\mathbb{P}(E) = \text{Area}(E) \quad \text{for} \quad E \subseteq \Omega$$

Ex 3: $\Omega = [0,1]$



$\mathbb{P}([a,b]) =$ area under line between a and b

$\mathbb{P}(\Omega) =$ area between 0 and 1 $= 1$

$$\mathbb{P}([a,b]) = \int_a^b 2x\,dx = x^2\Big|_a^b = b^2 - a^2.$$

Def: A continuous random variable X is a random var. such that there exists a function $f: \mathbb{R} \to \mathbb{R}$ with $f \ge 0$ satisfying

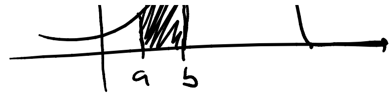
$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx \quad \text{for all } a < b.$$

$=$ area under f between a and b

We call this function f the probability density function (PDF).

Basic properties:

① $f(x) \ge 0$ for all $x \in \mathbb{R}$.



① $f(x) \geq 0$ for all $x \in \mathbb{R}$.

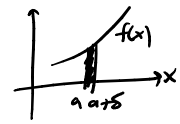
② $\mathbb{P}(a \leq X \leq b) \leq 1$ for all $a < b$

and as $a \to -\infty$ and $b \to \infty$

$\hookrightarrow$ $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

③ $\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)\,dx = 0$

for any $a \in \mathbb{R}$.



④ $\mathbb{P}(a \leq X \leq a + \delta) = \int_a^{a+\delta} f(x)\,dx \approx \delta f(a)$

for $a \in \mathbb{R}$, $\delta > 0$ small

Fact: $f(x) = \frac{d}{dx} \mathbb{P}(X \leq x)$

Why? $\mathbb{P}(X \leq x) = \mathbb{P}(-\infty \leq X \leq x) = \int_{-\infty}^{x} f(y)\,dy$

is an antiderivative of f by fundamental theorem of calculus, so its derivative is $f(x)$.

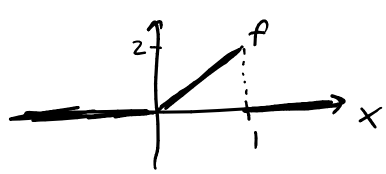
Also: $f(x) = \frac{d}{dx} \mathbb{P}(0 \leq X \leq x)$ for $x > 0$

- Think of $f(x)$ as the rate of change/acquisition of probability as we enlarge the interval on the right.

- WARNING: $f(x)$ is not the probability of any event!

Ex 1: (saw above)

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



This is a valid PDF because $f \geq 0$

and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 2x\,dx = x^2 \Big|_0^1 = 1 - 0 = 1$.

since $f = 0$ outside $[0,1]$

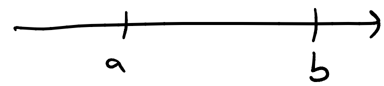
- For any $0 \le a < b \le 1$, as above,
$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx = \int_a^b 2x\,dx = x^2\Big|_a^b = b^2 - a^2.$$

- If $a < 0 < b \le 1$,
$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx = \cancel{\int_a^0 f(x)\,dx} + \int_0^b f(x)\,dx = \int_0^b 2x\,dx = b^2.$$
0 since $f = 0$ on $[a, 0]$

- Ex: $\mathbb{P}(-1 \le X \le 5) = \int_{-1}^5 f(x)\,dx = \int_0^1 2x\,dx = 1.$

- Note: The range of possible X values is the range where f is positive (strictly).

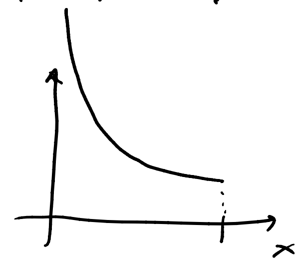
Ex 2: Let $a < b$. The uniform (continuous) r.v. $X \sim Unif[a, b]$ has PDF



$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

Note $f \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\, x\Big|_a^b = 1.$

Ex 3: Let
$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x \le 1 \\ 0 & \text{else.} \end{cases}$$



Then $f \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_0^1 = 1.$

Note: f does not have to be continuous!
In examples it will be discontinuous at at most finitely many points where it jumps. (piecewise continuous)

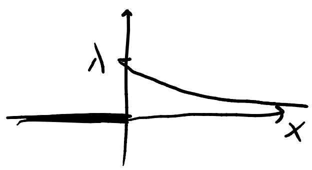
$$\mathbb{P}(X \ge \tfrac{1}{2}) = \mathbb{P}(\tfrac{1}{2} \le X \le \infty) = \int_{\frac{1}{2}}^{\infty} f(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_{\frac{1}{2}}^{1} = 1 - \frac{1}{\sqrt{2}}.$$

$$\mathbb{P}\left(X \geq \tfrac{1}{2}\right) = \mathbb{P}\left(\tfrac{1}{2} \leq X \leq \infty\right) = \int_{\frac{1}{2}}^{\infty} f(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\,\Big|_{\frac{1}{2}}^{1} = 1 - \frac{1}{\sqrt{2}}.$$

Ex 4: The exponential distribution with parameter $\lambda > 0$ (rate)

has PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$



Note: We could have written $x \geq 0$ and $x < 0$ instead. Doesn't make a difference since probability of the single point $x = 0$ is zero.

Check: $f \geq 0$ and $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 0 - (-e^{-\lambda \cdot 0})$

$= 1.$

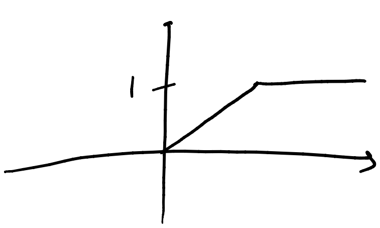
Cumulative Distribution Function (CDF)

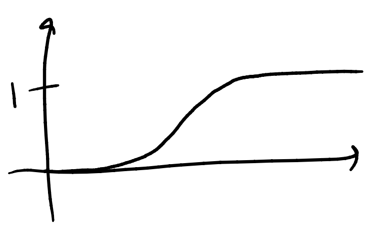
- The CDF of a r.v. X is the function sending x to $\mathbb{P}(X \leq x)$. Usually written $F_X(x) = \mathbb{P}(X \leq x)$.

  this is a probability, unlike PDF!

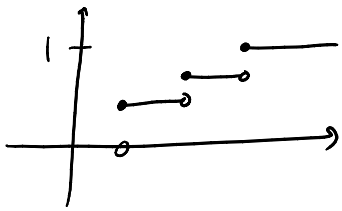
- Valid for continuous or discrete r.v.'s.
- If X is a continuous r.v., then $F_X' = f$ where f is the PDF of X. $\frac{d}{dx}\mathbb{P}(X \leq x) = f(x)$.

Plot of a typical CDF: F_X is an increasing function, increasing from 0 to 1.





Discrete r.v.'s have discontinuous CDFs,



Discrete r.v.'s have discontinuous CDFs,
continuous r.v.'s have continuous CDFs.