MATH-365 Week 2 HW - Stochastic Processes

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2.2 Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (1/2, 0, 1/2)$. Find the following:

(a) $P(X_2 = 1 | X_1 = 3)$

(b) $P(X_1 = 3, X_2 = 1)$

(c) $P(X_1 = 3 | X_2 = 1)$

(d) $P(X_9 = 1 | X_1 = 3, X_4 = 1, X_7 = 2)$

Sol

$P^n_{ij} = P(X_n = j \mid X_0 = i) \quad \forall\, i,j$

(a) $P_{31} = P(X_2 = 1 \mid X_1 = 3) = \boxed{\frac{1}{3}}$ Ans

(b) $P(X_1 = 3, X_2 = 1) = P(X_2 = 1 \mid X_1 = 3)\, P(X_1 = 3)$ [By conditional probability]

$P(X_1 = 3, X_2 = 1) = P_{31} \cdot P(X_1 = 3) = P_{31} \cdot (\alpha P)_3$

$\alpha P = \left(\frac{1}{2}, 0, \frac{1}{2}\right)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \left(\frac{1}{6}, \frac{5}{12}, \frac{5}{12}\right) \Rightarrow (\alpha P)_3 = \frac{5}{12}$

$\Rightarrow P(X_1 = 3, X_2 = 1) = \frac{1}{3} \times \frac{5}{12} = \boxed{\frac{5}{36}}$ Ans

(c) $P(X_1 = 3 \mid X_2 = 1) = \dfrac{P(X_1 = 3, X_2 = 1)}{P(X_2 = 1)} = \left(\dfrac{(5/36)}{(\alpha P^2)_1}\right)$

$\alpha P^2 = (\alpha P)P = \left(\frac{1}{6}, \frac{5}{12}, \frac{5}{12}\right)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \left(\frac{5}{9}, \frac{2}{9}, \frac{2}{9}\right) \Rightarrow (\alpha P^2)_1 = \frac{5}{9}$

$\Rightarrow P(X_1 = 3 \mid X_2 = 1) = \dfrac{(5/36)}{(5/9)} = \boxed{\frac{1}{4}}$ Ans

(d) $P(X_9 = 1 \mid X_1 = 3, X_4 = 1, X_7 = 2) = P(X_9 = 1 \mid X_7 = 2)$

$= P(X_2 = 1 \mid X_0 = 2) = (P^2)_{21} = \boxed{0}$ Ans

2.4 For the general two-state chain with transition matrix

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \boldsymbol{P} = \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, find the following:

(a) the two-step transition matrix

(b) the distribution of X_1

Sol

(a) $P^2 = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} = \begin{pmatrix} (1-p)^2 + pq & (1-p)p + p(1-q) \\ q(1-p) + q(1-q) & pq + (1-q)^2 \end{pmatrix}$

$P^2 = \boxed{\begin{pmatrix} (1+p^2+pq-2p) & (2p-p^2-pq) \\ (2q-q^2-pq) & (1+q^2+pq-2q) \end{pmatrix}}$ Ans

(b) Distribution of $X_1 = \alpha P = (\alpha_1\ \alpha_2)\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$

$= \boxed{\left((\alpha_1(1-p) + \alpha_2 q), (\alpha_2(1-q) + \alpha_1 p)\right)}$ Ans

2.5 Consider a random walk on $\{0, \ldots, k\}$, which moves left and right with respective probabilities q and p. If the walk is at 0 it transitions to 1 on the next step. If the walk is at k it transitions to $k-1$ on the next step. This is called *random walk with reflecting boundaries*. Assume that $k = 3$, $q = 1/4$, $p = 3/4$, and the initial distribution is uniform. For the following, use technology if needed.

(a) Exhibit the transition matrix.

(b) Find $P(X_7 = 1 | X_0 = 3, X_2 = 2, X_4 = 2)$.

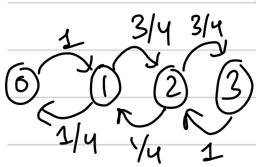
(c) Find $P(X_3 = 1, X_5 = 3)$.

Sol

(a)

Transition Matrix

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 \\ 2 & 0 & 1/4 & 0 & 3/4 \\ 3 & 0 & 0 & 1 & 0 \end{array}$$ Ans



(b) $P(X_7=1 | X_0=3, X_2=2, X_4=2) = P(X_7=1 | X_4=2) = P(X_3=1 | X_0=2)$

$= P^3_{21} = \boxed{0.296875}$ Ans

$$P^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0.4375 & 0 & 0.5625 \\ 0.109375 & 0 & 0.890625 & 0 \\ 0 & 0.296875 & 0 & 0.703125 \\ 0.0625 & 0 & 0.9375 & 0 \end{pmatrix}}$$

(c) $P(X_3=1, X_5=3) = P(X_5=3 | X_3=1)\, P(X_3=1)$

$= P(X_2=3 | X_0=1)\, P(X_3=1)$

$= P^2_{13}\, (\alpha P^3)_1 = \left(\frac{3}{4}\right)^2 (\alpha P^3)_1$

$= \frac{9}{16} (\alpha P^3)_1 = \frac{9}{16}\left(\frac{1}{4}(0.4375) + \frac{1}{4}(0.296875)\right)$

$= 0.10327148\ldots \approx \boxed{0.1033}$ Ans

$\alpha = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

as it says uniform distr for initial distr case.

2.16 Assume that $\boldsymbol{P}$ is a stochastic matrix with equal rows. Show that $\boldsymbol{P}^n = \boldsymbol{P}$, for all $n \geq 1$.

Sol: As per the question, P is a stochastic matrix with equal rows. $\Rightarrow \boxed{P_{ij} = p_j}$

We will prove this through mathematical induction,

$\mathbb{P}_1 : P^1 = P$ ✓ $\boxed{\text{TRUE}}$

Let's assume that $\mathbb{P}_k$ is true : $\mathbb{P}_k : P^k = P$

We have to prove that $\mathbb{P}_{k+1}$ is true ...

$\mathbb{P}_{k+1} : P^{k+1} = P^k \cdot P = P^2 \Rightarrow$ now, To prove $P^2 = P$

Since, P is a stochastic matrix with all rows same (let p_j)

P^2 is matrix with all rows $= \left(\sum_k p_k\right) p_j = p_j$ since sum of $\sum_k p_k$ is sum of all elements in a row of a stochastic matrix, which is 1.

$\Rightarrow P^2 = P$ ✓ $\boxed{\text{TRUE}}$. Hence Proved

$\Rightarrow \mathbb{P}_k$ is true, hence $\Rightarrow (\mathbb{P}_k \Rightarrow \mathbb{P}_{k+1})$ and

Hence, $\forall\, k \in \mathbb{N}$ $\mathbb{P}_k$ holds true by mathematical induction

In general, thus $\boxed{P^n = P}$. Hence Proved

Q2.27 in the rmd file continued from next page.....