7.2 A Markov chain on $\{1, 2, 3, 4\}$ has nonzero transition rates

$$q_{12} = q_{23} = q_{31} = q_{41} = 1 \quad \text{and} \quad q_{14} = q_{32} = q_{34} = q_{43} = 2.$$

(a) Exhibit the (i) generator, (ii) holding time parameters, and (iii) transition matrix for the embedded Markov chain.

(b) If the chain is at state 1, how long on average will it take before moving to a new state?

(c) If the chain is at state 3, how long on average will it take before moving to state 4?

(d) Over the long term, what proportion of visits will be to state 2?

Sol

(a) i) generator matrix Q:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & -3 & 1 & 0 & 2 \\ 2 & 0 & -1 & 1 & 0 \\ 3 & 1 & 2 & -5 & 2 \\ 4 & 1 & 0 & 2 & -3 \end{array}$$

$$Q = \begin{pmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ q_{31} & q_{32} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

ii) holding time parameters $\Rightarrow$ $\boxed{q_1 = 3 \quad q_2 = 1 \quad q_3 = 5 \quad q_4 = 3}$

iii) transition matrix: $\tilde{P} = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1/5 & 2/5 & 0 & 2/5 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix}$

(b) $\dfrac{1}{q_1} = \boxed{\dfrac{1}{3}\text{ hour}}$ Ans

(c) $\dfrac{1}{q_{34}} = \boxed{\dfrac{1}{2}\text{ hour}}$ Ans

(d) The stationary distribution of the embedded markov chain $(\pi) \Rightarrow$
$(0.161, 0.204, 0.376, 0.258)$

$\Rightarrow$ The long-term proportion of state 2 visits is $\boxed{20.4\%}$ Ans

7.4 During lunch hour, customers arrive at a fast-food restaurant at the rate of 120 customers per hour. The restaurant has one line, with three workers taking food orders at independent service stations. Each worker takes an exponentially distributed amount of time—on average 1 minute—to service a customer. Let X_t denote the number of customers in the restaurant (in line and being serviced) at time t. The process $(X_t)_{t \geq 0}$ is a continuous-time Markov chain. Exhibit the generator matrix.

Sol

customers arrive at rate of 120 customers per hour

one line

$\square$ worker 1, $\lambda = 1$ minute
$\square$ worker 2, $\lambda = 1$ minute
$\square$ worker 3, $\lambda = 1$ minute

exponential distributed amount of time

$$
Q = \begin{array}{c|c}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array} \\
\hline
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{array} &
\begin{pmatrix}
-2 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & -3 & 2 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & -4 & 2 & 0 & 0 & 0 & \cdots \\
0 & 0 & 3 & -5 & 2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 3 & -5 & 2 & 0 & \cdots \\
0 & 0 & 0 & 0 & 3 & -5 & 2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 3 & -5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

If we make cases we can observe that 2 customers arrive every minute but on average if all workers are available, the restaurant can serve 3 customers in a minute.

7.6 A Markov chain $(X_t)_{t\geq 0}$ on $\{1, 2, 3, 4\}$ has generator matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 \\ 2 & 2 & -4 & 0 \\ 1 & 2 & 3 & 6 \end{pmatrix}.$$

Use technology as needed for the following:

(a) Find the long-term proportion of time that the chain visits state 1.

(b) For the chain started in state 2, find the long-term probability that the chain visits state 3.

(c) Find $P(X_1 = 3|X_0 = 1)$.

(d) Find $P(X_5 = 1, X_2 = 4|X_1 = 3)$.

~~Sol~~ ~~(a)~~ The solution to this problem is on next page...