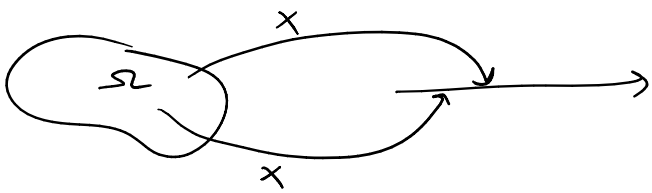
Class 8: Discrete RVs & Expectation

Thursday, September 26, 2019 11:39 AM

- Midterm 1 on Oct 8 (Tuesday)
- Optional HW 4 out today

Recap:

- A random variable (on a sample space Ω) is a function from Ω to $\mathbb{R}$.



- A discrete r.v. can take only finitely or countably many possible values.
- The probability mass function (PMF) of a discrete r.v. X is the function $p(k) = \mathbb{P}(X=k)$.
- Ex: $\mathbb{P}(X=1 \text{ or } X=2) = \mathbb{P}(X=1) + \mathbb{P}(X=2)$
$= p(1) + p(2)$.

Common RVs:

"has the distribution"

(1) Bernoulli: $X \sim Ber(p)$, $0 \le p \le 1$ parameter

PMF: $p(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$

success probability

(2) Binomial: $X \sim Bin(n,p)$, $0 \le p \le 1$, $n = 1,2,3,\dots$

\# successes

number of trials

- Ex: Flip n fair coins, set $X = \#$ heads. Then $X \sim Bin(n, \frac{1}{2})$.

PMF: $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0,1,2,\dots,n$.

Check that they add up to 1:

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \left(p + (1-p)\right)^n$$

$$= 1.$$

Binomial Theorem

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

Ex 2: Decisive Vote Probability.

- Suppose there are $2n+1$ (odd number) people voting, including myself.
- Each person votes Dem. (vs Rep.) independently with prob p, where $0 < p < 1$.
- Let X = # Dem. votes out of the $2n$ other people.

  Then $X \sim \text{Bin}(2n, p)$.

- Probability of casting a decisive vote is $\mathbb{P}(X=n)$.
- Suppose first $p = \frac{1}{2}$ (polarized). $X \sim \text{Bin}(2n, \frac{1}{2})$

$$\mathbb{P}(X=n) = \binom{2n}{n} \left(\tfrac{1}{2}\right)^n \left(1-\tfrac{1}{2}\right)^{2n-n}$$

$$= \frac{1}{2^{2n}} \binom{2n}{n}$$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2}.$$

For n large we can use Stirling's approximation:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Then

$$\mathbb{P}(X=n) \approx \frac{1}{2^{2n}} \frac{\sqrt{2\pi \cdot 2n} \left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)^2}$$

$$= \frac{2\sqrt{\pi n}\, 2^{2n}\, n^{2n}\, e^{-2n}}{2^{2n}\, 2\pi n\, n^{2n} e^{-2n}}$$

$$= \frac{1}{\sqrt{\pi n}}.$$

or $\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{\pi n}}$

This goes to zero pretty slowly as $n \to \infty$!

$2n+1 =$ population size	100,001	1,000,001
probab.	$\approx \frac{1}{400}$	$\approx \frac{1}{1250}$

- If $p \neq 1/2$, can show that $\mathbb{P}(X=n)$ decays <u>exponentially</u> as $n \to \infty$, like e^{-cn} for some $c>0$.
- From paper:

https://www.sciencedirect.com/science/article/pii/0022053181900223

③ <u>Geometric</u>: $X \sim \text{Geom}(p)$

- <u>Parameter</u> $0<p<1$ success probability
- <u>Models</u> the # of trials until first success (inclusive)
- <u>PMF</u>: $p(1) = \mathbb{P}(X=1) = \mathbb{P}(1^{st} \text{ trial is success})$

$$= p.$$

$$p(2) = \mathbb{P}(X=2) = \mathbb{P}(2^{nd} \text{ trial is success, } 1^{st} \text{ is fail})$$

$$= p(1-p)$$

$$\vdots$$

$$p(k) = \mathbb{P}(k^{th} \text{ is success, } 1^{st}-(k-1)^{th} \text{ are failures})$$

$$= p(1-p)^{k-1},$$

valid for $k=1,2,3,\dots$

<u>Check</u> that it adds up to 1:

$$\sum_{k=1}^{\infty} p(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p\sum_{k=1}^{\infty}(1-p)^{k-1}$$

$$(n=k-1)$$

$$= p\sum_{n=0}^{\infty}(1-p)^n$$

$$= p\,\frac{1}{1-(1-p)}$$

<u>Geometric series</u>:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

if $|r|<1$

$$= p \frac{1}{1-(1-p)}$$

$$= 1.$$

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ if } |r|<1}$$

④ Poisson: $X \sim Poiss(\lambda)$

- Parameter $\lambda > 0$ called the "rate"
- Models # of rare events
- PMF: $$p(k) = e^{-\lambda}\left(\frac{\lambda^k}{k!}\right) \quad \text{for } k = 0, 1, 2, \ldots$$
- Check: $$\sum_{k=0}^{\infty} p(k) = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{Taylor series for } e^{\lambda}} = e^{-\lambda} \cdot e^{\lambda} = 1.$$

- Rare Events: (Binomial limit theorem)

  Let $n > \lambda$ an integer. Let $X_n \sim Bin(n, \frac{\lambda}{n})$.

  As $n \to \infty$, more trials, but less chance of success.

  Fact: For $k = 0, 1, 2, \ldots$ we have
  $$\lim_{n\to\infty} \mathbb{P}(X_n = k) = e^{-\lambda}\frac{\lambda^k}{k!} = \mathbb{P}(X = k)$$
  where $X \sim Poiss(\lambda)$.

Transformations of RVs

- Let X be an RV with PMF p. $\left(p(k) = \mathbb{P}(X=k)\right)$
- How to find the PMF of $Y = 12X^2 - 4$?

  Or more generally, $Y = f(X)$ where $f: \mathbb{R} \to \mathbb{R}$.

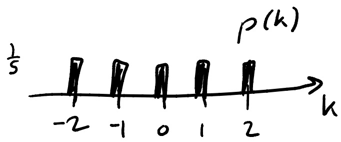
- In general: For any n,

  $\mathbb{P}(Y = \ \ ) = \mathbb{P}(f(X) = n)$

· **In general:** for any n,

$$\mathbb{P}(Y=n) = \mathbb{P}(f(X)=n) = \sum_{\substack{\text{all } k \text{ s.t.} \\ f(k)=n}} \mathbb{P}(X=k).$$

Ex 1: Suppose X has PMF $p(k)=\frac{1}{5}$ for $k=-2,-1,0,1,2$.



Let $Y=X^2$. Find PMF of Y:

The possible values for Y are $n=0,1,4$.

· $\mathbb{P}(Y=0) = \mathbb{P}(X^2=0) = \mathbb{P}(X=0)$
$= p(0) = 1/5$.

· $\mathbb{P}(Y=1) = \mathbb{P}(X^2=1) = \mathbb{P}(X=1 \text{ or } X=-1)$
$= \mathbb{P}(\{X=1\} \cup \{X=-1\})$ (just notation not a necessary step)
$= \mathbb{P}(X=1) + \mathbb{P}(X=-1)$
$= p(1) + p(-1)$
$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

· $\mathbb{P}(Y=4) = \mathbb{P}(X^2=4) = \mathbb{P}(X=2 \text{ or } X=-2)$
$= \mathbb{P}(X=2) + \mathbb{P}(X=-2)$
$= \frac{2}{5}$.

PMF of Y is
$$p_Y(n) = \begin{cases} \frac{1}{5} & \text{if } n=0 \\ \frac{2}{5} & \text{if } n=1,4 \\ 0 & \text{otherwise.} \end{cases}$$

Expectation or **Expected Value** or **Mean**

Warmup: Roll $n=1$ billion.

· Say #1 appears n_1 times

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- Say #1 appears n_1 times

  #2 appears n_2 times

  $\vdots$

  #6 appears n_6 times.

- Average roll $= \dfrac{\text{sum of all numbers that appeared}}{\text{number of rolls}}$

$$= \frac{1\cdot n_1 + 2\cdot n_2 + 3\cdot n_3 + 4 n_4 + 5 n_5 + 6 n_6}{n}$$

$$= 1\cdot \frac{n_1}{n} + 2\,\frac{n_2}{n} + \dots + 6\,\frac{n_6}{n}. \qquad = \sum \text{possible value} \times \text{prob of that value}$$

- So if n is large $\frac{n_1}{n} \simeq \frac{1}{6}$, $\frac{n_2}{n} \simeq \frac{1}{6}$, $\dots$, $\frac{n_6}{n} \simeq \frac{1}{6}$.

  and so average roll $\simeq 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + \dots + 6\cdot\frac{1}{6}$

$$= \tfrac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6} = 3.5.$$

<u>Def</u>: Let X be a RV with PMF p. The <u>expectation</u> of X is

$$\mathbb{E}[X] = \sum_k k\,\mathbb{P}(X=k) = \sum_k k\,p(k).$$

<u>Ex</u>: Let X have PMF $p(k) = \frac{1}{5}$ for $k = -2, -1, 0, 1, 2$. Then

$$\mathbb{E}[X] = (-2)\,p(-2) + (-1)\,p(-1) + 0\cdot p(0) + 1\cdot p(1) + 2\,p(2)$$

$$= -2\cdot\tfrac{1}{5} - 1\cdot\tfrac{1}{5} + 1\cdot\tfrac{1}{5} + 2\cdot\tfrac{1}{5}$$

$$= 0.$$

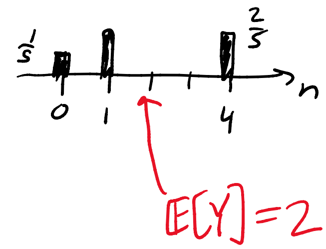
$= 0.$

Let $Y = X^2$. We found $\mathbb{P}(Y=n) = \begin{cases} \frac{1}{5} & \text{if } n=0 \\ \frac{2}{5} & \text{if } n=1,4 \end{cases}$.

So $\mathbb{E}[Y] = 0 \cdot \mathbb{P}(Y=0) + 1 \cdot \mathbb{P}(Y=1) + 4 \cdot \mathbb{P}(Y=4)$

$$= 0 \cdot \tfrac{1}{5} + 1 \cdot \tfrac{2}{5} + 4 \cdot \tfrac{2}{5} = \frac{2+8}{5}$$

$$= 2.$$



<u>Ex:</u> Let $X \sim Ber(p)$.

$$\mathbb{P}(X=1) = p$$
$$\mathbb{P}(X=0) = 1-p$$

So $\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1-p) = p.$

<u>Ex:</u> Let $X \sim Bin(n,p)$. $\quad \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$.

$$\mathbb{E}[X] = \sum_{k=0}^{n} k\, \mathbb{P}(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$k-1 \to k$

$$= np \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} p^k (1-p)^{(n-1)-k}}_{\mathbb{P}(\mathrm{Bin}(n-1,\,p)\,=\,k)}$$

$$= np \left(p + (1-p) \right)^{n-1}$$

(binomial theorem)

$$= np.$$