1. Stick broken into 2 pieces from a random place

(a) expected length of smaller piece?

Quick Wit
↳ smallest unif$(0, L/2)$
smallest piece ⇓ avg length $= L/4$

Sol $Y_1 = \min(X, 1-X)$ & $Y_2 = \max(X, 1-X)$

$$= \begin{cases} 1-X, & X > 1/2 \\ X, & X \le 1/2 \end{cases} \qquad = \begin{cases} 1-X, & X < 1/2 \\ X, & X \geq 1/2 \end{cases}$$

$$E[Y_1] = \int_0^{0.5} x\,dx + \int_{0.5}^1 (1-x)\,dx = \boxed{0.25}$$

$$E[Y_2] = \int_0^{0.5} (1-x)\,dx + \int_{0.5}^1 x\,dx = \boxed{0.75}$$

(b) expected ratio of length of smaller stick to larger one?

sol
$$R = \begin{cases} \frac{1-x}{x}, & x > 1/2 \\ \frac{x}{1-x}, & x \le \frac{1}{2} \end{cases} \Rightarrow E[R] = \int_{1/2}^1 \frac{1-x}{x}\,dx + \int_0^{1/2} \frac{x}{1-x}\,dx$$

$$\Rightarrow E[R] = [\ln x - x]_{1/2}^1 + \left(-\int_1^{1/2} \frac{1-t}{t}\,dt\right), \text{ where } t = 1-x,\ -dx = dt\ \&\ x = 1-t$$

$$\Rightarrow E[R] = \ln 1 - 1 - \ln\tfrac{1}{2} + \tfrac{1}{2} + [t - \ln t]_1^{1/2}$$

$$\Rightarrow E[R] = \cancel{\ln 1}^{0} - 1 - \ln\tfrac{1}{2} + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \ln\tfrac{1}{2} \cancel{-1} + \cancel{\ln 1}^{0}$$

$$\Rightarrow E[R] = -2\ln\tfrac{1}{2} - 1 = \boxed{2\ln 2 - 1}$$

(c) Let's do a generic problem instead. Say yo do 99 random cuts (100 pieces), now what is the expected length of smallest piece?

Sol
$X_1, \ldots, X_n \sim$ Unif$(0,1)$

$\underbrace{U_1, \ldots, U_n}_{= \text{cuts}} \Rightarrow$ order statistics

U_1 → smallest X_i; U_n → largest X_i

So, lengths:

$Y_1 = U_1$

$Y_2 = U_2 - U_1$

$\vdots$

$Y_n = U_n - U_{n-1}$

$Y_{n+1} = 1 - U_n$

PTO continued...

(c) continued...

$$\mathbb{P}(Y_1 > d_1, Y_2 > d_2, \ldots, Y_j > d_j) = (1 - d_1 - \cdots - d_j)^n$$

Hence, for $0 < c < \left(\frac{1}{n+1}\right)$, $\mathbb{P}(Y_1 > c, \ldots, Y_{n+1} > c) = (1-(n+1)c)^n$

$$E[\min\{Y_j\}] = \int_0^{1/(n+1)} (1-(n+1)c)^n \, dc$$

$$\left(\begin{array}{l} \text{substituting} \\ t = 1-(n+1)c \\ \Rightarrow dt = -(n+1)\,dc \end{array}\right)$$

$$= \frac{-1}{(n+1)} \int_1^0 t^n \, dt = \frac{-1}{(n+1)} \left[\frac{t^{n+1}}{n+1}\right]_1^0 = \frac{(-1)(-1)}{(n+1)^2}$$

$$= \boxed{\frac{1}{(n+1)^2}}$$ Ans

Hint: use PIE for $\boxed{E[\text{max-length}]}$ in general case