Week 10 HW Stochastic

— Dhyey

8.2 For standard Brownian motion, find

(a) $P(B_2 \le 1)$

(b) $E(B_4|B_1 = x)$

(c) $\mathrm{Corr}(B_{t+s}, B_s)$

(d) $Var(B_4|B_1)$

(e) $P(B_3 \le 5|B_1 = 2)$.

Sol (a) $P(B_2 \le 1) = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi(2)}} e^{-x^2/2(2)}\, dx = \boxed{0.760}$ Ans

(b) $E(B_4|B_1 = x) = E(B_4 - B_1) + x = \cancel{E(B_3)}^{\,0} + x$

$= \boxed{x}$ Ans

(c) $\mathrm{Corr}(B_{t+s}, B_s) = \frac{Cov(B_{t+s}, B_s)}{SD(B_{t+s})\, SD(B_s)}$

(from a derivation in class) $= \frac{s}{\sqrt{s+t}\,\sqrt{s}} = \boxed{\sqrt{\frac{s}{s+t}}}$ Ans

(d) $Var(B_4|B_1) = Var(B_4 - B_1 + B_1)$

$= Var(B_4 - B_1)$

$= Var(B_3) = \boxed{3}$ Ans

(e) $P(B_3 \le 5|B_1 = 2) = P(B_3 - B_1 \le 3|B_1 = 2)$

$= P(B_3 - B_1 \le 3) = P(B_2 \le 3)$

$= \int_{-\infty}^{3} \frac{1}{\sqrt{4\pi}} e^{-x^2/4}\, dx = \boxed{0.983}$ Ans

8.4 In a race between Lisa and Cooper, let X_t denote the amount of time (in seconds) by which Lisa is ahead when $100t$ percent of the race has been completed. Assume that $(X_t)_{0\le t\le 1}$ can be modeled by a Brownian motion with drift parameter 0 and variance parameter σ^2. If Lisa is leading by $\sigma/2$ seconds after three-fourths of the race is complete, what is the probability that she is the winner?

Sol

desired probability $= P(X_1 > 0 \mid X_{3/4} = \sigma/2)$

$= P(X_1 - X_{3/4} > -\sigma/2)$

$= P(X_{1/4} > -\sigma/2)$

$X_{1/4}$ is normally distributed with mean $= 0$ & variance $= \sigma^2/4$

$\Rightarrow P(X_{1/4} > -\sigma/2) = P(X_{1/4}/(\sigma/2) > -1)$

$= P(Z > -1) = \boxed{0.841}$ Ans

standard normal distribution

8.5 Consider standard Brownian motion. Let $0 < s < t$.

(a) Find the joint density of (B_s, B_t).

(b) Show that the conditional distribution of B_s given $B_t = y$ is normal, with mean and variance

$$E(B_s|B_t = y) = \frac{sy}{t} \quad \text{and } Var(B_s|B_t = y) = \frac{s(t-s)}{t}.$$

Sol

$$E[B_s|B_t=y] = \int_{-\infty}^{\infty} \left(\frac{\underbrace{x \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)}}_{f_{B_s,B_t}(x,y)}}{\underbrace{\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}}_{f_{B_t}(y)}} \right) dx$$

$$= \boxed{sy/t} \quad \text{(Mathematica)}$$

$$Var[B_s|B_t=y] = \int_{-\infty}^{\infty} \frac{\left(x - \frac{sy}{t}\right)^2 f_{B_s,B_t}(x,y)\,dx}{f_{B_t}(y)}$$

$$= \boxed{\frac{s(t-s)}{t}} \quad \text{(Mathematica)}$$

8.11 A standard Brownian motion crosses the t-axis at times $t = 2$ and $t = 5$. Find the probability that the process exceeds level $x = 1$ at time $t = 4$.

Sol

desired probability $= \underbrace{P(B_2 > 1 \mid B_3 = 0)}$

normal distr

(with mean 0 and variance 2/3.)

Ans $=$ 1 − pnorm(1, 0, sqrt(2/3))

$= \boxed{0.1103357}$ Ans

8.23 Let $0 < r < s < t$.

(a) Assume that standard Brownian motion is not zero in (r, s). Find the probability that standard Brownian motion is not zero in (r, t).

(b) Assume that standard Brownian motion is not zero in $(0, s)$. Find the probability that standard Brownian motion is not zero in $(0, t)$.

Sol

(a) Let $B_{a,b}$ is event that standard Brownian motion is not zero in (a,b).

$$\Rightarrow P(B_{r,t} \mid B_{r,s}) = \frac{P(B_{r,t})}{P(B_{r,s})}$$

$$= \frac{P(\text{none zero in } (r,t))}{P(\text{none zero in } (r,s))}$$

from notes

$$= \boxed{\frac{\sin^{-1}\left(\sqrt{\frac{r}{t}}\right)}{\sin^{-1}\left(\sqrt{\frac{r}{s}}\right)}} \quad \text{Ans}$$

(b)

$$\lim_{r \to 0} \left(\frac{\sin^{-1}\left(\sqrt{\frac{r}{t}}\right)}{\sin^{-1}\left(\sqrt{\frac{r}{s}}\right)} \right) = \lim_{r \to 0} \left(\frac{1/\left(\left(2\sqrt{1-r/t}\right)\left(\sqrt{tr}\right)\right)}{1/\left(\left(2\sqrt{1-r/s}\right)\left(\sqrt{rs}\right)\right)} \right)$$

using L'hopital rule

$$= \boxed{\sqrt{\frac{s}{t}}} \quad \text{Ans}$$

8.31 Consider standard Brownian motion started at $x = -3$.

(a) Find the probability of reaching level 2 before -7.

(b) When, on average, will the process leave the region between the lines $y = 2$ and $y = -7$?

Sol

(a) This is same as the probability of a standard Brownian motion started at 0 reaching $a=5$ before $b=4$

As derived in class, $\text{prob} = \frac{b}{a+b} = \boxed{\frac{4}{9}}$

(b) Expectation as derived in class $= ab$

$= \boxed{20}$ Ans