Simple Random Walk on a Graph

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February 24, 2022

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The objective of these exercises is to study a random walk on a small graph through a combination of simulations and analysis using R.

Problem set-up Consider the simple random walk on the graph shown below. We define a Markov chain which moves from vertex to vertex by randomly following an edge to an adjacent vertex. For example,

$$\mathbb{P}\{X_{n+1} = A \mid X_n = D\} = 1/2$$
$$\mathbb{P}\{X_{n+1} = C \mid X_n = A\} = 1/3$$
$$\mathbb{P}\{X_{n+1} = B \mid X_n = D\} = 0$$

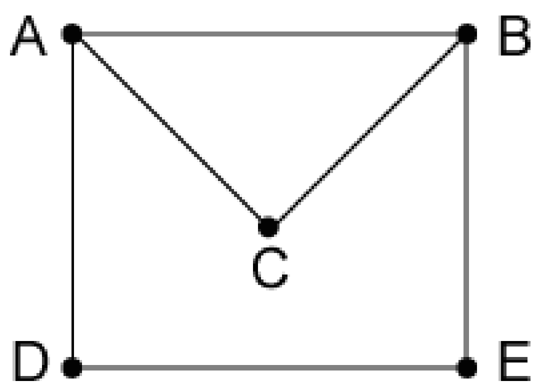


Figure 1: Graph

Determine the transition matrix $\mathbf{P}$ for this Markov chain with states $\{A, B, C, D, E\}$.

```
library(expm)
```

```
## Loading required package: Matrix

##
## Attaching package: 'expm'
```

```
## The following object is masked from 'package:Matrix':
##
##     expm
```

```
P <- matrix(0,5,5)
P[1,] <- c(0,1/3,1/3,1/3,0)
P[2,] <- c(1/3,0,1/3,0,1/3)
P[3,] <- c(1/2,1/2,0,0,0)
P[4,] <- c(1/2,0,0,0,1/2)
P[5,] <- c(0,1/2,0,1/2,0)
rownames(P)<-c('A','B','C','D','E')
colnames(P)<-c('A','B','C','D','E')
P
```

```
##           A         B         C         D         E
## A 0.0000000 0.3333333 0.3333333 0.3333333 0.0000000
## B 0.3333333 0.0000000 0.3333333 0.0000000 0.3333333
## C 0.5000000 0.5000000 0.0000000 0.0000000 0.0000000
## D 0.5000000 0.0000000 0.0000000 0.0000000 0.5000000
## E 0.0000000 0.5000000 0.0000000 0.5000000 0.0000000
```

Analysis of the Markov chain

1. Calculate the invariant probability vector $\bar{\pi}$. In the long run, about what fraction of the time is spent at vertex A?

Sol: 0.25

```
P %^% 100
```

```
##      A    B         C         D         E
## A 0.25 0.25 0.1666667 0.1666667 0.1666667
## B 0.25 0.25 0.1666667 0.1666667 0.1666667
## C 0.25 0.25 0.1666667 0.1666667 0.1666667
## D 0.25 0.25 0.1666667 0.1666667 0.1666667
## E 0.25 0.25 0.1666667 0.1666667 0.1666667
```

```
r <- eigen(t(P))
V<-r$vectors
pibar<-V[,1]/sum(V[,1])
pibar
```

```
## [1] 0.2500000 0.2500000 0.1666667 0.1666667 0.1666667
```

```
ans1 <- pibar[1]

ans1
```

```
## [1] 0.25
```

2. Suppose the random walk begins at vertex A. What is the expected number of steps until the walk returns to A?

```r
ans2 <- 1/pibar[1]
ans2
```

```
## [1] 4
```

We can use simulations to generate the distribution of return times, as the expected value doesn't tell the full story. First define a function that can randomly step from vertex to vertex:

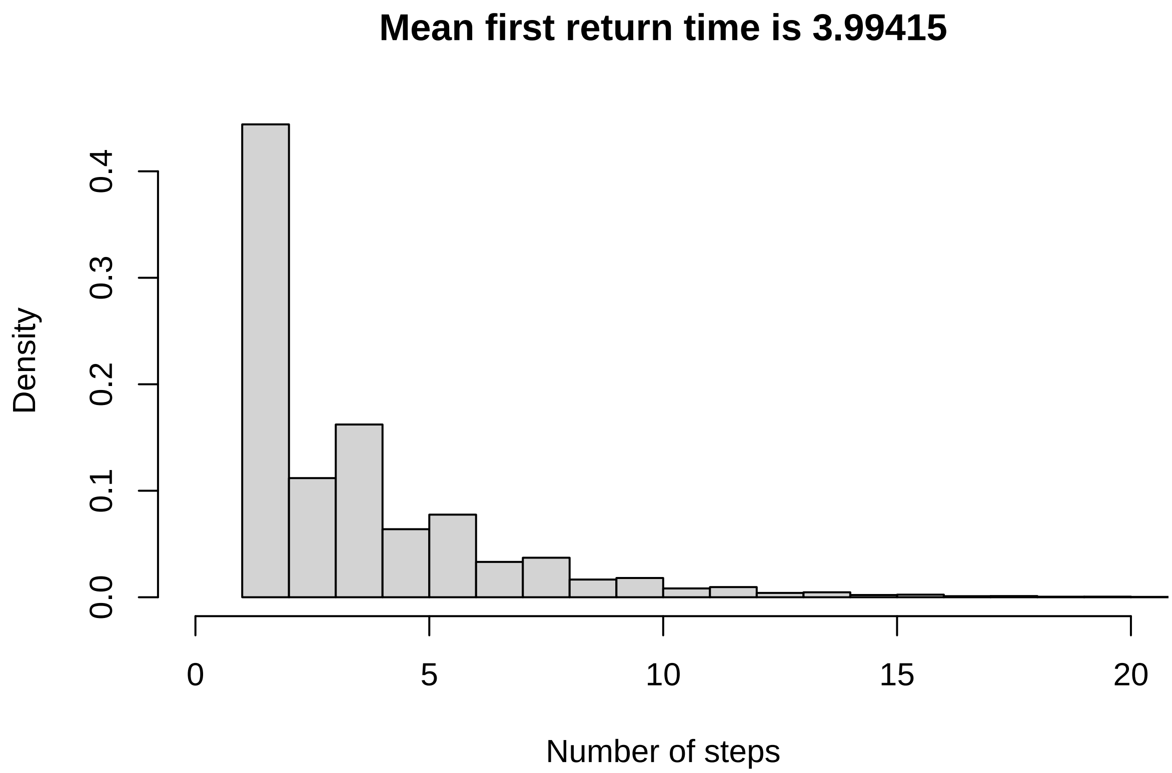
```r
takestep <- function(x) {
  switch(x,
         sample(c(2,3,4),1), #A
         sample(c(1,3,5),1), #B
         sample(c(1,2),1),   #C
         sample(c(1,5),1),   #D
         sample(c(2,4),1))   #E
}
```

Next define a function that finds the return time to a desired vertex for each simulation:

```r
waitingtime <- function(vertex) {
  x <- vertex
  for (j in 1:10000) {
    x <- takestep(x)
    ifelse(x==vertex, return(j),-1) # takes steps until hits vertex again
    } }
```

Run many simulations and plot a histogram of the return times:

```r
nsims <- 100000
sims <- matrix(0,nsims)
for (k in 1:nsims)  sims[k] <- waitingtime(1) # vertex A
hist(sims,(min(sims)-1):(max(sims)+1),freq=FALSE,
     xlab="Number of steps",xlim=c(0,20),
     main=paste("Mean first return time is",mean(sims)))
```



3. Suppose the random walk begins at vertex C. What is the expected number of steps until the walker reaches A? Hint: Make A an absorbing state, then calculate matrices $\mathbf{Q}$ and $\mathbf{M}$.

```
Q<-P[2:5,2:5]
Q
```

```
##     B         C   D         E
## B 0.0 0.3333333 0.0 0.3333333
## C 0.5 0.0000000 0.0 0.0000000
## D 0.0 0.0000000 0.0 0.5000000
## E 0.5 0.0000000 0.5 0.0000000
```

```
M<-solve(diag(4)-Q)
M
```

```
##           B         C         D         E
## B 1.6363636 0.5454545 0.3636364 0.7272727
## C 0.8181818 1.2727273 0.1818182 0.3636364
## D 0.5454545 0.1818182 1.4545455 0.9090909
## E 1.0909091 0.3636364 0.9090909 1.8181818
```

```
ans3<-sum(M[2,])
ans3
```

```
## [1] 2.636364
```

4. Suppose the random walk begins at vertex C. What is the expected number of visits to B before the walker reaches A?

```r
M[2,1]
```

```
## [1] 0.8181818
```

5. Suppose the random walk begins at vertex B. What is the probability that the walker reaches A before C?

```r
P[1,]<-c(1,0,0,0,0)
P[3,]<-c(0,0,1,0,0)
P
```

```
##           A   B         C   D         E
## A 1.0000000 0.0 0.0000000 0.0 0.0000000
## B 0.3333333 0.0 0.3333333 0.0 0.3333333
## C 0.0000000 0.0 1.0000000 0.0 0.0000000
## D 0.5000000 0.0 0.0000000 0.0 0.5000000
## E 0.0000000 0.5 0.0000000 0.5 0.0000000
```

```r
P_ac_absorbing_asymptotic<-P %^% 1000
P_ac_absorbing_asymptotic
```

```
##           A             B         C             D             E
## A 1.0000000  0.000000e+00 0.0000000  0.000000e+00  0.000000e+00
## B 0.5714286 3.136456e-191 0.4285714 3.136456e-191  0.000000e+00
## C 0.0000000  0.000000e+00 1.0000000  0.000000e+00  0.000000e+00
## D 0.8571429 4.704683e-191 0.1428571 4.704683e-191  0.000000e+00
## E 0.7142857  0.000000e+00 0.2857143  0.000000e+00 7.841139e-191
```

```r
P_ac_absorbing_asymptotic[2,1]
```

```
## [1] 0.5714286
```