MATH-365 Week 4 HW - Stochastic Processes

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3.37 Show that all two-state Markov chains, except for the trivial chain whose transition matrix is the identity matrix, are time reversible.

Sol

We have $P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-p & p \\ 2 & q & 1-q \end{array}$, assuming $\pi = [\pi_1 \ \pi_2]$

$\pi P = \pi$ gives us...

$$\left.\begin{aligned} \pi_1 &= \pi_1(1-p) + \pi_2 q \\ \pi_2 &= \pi_1 p + (1-q)\pi_2 \end{aligned}\right\} \Rightarrow \begin{aligned} &\pi_1 p = \pi_2 q \\ &\& \ \pi_1 + \pi_2 = 1 \Rightarrow \boxed{\begin{aligned} \pi_1 &= \frac{q}{p+q} \\ \pi_2 &= \frac{p}{p+q} \end{aligned}} \end{aligned}$$

$$\Rightarrow \pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$$

For the Markov chain to be time-reversible,

$\pi_1 P_{12} = \pi_2 P_{21}$, by substituting π_1, π_2, P_{12} & P_{21}...

$\left(\frac{q}{p+q}\right)(p) = \left(\frac{p}{p+q}\right)(q)$, which is True.

$$\Rightarrow \boxed{\pi_1 P_{12} = \pi_2 P_{21} = \frac{pq}{p+q}}$$

Hence proved that all two-state markov chains are time reversible except the trivial chain whose transition matrix is identity matrix because in that case $p = q = 0$, which signifies that we cannot get from one to another.

3.46 Given a Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$, the *time reversal* is a Markov chain with transition matrix $\widetilde{\boldsymbol{P}}$ defined by

$$\boxed{\widetilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i}, \quad \text{for all } i, j.} \rightarrow \text{used in next problem...}$$

(a) Show that a Markov chain with transition matrix $\boldsymbol{P}$ is reversible if and only if $\boldsymbol{P} = \widetilde{\boldsymbol{P}}$.

(b) Show that the time reversal Markov chain has the same stationary distribution as the original chain.

Sol

(a) To prove: chain is reversible iff $\tilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$ $\forall i,j$

We know that, a markov chain is time-reversible iff $\pi_i P_{ij} = \pi_j P_{ji}$ where $P_{ij}, P_{ji} \neq 0$.

Plugging in.. $\Rightarrow \tilde{P}_{ij} = \frac{\cancel{\pi_i} P_{ij}}{\cancel{\pi_i}} = \boxed{P_{ij}}$, Hence Proved

(b) As we know that π is the stationary distribution of original chain, $(\pi\tilde{P})_j = \sum_i \pi_i \tilde{P}_{ij}$, substituting from part (a)...

$$\Rightarrow (\pi\tilde{P})_j = \sum_i \cancel{\pi_i}\left(\frac{\pi_j P_{ji}}{\cancel{\pi_i}}\right) = \sum_i \pi_j P_{ji} = \pi_j\,(1).$$
$$= \boxed{\pi_j}$$

thus, π is the stationary distribution of reversal chain too

, Hence Proved

3.47 Consider a Markov chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}.$$

Find the transition matrix of the time reversal chain (see Exercise 3.46).

Sol

For the reversal markov chain $\left(\tilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i} \right)$

Firstly, lets calculate π for P..

assuming π to be $[\pi_1\ \pi_2\ \pi_3]$, $\pi P = \pi$

$$\Rightarrow \quad \pi_1 = \frac{\pi_1}{3} + \frac{\pi_2}{2} + \frac{\pi_3}{6} \Rightarrow \frac{2\pi_1}{3} = \frac{4\pi_1}{9} + \frac{2\pi_1}{9} \Rightarrow \text{verified!}$$

$$\pi_2 = \frac{\pi_2}{2} + \frac{\pi_3}{3} \Rightarrow \pi_2 = \frac{2\pi_3}{3} = \boxed{\frac{8\pi_1}{9}} = \boxed{\frac{8}{29}}$$

$$\pi_3 = \frac{2\pi_1}{3} + \frac{\pi_3}{2} \Rightarrow \boxed{\pi_3 = \frac{4\pi_1}{3}} = \boxed{\frac{12}{29}}$$

Also,

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 + \frac{8\pi_1}{9} + \frac{4\pi_1}{3} = 1$$

$$\frac{29\pi_1}{9} = 1$$

$$\Rightarrow \boxed{\pi_1 = \frac{9}{29}}$$

$$\Rightarrow \tilde{P} = \begin{pmatrix} \tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\ \tilde{P}_{21} & \tilde{P}_{22} & \tilde{P}_{23} \\ \tilde{P}_{31} & \tilde{P}_{32} & \tilde{P}_{33} \end{pmatrix} = \begin{pmatrix} \frac{\cancel{\pi_1} P_{11}}{\cancel{\pi_1}} & \frac{\pi_2 P_{21}}{\pi_1} & \frac{\pi_3 P_{31}}{\pi_1} \\ \frac{\pi_1 P_{12}}{\pi_2} & \frac{\cancel{\pi_2} P_{22}}{\cancel{\pi_2}} & \frac{\pi_3 P_{32}}{\pi_2} \\ \frac{\pi_1 P_{13}}{\pi_3} & \frac{\pi_2 P_{23}}{\pi_3} & \frac{\cancel{\pi_3} P_{33}}{\cancel{\pi_3}} \end{pmatrix} = \boxed{\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/3 & 4/9 & 2/9 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \end{array}}$$

Ans

3.59 Consider random walk on the weighted graph in Figure 3.20.

(a) If the walk starts in a, find the expected number of steps to return to a.

(b) If the walk starts in a, find the expected number of steps to first hit b.

(c) If the walk starts in a, find the probability that the walk hits b before c.

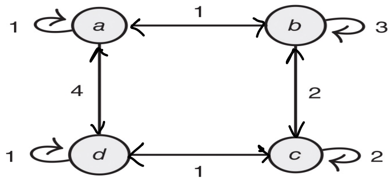

Figure 3.20

$$\begin{array}{c|cc} & b & c \\ \hline a & \frac{1}{6} & 0 \\ d & 0 & \frac{1}{6} \end{array}$$

Sol

(a)

Transition matrix $\Rightarrow$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1/6 & 1/6 & 0 & 4/6 \\ b & 1/6 & 3/6 & 2/6 & 0 \\ c & 0 & 2/5 & 2/5 & 1/5 \\ d & 4/6 & 0 & 1/6 & 1/6 \end{array}$$

as $\pi = \pi P$, assuming $\pi = [\pi_a, \pi_b, \pi_c, \pi_d]$, we get....

$$\left.\begin{aligned} \pi_a &= \frac{\pi_a}{6} + \frac{\pi_b}{6} + \frac{4\pi_d}{6} \\ \pi_b &= \frac{\pi_a}{6} + \frac{3\pi_b}{6} + \frac{2\pi_c}{5} \\ \pi_c &= \frac{2\pi_b}{6} + \frac{2\pi_c}{5} + \frac{\pi_d}{6} \\ \pi_d &= \frac{4\pi_a}{6} + \frac{\pi_c}{5} + \frac{\pi_d}{6} \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow 5\pi_a = \pi_b + 4\pi_d \\ &\Rightarrow 15\pi_b = 5\pi_a + 12\pi_c \\ &\Rightarrow 18\pi_c = 10\pi_b + 5\pi_d \\ &\Rightarrow 25\pi_d = 20\pi_a + 6\pi_c \end{aligned}$$

$$\boxed{\pi = \left[\frac{6}{23}, \frac{6}{23}, \frac{5}{23}, \frac{6}{23}\right]} \text{ Ans}$$

Expected number of steps to return to a $= E_{steps}(\text{return } a) = \frac{1}{\pi_a} = \frac{23}{6} \approx \boxed{3.83}$ Ans

(b) To find the expected number of steps to first hit b, lets model this by making b an absorbing state...

(continued on next page)...

part (b) continued...

The fundamental matrix in this case would be $(I_3 - Q)^{-1}$
where, Q = matrix with rows and columns labelled b deleted.

$$Q = \begin{array}{c|c} & \begin{array}{ccc} a & c & d \end{array} \\ \begin{array}{c} a \\ c \\ d \end{array} & \begin{pmatrix} 1/6 & 0 & 4/6 \\ 0 & 2/5 & 1/5 \\ 4/6 & 1/6 & 1/6 \end{pmatrix} \end{array} \quad \Bigg\| \quad \underbrace{(I_3 - Q)^{-1}}_{\text{expected value matrix}} = \begin{pmatrix} 5/6 & 0 & -4/6 \\ 0 & 3/5 & -1/5 \\ -4/6 & -1/6 & 5/6 \end{pmatrix}^{-1} = \begin{pmatrix} 42/11 & 10/11 & +36/11 \\ 12/11 & 45/11 & +15/11 \\ +36/11 & +25/22 & 45/11 \end{pmatrix}$$

$$E_{\text{steps}}(\text{first hit } b) = \frac{42}{11} + \frac{10}{11} + \frac{36}{11} = \boxed{8} \text{ Ans}$$

(c) Similar to the logic used in part (b), in this case let's make both states b and c absorbing....

$$Q = \begin{array}{c|c} & \begin{array}{cc} a & d \end{array} \\ \begin{array}{c} a \\ d \end{array} & \begin{pmatrix} 1/6 & 4/6 \\ 4/6 & 1/6 \end{pmatrix} \end{array} \quad \Bigg\| \quad (I_2 - Q)^{-1} = \begin{pmatrix} 5/6 & -4/6 \\ -4/6 & 5/6 \end{pmatrix}^{-1} = \begin{pmatrix} 5/9 & +4/9 \\ +4/9 & 5/9 \end{pmatrix}$$

$$\text{Then, } \underbrace{(I_2 - Q)^{-1} R}_{\text{probability matrix}} = \begin{pmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{pmatrix} \begin{array}{c|c} & \begin{array}{cc} b & c \end{array} \\ \begin{array}{c} a \\ d \end{array} & \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix} \end{array} = \boxed{\begin{array}{c|c} & \begin{array}{cc} b & c \end{array} \\ \begin{array}{c} a \\ d \end{array} & \begin{pmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{pmatrix} \end{array}}$$

The probability that walk hits b before c $= \boxed{\frac{5}{9}}$ Ans