Week 7: HW Dhyey Mavani

6.1 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 1.5$. Find the following:

(a) $P(N_1 = 2, N_4 = 6)$

(b) $P(N_4 = 6 | N_1 = 2)$

(c) $P(N_1 = 2 | N_4 = 6)$

Sol

(a) $P(N_1=2, N_4=6) = P(N_1=2, N_4-N_1=4)$

$= P(N_1=2)\,P(N_4-N_1=4) = P(N_1=2)\,P(N_3=4)$

$= \left(e^{-1.5}\cdot\frac{(1.5)^2}{2!}\right)\left(e^{-4.5}\cdot\frac{(4.5)^4}{4!}\right) = \frac{e^{-6}(3)^4(1.5)^6}{(2!)(4!)} = \frac{e^{-6}(1.5)^{10}}{3} = \boxed{0.048}$ Ans

(b) $P(N_4=6 | N_1=2) = P(N_4-N_1=4 | N_1=2) = P(N_4-N_1=4)$

$= P(N_3=4) = \left(e^{-4.5} * \frac{(4.5)^4}{4!}\right) = \boxed{0.1898}$ Ans

(c) $P(N_1=2 | N_4=6) = \frac{P(N_1=2, N_4=6)}{P(N_4=6)} = \frac{0.048}{\left(\frac{e^{-6}(6)^6}{(6!)}\right)} = \boxed{0.297}$ Ans

6.3 Calls are received at a company call center according to a Poisson process at the rate of five calls per minute.

(a) Find the probability that no call occurs over a 30-second period.

(b) Find the probability that exactly four calls occur in the first minute, and six calls occur in the second minute.

(c) Find the probability that 25 calls are received in the first 5 minutes and six of those calls occur in the first minute.

Sol

(a) $P(N_{0.5}=0) = e^{-5(0.5)} = e^{-2.5} = \boxed{0.082}$ Ans

(b) $P(N_1=4, N_2-N_1=6) = P(N_1=4)\,P(N_2-N_1=6) = P(N_1=4)\,P(N_1=6)$

$= \left(\frac{(e^{-5})(5)^4}{4!}\right)\left(\frac{(e^{-5})(5)^6}{6!}\right)$

$= \boxed{0.0257}$ Ans

(c) $P(N_1=6, N_5=25) = P(N_1=6, N_5-N_1=19) = P(N_1=6)\;P(N_4=19)$

$= \left(\frac{e^{-5}(5)^6}{6!}\right)\left(\frac{e^{-20}(20)^{19}}{19!}\right)$

$= \boxed{0.01299}$ Ans

6.6 Occurrences of landfalling hurricanes during an El Niño event are modeled as a Poisson process in Bove et al. (1998). The authors assert that "During an El Niño year, the probability of two or more hurricanes making landfall in the United States is 28%." Find the rate of the Poisson process.

Sol

letting $0.28 = P(N_1 \geq 2) = 1 - P(N_1 = 0) - P(N_1 = 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$

$\Rightarrow (\lambda + 1)e^{-\lambda} = 0.72 \Rightarrow \boxed{\lambda = 1.043}$ Ans

using wolfram alpha

6.7 Ben, Max, and Yolanda are at the front of three separate lines in the cafeteria waiting to be served. The serving times for the three lines follow independent Poisson processes with respective parameters 1, 2, and 3.

(a) Find the probability that Yolanda is served first.

(b) Find the probability that Ben is served before Yolanda.

(c) Find the expected waiting time for the first person served.

Sol

(a) $P(\min(B,M,Y)=Y) = \dfrac{\lambda_Y}{\lambda_B+\lambda_M+\lambda_Y} = \dfrac{3}{1+2+3} = \boxed{\dfrac{1}{2}}$

(b) $$P(B<Y) = \int_0^\infty P(B<y \mid Y=y)\, 3e^{-3y}\,dy = \int_0^\infty P(B<y)\,3e^{-3y}\,dy$$
$$= \int_0^\infty (1-e^{-y})\,3e^{-3y}\,dy$$
$$= \int_0^\infty \left(3e^{-3y} - 3e^{-4y}\right)dy$$
$$= \left[\frac{3e^{-3y}}{(-3)} - \frac{3e^{-4y}}{(-4)}\right]_0^\infty = 0 - \left[\frac{3}{-3} - \frac{3}{-4}\right]$$
$$= 1 - \frac{3}{4} = \boxed{\frac{1}{4}}$$
Ans

(c) $\lambda_B+\lambda_M+\lambda_Y = 1+2+3 = 6 \Rightarrow$ expected waiting time $= \dfrac{1}{\lambda_B+\lambda_M+\lambda_Y} = \boxed{\dfrac{1}{6}}$
Ans

6.8 Starting at 6 a.m., cars, buses, and motorcycles arrive at a highway toll booth according to independent Poisson processes. Cars arrive about once every 5 minutes. Buses arrive about once every 10 minutes. Motorcycles arrive about once every 30 minutes.

(a) Find the probability that in the first 20 minutes, exactly three vehicles—two cars and one motorcycle—arrive at the booth.

(b) At the toll booth, the chance that a driver has exact change is 1/4, independent of vehicle. Find the probability that no vehicle has exact change in the first 10 minutes.

(c) Find the probability that the seventh motorcycle arrives within 45 minutes of the third motorcycle.

(d) Find the probability that at least one other vehicle arrives at the toll booth between the third and fourth car arrival.

Sol (a) $P(C_{20}=2, B_{20}=0, M_{20}=1) = P(C_{20}=2) \cdot P(B_{20}=0) \cdot P(M_{20}=1)$

$\Rightarrow$ using R: dpois (2, 20/5) * dpois(0, 20/10) * dpois(1, 20/30) = $\boxed{0.00678738}$ Ans

(b) Arrival of vehicles is a Poisson's process with parameter $= \frac{1}{5} + \frac{1}{10} + \frac{1}{30} = \boxed{\frac{1}{3}}$

The thinned process of cars with exact change is a Poisson's Process with parameter $= \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \boxed{\frac{1}{12}}$

desired probability is $e^{-10/12} = \boxed{0.435}$ Ans

(c) $P(S_7 - S_3 < 45) = P(\cancel{S_3} + X_4 + X_5 + X_6 + X_7 - \cancel{S_3} < 45)$

$= P(X_4 + X_5 + X_6 + X_7 < 45) = P(Z_4 < 45)$

$\Rightarrow$ here we can say that Z is a gamma distribution essentially

$\Rightarrow$ using R: pgamma (45, 4, 1/30) = $\boxed{0.06564245}$ Ans

(d) Let Z_t be number of buses and motorcycles arriving in an interval of length t.

parameter $= \frac{1}{10} + \frac{1}{30} = \boxed{\frac{2}{15}}$

The length of interval T between 3rd & 4th car arrival is exponentially distributed with parameter $\boxed{\frac{1}{5}}$

Conditioning on the length of the interval gives...

$$P(Z_T > 0) = \int_0^\infty P(Z_T > 0 | T = t) \left(\frac{1}{5}\right) e^{-t/5} dt = \frac{1}{5}\int_0^\infty (1 - e^{-2t/15}) e^{-t/5} dt = 1 - \frac{1}{5}\int_0^\infty e^{-t/3} dt = 1 + \frac{3}{5}\left[e^{-t/3}\right]_0^\infty$$

$$= \boxed{\frac{2}{5}} \text{ Ans}$$

Alternative way

Ans $= \dfrac{(2/15)}{\left(\frac{2}{15}\right) + \frac{3}{15}} = \frac{2}{5}$

6.15 Failures occur for a mechanical process according to a Poisson process. Failures are classified as either major or minor. Major failures occur at the rate of 1.5 failures per hour. Minor failures occur at the rate of 3.0 failures per hour.

(a) Find the probability that two failures occur in 1 hour.

(b) Find the probability that in half an hour, no major failures occur.

(c) Find the probability that in 2 hours, at least two major failures occur or at least two minor failures occur.

Sol

The failure process is Poisson with parameter $1.5+3.0=4.5$

Let, M_t and m_t denote thinned major and minor processes respectively

(a) $P(N_1=2) = \left(e^{-4.5}\dfrac{(4.5)^2}{2!}\right) = \boxed{0.112}$ Ans

(b) $e^{-1.5/2} = \boxed{0.472}$ Ans

(c)

$$P(M_2 \geq 2 \text{ or } m_2 \geq 2) = 1 - P(M_2 \leq 1, m_2 \leq 1) = 1 - P(M_2 \leq 1)P(m_2 \leq 1)$$

$$= 1-(e^{-3}+3e^{-3})(e^{-6}+6e^{-6})$$

$$= 1-28e^{-9} = \boxed{0.997}$$ Ans