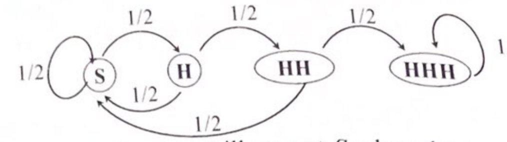
Coin triplets

Part A. If you keep on tossing a fair coin, what is the expected number of tosses such that you can have HHH (heads heads heads) in a row? What is the expected number of tosses to have THH (tails heads heads) in a row?

Solution: The most difficult part of the Markov chain is, again, to choose the right state space. For the HHH sequence, the state space is straightforward. We only need four states: S (for the starting state when no coin is tossed or whenever a T turns up before HHH), H, HH, and HHH. The transition graph is



At state S, after a coin toss, the state will stay at S when the toss gives a T. If the toss gives an H, the state becomes H. At state H, it has 1/2 probability goes back to state S if the next toss is T; otherwise, it goes to state HH. At state HH, it also has 1/2 probability goes back to state S if the next toss is T; otherwise, it reaches the absorbing state HHH.

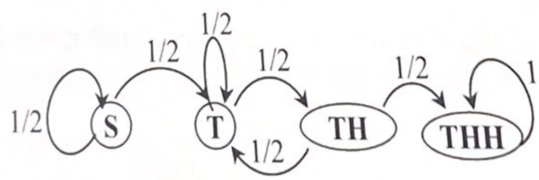
So we have the following transition probabilities: $P_{S,S}=\frac{1}{2}$, $P_{S,H}=\frac{1}{2}$, $P_{H,S}=\frac{1}{2}$, $P_{H,HH}=\frac{1}{2}$, $P_{HH,S}=\frac{1}{2}$, $P_{HH,HHH}=\frac{1}{2}$, and $P_{HHH,HHH}=1$.

We are interested in the expected number of tosses to get HHH, which is the expected time to absorption starting from state S. Applying the standard equations for the expected time to absorption, we have

$$\left.\begin{aligned}\mu_S&=1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_H\\ \mu_H&=1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_{HH}\\ \mu_{HH}&=1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_{HHH}\\ \mu_{HHH}&=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}\mu_S&=14\\ \mu_H&=12\\ \mu_{HH}&=8\\ \mu_{HHH}&=0\end{aligned}\right.$$

So from the starting state, the expected number of tosses to get HHH is 14.

Similarly, for the expected time to reach THH, we can construct the following transition graph and estimate the corresponding expected time to absorption:



$$\left.\begin{aligned}\mu_S&=1+\tfrac{1}{2}\mu_S+\tfrac{1}{2}\mu_T\\ \mu_T&=1+\tfrac{1}{2}\mu_T+\tfrac{1}{2}\mu_{TH}\\ \mu_{TH}&=1+\tfrac{1}{2}\mu_T+\tfrac{1}{2}\mu_{THH}\\ \mu_{THH}&=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}\mu_S&=8\\ \mu_T&=6\\ \mu_{TH}&=2\\ \mu_{THH}&=0\end{aligned}\right.$$

So from the starting state S, the expected number of tosses to get THH is 8.

Part B. Keep flipping a fair coin until either HHH or THH occurs in the sequence. What is the probability that you get an HHH subsequence before THH?

Solution: Let's try a standard Markov chain approach. Again, the focus is on choosing the right state space. In this case, we begin with starting state S. We only need ordered subsequences of either HHH or THH. After one coin is flipped, we have either state T or H. After two flips, we have states TH and HH. We do not need TT (which is equivalent to T for this problem) or HT (which is also equivalent to T as well). For three-coin sequences, we only need THH and HHH states, which are both absorbing states. Using these states, we can build the following transition graph:

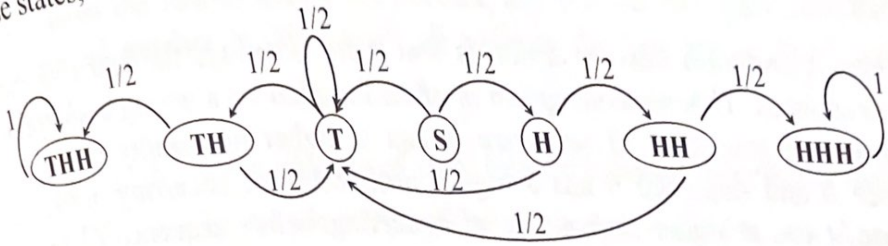


Figure 5.4 Transition graph of coin tosses to reach HHH or THH

We want to get the probability to reach absorbing state HHH from the starting state S. Applying the equations for absorption probability, we have

$$\left.\begin{array}{l} a_{HHH}=1,\ a_{THH}=0 \\ a_S=\frac{1}{2}a_T+\frac{1}{2}a_H \\ a_T=\frac{1}{2}a_T+\frac{1}{2}a_{TH},\ a_H=\frac{1}{2}a_T+\frac{1}{2}a_{HH} \\ a_{TH}=\frac{1}{2}a_T+\frac{1}{2}a_{THH},\ a_{HH}=\frac{1}{2}a_T+\frac{1}{2}a_{HHH} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_T=0, a_{TH}=0 \\ a_S=\frac{1}{8} \\ a_H=\frac{1}{4} \\ a_{HH}=\frac{1}{2} \end{array}\right.$$

So the probability that we end up with the HHH pattern is 1/8.

This problem has a special feature that renders the calculation unnecessary. You may have noticed that $a_T=0$. Once a tail occurs, we will always get THH before HHH. The reason is that the last two coins in THH is HH, which is the first two coins in sequence HHH. In fact, the only way that the sequence reaches state HHH before THH is that we get three consecutive Hs in the beginning. Otherwise, we always have a T before the first HH sequence and always end in THH first. So if we don't start the coin-flipping sequence with HHH, which has a probability of 1/8, we will always have THH before HHH.

Part C. (Difficult) Let's add more fun to the triplet game. Instead of fixed triplets for the two players, the new game allows both to choose their own triplets. Player 1 chooses a triplet first and announces it; then player 2 chooses a different triplet. The players again toss the coins until one of the two triplet sequences appears. The player whose chosen triplet appears first wins the game.

If both player 1 and player 2 are perfectly rational and both want to maximize their probability of winning, would you go first (as player 1)? If you go second, what is your probability of winning?[2]

Solution: A common misconception is that there is always the best sequence that beats other sequences. This misconception is often founded on a wrong assumption that these sequences are transitive: If sequence A has a higher probability of occurring before sequence B and sequence B has a higher probability of occurring before sequence C, sequence A has a higher probability of occurring before sequence C. In reality, such transitivity does not exist for this game. No matter what sequence player 1 chooses, player 2 can always choose another sequence with more than 1/2 probability of winning. The key, as we have indicated in Part B, is to choose the last two coins of the sequence as the first two coins of player 1's sequence. We can compile the following table for each pair of sequences:

2's winning probability		Player 1: HHH	THH	HTH	HHT	TTH	THT	HTT	TTT
Player 2	HHH	/	1/8	2/5	1/2	3/10	5/12	2/5	1/2
	THH	**7/8**	/	1/2	**3/4**	1/3	1/2	1/2	3/5
	HTH	3/5	1/2	/	1/3	3/8	1/2	1/2	7/12
	HHT	1/2	1/4	**2/3**	/	1/2	5/8	**2/3**	7/10
	TTH	7/10	**2/3**	5/8	1/2	/	**2/3**	1/4	1/2
	THT	7/12	1/2	1/2	3/8	1/3	/	1/2	3/5
	HTT	3/5	1/2	1/2	1/3	**3/4**	1/2	/	**7/8**
	TTT	1/2	2/5	5/12	3/10	1/2	2/5	1/8	/

Table 5.1 Player 2's winning probability with different coin sequence pairs

As shown in Table 5.1 (you can confirm the results yourself), no matter what player 1's choices are, player 2 can always choose a sequence to have better odds of winning. The best sequences that player 2 can choose in response to 1's choices are highlighted in bold. In order to maximize his odds of winning, player 1 should choose among HTH, HTT, THH, and THT. Even in these cases, player 2 has a 2/3 probability of winning.

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[2] This problem is a difficult one. Interested reader may find the following paper helpful: "Waiting Time and Expected Waiting Time-Paradoxical Situations" by V. C. Hombas, *The American Statistician*, Vol. 51, No. 2 (May, 1997), pp. 130-133. In this section, we will only discuss the intuition.