Chapter 9 Stochastic Calculus

Two flavors of stochastic integral:

$$\int_0^t B_s\,ds \quad \text{and} \quad \int_0^t f(s, B_s)\,dB_s$$

integrate wrt time

integrate wrt standard Br. motion

What does it mean to integrate a stochastic process?

Example: $Z_t = \int_0^t Y_s\,dB_s$

B_t = price of one share of stock at time t

Y_t = # of shares of that stock held at time t
(continuous trading allowed)

How much money did you make or lose?

$$Z_t = \int_0^t Y_s \, dB_s$$

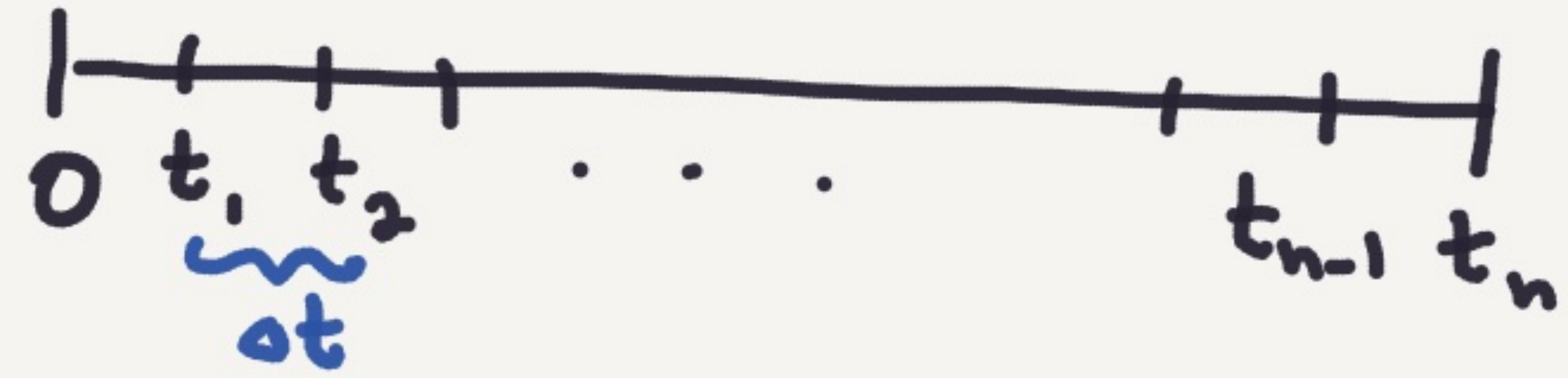
B_t = price of one share of stock at time t

Y_t = # of shares of that stock held at time t (continuous trading allowed)

On time interval $(t, t+\Delta t]$, the price changes by $\Delta B_t = B_{t+\Delta t} - B_t$, with $\approx Y_t$ shares, so change in equity is $Y_t \Delta B_t = Y_t (B_{t+\Delta t} - B_t)$.

Total net change over time interval divided into small Δt increments is

$$\approx \sum_{k=1}^{n} Y_{t_k} \Delta B_{t_k}$$



$t_k = k\Delta t$

Itô def'n of $\int_0^t Y_s \, dB_s$ is $\lim_{\Delta t \to 0} \sum_{k=1}^{n} Y_{t_k} \Delta B_{t_k}$

note this limit isn't straightforward like a Riemann sum def'n of a regular integral

We need some tools for evaluating this expression.

One property we'd like to have the integral satisfy is that $(Z_t)_{t\geq 0}$ be a martingale wrt the Brownian motion $(B_t)_{t\geq 0}$

$$Z_t = \int_0^t Y_s \, dB_s$$

In our stock example, that implies there's no way to make money.

One idea: Let $Y_t = B_t$ (price goes up, you buy; price goes down, you sell)

A naive calculation might be $\int_0^t B_s \, dB_s = \frac{1}{2}B_t^2 - \cancel{\frac{1}{2}B_0^2}^{\,0} = \frac{1}{2}B_t^2$

but $\mathbb{E}[\frac{1}{2}B_t^2] = \frac{1}{2}\mathrm{Var}[B_t] = \frac{1}{2}t$

This can't be right – it claims you steadily make money!

We need a "correction" term.

Recall the Fundamental Theorem of Calculus: $f(t) - f(0) = \int_0^t f'(s)\,ds$

We want an equivalent formula for $\int_0^t f'(B_s)\,dB_s$

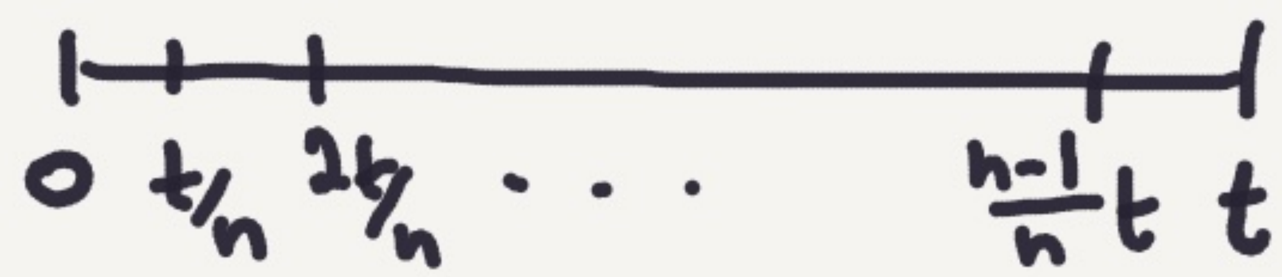
Intuition: we can't ignore terms with dB_t^2 because $\mathrm{Var}[B_t] = t$

(in regular calc, we ignore dx^2 as going to 0 much faster than dx)

Taylor expansion about $x = x_0$:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + o((x-x_0)^2)$$

Subdivide $[0,t]$ into n intervals $\left[\frac{k}{n}t, \frac{k+1}{n}t\right]$



Telescoping sum: $f(B_t) = f(B_0) + \sum_{k=0}^{n-1} \left(f(B_{\frac{k+1}{n}t}) - f(B_{\frac{k}{n}t}) \right)$

Taylor expansion about $B_{\frac{k}{n}t}$: $f(B_{\frac{k+1}{n}t}) = f(B_{\frac{k}{n}t}) + f'(B_{\frac{k}{n}t})(B_{\frac{k+1}{n}t} - B_{\frac{k}{n}t})$
$+ \frac{1}{2} f''(B_{\frac{k}{n}t})(B_{\frac{k+1}{n}t} - B_{\frac{k}{n}t})^2 + t \cdot o(\frac{1}{n})$

Plug into telescoping sum:

$$f(B_t) - f(B_0) = \sum_{k=0}^{n-1} f'(B_{\frac{k}{n}t}) \underbrace{(B_{\frac{k+1}{n}t} - B_{\frac{k}{n}t})}_{dB_s \text{ as } n \to \infty} + \frac{1}{2} \sum_{k=0}^{n-1} f''(B_{\frac{k}{n}t}) \underbrace{(B_{\frac{k+1}{n}t} - B_{\frac{k}{n}t})^2}_{\substack{\text{looks like variance,} \\ t/n \to ds \text{ as } n \to \infty}} + t \cdot o(\tfrac{1}{n})$$

Let $n \to \infty$,

$$\int_0^t f'(B_s)\,dB_s \quad + \quad \underbrace{\frac{1}{2}\int_0^t f''(B_s)\,ds}_{\text{"correction" term}}$$

Itô's Lemma

If $f(x)$ is twice differentiable and $(B_t)_{t\geq 0}$ is a standard Brownian motion, then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$$

Return to stock example: $Z_t = \int_0^t B_s\,dB_s$

using $f'(x) = x$,
so $f(x) = \frac{1}{2}x^2$ and $f''(x) = 1$

$$= f(B_t) - f(B_0) - \frac{1}{2}\int_0^t f''(B_s)\,ds$$

$$= \frac{1}{2}B_t^2 - \frac{1}{2}\cancel{B_0^2}^{\,0} - \frac{1}{2}\int_0^t 1\,ds$$

$$= \frac{1}{2}B_t^2 - \frac{1}{2}t$$

$$\mathbb{E}[Z_t] = \frac{1}{2}\mathrm{Var}[B_t] - \frac{1}{2}$$

$$= \frac{1}{2}t - \frac{1}{2}t$$

$$= 0$$

$\leftarrow$ makes more sense - not so easy to make money

Itô's Lemma If $f(x)$ is twice differentiable and $(B_t)_{t\geq 0}$ is a standard Brownian motion, then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$$

Shorthand differential form:

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2} f''(B_t)\,dt$$

Extension of Itô's Lemma If $g(t,x)$ is a real-valued function with continuous 2^{nd} order partial derivatives, then

$$g(t,B_t) - g(0,B_0) = \int_0^t \left(\frac{\partial}{\partial t} g(s,B_s) + \frac{1}{2}\frac{\partial^2}{\partial x^2} g(s,B_s)\right) ds + \int_0^t \frac{\partial}{\partial x} g(s,B_s)\,dB_s$$

Shorthand:

$$dg = \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right) dt + \frac{\partial g}{\partial x}\,dB_t$$

(informally, can think of as combination of chain rule and Itô's Lemma)

Itô's Lemma: $f(B_t) - f(B_0) = \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$

Example: $\int_0^t B_s\,dB_s =$

Example: $\int_0^t B_s^2\,dB_s =$

Example: $\int_0^t (B_s^2 - s)\,dB_s =$

Extension: $g(t, B_t) - g(0, B_0) = \int_0^t \left(\frac{\partial}{\partial t} g(s, B_s) + \frac{1}{2} \frac{\partial^2}{\partial x^2} g(s, B_s) \right) ds$
$+ \int_0^t \frac{\partial}{\partial x} g(s, B_s)\,dB_s$

Generalized Itô's Lemma: If $g(t,x)$ has two continuous derivatives in x and one continuous derivative in t, and

$dZ_t = X_t\,dt + Y_t\,dB_t$, then

$$g(t,Z_t) - g(0,Z_0) = \int_0^t \frac{\partial g}{\partial x}(s,Z_s)\,Y_s\,dB_s + \int_0^t \left(\frac{\partial g}{\partial t}(s,Z_s) + \frac{\partial g}{\partial x}(s,Z_s)X_s + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(s,Z_s)Y_s^2 \right) ds$$

Shorthand: $$dg(t,Z_t) = \frac{\partial g}{\partial x}(t,Z_t)Y_t\,dB_t + \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}Y_t^2 \right) dt$$

General properties of Itô's stochastic integral

1) Linearity: $\int_0^t (aX_s + bY_s)\,dB_s = a\int_0^t X_s\,dB_s + b\int_0^t Y_s\,dB_s$

2) For $0 < r < t$, $\int_0^t X_s\,dB_s = \int_0^r X_s\,dB_s + \int_r^t X_s\,dB_s$

3) $(Z_t)_{t\geq 0}$ defined as $Z_t = \int_0^t X_s\,dB_s$ is a martingale wrt $(B_t)_{t\geq 0}$

4) $\mathbb{E}[Z_t] = 0$ for all $t \geq 0$ (because $Z_0 = 0$ and $(Z_t)_{t\geq 0}$ is a martingale)

5) $\mathrm{Var}[Z_t] = \mathbb{E}\left[\left(\int_0^t X_s\,dB_s\right)^2\right] = \int_0^t \mathbb{E}[X_s^2]\,ds$ (very messy to prove)

Note: Itô integral does not satisfy most Calc 1 rules – have to be careful and remember the correction term.

Generalized Itô's Lemma:

$$dZ_t = X_t\,dt + Y_t\,dB_t \Rightarrow$$

$$dg(t,Z_t) = \frac{\partial g}{\partial x}(t,Z_t)Y_t\,dB_t + \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}Y_t^2\right)dt$$

→ Example: Geometric Brownian motion

Suppose stochastic process $(S_t)_{t\geq 0}$ satisfies

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$$

$X_t = \mu S_t$

$Y_t = \sigma S_t$

Apply generalized Itô's Lemma to $\ln(S_t)$:

$$d(\ln S_t) = \frac{1}{S_t}\cdot \sigma S_t\,dB_t + \left(0 + \frac{1}{S_t}\cdot\mu S_t + \frac{1}{2}\cdot\frac{-1}{S_t^2}\sigma^2 S_t^2\right)dt$$

using $g(t,x) = \ln x$

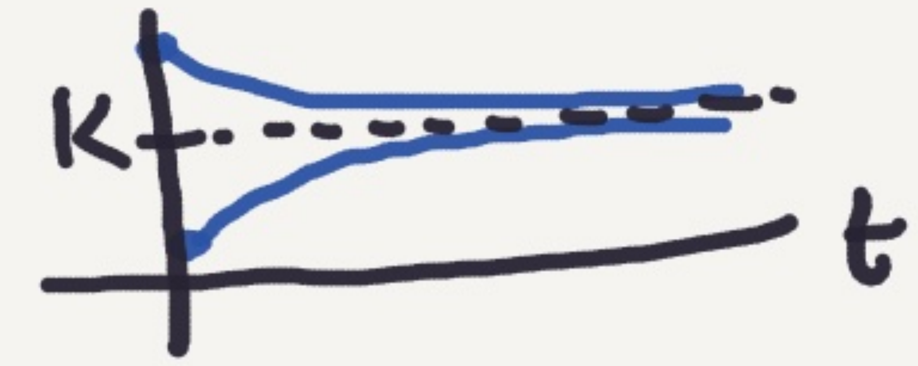
$$\ln(S_t) - \ln(S_0) = \sigma(B_t - B_0) + (\mu - \tfrac{1}{2}\sigma^2)(t-0)$$

$$\Rightarrow S_t = S_0\, e^{(\mu-\frac{1}{2}\sigma^2)t + \sigma B_t}$$

Logistic equation (model growth of population $P(t)$
where environment has carrying capacity K)

Ordinary differential eq $\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)$

growth rate



Stochastic DE $dP_t = \frac{r}{K}P_t(K-P_t)\,dt + \sigma P_t\,dB_t$

noise parameter

First apply transformation $Q_t = 1/P_t$ to yield a linear SDE:

$$dQ_t = d(1/P_t) = -\frac{1}{P_t^2}\cdot\sigma P_t\,dB_t + \left(-\frac{1}{P_t^2}\cdot\frac{r}{K}P_t(K-P_t) + \frac{1}{2}\cdot\frac{2}{P_t^3}\sigma^2 P_t^2\right)dt$$

$g(t,x) = 1/x$

$$= -\sigma Q_t\,dB_t + \left(-rQ_t + \frac{r}{K} + \sigma^2 Q_t\right)dt$$

$$\boxed{dQ_t = \left((\sigma^2-r)Q_t + \frac{r}{K}\right)dt - \sigma Q_t\,dB_t}$$

Linear SDEs $dX_t = (aX_t+b)\,dt + c\,dB_t$ have known solution

$$X_t = G_t\left(X_0 + b\int_0^t \frac{1}{G_s}\,ds\right), \text{ where } G_t = e^{(a-c^2/2)t + cB_t}$$

(geometric Br. motion)

Black-Scholes formula for option pricing

Black-Scholes Formula for option pricing

Let S_t = price of a stock at time t and suppose S_t satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \qquad \text{geometric Br. motion}$$

This is a linear SDE, so has sol'n $S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$

Also assume some investment Y_t in bonds whose value grows at rate r:

$$dY_t = rY_t \, dt$$

European call option (strike price K at time T) is opportunity to buy stock at time t for price K, hoping that $S_T > K$.

Black-Scholes formula is a way of predicting S_T to determine a good value for K.

Let $V_t = V(t, S_t)$ be the value of of option at time $t \le T$.

Total assets are $U_t = -V_t + X_t S_t + Y_t$ (X_t = # shares in stock at time t)

$$X_t = G_t\left(X_0 + b\int_0^t \frac{1}{G_s}\,ds\right), \text{ where } G_t = e^{(a - c^2/2)t + cB_t}$$

The transformed logistic SDE has $a = \sigma^2 - r$

$b = r/K$

$c = -\sigma$

$$\Rightarrow Q_t = G_t\left(Q_0 + \frac{r}{K}\int_0^t \frac{1}{G_s}\,ds\right) \text{ where } G_t = e^{(\sigma^2 - r - \sigma^2/2)t - \sigma B_t}$$

$$= e^{(-r + \sigma^2/2)t - \sigma B_t}$$

$$\Rightarrow P_t = \frac{1}{Q_t} = \frac{e^{(r - \sigma^2/2)t + \sigma B_t}}{1/P_0 + \frac{r}{K}\int_0^t e^{-(r-\sigma^2/2)s - \sigma B_s}\,ds}$$

$$\boxed{P_t = \frac{P_0 K e^{(r-\sigma^2/2)t + \sigma B_t}}{K + P_0 r \int_0^t e^{-(r-\sigma^2/2)s - \sigma B_s}\,ds}}$$

Solution to logistic SDE

P_0 = initial population

r = growth rate

K = carrying capacity

σ = noise parameter

Using a bunch of assumptions like no arbitrage opportunities, self-financing, etc, obtain a stochastic PDE for $V(t, S_t)$:

$$\frac{\partial}{\partial t}V(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2}{\partial x^2}V(t,x) + r\,x\,\frac{\partial}{\partial x}V(t,x) - r\,V(t,x) = 0$$

Note μ has disappeared — value of option only depends on bond rate r and volatility σ^2

The Black-Scholes formula is the solution to this stochastic PDE

(will let those doing projects on Black-Scholes provide more details in their presentations)

Reference: Intro to Stochastic Processes by G. Lawler, 2nd Edition pages 223-227

Ornstein-Uhlenbeck Process - "mean-reverting"

SDE $dX_t = -r(X_t - \mu)\,dt + \sigma B_t$ (assume $r, \sigma > 0$)

Solve using generalized Itô's Lemma (or sol'n for linear SDEs):

$$d(e^{rt}X_t) = (re^{rt}X_t - e^{rt}r(X_t-\mu))\,dt + e^{rt}\sigma\,dB_t$$

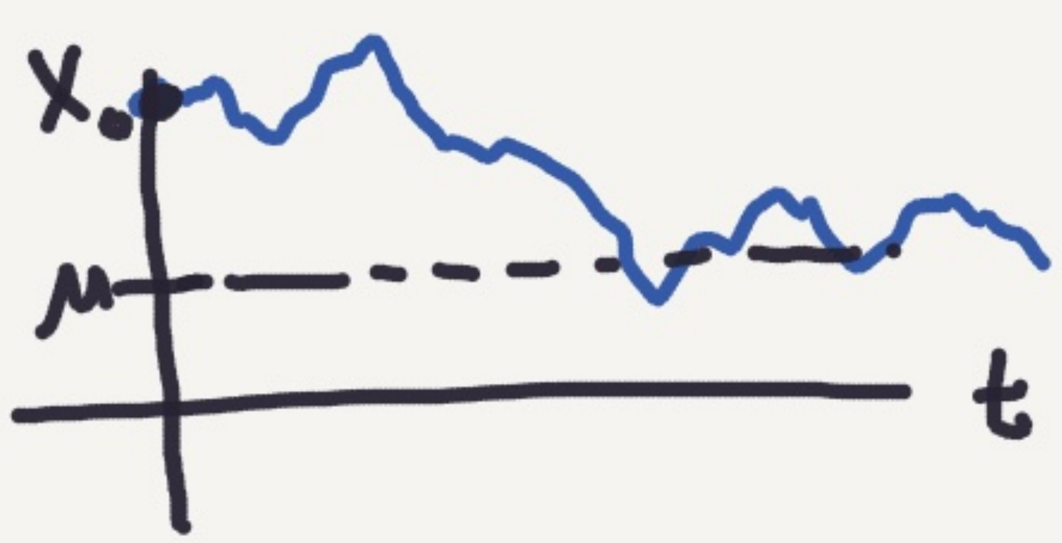
$g(t,x) = e^{rt}x$

$$= r\mu e^{rt}\,dt + e^{rt}\sigma\,dB_t$$

$$\Rightarrow e^{rt}X_t - e^{r\cdot 0}X_0 = r\mu\int_0^t e^{rs}\,ds + \sigma\int_0^t e^{rs}\,dB_s$$

$$= r\mu\cdot\frac{1}{r}(e^{rt} - e^{r\cdot 0}) + \sigma\int_0^t e^{rs}\,dB_s$$

$$e^{rt}X_t - X_0 = \mu(e^{rt}-1) + \sigma\int_0^t e^{rs}\,dB_s$$

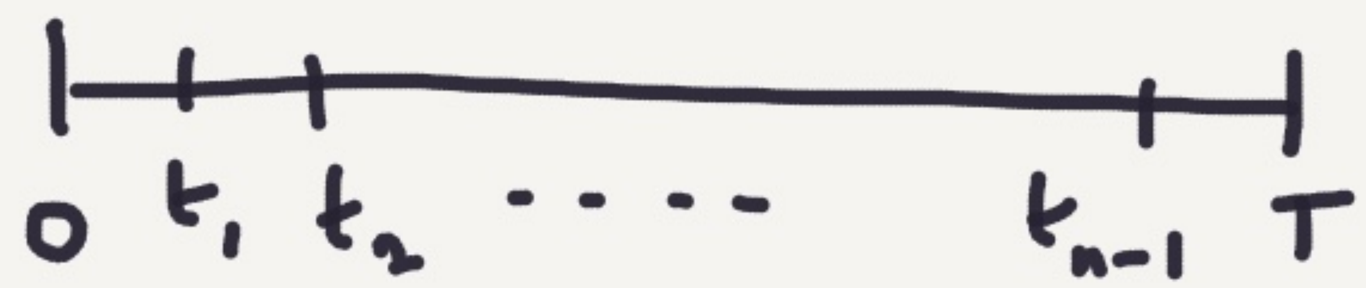


$$\Rightarrow X_t = \mu + (X_0-\mu)e^{-rt} + \sigma\int_0^t e^{-r(t-s)}\,dB_s$$

$\mathbb{E}[X_t] = \mu + (X_0-\mu)e^{-rt}$ tends toward μ as $t \to \infty$

Euler-Maruyama method for numerical approximation

Partition time interval $[0,T]$ into n equally spaced points $t_k = \frac{kT}{n}$



$\Delta t = \frac{T}{n}$

$\Delta B_{t_k} = B_{t_k} - B_{t_{k-1}}$

is normally distributed with mean 0 and variance Δt

Given initial value X_0,

can numerically approximation sol'n to SDE $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$ using iteration

$$X_{k+1} = X_k + a(t_k, X_k)\Delta t + b(t_k, X_k)\sqrt{\Delta t}\, Z_k \quad \text{for } k = 0, 1, 2, \cdots, n-1$$

where Z_k are independent samples from the standard normal distribution

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