# Class 12: Joint distributions & conditioning

Tuesday, October 15, 2019 11:38 AM

NO CLASS TUESDAY Oct 22

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## Conditioning on an event

Let $X$ be a r.v. and let $A$ be an event.

The conditional PMF of $X$ given $A$

$$p_{X|A}(x) = \mathbb{P}(X=x \mid A) = \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}, \quad \text{for } x \in \mathbb{R}.$$

- This function $p_{X|A}$ is itself a PMF.

  i.e. $p_{X|A} \geq 0$ and $\sum_x p_{X|A}(x) = 1$.

- Define the conditional expectation of $X$ given $A$ by

$$\mathbb{E}[X|A] = \sum_x x\, p_{X|A}(x) = \sum_x x\, \mathbb{P}(X=x|A),$$

and likewise

$$\mathbb{E}[f(X)|A] = \sum_x f(x) p_{X|A}(x) = \sum_x f(x)\, \mathbb{P}(X=x|A),$$

for functions $f: \mathbb{R} \to \mathbb{R}$.

Ex 1: Roll a die. $X$ = value rolled. $\to \mathbb{P}(X=x) = \frac{1}{6}$ for $x = 1, 2, \ldots, 6$.

Let $A$ = "even number" $= \{X \in \{2,4,6\}\}$.

Then $p_{X|A}(1) = \mathbb{P}(X=1|A) = \mathbb{P}(X=1 \mid X \text{ even}) = 0$

$$p_{X|A}(2) = \mathbb{P}(X=2 \mid X \text{ even}) = \frac{\mathbb{P}(X=2,\ X \text{ even})}{\mathbb{P}(X \text{ even})}$$

$$= \frac{\mathbb{P}(X=2)}{\mathbb{P}(X \text{ even})} = \frac{1/6}{1/2} = \frac{1}{3}.$$

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p_X$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{6}$ |
| $p_{X\|A}$ | $0$ | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ |

Also $\mathbb{E}[X|A] = \sum_x x\, p_{X|A}(x) = 2\, p_{X|A}(2) + 4\, p_{X|A}(4) + 6\, p_{X|A}(6)$

$= 2\cdot\frac{1}{3} + 4\cdot\frac{1}{3} + 6\cdot\frac{1}{3}$

$= 4.$

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**Def:**

Let $X$ and $Y$ be r.v.'s with joint PMF $p_{X,Y}$.

(meaning $p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$ for all $x,y$)

Recall that the marginal distributions are

$$p_X(x) = \sum_y p_{X,Y}(x,y), \quad \text{and} \quad p_Y(y) = \sum_x p_{X,Y}(x,y).$$

The conditional PMF of $X$ given $Y$ is defined by

$$p_{X|Y}(x|y) = \mathbb{P}(X=x \mid Y=y)$$

for all $x$ and $y$ such that $\mathbb{P}(Y=y) > 0$.

Notes:
- For fixed $y$, $p_{X|Y}(x|y)$ is a PMF, as a function of $x$. That is, $p_{X|Y} \geq 0$, and $\sum_x p_{X|Y}(x|y) = 1$.
- $$p_{X|Y}(x|y) = \mathbb{P}(X=x \mid Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

$= p_{X,Y}(x,y)$

$$\mathbb{P}(Y=y)$$

$$= \frac{p_{X,Y}(x,y)}{p_Y(y)}.$$

- Likewise, the conditional PMF of $Y$ given $X$ is

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \mathbb{P}(Y=y \mid X=x).$$

<u>Ex 2</u>: Roll two 4-sided dice. $\Omega = \{1,2,3,4\}^2$, equally likely outcomes with probabilities $1/16$ each.

Let $X=$ minimum, $Y=$ maximum.

Joint PMF $p_{X,Y}$

can't have $X>Y$

| | Y \ X | 1 | 2 | 3 | 4 | $p_Y$ |
|---|---|---|---|---|---|---|
| (1,1) | 1 | $\frac{1}{16}$ | 0 | 0 | 0 | $p_Y(1)=\frac{1}{16}$ |
| (1,2) or (2,1) | 2 | $\frac{2}{16}$ | $\frac{1}{16}$ | 0 | 0 | $p_Y(2)=\frac{3}{16}$ |
| (1,3) or (3,1) | 3 | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ | 0 | $p_Y(3)=\frac{5}{16}$ |
| | 4 | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ | $p_Y(4)=\frac{7}{16}$ |
| | $p_X$ | $p_X(1)=\frac{7}{16}$ | $p_X(2)=\frac{5}{16}$ | $\frac{3}{16} = p_X(3)$ | $\frac{1}{16} = p_X(4)$ | |

Condition PMF of $X$ given $Y$: $\quad p_{X|Y}(x|y)$ for all $x,y=1,2,3,4$ for $p_Y(y)>0$.

We know $p_{X|Y}(x|y)=0$ if $x>y$.

For <u>$y=1$</u>: $\quad p_{X|Y}(1|1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/16}{1/16} = 1.$

For <u>$y=2$</u>: $\quad p_{X|Y}(1|2) = \frac{p_{X,Y}(1,2)}{p_Y(2)} = \frac{2/16}{3/16} = \frac{2}{3}.$

$$p_{X|Y}(2|2) = \frac{p_{X,Y}(2,2)}{p_Y(2)} = \frac{1/16}{3/16} = \frac{1}{3}.$$

$$p_{X|Y}(2|2) = \frac{p_{X,Y}(2,2)}{p_Y(2)} = \frac{1/16}{3/16} = \frac{1}{3}.$$

For $\underline{y=3}$: $p_{X|Y}(1|3) = \dfrac{p_{X,Y}(1,3)}{p_Y(3)} = \dfrac{2/16}{5/16} = \dfrac{2}{5}.$

... and so on ...

To be clear:

not $p_{X,Y}(3,2)$

$$p_{Y|X}(3|2) = \frac{p_{X,Y}(2,3)}{p_X(2)} = \frac{2/16}{5/16} = \frac{2}{5}.$$

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<u>Conditional Expectations</u>: For each $y$ with $p_Y(y)>0$, define

$$\mathbb{E}[X|Y=y] = \sum_x x\, p_{X|Y}(x|y) = \sum_x x\, \mathbb{P}(X=x|Y=y).$$

<u>Ex 2 revisited</u>:

$$\mathbb{E}[X|Y=1] = 1\cdot p_{X|Y}(1|1) + 2\cdot \cancel{p_{X|Y}(2|1)}^{0} + 3\,\cancel{p_{X|Y}(3|1)}^{0} + 4\,\cancel{p_{X|Y}(4|1)}^{0}$$
$$= 1.$$

$$\mathbb{E}[X|Y=2] = 1\cdot p_{X|Y}(1|2) + 2\,p_{X|Y}(2|2)$$

$p_{X|Y}(3|2) = p_{X|Y}(4|2) = 0$

$$= 1\cdot\frac{2}{3} + 2\cdot\frac{1}{3}$$
$$= \frac{4}{3}.$$

$$\mathbb{E}[X|Y=3] = \ldots$$

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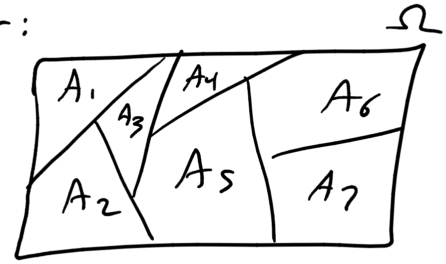
<u>Properties</u>:

① Multiplication rule: $p_{X,Y}(x,y) = p_{X|Y}(x|y)\,p_Y(y).$

① Multiplication rule: $p_{X,Y}(x,y) = p_{X|Y}(x|y)\, p_Y(y)$.

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x \mid Y=y)\,\mathbb{P}(Y=y).$$

② Law of total probability for expectations:




(★) $$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y=y]\, p_Y(y).$$

More generally: Let $A_1, \dots, A_n$ be mutually exclusive events, with $\bigcup_{i=1}^{n} A_i = \Omega$. (partition)

Then (★★) $$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i]\, \mathbb{P}(A_i).$$

If $Y$ can take values $y_1, y_2, \dots, y_n$, then (★) follows from (★★) by taking $A_i = \{Y = y_i\}$.

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Ex 3: Each class I am asked 0, 1, or 2 questions, each with prob. $\frac{1}{3}$. Each question I answer correctly with prob. $\frac{3}{4}$, independently.

Let $Y$ = # questions asked

$X$ = # questions answered correctly

Lets find Joint PMF.

What are we given?

① Marginal of $Y$: $p_Y(0) = p_Y(1) = p_Y(2) = \frac{1}{3}$.

② Conditional PMF of $X$ given $Y$:

- If $Y = 0$ then also $X=0$, so $p_{X|Y}(0|0) = 1$.

- If $Y=0$ then also $X=0$, so $p_{X|Y}(0|0)=1$.
- If $Y=1$ then $X=0$ or $X=1$,

  $p_{X|Y}(1|1)=\frac{3}{4}$ and $p_{X|Y}(0|1)=\frac{1}{4}$.
- If $Y=2$, then $X=0, 1,$ or $2$.

  $p_{X|Y}(2|2)=\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}$

  $p_{X|Y}(1|2)=\frac{3}{4}\cdot\frac{1}{4}\cdot 2=\frac{3}{8}$

  $p_{X|Y}(0|2)=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$.

(Given $Y=2$, $X$ is $\mathrm{Bin}(n=2, p=\frac{3}{4})$.)

**Joint PMF:** Use multiplication rule,

$p_{X,Y}(0,0)=p_{X|Y}(0|0)\,p_Y(0)=\frac{1}{3}$

$p_{X,Y}(0,1)=p_{X|Y}(0|1)\,p_Y(1)=\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}$

$p_{X,Y}(1,1)=\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{4}$

$p_{X,Y}(0,2)=\frac{1}{16}\cdot\frac{1}{3}=\frac{1}{48}$

$p_{X,Y}(1,2)=\frac{3}{8}\cdot\frac{1}{3}=\frac{1}{8}$

$p_{X,Y}(2,2)=\frac{9}{16}\cdot\frac{1}{3}=\frac{3}{16}$

$X \to$ #correct

$Y \to$ #asked

| $Y \backslash X$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $1/3$ | 0 | 0 |
| 1 | $\frac{1}{12}$ | $\frac{1}{4}$ | 0 |
| 2 | $\frac{1}{48}$ | $\frac{1}{8}$ | $\frac{3}{16}$ |
| $p_X$ | $\frac{1}{3}+\frac{1}{12}+\frac{1}{48}$ | $\frac{1}{4}+\frac{1}{8}$ | $\frac{3}{16}$ |

Marginal of $X$: $\xrightarrow{\text{column sums}}$

$p_X(1)=\mathbb{P}(X=1)=\mathbb{P}(\text{answer 1 question correctly})$.

① $\mathbb{E}[X]$ = expected # questions I answer correctly

$=0\cdot p_X(0)+1\cdot p_X(1)+2\cdot p_X(2)$

$=\left(\frac{1}{4}+\frac{1}{8}\right)+2\cdot\frac{3}{16}=\frac{3}{4}$.

using our original definition of $\mathbb{E}[X]$.

(2) Alternatively, even without finding $p_{X,Y}$ or $p_X$, we can find $\mathbb{E}[X]$ from $p_{X|Y}$ and $p_Y$ using the law of total prob:

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y=y]\, p_Y(y).$$

Note that given $Y=n$ (for $n=0,1,2$) we have $X \sim \mathrm{Bin}(n, \frac{3}{4})$ which has mean $n \cdot \frac{3}{4}$. So $\mathbb{E}[X \mid Y=n] = \frac{3}{4}n$.

Since $Y$ is $0, 1,$ or $2$:

$$\mathbb{E}[X] = \sum_{n=0}^{2} \underbrace{\mathbb{E}[X \mid Y=n]}_{\frac{3}{4}n} \underbrace{p_Y(n)}_{\frac{1}{3}}$$

$$= \sum_{n=0}^{2} \frac{3}{4}n \cdot \frac{1}{3} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Exercise: Extend to let $Y = 0, 1, \dots, N$ with prob $\frac{1}{N+1}$.

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## Independence:

Def: We say $X$ and $Y$ are *independent* r.v.'s if for each $x$ and $y$ the events $\{X=x\}$ and $\{Y=y\}$ are independent (as events).

$\hookrightarrow$ $\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y)$ for all $x, y$.

$\hookrightarrow$ $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$ for all $x, y$.

Properties:

If $X$ and $Y$ are independent:

$p_{X|Y}(x|y) = p_X(x)$ for all $x, y$ with $p_Y(y) > 0$

If $X$ and $Y$ are independent:

(1) $p_{X|Y}(x|y) = p_X(x)$ for all $x, y$, with $p_Y(y) > 0$.

That is, $\mathbb{P}(X=x \mid Y=y) = \mathbb{P}(X=x)$.

(2) If $A, B \subseteq \mathbb{R}$ then

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

e.g.,

$$\mathbb{P}(3 \le X \le 10, Y > 2) = \mathbb{P}(3 \le X \le 10)\,\mathbb{P}(Y > 2).$$