Statistical Methods in Finance Final Project - Spring 2023

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1. Summary

- Analysis was conducted on 15 assets with tickers AAPL, MSFT, AMZN, GOOGL, META, TSLA, BRK-B, JNJ, JPM, V, PG, UNH, MA, NVDA, and DIS. The equity curves for all assets generally have a positive slope and are gradually increasing in nature. The returns distributions were stationary for all assets except TSLA, and the normal distribution was found to be the best fit for most assets.
- The Sharpe slope for each asset was calculated, ranging from 0.1 to 0.5, with MSFT possessing the highest Sharpe slope of around 0.47. The S&P 500 equity curve followed a similar trend to most assets, except TSLA and NVDA. AMZN and TSLA had the most outliers among the assets under consideration.
- When shorting is not allowed, the portfolio heavily weights on PG stocks, with some stocks having weights close to zero. When shorting is allowed, the MVP is still weighted towards PG but with assets with higher volatility shorted.
- MVPs with and without short selling have similar returns, but the MVP without short selling has slightly higher standard deviation. Most individual tickers have higher annual expected return compared to the MVP, but all tickers have higher annual risk.
- The tangency portfolio in both cases has the highest Sharpe ratio compared to all individual assets, with the tangency portfolio with shorting having a higher expected return but also a higher expected risk.
- Efficient portfolio analysis was conducted with a target expected monthly return of 0.5% using only risky assets, and the optimal allocation had weights assigned to seven assets, achieving the expected monthly return with a standard deviation of 0.03652; further analysis was done with a combination of T-bills and the tangency portfolio to achieve the same target, resulting in investing in five risky assets with a monthly risk of 0.1049 and comparable monthly VaR values to individual risky assets.
- Asset pairs with high correlations (>0.6) are (V and MA), (AMZN and MSFT), (GOOG and MSFT), (JPM and BRK-B), (BRK-B and DIS), (DIS and JPM), and (DIS and MA), while asset pairs with low correlations (<0.2) are (PG and AMZN), (PG and TSLA), (NVDA and JNJ), (PG and JPM), (UNH and NVDA), and (NVDA and DIS), with only one negative correlation found between (PG and NVDA). Diversification of certain assets may reduce risk, while shorting some assets may also manage risks.
- PCA indicated that 9 principal components were needed to explain 90%+ (cumulative) of the variance in the data. After running factor analysis we saw that we needed at least four factors but more than 10 was too many for our number of assets. Four factors yielded slightly more than 60% of explained variance (cumulative) and 9 gave 77.5%. The four factor model's loadings had a meaningful interpretation as mastercard and visa (our most highly correlated pair) were msot heavily and lightly loaded onto the same factors. Even more significantly, we saw this grouping with MSFT, AMZN, AAPL, and

NVDA which are all in the tech sector. GOOGL was on the verge of being included in this grouping.
- With respect to risk managment through VaR, TSLA had the highest risk among all assets under consideration while PG had the lowest risk based on the Gaussian method, and BRK-B had the lowest risk while TSLA had the highest risk based on the historical method. S&P500 had the lowest risk compared to all the assets in the case based on the Gaussian method, but not based on the historical method.
- The best fitting dependence structure for all assets together was given by T-Copula but pairwise fits showed some asymmetric dependencies between assets also. The symmetric options of Normal and T-copula did noticeably better than the asymmetric ones (Clayton and Gumbel) for all of the assets together.

2. Descriptive Statistics

We decided to use the assets tickers "AAPL", "MSFT", "AMZN", "GOOGL", "META", "TSLA", "BRK-B", "JNJ", "JPM", "V", "PG", "UNH", "MA", "NVDA", "DIS" for our analysis. The Market sample statistics (Mean, Standard deviation, skewness coefficient, kurtosis coefficient, beta) are (0.1373691, 0.1448946, -0.4261589, 4.3454827, 1.0000000). The detailed breakdown of summary statistics for each of the other tickers can be found in Table 1 of the appendix. However, it is worth mentioning that all the annual means were greater than 0.10 and interestingly, the skewness and kurtosis values are generally similar across tickers, while the annual mean and standard deviation vary more widely. Additionally, the beta value for the market is 1, indicating that it is used as a baseline for comparison with the tickers.

For further visualization we generated the equity curve for each asset, that is, a curve that shows the growth of $1 in each of the assets over the time period of choice. These curves can be found in Figure 1 of the appendix. We can see that the equity curves for all the assets are generally having a positive slope and are gradually increasing in nature. It is important to note that TSLA and NVDA have an evidently faster climb to its peak compared to the other tickers in consideration. Finally, it is interesting to note that the equity curves for V and MA are very similar in their structure, and we will see in the future analysis that this is justified by the exceptionally high correlation between these assets. This makes sense financially since both are credit-providing companies working with banks and thus whenever the demand for credit rises, the prices associated with both of these tickers go up as part of the market reaction.

We also generated an equity curve for the S&P 500, which can be found in Figure 2, so that we can compare it with the assets we have under consideration. Comparing the equity curve of S&P500 with the assets in consideration, we can see that it follows a similar trend of gradual increase like most other assets except TSLA and NVDA.

Furthermore, we ran ADF and KPSS tests for the stationarity of the distributions. We were able to conclude that the returns distributions are stationary for all the assets under consideration except TSLA with 95% confidence. In addition to this, by performing the normality test, we got to know that we cannot reject the null hypothesis that the returns follow a normal distribution even with 95% confidence in the case of all the assets except TSLA.

We then decided to look into the data to check for outliers including additive outliers, level shifts, and temporary changes (R documentation). Additive outliers are observations which deviate enough from thee surrounding data points to not be considered a continuation of the time series. Such outliers can be caused by errors in measurement or random ocurrences. Level shifts (LS) are defined differently and more even more rare. LS anomalies are sudden changes that persist over time instead of being discrete jumps and tend to be caused by abrupt changes in the underlying process that generates each time series. Temporary changes are short-lived fluctuations in thee level of the series that can be caused by transient events that have an effect on the underlying process. After our analysis, we got to know that AMZN and TSLA had the most outliers (= 5), while the other assets in consideration had 0,1 or 2 outliers. Moreover, after trying to assess the fit of various distributions including sstd, ged, and normal based on the AIC and BIC values, we found that for all the assets in consideration except TSLA, DIS, and “Market”, normal distribution turned out to be the best fit. Among the other exceptions, we got to know that sstd distribution was the best fit for TSLA and ged distribution was the best fit for DIS and “Market.”

Finally, upon calculating Sharpe’s slope for each asset, we realized that for the assets under consideration, it ranged from 0.1 to 0.5. Furthermore, the highest Sharpe slope of around 0.47 was possessed by MSFT. Following is the table summary of all the tickers we considered alongside the S&P 500 “Market”:

Asset	**Mean Return**	**Standard Deviation**	**Sharpe Ratio**	**Historical VaR(0.05), S0 = 100000**	**HistoricalES (0.05)**	**Beta**	**Distribution**
AAPL	31.8%	28.3%	0.315	11425.43	13534.63	1.258	Normal
MSFT	35.1%	20.7%	0.475	6726.91	7820.60	0.966	Normal
AMZN	36.8%	28.3%	0.365	7779.89	12224.97	1.173	Normal
GOOGL	27.2%	22.5%	0.336	7576.43	9655.74	1.061	Normal
META	25.3%	26.3%	0.268	10311.91	11226.5	1.141	Normal

TSLA	64.2%	61.4%	0.298	16512.77	20687.69	1.798	sstd
BRK-B	12.0%	16.4%	0.195	6003.96	8178.60	0.863	Normal
JNJ	11.9%	15.9%	0.198	6502.60	8473.85	0.705	Normal
JPM	21.0%	23.2%	0.249	8476.06	12984.48	1.130	Normal
V	20.6%	20.3%	0.280	8320.34	9798.40	0.981	Normal
PG	13.6%	15.1%	0.242	7100.95	8253.56	0.454	Normal
UNH	26.4%	21.0%	0.351	6380.37	8449.46	0.810	Normal
MA	24.9%	23.0%	0.300	10097.51	12797.65	1.142	Normal
NVDA	70.2%	42.3%	0.473	10965.76	20381	1.315	Normal
DIS	12.2%	26.4%	0.123	8725.25	14207.54	1.228	ged
“Market” or S&P 500	13.7%	14.5%	0.274	6545.80	8723.76	1.000	ged

3. Portfolio Theory

The Table 3 in Appendix B provides the weights of each asset in the MVP in each of the cases: when short sales are allowed and when they’re not allowed. The assets in the table follow the ascending order with respect to the Asset risk, which is measured by its volatility or standard deviation of its return. We can see that the MVP has assigned 0 weights to the assets GOOGL, MA, META, DIS, AAPL, AMZN and TSLA in the case when shorting is not allowed. On the other hand, when the shorting is allowed, our MVP takes non-zero position on every asset under consideration.

Specifically, when short positions is not allowed, the portfolio is heavily weighted on PG stocks, which has least volatility among all the assets in the portfolio, with a normalized weight of 0.458. This is in line with the logic of minimization of the variance of the portfolio. Moreover, some of the stocks have weights close to zero. This can happen if these assets have high correlations with other assets in the portfolio, or if they have high individual variances that make them unattractive from a risk/return perspective.

Conversely, when short positions are allowed, based on the weights printed in the same table, we can see that the MVP is heavily weighted towards PG again, with a weight of 0.496. Since the MVP is designed to minimize portfolio variance, we still want to hold a long position with a least volatile asset. Likewise, assets with higher volatility are shorted, but still with weights less than 10%.

Now upon comparing the returns and risks for MVP with/without short selling thus constructed, we can see that both have similar returns, but MVP without short sales posses slightly higher standard deviation. Further statistics detailed statistics of this analysis can be found in Appendix B Table 4.

Next , we decided to compare the annual expected return and risk of MVPs with respect to the individual stock tickers. In both of the case when short sales are allowed and not allowed, we can see in the output detailed in Appendix B Table 5 and Table 6 respectively that most tickers except BRK-B, JNJ, PG and DIS had higher annual expected return compared to the MVP. But, it is also evident that all the constituent tickers had higher annual risk compared to our MVPs.

This confirms that diversification by including multiple assets into a portfolio lowers risks of a portfolio. However, comparing the annual expected returns of the assets to the MVP's annualized expected return we can see that 11 out of 15 assets have higher individual returns than the MVP mean. This confirms that minimizing the risk does not necessarily lead to the maximizing the expected return.

Now assuming that we have $100,000 to invest, we found that in the case of MVP without short sales, VaR(alpha = 5%, T = 1 month) was calculated to be -$5096.143. For the case when MVP with short sales allowed, VaR(alpha = 5%, T = 1 Month) was calculated to be -$4015.542. Moreover, as we can see in the Appendix B Table 7 that assets are potentially more profitable individually but have very high variance/risk, which may be leading the VaR for individual assets to be so much higher compared to the portfolios constructed.

After this inspection, we decided to plot the efficient frontier to locate the MVP and in the effort towards construction of a tangency portfolio. The graph of efficient frontier in the case of short sales not allowed can be found in Appendix A, Figure 4, and the same in the case of short sales allowed can be found in Appendix A, Figure 5. Upon further calculations, we found that the

Sharpe ratio of the tangency portfolio in case when short sales are not allowed is 0.619, and the same in the case when short sales are allowed is 0.71.

We then compared the sharpe ratio of the tangency portfolio with that of all the individual assets and found that tangency portfolio had the highest sharpe ratio compared to all the assets individually in both the scenarios. The detailed comparison of the same can be seen in Appendix B Table 8.

As far as the expected return and risk of the constructed Tangency portfolio are concerned, we can see that tangency portfolio with shorting had higher expected return, but also had a higher expected risk. These measurements are recorded in Appendix B Table 9. Furthermore, the weights of different assets in the tangency portfolio thus constructed in both cases can be found in Appendix B Table 10.

4. Asset Allocation

For the Asset Allocation aspect of our analysis, we started with considering a situation where we targeted an expected return of 6% annually (or 0.5% monthly) only using risky assets and no short sales allowed. The efficient portfolio that met this objective had assigned the weights of 0.056 to MSFT, 0.01 to GOOGL, 0.038 to BRK-B, 0.242 to JNJ, 0.142 to V, 0.510 to PG, 0.002 to MA, and almost 0 weights to the rest of the assets under consideration. This allocation achieved the expected monthly return of 0.5% with an associated standard deviation of 0.03652.

Moreover, for further analysis of the monthly risk associated with the allocation described above, we decided to compute the monthly 5% VaR and expected shortfall based on an initial $100,000 investment. We found that this allocation turned out to have a 5% VaR of around $4717 and an Expected shortfall of -$8202.

Now, we decided to analyse a situation where we want to achieve a target expected return of 6% a year (or 0.5% per month) using a combination of T-bills (annual return 0.94%) and the tangency portfolio (that does not allow short selling). Our results show that we would assign a weight of 2.262 to our tangency portfolio and -1.262 to the T-Bills. This meant that we would invest only in the following risky assets with the corresponding weights: MSFT (0.5768), TSLA (0.0701), PG (0.6130), UNH (0.5338), and NVDA (0.4682). To further get some more statistics, we decided to calculate the monthly risk on this efficient portfolio as well as the monthly and 5% VaR and ES based on an initial $100,000 investment. As part of the results, we observed that the monthly risk for the efficient portfolio was 0.1049, the monthly 5% VaR for the allocation was -$10583.55, and the monthly expected shortfall was -$21637.54. We can see that these VaR values are, generally speaking, comparable to that of individual risky assets.

5. Principal Component Analysis

To start with, we calculated the sample correlation matrix of returns on our assets, which can be found in Figure 3 and Table 2.

The returns of asset pairs (V and MA), (AMZN, MSFT), (GOOG, MSFT), (JPM, BRK-B), (BRK-B, DIS), (DIS, JPM), and (DIS, MA) all had correlations higher than 0.60. That of V and MA was the largest at 0.904 with the second largest being noticeably lower at 0.7 between DIS and JPM. On the other end of the spectrum, the returns of asset pairs (PG and AMZN), (PG and TSLA), (NVDA and JNJ), (PG and JPM), (UNH and NVDA), (NVDA and DIS), and (PG and NVDA) possess correlations lower than 0.2. Among them, the asset returns of NVDA and DIS have the lowest positive correlation of around 0.044. Also, we found that only one asset pair of PG and NVDA had a negative correlation, and it was of a relativewly small magnitude given the value -0.071 . Based on these estimated correlations, we think that diversification of the portfolio to certain assets may reduce the overall risk. Then, we ran PCA analysis on the asset returns data as a matrix to find the number of principal components that are required to capture above 90% (cumilative) of the variance. From our results we can see that the top 9 principal components yield 90.5% cumulative explained variance and 8 capture slightly less with 87.7%. We have also obtained the projection of the data onto the principal components by extracting the matrix of ‘rotation’ values from our PCA analysis and multiplying the data by its transpose.

Additionally, we ran factor analysis on the covariance matrix using the factanal() function which relies on maximum likelihood. The p-value in factanal(factors=k) are testing the null hypothesis that the k-factor model fits the data well (via Chi-square statistic). These p-values can be used to see how well k-factor model fits the data using the rule: if this PVAL < 0.05 we reject H0 so need to test different number of factors. At 5% our results showed we should reject the 1, 2, and 3 factor models since these p-values were too small. Starting with the four factor model the p-values indicated acceptable fit. The 9 factor model yields 77.5% cumulative explained variance and 10 yield 79.2%, with more factors yielding the output that its too many for 15 assets. Additionally, we saw that 7 factors were enough to capture above 70% of the variance (72.3%). In the nine factor model, we saw the first two factors have approximately the same proportion of explained variance and loadings (13.6% each on average), the third and fourth were also similar and the fifth and sixth were with this paired structure continuing until the final three which were all similar as a trio. In the four factor model we saw AAPL, MSFT, AMZN, NVDA and potentially GOOGL also (all tech) load most heavily onto the first factor together and least heavily onto the third factor (not GOOGL though). It was interesting to see some assets paired in the sense of having higher or lower loadings for different factors in tandem. For instance, visa and mastercard were most heavily loaded onto the second factor in the four factor model (and the least onto the fourth for both) and onto the 1st factor in the nine factor model (and least onto the 5th). Overall, the results suggest that the original variables can be meaningfully summarized by a small number of factors. The loadings suggest that the variables are that are highly correlated with each other, can be grouped together.

6. Risk Management

To get started with analyzing the risk, we decided to consider the problem of investing $100,000 in each of the assets and the market separately over a month investment horizon and calculating the 5% VaR and Expected Shortfall (ES). We employed two methods: gaussian and historical.

From the Gaussian method, we got to know that TSLA had the highest 5% VaR of 23605.88 and the highest 5% ES of 30961.76. On the other hand, PG had the lowest 5% VaR of 5977.72 and the lowest 5% ES of 7783.80 among the assets under consideration. It is important to note that S&P500 or "Market" had an even lower 5% VaR of 5693.69 and an even lower 5% ES of 7430.93.

From the historical or non-parametric method, we got to know that TSLA had the highest 5% VaR of 16512.77 and the highest 5% ES of 20687.69. On the other hand, BRK-B had the lowest 5% VaR of 6003.96, but MSFT had the lowest 5% ES of 7820.60 among the assets under consideration. It is important to note that S&P500 or "Market" had a 5% VaR of 8723.76, and a 5% ES of 7430.93, which is not the lowest compared to all the assets in this case.

7. Copulas

To first explore the pairwise relationships between assets we fit pairwise copulas to all different pairs of assets. Most of the pairs showed a best fit with normal copulas but a noticeable number also had best fits of Clayton/Gumbel copulas indicating some of the pairs of assets had higher dependence during extreme negative/positive events. For the full joint distribution of all 15 assets, as with the pairwise checks initially, we fit the normal copula, t-Copula, and the Gumbel & Clayton copulas. As discussed in class, the first two of these are more well suited for situations where the dependence between assets behaves similarly in the upper and lower tails (since they are symmetric). In contrast, the latter two are asymmetric and can yield better results when there is a difference in the dependence between assets in the upper and lower tails. Specifically, the Clayton copula captures lower tail dependence and the Gumbel copula captures upper tail dependence. Table 11 in Appendix B below shows the results (all rounded to 1 d.p.) of these fits compared by AIC and (Max.) likelihood.

Based on the results of fitting distributions to the assets individually, it was not surprising that the normal and T-copula did better than the clayton and Gumbel copulas, since most of our assets had best fits of normal distribution. However, the T-copula had the best fit for all the assets together which is indicative of tail dependence and is something not captured by the normal copula since it assumes tail independence. In other words, the normal copula does not account for the possibility of extreme co-movements of the assets during periods of high volatility in the market but our data seems to have signs of this. The worst fitting copula was Gumbel which had an AIC of -316.3, which is a lot worse than the T-copula's -520.9 indicating that the difference between the worst and best fitting copula is noticeably large. Moreover, the difference between the second best (normal) and the third best (Clayton) was also noticeably large since they had fits with -520.9 and -362.3 AICs respectively. A potential concern about our procedure is that we are checking which copula fits many assets (15) at once, which makes the symmetric copulas more likely to have a better fit overall since it is less likely for a larger number of assets to demonstrate the asymmetry of Clayton or Gumbel copula together. With a larger number of assets the overall depencende structure across them is more likely to be symmetrical than not.

8. Conclusion

In conclusion, the analysis of the 15 assets revealed a generally positive trend in equity curves with stationary returns distributions, except for the potential non-stationarity with TSLA. The Sharpe slope varied among assets with MSFT having the highest Sharpe slope. The portfolio without shorting heavily weights on PG stocks, while the portfolio with shorting still leans towards PG but with assets with higher volatility shorted. The tangency portfolio had the highest Sharpe ratio and investing in risky assets in combination with T-bills achieved the target expected monthly return with comparable monthly VaR values to individual risky assets. Correlations between assets were analyzed, and diversification and shorting were found to be potential risk management strategies. VaR analysis revealed TSLA to have the highest risk and PG to have the lowest risk based on the Gaussian method, while BRK-B had the lowest risk and TSLA had the highest risk based on the historical method. The S&P 500 had the lowest risk compared to all assets based on the Gaussian method but not the historical method. Overall, these findings can assist in making informed investment decisions and managing risk in investment portfolios.

Appendix A

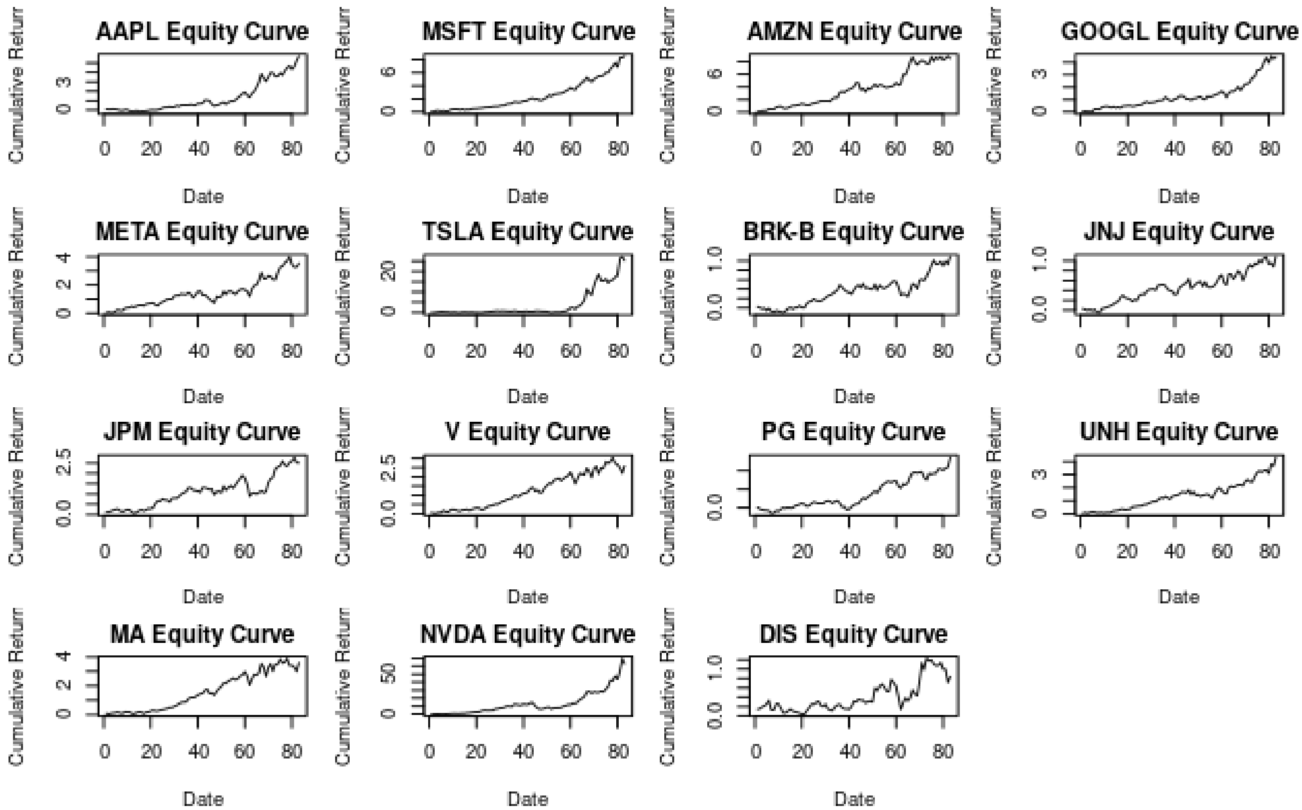

Figure 1: Equity curves for all the tickers under consideration

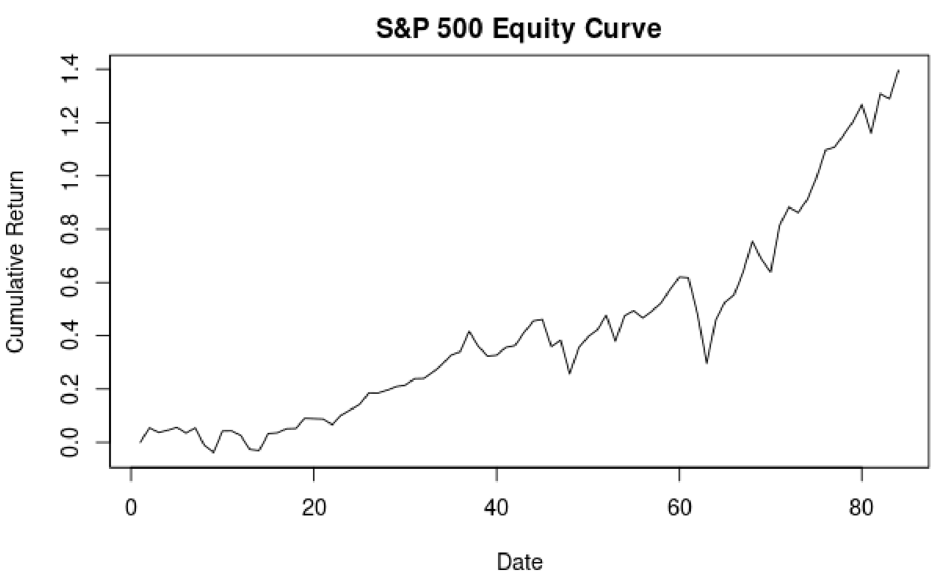

Figure 2: S&P500 Equity curve

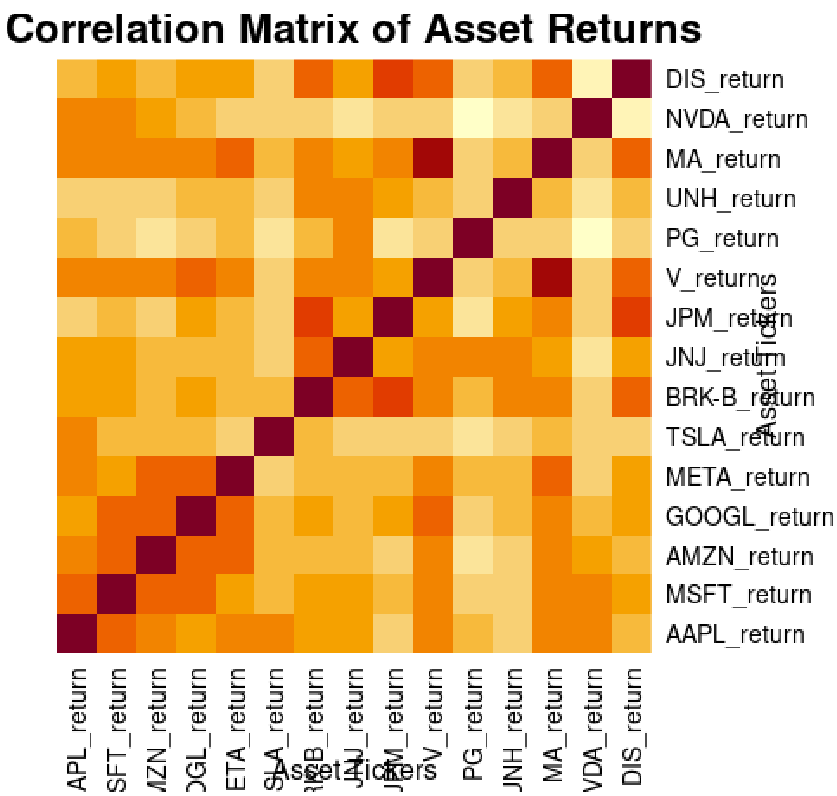

Figure 3: Correlation heatmap between different assets

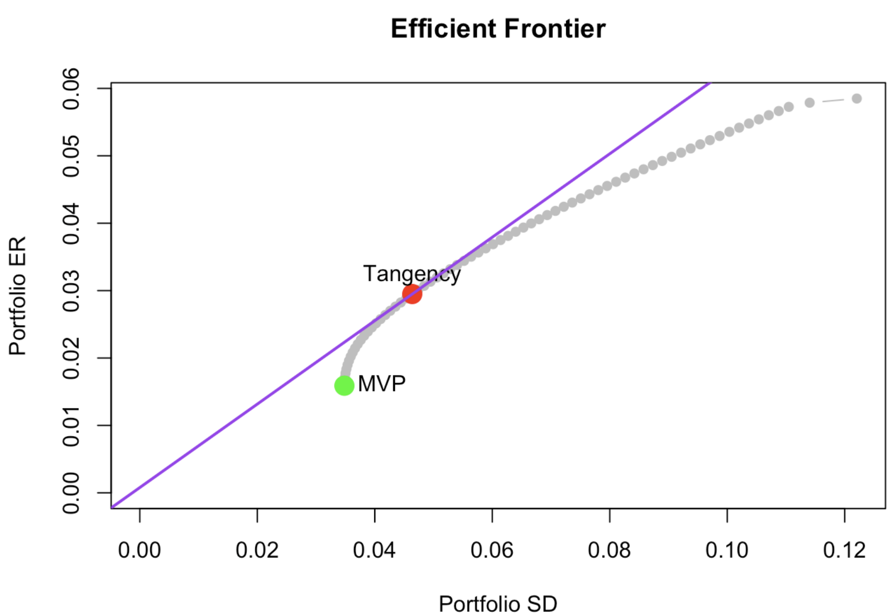

Figure 4: Efficient Frontier in the case when short selling is not allowed

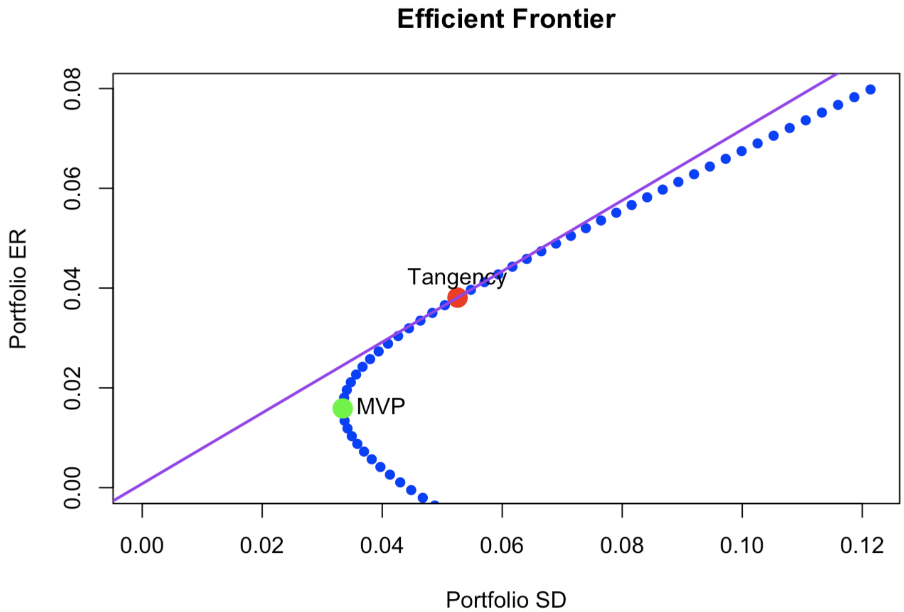

Figure 5: Efficient Frontier in the case when short selling is allowed

Appendix B

	AAPL	MSFT	AMZN	GOOGL	META	TSLA	BRK-B	JNJ	JPM	V	PG	UNH	MA	NVDA	DIS	Market
Annual Mean	0.3177667	0.3509460	0.3682403	0.2717904	0.2534126	0.6419621	0.1197883	0.11873231	0.2095210	0.2060311	0.13580716	0.2644403	0.2485565	0.701958639	0.1218242	0.1373691
Annual SD	0.2830103	0.2073802	0.2834770	0.2254505	0.2627147	0.6135183	0.1637776	0.15933970	0.2315432	0.2026455	0.15063680	0.2096051	0.2302275	0.422813679	0.2636638	0.1448946
Skewness	-0.1673713	0.3279953	0.4114870	0.2654103	0.5288971	1.2989028	0.0810402	-0.03972393	-0.3125785	-0.1025479	-0.04536545	0.3161312	-0.3764796	0.001196319	0.5591020	-0.4261589
Kurtosis	2.6098401	3.3768192	3.9625319	3.1766764	3.7034564	5.4134494	2.8991613	3.53436026	4.5596020	2.6976009	3.13385740	3.0612408	3.3213930	3.424708144	4.3500892	4.3454827
Beta	1.2578217	0.9659538	1.1729411	1.0607543	1.1410665	1.7983521	0.8625411	0.70485711	1.1300957	0.9808621	0.45449988	0.8099283	1.1419964	1.314771455	1.2278552	1.0000000

Table 1: Summary Statistics Table for all the tickers and the market

	AAPL_return	MSFT_return	AMZN_return	GOOGL_return	META_return	TSLA_return	BRK-B_return	JNJ_return	JPM_return	V_return	PG_return	UNH_return	MA_return	NVDA_return	DIS_return
AAPL_return	1.0000000	0.5829380	0.4665927	0.4395939	0.5048647	0.4757889	0.3814347	0.3975770	0.2754842	0.4914613	0.35216777	0.2608884	0.5208303	0.52382870	0.33573338
MSFT_return	0.5829380	1.0000000	0.6390587	0.6329232	0.4052215	0.3658509	0.3979647	0.4087626	0.3734390	0.5310791	0.23703910	0.2349800	0.5041658	0.48319118	0.39552143
AMZN_return	0.4665927	0.6390587	1.0000000	0.5774511	0.5558888	0.3004167	0.3233253	0.3564858	0.2734601	0.5317983	0.12410384	0.2764696	0.5153931	0.42715428	0.30764197
GOOGL_return	0.4395939	0.6329232	0.5774511	1.0000000	0.5722067	0.2914563	0.4240117	0.3423280	0.4278543	0.5947282	0.25243007	0.3014662	0.5247505	0.35405235	0.40777818
META_return	0.5048647	0.4052215	0.5558888	0.5722067	1.0000000	0.2565702	0.3734350	0.3383714	0.3115576	0.5419813	0.34860256	0.3663611	0.5615216	0.27753068	0.41084651
TSLA_return	0.4757889	0.3658509	0.3004167	0.2914563	0.2565702	1.0000000	0.2870671	0.2793280	0.2216824	0.2563806	0.17044964	0.2392575	0.3020523	0.23054813	0.24684520
BRK-B_return	0.3814347	0.3979647	0.3233253	0.4240117	0.3734350	0.2870671	1.0000000	0.5894949	0.6933717	0.5522695	0.36662968	0.4872596	0.5362767	0.27090125	0.60405342
JNJ_return	0.3975770	0.4087626	0.3564858	0.3423280	0.3383714	0.2793280	0.5894949	1.0000000	0.4003573	0.4776287	0.49226990	0.4739188	0.4187478	0.12162531	0.46324237
JPM_return	0.2754842	0.3734390	0.2734601	0.4278543	0.3115576	0.2216824	0.6933717	0.4003573	1.0000000	0.4389572	0.11349922	0.3800062	0.4827482	0.22154625	0.69002928
V_return	0.4914613	0.5310791	0.5317983	0.5947282	0.5419813	0.2563806	0.5522695	0.4776287	0.4389572	1.0000000	0.28505882	0.3182414	0.9036023	0.23127522	0.60629455
PG_return	0.3521678	0.2370391	0.1241038	0.2524301	0.3486026	0.1704496	0.3666297	0.4922699	0.1134992	0.2850588	1.00000000	0.2530442	0.2700387	-0.07123758	0.26788852
UNH_return	0.2608884	0.2349800	0.2764696	0.3014662	0.3663611	0.2392575	0.4872596	0.4739188	0.3800062	0.3182414	0.25304420	1.0000000	0.3290819	0.16417715	0.35789840
MA_return	0.5208303	0.5041658	0.5153931	0.5247505	0.5615216	0.3020523	0.5362767	0.4187478	0.4827482	0.9036023	0.27003868	0.3290819	1.0000000	0.26483463	0.61325559
NVDA_return	0.5238287	0.4831912	0.4271543	0.3540524	0.2775307	0.2305481	0.2709012	0.1216253	0.2215462	0.2312752	-0.07123758	0.1641771	0.2648346	1.00000000	0.04408586
DIS_return	0.3357334	0.3955214	0.3076420	0.4077782	0.4108465	0.2468452	0.6040534	0.4632424	0.6900293	0.6062945	0.26788852	0.3578984	0.6132556	0.04408586	1.00000000

Table 2: Correlation matrix between different assets

Asset	MVP weights without shorting	MVP weights with shorting	Asset risk
PG	0.458	0.496	0.043
JNJ	0.122	0.12	0.046
BRK-B	0.102	0.087	0.047
V	0.04	0.197	0.058
MSFT	0.083	0.128	0.06
UNH	0.095	0.106	0.061
GOOGL	0	-0.015	0.065
MA	0	-0.081	0.066
JPM	0.063	0.125	0.067
META	0	-0.02	0.076
DIS	0	-0.092	0.076
AAPL	0	-0.093	0.082
AMZN	0	0.008	0.082
NVDA	0.037	0.054	0.122
TSLA	0	-0.019	0.177

Table 3: Weights of assets in the Minimum Variance Portfolio (MVP) in the cases when shorting is allowed and not allowed.

Measurement	MVP without short sales	MVP with short sales
Annual Mean Return	0.191	0.190
Annual Standard Deviation	0.121	0.116
VaR(5%, t=1month)	-0.051	-0.040
ES(5%, t=1month)	-0.063	-0.059

Table 4: Comparison of annualized asset returns and risks of the MVPs with and without short selling permitted.

Asset	Annual Expected Return	Annual Expected Return of MVP	Higher than MVP return?	Annual Asset Risk (st.dev)	Annual MVP Risk (st.dev)	Higher than MVP risk?
AAPL	0.318	0.191	Yes	0.283	0.121	Yes
MSFT	0.351	0.191	Yes	0.207	0.121	Yes
AMZN	0.368	0.191	Yes	0.283	0.121	Yes
GOOGL	0.272	0.191	Yes	0.225	0.121	Yes
META	0.253	0.191	Yes	0.263	0.121	Yes
TSLA	0.642	0.191	Yes	0.614	0.121	Yes
BRK-B	0.120	0.191	**No**	0.164	0.121	Yes
JNJ	0.119	0.191	**No**	0.159	0.121	Yes
JPM	0.210	0.191	Yes	0.232	0.121	Yes
V	0.206	0.191	Yes	0.203	0.121	Yes
PG	0.136	0.191	**No**	0.151	0.121	Yes
UNH	0.264	0.191	Yes	0.210	0.121	Yes
MA	0.249	0.191	Yes	0.230	0.121	Yes
NVDA	0.702	0.191	Yes	0.423	0.121	Yes
DIS	0.122	0.191	**No**	0.264	0.121	Yes

Table 5: Comparison of returns and risks of assets and MVP without short sales allowed

Asset	Annual Expected Return	Annual Expected Return of MVP	Higher than MVP return?	Annual Asset Risk (st.dev)	Annual MVP Risk (st.dev)	Higher than MVP risk?
AAPL	0.318	0.19	Yes	0.283	0.116	Yes
MSFT	0.351	0.19	Yes	0.207	0.116	Yes
AMZN	0.368	0.19	Yes	0.283	0.116	Yes
GOOGL	0.272	0.19	Yes	0.225	0.116	Yes
META	0.253	0.19	Yes	0.263	0.116	Yes
TSLA	0.642	0.19	Yes	0.614	0.116	Yes
BRK-B	0.120	0.19	**No**	0.164	0.116	Yes
JNJ	0.119	0.19	**No**	0.159	0.116	Yes
JPM	0.210	0.19	Yes	0.232	0.116	Yes
V	0.206	0.19	Yes	0.203	0.116	Yes
PG	0.136	0.19	**No**	0.151	0.116	Yes
UNH	0.264	0.19	Yes	0.210	0.116	Yes
MA	0.249	0.19	Yes	0.230	0.116	Yes
NVDA	0.702	0.19	Yes	0.423	0.116	Yes
DIS	0.122	0.19	**No**	0.264	0.116	Yes

Table 6: Comparison of returns and risks of assets and MVP with short sales allowed

Asset	Asset VaR	No short sales allowed		Short sales allowed	
		MVP VaR	Higher than MVP VaR?	MVP VAR_S	Higher than MVP VaR?
AAPL	-0.111	-0.051	Yes	-0.0402	Yes
MSFT	-0.071	-0.051	Yes	-0.0402	Yes
AMZN	-0.107	-0.051	Yes	-0.0402	Yes
GOOGL	-0.086	-0.051	Yes	-0.0402	Yes
META	-0.106	-0.051	Yes	-0.0402	Yes
TSLA	-0.251	-0.051	Yes	-0.0402	Yes

BRK-B	-0.069	-0.051	Yes	-0.0402	Yes
JNJ	-0.067	-0.051	Yes	-0.0402	Yes
JPM	-0.095	-0.051	Yes	-0.0402	Yes
V	-0.081	-0.051	Yes	-0.0402	Yes
PG	-0.061	-0.051	Yes	-0.0402	Yes
UNH	-0.079	-0.051	Yes	-0.0402	Yes
MA	-0.091	-0.051	Yes	-0.0402	Yes
NVDA	-0.149	-0.051	Yes	-0.0402	Yes
DIS	-0.118	-0.051	Yes	-0.0402	Yes

Table 7: Comparing asset VaR values with the MVP VaR values both with and without shorting

Asset	**Asset SR**	**Without short sales**		**With short sales**	
		Tangency Port SR	**Higher than tangency port SR?**	**Tangency Port SR**	**Higher than tangency port SR?**
AAPL	0.3145	0.619	No	0.71	No
MSFT	0.4754	0.619	No	0.71	No
AMZN	0.3654	0.619	No	0.71	No
GOOGL	0.3360	0.619	No	0.71	No
META	0.2681	0.619	No	0.71	No
TSLA	0.2976	0.619	No	0.71	No
BRK-B	0.1946	0.619	No	0.71	No
JNJ	0.1981	0.619	No	0.71	No
JPM	0.2495	0.619	No	0.71	No
V	0.2801	0.619	No	0.71	No
PG	0.2422	0.619	No	0.71	No
UNH	0.3513	0.619	No	0.71	No
MA	0.2999	0.619	No	0.71	No
NVDA	0.4728	0.619	No	0.71	No
DIS	0.1231	0.619	No	0.71	No

Table 8: Comparison of Sharpe Ratios of individual assets with the tangency portfolio

Situation	Expected return	Risk	Sharpe Ratio
Without Shorting	0.029	0.046	0.619
With Shorting	0.038	0.053	0.710

Table 9: Comparison and depiction of expected returns and risks of the tangency portfolios

Asset	Weights w/o short sales	Weights w/ short sales
AAPL	0	-0.254
MSFT	0.255	0.373
AMZN	0	0.035
GOOGL	0	-0.128
META	0	-0.040
TSLA	0.031	0.073
BRK-B	0	-0.590
JNJ	0	-0.216
JPM	0	0.346
V	0	0.252
PG	0.271	0.663
UNH	0.236	0.370
MA	0	0.022
NVDA	0.207	0.274
DIS	0	-0.181

Table 10: Weights of assets in the Tangency Portfolios with and without shorting

Copula	Normal	T-Copula	Clayton	Gumbel
Likelihood	365.4	382.9	182.1	159.2
AIC	-520.9	-553.7	-362.3	-316.3

Table 11: Copula Fits AIC and Likelihood Comparisons

Loadings:

	Factor1	Factor2	Factor3	Factor4
AAPL_return	0.639	0.222		0.325
MSFT_return	0.714	0.249	0.187	0.188
AMZN_return	0.674	0.308		0.125
GOOGL_return	0.571	0.358	0.258	0.162
META_return	0.469	0.331	0.153	0.297
TSLA_return	0.404		0.119	0.249
BRK.B_return	0.224	0.271	0.599	0.434
JNJ_return	0.206	0.221	0.290	0.651
JPM_return	0.201	0.145	0.963	
V_return	0.311	0.889	0.240	0.225
PG_return		0.110		0.700
UNH_return	0.192	0.103	0.307	0.417
MA_return	0.342	0.766	0.299	0.198
NVDA_return	0.686			
DIS_return	0.132	0.401	0.607	0.285

	Factor1	Factor2	Factor3	Factor4
SS loadings	2.957	2.151	2.145	1.791
Proportion Var	0.197	0.143	0.143	0.119
Cumulative Var	0.197	0.340	0.483	0.603

objective
0.07940241

Table 12: Four Factor model output

Loadings:

	Factor1	Factor2	Factor3	Factor4	Factor5	Factor6	Factor7	Factor8	Factor9
AAPL_return	0.280	0.123	0.411	0.275	0.322		0.227	0.440	
MSFT_return	0.237	0.211	0.282	0.696	0.169			0.231	0.239
AMZN_return	0.283		0.211	0.564		0.224	0.339	0.147	0.197
GOOGL_return	0.260	0.221	0.150	0.318	0.112	0.114	0.235	0.127	0.811
META_return	0.286	0.147	0.126	0.165	0.199	0.166	0.769	0.116	0.194
TSLA_return		0.107		0.127		0.144		0.722	
BRK.B_return	0.271	0.589	0.177		0.240	0.451		0.106	
JNJ_return	0.215	0.262		0.236	0.401	0.571		0.131	
JPM_return	0.140	0.903	0.102			0.182			0.127
V_return	0.879	0.242		0.210	0.127	0.186	0.141		0.192
PG_return	0.103				0.795	0.197	0.126		
UNH_return		0.261			0.117	0.522	0.191	0.135	
MA_return	0.775	0.318	0.116	0.178	0.112	0.114	0.226	0.164	
NVDA_return			0.948	0.204				0.119	0.101
DIS_return	0.382	0.642		0.149	0.146	0.164	0.142	0.123	

	Factor1	Factor2	Factor3	Factor4	Factor5	Factor6	Factor7	Factor8	Factor9
SS loadings	2.054	2.024	1.316	1.233	1.113	1.073	0.977	0.941	0.896
Proportion Var	0.137	0.135	0.088	0.082	0.074	0.072	0.065	0.063	0.060
Cumulative Var	0.137	0.272	0.360	0.442	0.516	0.588	0.653	0.715	0.775

objective
0.8940836

Table 13: Nine Factor moodel output