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REVIEW ARTICLE

The Structure of Ill-Structured (and Well-Structured) Problems Revisited

Stephen K. Reed [1,2,3]

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Abstract In his 1973 article *The Structure of ill structured problems*, Herbert Simon proposed that solving ill-structured problems could be modeled within the same information-processing framework developed for solving well-structured problems. This claim is reexamined within the context of over 40 years of subsequent research and theoretical development. Well-structured (puzzle) problems can be represented by a problem space consisting of well-defined initial and goal states that are connected by legal moves. In contrast, the initial, goal, and intermediate states of ill-structured (design) problems are incompletely specified. This article analyzes the similarities and differences among puzzles, insight puzzles, classroom problems, and design problems within Gick's (*Educational Psychologist, 21*, 99–120, 1986) theoretical framework consisting of representation construction, schema activation, and heuristic search. The analysis supports Simon's (*Artificial Intelligence, 4*, 181–201, 1973) claim that information-processing principles apply to all problems but apply differently as problems become more ill structured.

Keywords Problem solving · Representation · Search · Analogy · Schema

Problems come in many guises (Jonassen 2000). There are puzzles (Rubik's Cube), algorithmic problems (factoring quadratic equations), story problems (car A overtaking car B), decision-making problems (changing jobs), trouble-shooting (car won't start), diagnosis (medical), design problems (a bridge), and dilemmas (resolve a crises). Problems vary from well structured (puzzles, algorithmic problems) to ill structured (design, dilemmas). Ill-structured problems possess multiple solutions and uncertainty about which concepts, rules, and principles are necessary for the solution (Jonassen 1997).

✉ Stephen K. Reed
sreed@mail.sdsu.edu

[1] Psychology, San Diego State University, San Diego, CA, USA

[2] CRMSE, San Diego State University, San Diego, CA, USA

[3] Department of Psychology, University of California, San Diego, La Jolla, CA 92093, USA

In his seminal article *The Structure of Ill Structured Problems*, Simon (1973) declared that an ill-structured problem (ISP) is a residual concept—it is *not* a well-structured problem (WSP). Simon found it impossible to define a WSP but instead proposed that it possesses some or all of the characteristics listed in Table 1.

Although Simon (1973) listed the characteristics of a WSP, he concluded "there is no real boundary between WSPs and ISPs, and no reason to think that new and hitherto unknown types of problem solving processes are needed to enable artificial intelligence systems to solve problems that are ill structured" (p. 182). The distinction between WSPs and ISPs is one of degree. Designing a house can be considered a problem that lies at the ill-structured end of the continuum. There is initially no specific criterion to test a proposed solution and the search for a solution is not well defined because it would have to contain at each stage all kinds of structures that the architect might consider. In selecting this example, Simon acknowledged the similarity to Reitman's (1965) prior claim that writing a fugue is an ISP.

My purpose in writing this article is to examine the claims made by Simon (1973). Part of my motivation is that, although I collaborated with Simon a few years later (Simon and Reed 1976), I have never reflected on whether our work on a WSP had any relevance to solving ISPs. It now seems to me that many of the theoretical concepts used in our article are also relevant to ISPs. My objective therefore is to reexamine Simon's 1973 claims within the context of over four decades of subsequent research on WSPs and ISPs.

My plan is to begin by providing some background information that will form the foundation for my analysis. I will then review some of the major empirical and theoretical findings from research on both WSPs and ISPs. The best examples of WSPs consist of puzzles such as logical proofs, missionaries and cannibals, the tower of Hanoi, and water jars. I will next discuss the insight problems initially studied by Gestalt psychologists. I will then consider research on classroom problems followed by design problems. In each case, I will examine the similarities and differences between WSPs and ISPs. This comparison should be revealing because chapters on problem solving typically focus only on WSPs (Bassok and Novick 2012; Reed 2015).

Table 1 Characteristics of well-structured problems (Simon 1973)

1. There is a definite criterion for testing any proposed solution, and a mechanizable process for applying the criterion.
2. There is at least one problem space in which can be represented the initial problem state, the goal state, and all other states that may be reached, or *considered*, in the course of attempting a solution of the problem.
3. Attainable state changes (legal moves) can be represented in a problem space, as transitions from given states to the states directly attainable from them. But considerable moves, whether legal or not, can also be represented—that is, all transitions from one considerable state to another.
4. Any knowledge that the problem solver can acquire about the problem can be represented in one or more problem spaces.
5. If the actual problem involves acting on the external world, then the definition of state changes and of the effects upon the state of applying any operator reflect with complete accuracy in one or more problem spaces the laws (laws of nature) that govern the external world.
6. All of these conditions hold in the strong sense that the basic processes postulated acquire only practicable amounts of computation, and the information postulated is effectively available to the processes—i.e., available with the help of only practicable amounts of search.

From Simon (1973)

Background

Two key components of Newell and Simon's (1972) theory of problem solving are the task environment, represented as a problem space, and the strategies used to search the problem space. The next section describes the problem space and the following section describes strategies used to explore it.

Problem Spaces

A theoretical construct listed in Table 1 that is helpful for making comparisons across problems is a problem space. As stated by Newell and Simon (1972, p. 428) "The problem space was viewed as the space that is generated by starting with a set of initial objects and working outwards from these to other objects that can be reached from them, without imposing any particular direction on the search". One advantage of using puzzles during the initial study of problem solving is that puzzles have well-defined problem spaces. An example is a variation of the missionary/cannibal (MC) problem that requires transporting five missionaries and five cannibals from the left bank to the right bank of a river using a boat that can hold three people. Cannibals cannot outnumber missionaries in the boat or on either bank of the river. The instructions are:

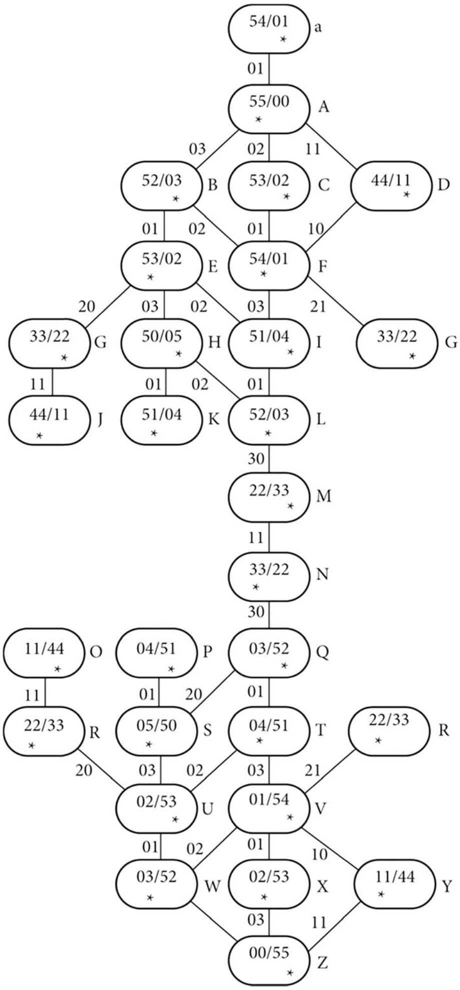
> Five missionaries and five cannibals who have to cross a river find a boat, but the boat is so small that it can hold no more than three persons. If the missionaries on ether bank of the river or in the boat are outnumbered at any time by cannibals, they will be eaten. Find the simplest schedule of crossings that will allow everyone to cross safely. At least one person must be the boat at each crossing.

Figure 1 shows the legal problem states that connect the initial state (A) to the goal state (Z) in the 5MC problem (Simon and Reed 1976). The two numbers to the left of the slash represent the number of missionaries and cannibals on the left bank. The two numbers to the right of the slash represent the number of missionaries and cannibals on the right bank. The asterisk shows the location of the boat. The problem space reveals important characteristics of the problem such as the minimum solution consists of 11 moves, more than one pathway produces this minimum solution, and a blind alley (J) requires problem solvers to move backward before they can move forward.

In principle, the problem space for a WSP can be constructed from analyzing the problem although in practice it may be very large. In 2007 Jonathan Schaeffer, a computer scientist at the University of Alberta, announced a computer checkers program that would never lose a game (Cho 2007). The proof required hundreds of computers running for nearly two decades. The long time period was caused by the 500 billion–billion possible arrangements of checkers on the board. However, the game of checkers satisfies the criteria of a WSP. It consists of clearly specified legal moves and a goal state indicating when the game is over.

In contrast to the problem space, a search space consists of the (legal and illegal) moves considered by the problem solver. A large problem space is not a detriment if the problem solver has an effective strategy for searching it. As stated by Newell and Simon (1972, p. 428) "The important point is that, so long as the search need never extend beyond a relatively narrow band joining initial with terminal objects, the size of the search space remains small and *independent* of the size of the problem space". The next section describes a general model of how people search the problem space.

Fig. 1 Problem space for the five missionaries/cannibals puzzle. From Simon and Reed (1976)



Strategies

Generic frameworks are helpful for making comparisons across different types of problems. The one shown in Fig. 2 is a general model of problem-solving strategies proposed by Gick (1986) that continues to be used as a foundation for developing models of problem solving (Nokes-Malach and Mestre 2013). The problem solver begins by constructing a representation to understand the problem by focusing on the goal, the constraints, and provided information. This construction may activate a schema if the problem is familiar. Schemas provide knowledge structures for encoding and interpreting particular experiences, greatly reducing the amount of search required to find a solution (Brewer and Nakamura 1984). Invoking a schema is an active process in which a particular experience is matched to a schema that best fits that experience.

If schematic knowledge is not activated, the problem solver must search for a solution by using general methods such as means/end analysis and forming subgoals. Means/end analysis

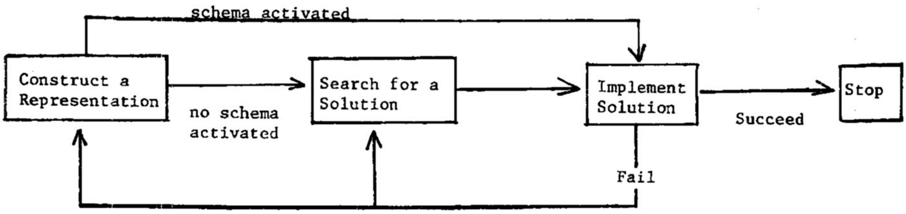


Fig. 2 Problem solving stages. From Gick (1986)

guides the search process by identifying moves that reduce the difference between the current problem state and the goal state. Subgoals are intermediate problem states between the initial and goal states. General search strategies are typically labeled *heuristics* because they apply to many types of problems but do not guarantee a successful solution.

I will use Gick's (1986) model of problem-solving strategies to compare how representation, search, and schema activation apply to the four types of problems (puzzles, insight puzzles, classroom problems, design problems). I also include analogy as a fourth strategy because of its important role in relating a general strategy (search for an analogy) to information in LTM (the solution of the analogous problem). Analogous problems also play an important role in providing the foundation for schema abstraction (Gick and Holyoak 1983). Combining the four problem types with the four strategies produces the 4×4 taxonomy shown in Table 2. Entries in the table illustrate representative studies for each of the 16 cells.

Summary

Simon (1973) proposed that WSPs differ from ISPs in degree depending on how many characteristics they satisfy for WSPs (Table 1). WSPs can be represented by a problem space specifying an initial state, a goal state in which there is a definite criterion for testing any proposed solution, and legal options connecting the two states (Fig. 1). The problem space specifies the legal moves that are available at each stage in solving a problem while the search space represents the (legal and illegal) moves that the problem solver evaluates. Strategies interact with the problem space to determine success in solving both WSPs and ISPs. Problem solving begins with forming a representation of the problem that can be implemented with minimal search when the representation activates schematic knowledge (Fig. 2). Failure to activate schematic knowledge requires the use of general strategies (subgoals, analogy) to search for a solution.

Table 2 A strategy×problem taxonomy

Strategy	Puzzle	Insight puzzle	Classroom	Design
Representation	Simon (1975)	Knoblich et al. (1999)	Greeno et al. (1993)	Carroll et al. (1980)
Search	Atwood and Polson (1976)	MacGregor et al. (2001)	Sweller et al. (1983)	Klahr and Simon (1999)
Analogy	Reed et al. (1974)	Duncker (1945)	Richland et al. (2004)	Smith et al. (1993)
Schema	Typically unavailable	Gick and Holyoak (1983)	Silver (1981)	Goel and Pirolli (1992)

Note: Entries show representative studies.

Puzzles

In his article on instructional design models for well-structured and ill-structured problems, Jonassen (1997, pp. 67-68) claimed that “while puzzle problems are interesting testbeds for research, they are not relevant either to school learning or everyday practice and so will not be treated further in this paper”. I disagree. As I will argue here, many of the important theoretical discoveries regarding *well-structured puzzles* in the research on problem solving in the 1970s provide a theoretical foundation for understanding other types of problems.

Representation

Different types of problem representations were not emphasized in Newell and Simon’s (1972) information-processing theory because they represented problems as symbols that a computer could simulate. Perception, however, did influence some search strategies as illustrated by the Tower of Hanoi (TOH), which consists of three pegs with a stack of disks on one of the pegs. The disks vary in size from the largest at the bottom of the stack to the smallest at the top of the stack. The problem requires moving the disks, one at a time, to another peg under the constraint that a larger disk cannot be placed on a smaller disk.

Simon (1975) analyzed the computational demands of four strategies that can be used to solve the problem. One strategy is primarily goal driven and uses subgoals to select the next move. A second alternative, a move-pattern strategy, requires cycling a disk through the pegs in a particular order. Two other strategies are perceptually driven such as noticing that the largest disk is blocking the movement of another disk. The four strategies place different burdens on perception and short-term memory, and differ in their potential for transfer to another problem.

Response latencies, action patterns, and eye movements provide evidence for the use of perceptual strategies in solving the TOH (Patsenko and Altmann 2010). Their selective-attention model recognizes towers of disks as objects relevant to performance. The assembly and disassembly of towers is guided by the recognition of a new tower, which initiates planning how to move it to another peg.

Search

Puzzles such as Missionaries/Cannibals and the Tower of Hanoi are WSPs that provide a convenient means to study general search strategies because people lack schematic knowledge about how to solve such problems (VanLehn 1989). Means/end analysis and subgoals are two strategies that are particularly helpful for guiding search.

Means/End Analysis Problems that consist of an initial state, a specific goal state, and well-defined moves connecting the two have been labeled *transformation* problems because they require transforming the initial state into the goal state through a sequence of legal moves (Greeno 1978). The transformation is guided by a means/end analysis heuristic that selects a move to maximally reduce the distance between the current problem state and the goal state (Ernst and Newell 1969; Greeno 1978; Newell and Simon 1972). Using means/end analysis to solve the 5MC problem in Fig. 1 would take three people to the right bank because that is the maximum capacity of the boat. It would return one person to the left bank because at least one person must be in the boat classroom and design problems (Simon and Reed 1976).

Means/end analysis has been used to solve other transformation problems such as water jug problems (Atwood and Polson 1976). A typical problem consisted of a filled large (8-gallon) jug and two empty smaller (5-gallon, 3-gallon) jugs. The goal in this problem was to obtain 4 gallons of water in the two larger jugs. The means/end heuristic reduced the difference between the actual and desired amount of water in these two jugs. A model based on the means/end heuristic that made assumptions about the use of short- and long-term memory accounted for the detailed performance of people on different problems.

A table of connections is helpful to carry out means/end analysis when there are many different moves or operators (Ernst and Newell 1969; Greeno 1978; Newell and Simon 1972). For constructing logical proofs, the table related 12 transformation rules to the differences (such as sign and position) that could be eliminated between the current state and the goal state (Newell and Simon 1972). The table of connections was provided to AI systems such as the General Problem Solver (Ernst and Newell 1969) but humans constructed proofs that were consistent with predictions based on their generation of a table of connections to guide means/end analysis (Newell and Simon 1972).

A limitation of heuristics is that they are not always helpful. For instance, means/end analysis is of limited use in solving the problem of producing the number 56 by combining four 7 s with the standard arithmetic operations of adding, subtracting, multiplying, dividing, and using parentheses. The difficulty is that there are so many combinations of these symbols. Another heuristic—working backwards—is effective when there are many paths leading from the initial state and relatively few paths connected to the goal state (Wickelgren 1974). The factors of 56 (the goal state) are 28×2, 14×4, and 7×8. The latter is particularly promising because it uses one of the 7 s, creating the subgoal of combining three 7 s to make the number 8.

Subgoals Subgoals are problem states on the solution path between the initial state and goal state. They reduce the size of the problem space by decomposing a problem into its parts (Wickelgren 1974). They are also helpful because fewer moves are required to reach a subgoal than the goal so it should be easier to use the means/end strategy to reduce distances.

Using subgoals to reduce the size of the problem space should be more effective for larger problem spaces than for smaller problem spaces. Providing a subgoal of three cannibals across the river, by themselves, without the boat (state L in Fig. 2) for the 5MC problem reduced the number of legal moves, illegal moves, and solution time. Providing a corresponding subgoal for the simpler 3MC problem had no effect (Reed and Abramson 1976). The problem space of this problem lacked the potential hazards of the larger problem space.

One way in which subgoals can be helpful is to encourage people to use a more productive search strategy. People who solved the 5MC problem without a subgoal required an average of 30 moves compared to 20 moves for people given the subgoal (Reed and Abramson 1976). A strategy-shift model provided a detailed simulation of how people navigated the problem space in Fig. 1 by assuming that the subgoal caused solvers to shift from an unproductive balance strategy to a more productive means/end strategy (Simon and Reed 1976). The balance strategy attempted to equalize the missionaries and cannibals on each bank, which resulted in entering the blind alley that ended in State J. Without knowledge of the problem space we would have no idea why a balance strategy is ineffective.

One limitation in generating a subgoal is that it may not be obvious what constitutes a good subgoal when first encountering a problem. However, memory for subgoals after solving a problem can aid subsequent attempts. Students in an intentional learning group who knew they would have to solve the 5MC problem twice did better on their second attempt than students in an incidental group who were not told they would have to resolve the problem (Reed and Johnsen 1977). The intentional learning group was better at recalling useful subgoals than at recalling or recognizing individual moves.

A second limitation of subgoals is that they may be too far from the current problem state to be helpful. As problem solvers make moves that bring them closer to achieving the subgoal, correct moves become more obvious and errors decline (Egan and Greeno 1974). A third limitation is that reaching a subgoal can create confusion about what to do next. Some problems actually took longer with a subgoal because it took a longer to figure out what to do after reaching the subgoal (Hayes 1966). Subgoals are nonetheless usually helpful, as we will see for many types of problems.

Analogy

One conception of analogy that has had a major influence on research in cognitive psychology defines analogical reasoning as the ability to detect that two situations share a common structure even though they have different surface characteristics (Gentner and Markman 1997). A formal representation of this perspective is based on the premise that the two situations share *relations* among dissimilar attributes. Establishing an analogy requires mapping relations in one situation onto corresponding relations in another situation (Gentner 1983). The analogy that an atom is like the solar system maps shared relations such as the sun *attracts* the planets and *causes* the planets to *revolve around* the sun. The analogy does not map dissimilar attributes such as the sun is hot and yellow.

The study of transfer across problems has typically investigated isomorphic problems that, like the atom and solar system, are structurally identical but differ in objects/attributes. An example is variations of the Tower of Hanoi that are stated as monster problems (Hayes and Simon 1977). In one variation, both monsters and globes come in different sizes and globes are transferred from one monster to another. In another variation, the globes change size. Although there are corresponding moves in the two variations, problems involving transfer of globes were easier than problems involving size changes. These two different representations of the problem also influenced transfer of knowledge across problems.

Isomorphic problems can be solved by a one-to-one mapping of the moves in one problem onto the moves in the other problem. Another type of analogy involves a one-to-many mapping of moves across problems (Reed 2012). An example is transfer from the 3MC problem to the Jealous Husbands (JH) problem (Reed et al. 1974). The JH problem requires transporting three couples across a river under the constraint that a wife cannot be left in the presence of another man unless her husband is present. Moves in the JH problem map onto unique moves in the MC problem and there was transfer in this direction. Moves in the MC problem constrain moves in the JH problem but do not map onto a unique move. There was no transfer in this direction.

Schema

Heuristics such as means/end analysis, subgoals, and analogy may be unnecessary when the representation of a problem activates a familiar schema that guides the solution (Fig. 2). In her book *Schemas in Problem Solving* Marshall (1995) began by tracing the historical development of the concept in the writings of Plato, Aristotle, Kant, Bartlett, and Piaget. She defined a schema as a memory organization that can (1) recognize similar experiences, (2) access a general framework that contains essential elements of those experiences, (3) use the framework to draw inferences, create goals, and develop plans, and (4) provide skills and procedures for solving problems in which the framework is relevant.

Puzzles are typically novel problems so they do not activate a familiar schema to guide the search. Comparing analogous puzzles can nonetheless encourage the creation of schemas that are helpful in solving subsequent problems that fit the schema. This process was discovered by Gick and Holyoak (1983) and likely was influential in Gick's (1986) formulation several years later of her general problem-solving model. Because Gick and Holyoak (1983) studied whether schema abstraction would facilitate solving an insight puzzle, I will present this material in the next section.

Summary

WSPs such as Missionaries/Cannibals and the Tower of Hanoi played an important role in the initial development of Newell and Simon's information-processing theory based on a problem space consisting an initial state, a goal state, and well-defined legal moves connecting the two states. Heuristics such as means/end analysis, subgoals, and analogy help guide the search. Means/end analysis reduces the distance between the current problem state and the goal state. Subgoals are intermediate problem states that reduce the size of the search space. Analogy uses the solution of a similar problem to guide the search. Perception also helps guide the search in problems such as the Tower of Hanoi. However, as discussed in the next section, perception took center stage in the work of the Gestalt psychologists on insight puzzles.

Insight Puzzles

Perceptual reorganization and insight are two key theoretical constructs developed by Gestalt psychologists (Kohler 1947). Gestalt psychologists had emphasized the importance of structure in perception and continued this emphasis when they began to study problem solving. Insight refers to the sudden discovery of a solution following a search that appears to be making no progress. It typically involves a restructuring that identifies the correct relations among the components of the problem. Although perceptual reorganization and insight are two "trademarks" of Gestalt psychology, more recent research has investigated how these two theoretical constructs interact with search strategies (Bassok and Novick 2012).

Representation

The finding that Gestalt problems are solved through a sudden reorganization is a consequence of the problem space. The problem space is broad but shallow. There are many possible paths leading from the initial state but those leading to the solution are typically short. As stated by Goel (2014):

> Insight problems differ from the broader class of well-structured problems in that the goal state lies in a part of the problem space that is unconnected (or remotely connected) to, or not "visible" from, the current state of the problem solver. The phenomenological

> experience of the problem solver is one of being suddenly transferred from the current node in the problem space to a node that is connected to or near the goal state (p. 4).

One means of suddenly forming a new representation of a problem is to remove unnecessary constraints that are imposed by the problem solver rather than the problem. An example is a match problem that requires moving one match to make a correct arithmetic statement (Knoblich et al. 1999). Moving the first match to right of the V makes a correct statement for the problem IV=III+III. The most difficult problems involved moving a match to create two equal signs such as for the problem III=III+III. When informed that this was a permissible move, people solved these problems as easily as the others.

Another problem that requires constraint removal is the nine-dot problem, which consists of a matrix of three rows and three columns of equally spaced dots. The problem requires connecting all nine dots by drawing four straight lines without retracing a line or lifting the pencil from the paper. Problem solvers typically constrain the lines to fall within the dot matrix but the solution requires constructing one line that extends beyond the boundary of the outer dots.

A variable that can influence reorganization through insight is the inclusion of an incubation period that produces a temporary shift away from the unsolved problem. Sio and Ormerod (2009) conducted a meta-analytic review of incubation effects to shed light on three different unconscious processes that have been proposed as explanations of the effectiveness of an incubation period. The first is that is that activation will spread over time to previously ignored but relevant memory items. The second is that an incubation period will weaken the activation of incorrect solutions. The third is that an individual's mental representation of the problem will be reorganized into a more appropriate form. The analysis supported the positive influence of incubation and showed that the explanatory value of the three hypotheses (spreading activation to promising solutions, weakened activation of incorrect solutions, reorganization of existing solutions) depended on the specific tasks.

Evidence for the unconscious discovery of some solutions is also supported by research that relates mind wandering to finding answers for compound remote associates (Zedelius and Schooler 2015). The task provides a series of mini-insight problems that require finding a word (such as *head*) that combines with three other words (*ache*, *hunter*, *cabbage*). After entering a word, the participant rated the extent to which the answer was found through a strategic search or through insight. An increased tendency to mind wander was associated with increased insights.

A prediction resulting from this finding would be that work on an alternative task during the incubation period should be very different from the initial problem. Evidence supporting this prediction occurred in an experiment in which incubation following a visual problem benefitted from a verbal task during the incubation interval and incubation following a verbal problem benefitted from a visual task (Gilhooly et al. 2013). The practical advice is, when stuck during creative problem solving, do something very different before returning to the problem.

Search

Gick's (1986) distinction between representation and search in Fig. 2 provides a helpful framework that Bassok and Novick (2012) adopt in their chapter on problem solving by contrasting problem representation (the Gestalt legacy) with search in a problem space (the

Newell and Simon legacy). A difference between the arrangement problems studied by Gestalt psychologists and the transformation problems studied by Newell and Simon is that transformation problems support evaluation of progress (Greeno 1978). Ratings of progress did increase for transformation problems as solvers approached the goal (Metcalfe and Wiebe 1987). In contrast, ratings of progress only increased shortly before reaching the goal in representation problems, a finding that is consistent with the Gestalt concept of insight.

Two characteristics of the problem space for insight puzzles are helpful in explaining the progress ratings for arrangement problems. One characteristic, already discussed, is that the problem space is broad but shallow. Another characteristic, in contrast to problems amenable to means/end analysis, arrangement problems do not specify a specific goal state. A chimp that wants to obtain bananas hanging from the ceiling has to *discover* that the goal state consists of climbing stacked crates and using a stick to hit the banana (Kohler 1925). Solvers of the candle problem had to discover the new relations among objects to mount the candles on the door (Duncker 1945). Using means/end analysis to reduce the differences between the current state and goal state requires a specific goal state, such as all the missionaries and cannibals are across the river (Simon and Reed 1976) or there is a tower of disks on peg C (Simon 1975). Discovering the goal state in Gestalt problems requires relying on skills such as fluency in generating many possible solutions, retrieval of solution patterns, and knowledge of principles to constrain search (Greeno 1978).

An important contribution of Bassok and Novick's (2012) chapter is their discussion of examples in which the representation and the search perspectives can be combined to enhance our understanding of insight puzzles. For instance, the nine-dot problem falls squarely within the Gestalt tradition with its emphasis on perception and insight. Nonetheless, one model proposes that its solution can be analyzed within a search framework using means/end analysis (MacGregor et al. 2001).

The model assumes that means/end analysis models progress by counting how many unconnected dots a new line connects. According to the model, insight is evoked by violation of the progress-monitoring criterion, not the perceptual organization of the array. Constraint relaxation—extending a line outside the boundary of the dots—occurs when drawing lines within the boundary connects an insufficient number of dots. The model has been extended to explain searching for a solution to the eight-coin problem in which two coins in a starting arrangement must be moved so that each coin touches exactly three others (Ormerod et al. 2002). Solving the problem requires removing the self-imposed constraint that coins cannot be stacked on other coins.

Further evidence for both progress monitoring and representation changes occur in the car park problem in which cars blocking a taxi must be moved before the taxi can exit (Jones 2003). Solving the problem requires realizing that the taxi, along with other cars, must initially be moved to other parking spaces to clear the exit. Jones evaluated both the progress monitoring theory based on failing to achieve a minimal distance toward the goal (MacGregor et al. 2001) and the representation change theory based on removing unnecessary constraints (Knoblich et al. 1999). He found evidence for both theories but experimental manipulations suggested that changes in representations provided the better explanation.

We have reviewed evidence that unconscious processing supports representation changes during incubation but it would seem unlikely that unconscious processing supports monitoring a more-extended search (Wiley and Jarosz 2012). Results supporting this hypothesis occurred in a study in which students solved two variations of the eight-coin problem (Ormerod et al. 2002) that required either a demanding search phase prior to restructuring or only restructuring. Working memory span predicted success on problems requiring search but did not predict performance on problems requiring restructuring (Ash and Wiley 2006). The finding is consistent with an interpretation that, in contrast to search, restructuring occurs through an automatic redistribution of activation. Indeed, insight may be just one of a number of high-level functions that can be performed without consciousness (Hassin 2013).

The interplay between conscious and unconscious thinking was modeled by the explicit–implicit interaction theory (Helie and Sun 2010) that is implemented in a cognitive architecture called CLARION (Sun and Zhang 2006). The theory analyzes four stages of problem solving discussed in the book *The Art of Thought* (Wallas 1926). It proposes that the initial preparation phase is predominately rule-based as people respond to verbal instructions, form representations of the problem, and establish goals. In contrast, the incubation stage is predominately implicit processing in which people do not consciously think about the problem. The third stage, insight, occurs when an activation level crosses a threshold that makes the results available for verbal report. The final verification stage, like the initial stage, requires explicit processing to evaluate the discovered solution.

Analogy

The use of puzzles to study analogy within the information-processing tradition was founded on the hope that a previous solution would be helpful in finding a solution to the current problem (Reed 2015). In contrast, analogical reasoning in the Gestalt tradition typically had an adverse effect because functional fixedness on prior usage prevented a reorganization of the components to solve the problem.

A classic example is the candle problem studied by Duncker (1945). The goal is to place three small candles on a door at eye level. Among the objects on a table are three boxes about the size of matchboxes. In one condition, the boxes are filled with candles, tacks, and matches. In another condition, the candles, tacks, and matches are on the table beside the empty boxes. More participants used the boxes as platforms to solve the problem when the boxes were empty. The filled boxes reminded participants that boxes typically function as containers and made it more difficult to recognize their novel function as platforms.

As discussed previously, one of the difficulties in solving Gestalt problems is that self-imposed constraints block the solution (Knoblich et al. 1999). Functional fixedness is a type of self-imposed constraint in which considering only the most typical uses of objects prevents consideration of novel uses. Problems such as the candle problem nonetheless are WSPs in the sense that it is possible to judge both legal moves and the correctness of a proposed solution. Using the box as a platform is clearly a legal move and the solution works if attaching the platform to the door supports the candles.

However, other Gestalt problems are ISPs if there is ambiguity regarding legal moves and the correctness of a solution. Moving a match in IV=III+III is a WSP because 6=3+3 is a typical arithmetic equation. However, moving a match in III=III+III is an ISP because is 3=3=3 is a questionable arithmetic equation. These problems became easy only after students learned that creating two equal signs is a permissible move (Knoblich et al. 1999). The eight-coin problem, in which two coins in a starting arrangement must be moved so that each coin touches exactly three others (Ormerod et al. 2002), is another example of an ISP. Solving the problem requires stacking one coin on another but many problem solvers may regard stacking as "cheating". Gestalt problems therefore vary in their location on the well-defined versus ill-defined continuum depending on the difficulty in evaluating permissible moves and the correctness of a solution.

One method for overcoming functional fixedness requires discovering features of objects that were previously overlooked but can be made more transparent (McCaffrey 2012). The technique encourages the participant to create a hierarchy by decomposing an object into parts that may be further decomposed. One problem evaluated in the study required fastening together two steel rings using only a long candle, a match, and a 2-in cube of steel. Solving the problem requires that the wick of the candle can be used as a string to tie together the two rings. Training on the decomposition technique helped participants list more target features, list more key obscure features, and solve more insight problems that evoked functional fixedness.

Another study (Gick and Holyoak 1980) attempted to encourage the positive use of analogical reasoning to find a solution to Duncker's (1945) radiation problem. The problem requires using radiation to destroy a tumor in the interior of the body without destroying the healthy tissue that surrounds it. High-intensity rays could not be applied directly because they would destroy healthy tissue in their pathway. One solution, the convergence solution, destroys the tumor by dividing the rays so they converge with maximum intensity at the site of the tumor. Gick and Holyoak (1980) investigated whether students would use analogical reasoning to discover the convergence solution if they initially read several stories, one of which described a general who divided his army to capture a fortress by converging on it from multiple directions. Many students did successfully use the analogy when prompted but did not spontaneously recognize the similarity between the two problems.

Schema

The challenge in encouraging students to spontaneously use a potentially helpful analogy is that the two analogous problems had very different story content. One story described dividing radiation to converge on a tumor and the other described dividing an army to converge on a fortress. Subsequent studies (Catrambone and Holyoak 1989; Gick and Holyoak 1983) established that describing the similarities between two convergence stories encouraged abstraction of a generic convergence schema that was spontaneously applied to the radiation problem. The creation of a (convergence) schema enabled the problem solvers to recognize the radiation problem as another example of a convergence problem.

The activation of relevant schema is important for solving many problems encountered in classrooms, as we will see in the next section. These problems rely on knowledge of particular domains such as mathematics and physics so general search heuristics are insufficient unless accompanied by domain knowledge that can be organized by schemas.

Summary

Puzzles studied by the Gestalt psychologists consist of both WSPs and ISPs that require finding a solution by reorganizing an initial perceptual configuration. Some reorganization requires removing self-imposed constraints, such as believing that a line cannot extend beyond the configuration of dots or a coin cannot be placed on top of another coin. Although the Gestalt perspective emphasized representations rather than search, subsequent theories proposed that representation changes are triggered by a failure to make sufficient progress based on means/end analysis. Changes in representations that occur suddenly are labeled "insight" and can be facilitated by an incubation period. Unconscious processing contributes to representation shifts through spreading activation to promising solutions or weakened activation of incorrect solutions. In contrast, search heuristics involve conscious processing based on a limited-capacity STM because progress is incremental. Unlike the positive influence of analogies in solving puzzles, analogical reasoning can interfere with novel reorganization because of functional fixedness. The comparison of two related solutions, however, can help solvers by representing perceptual organization at a more generic, schematic level. Schema acquisition is particularly important for solving classroom problems, as we will see in the next section.

Classroom Problems

The focus on puzzles as problems in both the Gestalt (Kohler 1947) and Information Processing (Newell and Simon 1972) traditions had the advantage that domain knowledge did not play a major role in the solution. Searching a problem space or reorganizing the components of the problem to achieve a new structure did not require knowledge of specific subjects. VanLehn (1989) referred to this research as knowledge-lean problem solving. In contrast, solving a mathematics or physics problem requires knowledge about mathematics or physics. VanLehn referred to this research as knowledge-rich problem solving.

The study of knowledge-rich problem solving began to gather momentum in the 1980s as researchers became more interested in studying problems encountered in the classroom. General search heuristics, without the aid of domain knowledge, were insufficient for solving these problems. Such problems required schema activation but the depiction in Fig. 2 of implementing the solution following schema activation is oversimplified unless the schema contained all the details of the solution. In other cases, schema activation could help guide a search so heuristics such as forming subgoals and retrieving analogous solutions might still be required.

Representation

A discussion of the representation of classroom problems could easily fill an entire book, so let me focus on one particular issue that contrasts with the Newell and Simon (1972) theory of problem solving. Although problems such as the Tower of Hanoi and Missionaries and Cannibals required moving disks and tokens, these tasks were theoretically represented as operations on symbols in a problem space. As argued by Barsalou (1999) in his classic article on perceptual symbol systems, perception and action had traditionally been treated as input and output systems that supported cognition where the real thinking occurred. He proposed that perception and action should, instead, be included as central aspects of thinking.

Greeno et al. (1993) had earlier advocated a theory of transfer in which perception and action received greater emphasis than the creation of symbols. They argued that the similarity between two problems could be based on simulated models. As they stated:

> Similar properties can include spatial relations, such as objects that move similarly in relation to each other, or functional roles, such as causal interactions or properties to avoid some consequence. The most important difference between this hypothesis of mental models and the more common hypothesis of abstract symbolic schemata is that properties needed for analogical problem solving may be properties of simulated events or objects rather than being denoted by symbols (p. 149).

An example of a study supporting this view instructed third-graders to use either concrete (toy fish) or abstract (Lego blocks) manipulatives to solve arithmetic story problems (Glenberg et al. 2007). A control group did not use manipulatives but read the problem twice. The concrete manipulatives were very effective in increasing the number of correct solutions. The children in this group also did very well with imagined manipulation of the objects after they had learned how to solve problems with physical manipulatives a week earlier.

When used correctly, manipulatives can be very effective in enhancing learning (Reed 2010). I say “used correctly” because some teachers have used manipulatives as “fun math” without connecting the activities to working with symbols (Moyer 2002). One group of teachers made subtle distinctions between real math based on rules, procedures, and paper-and-pencil tasks and fun math that used manipulatives. The importance of connecting real math with fun math falls within the perspective labeled ‘representational pluralism’. It advocates the use of multiple representations because different representations have different strengths that complement each other (Dove 2009). The challenge is to select the best representation for a particular task and to carefully integrate different representations when more than one is selected.

Search

The importance of general search heuristics such as means/end analysis is diminished, but not eliminated, when solving classroom problems. Sweller et al. (1983) studied the application of means/end analysis to physics problems that required using the equations: (1) Distance=Rate×Time, (2) Rate=0.5×Final rate, and (3) Final rate=Acceleration×Time. Means/end analysis requires beginning with the equation that contains the variable stated in the goal, such as Equation 1 if the problem requires solving for distance. If the value of a variable such as rate is not stated in the problem, the next step is to search for an equation that enables the calculation of the required variable. The means/end search strategy is effective in finding the solution but it imposes a high cognitive load that can prevent the problem solver from learning the solution. An alternative instructional procedure, in which problem solvers solved for whatever variables they could, reduced cognitive load and increased procedural learning (Sweller et al. 1983).

Another previously discussed topic for solving puzzles was the interaction between search and perception (Bassok and Novick 2012). This interaction is also important for solving classroom problems such as those encountered in a geometry class. The interaction between perception, subgoals, and strategies occurs when a problem solver must choose among several possible methods (angle–side–angle, side–angle–side, side–side–side) for proving the congruence of two triangles (Greeno 1976). Identifying congruent pairs of sides and angles is driven by perception. Finding that two pairs of angles are congruent could result in establishing the subgoal of determining whether the pair of sides between the angles is also congruent to solve the problem by the angle-side-angle method. The choice of a solution method and appropriate subgoals is therefore delayed until the collection of perceptual information. Greeno (1976) concludes “with this modest extension to the theory of well-structured problems, important aspects of problem solving formerly considered characteristic of ill-structured problems can be included in the domain of well-structured problems” (p. 479).

A major effort in the study of classroom problem solving has been devoted to developing effective instructional methods and some these studies extend effective methods for solving puzzles. As was found for solving the MC problem (Reed and Abramson 1976) providing subgoals can be helpful for organizing procedures for solving classroom problems. For instance, students studied a worked example that showed how to solve a probability problem that required finding total frequency as one of the subgoals. Students did better in transferring the solution to other problems when this step was explicitly labeled as *Total Frequency* (Catrambone 1995). The transfer problems required a different method for finding total frequency so students could not rotely apply the same procedures. Learning required understanding the example and explicitly labeling the subgoal increased understanding.

Analogy

A characteristic that classroom problems share with puzzles is the potential helpfulness of basing search on an analogous solution. And indeed, models based on mapping concepts and relations across problems (Gentner 1983; Gentner and Markman 1997) can be applied to classroom problems as they were to puzzles (Gick and Holyoak 1980; Reed et al. 1974).

An example is the application of Gentner's (1983) structure-mapping model to predict the relative effectiveness of analogous solutions to solve algebra word problems (Reed 1987). Students received two worked examples (such as two pipes filling a tank), each of which served as an analogous problem for four test problems. Success in using a worked example to solve each test problem could be predicted by students' ability in identifying corresponding concepts and relations between the worked example and the test problems.

The differential effectiveness of analogous solutions raises the issue of how teachers provide helpful analogies to students. Richland et al. (2004) sought to answer this question by studying videotapes of 25 eighth-grade mathematics classrooms included in the Third International Mathematics and Science Study. When students experienced difficulty solving a problem, teachers typically selected analogous problems that had highly transparent mappings to the test problem. The authors concluded that the teachers made consistent efforts to encourage student inferences but may have maintained too much control over guiding the reasoning process.

Although these studies fit within the cognitive tradition, other perspectives propose a broader view of transfer. The actor-oriented perspective (Lobato 2012) differs from the cognitive approach by focusing on how individual students, rather than experts, perceive analogous tasks. Lobato (2012) proposed that there is extensive transfer across problems but transfer of some information is incorrect. For instance, she asked students to calculate the slope of a slide after they had learned how to calculate the slope of the hypotenuse of a right triangle by dividing rise (vertical change) by run (horizontal change). Some students incorrectly selected as 'run' the horizontal line representing the platform at the top of the slide. Identifying concepts that are incorrectly mapped across problems is valuable for designing instruction to eliminate errors.

A difference between laboratory and classroom studies is that classroom learning occurs over long periods of time. An instructional objective should be to frame the delivery of information so that students expect that information learned today will remain relevant for future learning (Engle 2006). A curriculum on endangered species framed discussion about whales to make it relevant to students when they later discussed another endangered species. The idea that there are often multiple causes remained consistent across species but the types of causes for endangerment differed. Although there are different perspectives on analogical reasoning, a consistent theme is the mapping of information across situations (Reed 2012).

Engle's design of an endangered species curriculum differs from other instruction in mathematics and science that typically include problems in which a correct solution is easier to evaluate. ISPs, such as searching for causes of endangered species, require identifying alternative views or perspectives that might be more easily modeled in multiple problem spaces (Jonassen 1997). The learner needs to gather evidence to support or reject various perspectives by making claims about probable causes and consequences. Discussing such claims occurs not only for students in the classroom but also for scientists in areas of "unsettled" science,

Schema

A finding from the puzzles literature is that people miss the opportunity to use an analogous problem because of different story content (Gick and Holyoak 1980). Students failed to recognize the relevance of dividing an army to converge on a fortress when another solution required dividing radiation to converge on a tumor. Research on classroom problems has established that it takes considerable expertise to find common solutions in problems that differ in story content. One of the first studies to demonstrate this finding asked seventh graders to sort problems into groups based on common solutions. Students performing well in a mathematics class could follow the instructions but students having difficulty with mathematics sorted the problems according to story content (Silver 1981).

Many studies have replicated this crucial role of expertise in finding common principles in problems that on the surface appear quite different. One of the earliest, and most cited, asked eight undergraduates who had recently completed a physics course and eight physics doctoral students to sort physics problems into categories based on common solutions (Chi et al. 1982). The undergraduates primarily classified problems on the basis of shared objects such as springs or inclined planes. The doctoral students classified problems on the basis of common principles such as the conservation-of-energy law or Newton's second law ($F=MA$). The performance of experts depends on their ability to see shared relations among the objects in the problem (Chi and VanLehn 2012).

The finding that comparing two analogous puzzles resulted in a more generic (convergence) schema (Gick and Holyoak 1983) is another example that has been extended to classroom problems. This technique can be scaled up to classroom instruction as demonstrated by its use in a class at Northwestern University. Comparing examples helped management consultants learn to recognize principles of a contingent contract and recall creation of contingent contracts from their own experiences (Gentner et al. 2009).

Rittle-Johnson and Star (2009) initiated an extensive program of research to discover the conditions in which comparing mathematics problems would be helpful. Their findings were included in a meta-analysis of 57 experiments on a variety of topics that found that four of the 15 evaluated variables were most predictive of learning (Alfieri et al. 2013). Greater learning occurred when learners judged only similarities rather than similarities and differences, compared perceptual rather than procedural content, were tested immediately rather than on a later day, and received feedback on principles after making comparisons.

Summary

Solving classroom problems requires knowledge-rich problem solving in which domain knowledge is combined with heuristics such as analogical reasoning and means/end analysis. A limitation of the use of means/end analysis for classroom problems is that cognitive load may prevent solvers from learning the solutions. As is the case for solving puzzles, providing subgoals can be an effective instructional technique for helping students remember and apply solutions. Analogies are also helpful because they combine a general search heuristic (look for an analogy) with domain knowledge stored in the analogy. Instruction benefits from teachers selecting appropriate analogies and from researchers analyzing the causes of both appropriate and inappropriate transfer. Student selection of helpful analogies based on shared structure typically requires considerable expertise that is stored as schematic knowledge. As occurred for puzzles, research is identifying conditions in which problem comparisons aid analogical reasoning through the abstraction of schematic structure. Although this section focused on well-structured classroom problems, the next section examines ill-structured design problems. The solutions of some classroom problems, such as identifying causes for endangered species, are difficult to evaluate for correctness and therefore share characteristics of ISPs.

Design Problems

In the revision of his popular book, *The Design of Everyday Things*, Don Norman (2013) writes how his experience in industry taught him about the complexities of the real world. Designers must adhere to cost and schedules, pay attention to the competition, and work in multidisciplinary teams. Their products are typically the result of incremental innovation. Radical innovation is rare and can take a long time for acceptance.

Although the study of design is discipline specific, Goel and Pirolli (1992) argued that commonalities can be studied as generic design. They proposed a design problem-space hypothesis that “problem spaces exhibit major invariants across design problem-solving situations and major variants across design and nondesign problem-solving situations” (p. 399).

Table 3 summarizes the major invariants of design problems (Goel and Pirolli 1992). Many of the characteristics reflect the characteristics of ISPs. The initial, goal, and intermediate states are incompletely specified. There are no right or wrong answers. Constraints come from a variety of sources: physical, social, legal, and economic. The size and complexity of the problems require decomposition into smaller problems. The designer must request feedback and the cost of errors is high.

Tables 1 and 3 illustrate the differences between WSPs (Simon 1973) and design problems (Goel and Pirolli 1992). These differences are important but design problems also share

Table 3 Characteristics of design problems (Goel and Pirolli 1992)

1. The three components of design problems—the start state, the goal state, and the intermediate states—are incompletely specified.
2. There are typically physical, social, economic, political, and legal constraints on the design.
3. The problems are large and complex.
4. Decomposition into smaller parts is determined more by the designer than by the structure of the problem.
5. There are many contingent interactions among the components.
6. Design problems do not have right or wrong answers, only better and worse ones.
7. Functional information mediates in many ways between the input and output information.
8. Feedback must be simulated or generated by the designer during the problem-solving process.
9. The cost of errors can be high.
10. The product is required to function independently of the designer.
11. There is a distinction between specification of the product and its construction.
12. Specification precedes delivery of the product.

Based on Goel and Pirolli (1992)

characteristics with puzzles such as Duncker's radiation problem. An Associated Press article described a new treatment procedure at UCLA that allows doctors to target beams of radiation to exactly fit a tumor's dimensions (*San Diego Union-Tribune*, October 2, 1998). The title of the article is Radiation Beams Target Tumors, Bypass Healthy Tissue.

Representation

As previously discussed, the Gestalt psychologists were particularly interested in the perceptual representation of problems. An example that combines design with perceptual organization is Japanese gardens (van Tonder and Vishwanath 2015). Classical Japanese design emphasized the relation between human perception and natural form. The placement of rocks on the ground should not be aligned but follow an irregular, winding pattern. Rocks should be separated by unequal distances and adjacent rocks should be of unequal size (van Tonder and Vishwanath 2015). These principles of organization differ from classic Gestalt principles by emphasizing irregularity rather than good form that would be more characteristic of symmetric formal gardens. In this case, analogy trumps Gestalt principles of organization. The source of the analogy is the natural landscape.

Occasionally designs are so spectacular that they become an international sensation. One such occasion occurred when a young architect, Jorn Utzon, received a commission to design an opera house on a point of land jutting out into the harbor of Sydney, Australia. Goel (2014) combined this case study with design research to document the different kinds of symbol systems used by designers. The symbols range from vague, imprecise, and abstract (such as the ones used in conceptual sketches) to unambiguous, precise, and concrete (such as the ones used in contract documents). The former are more lateralized in the left prefrontal cortex and support associations that broaden the problem space. The latter are more lateralized in the right prefrontal cortex and support transformations that deepen the problem space.

Goel's (2014) distinction between imprecise symbols used in conceptual sketches and precise ones used in contract documents is mirrored in two different computer design environments (Yu et al. 2013). Traditional geometry modeling environments enable architects to sketch their designs on the screen. In contrast, parametric design environments are rule-based and used primarily for complex building forms and optimization of solutions. A study of five professional architects using both environments revealed that reformulation of the problem was more encouraged and better supported in the geometry modeling environments.

Although the usefulness of sketches is obvious for buildings, their usefulness is not as transparent for other designs. One of the early studies that demonstrated the importance of representation compared a spatial and a temporal version of a design task that had corresponding constraints (Carroll et al. 1980). The spatial version of the task required designing a business office for seven employees to satisfy multiple constraints. The temporal version of the task required designing a manufacturing process that consisted of seven stages. Students who designed the business office by drawing diagrams satisfied more of the constraints than students who worked on the manufacturing process and did not draw diagrams. However, the two groups were equally successful when both were instructed to use diagrams.

Search

Although Simon (1973) referred to designing a house as an example at the ill-structured end of the continuum, he nonetheless believed that ISPs could be solved using basic information-processing principles. Design is composed of a combination of using the General Problem Solver to guide search on well-structured subproblems and a retrieval system that continually modifies the problem space by retrieving from long-term memory new constraints, new subgoals, and new design alternatives as that information becomes relevant. The potentially large amount of information required to solve most design problems prevents the problem solver from construction of a well-structured problem space at the beginning of the problem.

Decomposition was apparent in the tasks studied by Goel and Pirolli (1992) that contributed to their formulation of the design characteristics shown in Table 3. Their tasks required three experts to either design a post office, a bank teller machine, or technical training material. The nondesign tasks required undergraduates to solve WSPs that were studied by Newell and Simon (1972). A major difference between the design and nondesign verbal protocols regarded the incremental development of knowledge (Goel and Pirolli 1992). Most of the search paths in the nondesign problems were wrong. Abandoning an unproductive search path therefore seldom required reusing the knowledge gained from exploring that path. In contrast, they found considerable incremental development when experts returned to a previous module within a design task.

Another difference between design and nondesign tasks concerns the Gestalt concept of insight. Goel (2014) listed two differences between classic insight problems and real-world design problems. First, insight in Gestalt problems typically reveals the solution whereas insight in the more complex design problems typically reveals only an important step toward achieving the solution. Second, insight problems are usually well structured while design problems have both well-structured and ill-structured components. As one progresses from the preliminary to the final design, the problem becomes more structured.

Scientific discovery is another domain that shares characteristics of both WSPs and ISPs. Klahr and Simon (1999) based their review of this domain on four major methods—historical accounts, laboratory experiments with nonscientists, direct observation of ongoing scientific laboratory work, and computational modeling. The results from using these diverse methods converged on key aspects of the discovery process:

> In our exploration of scientific discovery, we have seen that (a) it is based on heuristic search in a set of problem spaces: spaces of instances, of hypotheses, of representations, of strategies, of instruments, and perhaps others; (b) the control structures for search are such general mechanisms as trial and error, hill climbing, means-end analysis, and responses to surprise; and (c) recognition processes evoked by familiar patterns recognized in phenomena, evoke knowledge and strong methods from memory, thereby linking the weak methods to the mechanism that are domain specific (pp. 539-540).

Retrieving specific cases from memory includes analogical reasoning, as discussed in the next section.

Analogy

One of the methods used by Klahr and Simon in their review was ongoing laboratory work, including studies by Dunbar that are summarized in Dunbar and Blanchette (2001). Dunbar's initial studies focused on the use of analogy in four different laboratories of leading molecular biologists and immunologists. The investigators identified over 99 analogies mentioned during a total of 16 meetings. Analogies based on the physical similarity of experiments, such as ones varying incubation time, were useful for modifying experiments to make them more effective.

Analogies based on structural features, such as the underlying genetic sequence, were useful for formulating hypotheses. Although only 25 % of the scientific analogies were based on structural features, over 80 % of these analogies were used to formulate hypotheses. Dunbar's findings are consistent with laboratory studies demonstrating the role of expertise in identifying structural features (Chi and VanLehn 2012) but also show that analogies based on physical similarities can be helpful.

We should not lose sight of the fact, however, that analogy use is a heuristic and heuristics are often, but not always, helpful. A drawback of analogy is that it can stifle creativity for producing novel designs. Imagine that your task is to design a novel toy and you have an option of viewing other toys as examples. The toys designed by people who were shown examples were similar to the examples, even when otherwise instructed (Smith et al. 1993). The same result occurred when asked to draw creatures that inhabited another planet. The drawings looked like the examples. A related study asked students to generate animals that might live on a planet that was very different from earth (Ward et al. 2004). Those students who were told to consider attributes that animals would need to live on this new environment were less likely to draw typical animal properties such as eyes, ears, noses, and legs.

A method for encouraging novel creations is to place multiple constraints on the design (Finke 1990). Participants in one of the studies had to invent a product that would fit into one of eight categories (furniture, personal items, scientific instruments, appliances, transportation, utensils, toys, weapons). The experiment compared four groups that differed in whether they were assigned, or allowed to select, the parts and the category. The originality of the invention was greatest for the most constrained condition, such as inventors who had to design a toy using a half sphere, bracket, and hook. Multiple constraints made it more difficult to base the design on a preexisting toy.

Another approach that avoids overreliance on a single analogy is to borrow the best characteristics from multiple analogies. Ted Koppel in his *Nightline* television program gave a design team one week to develop a new shopping cart (Weisberg 2009). Their new design had rotating wheels to maneuver in narrow aisles, several small removable baskets to replace the large metal basket, a child's seat equipped with a safety bar, a microphone to allow communication with customer service, and a scanner for self-checkout. Multiple analogies aided the design. Office chairs inspired the use of better wheels and amusement park rides inspired the use of a safety bar.

Schema

Simon's (1973) model for solving ISPs required search of an immediate problem space combined with a recognition system that evoked retrieval of information from LTM. Most of the information stored in LTM would be irrelevant during any brief interval of processing. Nonetheless, expert schematic structures stored in LTM are crucial for implementing a good design.

To illustrate design as an organizational process Simon (1973) selected as a case study the design of a new battleship described 50 years earlier (Murray 1923). The Director of Naval Construction and his officers had a good general knowledge to outline the desired ship. Then specialists such as the Engineer-in-Chief, the Director of Torpedoes, the Director of Electrical Engineering, the Director of Naval Equipment planned the details. The planning progressed in a tentative manner until a more-or-less complete plan was eventually developed.

Planning, of course, also occurs during individual design with the efficiency of the plan dependent on the expertise of the individual. The protocols of three experienced computer programmers, who differed in their knowledge of the assigned task, illustrates differences in planning (Atwood et al. 1978). The task knowledge of one programmer was sufficiently developed to formulate a plan for indexing terms and page numbers in a book. The knowledge of the second programmer was less well developed so required backtracking to correct deficiencies. The knowledge of the third programmer was developed to such an extent that he was able to retrieve most of the design directly from memory and therefore required less planning than the others. This programmer was able to implement a schema (Fig. 2) while the others required more search of the problem space.

The three people studied by Goel and Pirolli (1992) also had considerable expertise in their respected fields of architecture, mechanical engineering, and instructional design. The experimental protocols revealed that not only did the designers interpret the problem situation through their personal experiences, but they would occasionally try to explicitly change the problem to more closely fit their expertise, knowledge and experience. They would attempt to negotiate changes in the initial state and goal state that would be more easily achievable or might lead to a more effective design. The detailed verbal protocols were instrumental in identifying the characteristics of design problems listed in Table 3.

Summary

Design problems share characteristics that distinguish them from WSPs (Table 3). The initial, goal, and intermediate states are incompletely specified. They require decomposition into smaller problems and must satisfy constraints from multiple sources. Decomposition results in incremental development in which designers return to previous modules. According to Simon, design requires means/end analysis to guide search on well-structured subproblems and continually modifying the problem space by retrieving from LTM new constraints, new subgoals, and new design alternatives. Insights create both well- and ill-structured components and typically contribute to the solution, rather than produce it. Success is influenced by the choice of a helpful representation such as sketching. Unlike problems in which creativity is not required, design problems suffer from overreliance on an analogous problem. Placing many constraints on the design and using multiple analogies enhances creative solutions.

Concluding Remarks

Writing this article gave me the opportunity to reflect on the highlight of my academic career—the 6 months that I spent working on a WSP with Herb Simon in 1975. My experience in working on this project motivated me to evaluate Simon's (1973) claim that information-processing models of WSPs could also be applied to ISPs. I therefore used Gick's (1986) model of problem solving strategies to examine whether work on WSPs such as puzzles could be extended to practical problems. My conclusion is that the theoretical concepts developed from studying puzzles in the 1970s formed a foundation for extending these ideas although these concepts sometimes had different implications when applied to classroom and design problems.

Research on the application of search heuristics to classroom problems have successfully developed techniques to reduce cognitive load (Sweller et al. 1983) and improve instructional examples by explicitly labeling subgoals (Catrambone 1995). Analyzing films of classroom instruction found that teachers select helpful analogous problems to match student ability

(Richland et al. 2004). Studies have identified how expertise enables students to organize classroom problems by common principles and solutions rather than by story content (Chi et al. 1982; Silver 1981). A meta-analysis identified when problem comparisons are most effective for improving learning (Alfieri et al. 2013).

There is also evidence to support Simon's (1973) claim that the same information-processing framework developed for puzzles also apply to design problems. He proposed that design consists of a combination of using means/end analysis to guide search on well-structured subproblems and continually modifying the search space by retrieving from LTM new constraints, new subgoals, and new design alternatives. Decomposition of design tasks into modules is similar to forming subgoals although a difference is the incremental development in design requires return to previous modules (Goel and Pirolli 1992). Similar to the problems studied by Gestalt psychologists, design tasks often require spatial reorganization (Carroll et al. 1980). Insights in design problems, however, typically only result in progress toward the goal because of their greater complexity (Goel 2014). Both physical and structural analogies are helpful in science laboratories (Dunbar and Blanchette 2001) although a limitation of analogical reasoning in product design is that it reduces creativity when the designs are too similar to the examples (Smith et al. 1993). Creative designs can be encouraged by introducing multiple constraints (Finke 1990) and using multiple analogies (Weisberg 2009).

Most of the research on comparing instructional methods for improving problem solving has studied WSPs such as those found in mathematics and science learning (Richey and Nokes-Malach 2015). It would be beneficial in future research to also compare the impact of such standard instructional techniques as practice, worked examples, analogical comparison, and self-explanation on improving design problem solving.

My discussion of ISPs raises the question of whether some ISPs are more difficult than the ones I considered. In the same year as Simon (1973) published *The structure of ill structured problems* Rittel and Webber (1973) published *Dilemmas in a general theory of planning*. The authors argued that the society problems faced by planners are inherently different from the types of problems faced by scientists. They identified ten characteristics of planning problems such as determining the location of a freeway, adjusting a tax rate, and modifying a school curriculum. Many of these characteristics overlap with the ones identified in Table 3 for design problems (Goel and Pirolli 1992). But they also include additional ones such as every problem can be considered a symptom of another problem. Thus crime in the streets can be considered a symptom of either general moral decay, permissiveness, a lack of opportunity, or other possible causes. Attempts to solve such challenging problems would benefit from a variety of theoretical approaches. For instance, a dynamical systems approach is helpful for thinking about resolving intractable conflict (Vallacher et al. 2010).

At the most general level, all problems require a search for a solution. In their article *Exploration versus exploitation in space, mind, and society* a group of scientists claimed that search is a ubiquitous property of life (Hills et al. 2015). It is prevalent in animal foraging, visual search, information search, search in memory, search in problem solving, and social learning. At the top of their list of outstanding questions, the authors ask "What cognitive representations of search spaces are there, what are their features, how do we switch between them, and how might they change over time?" (Hills et al. 2015, p. 52). I hope my article makes a contribution toward answering that question for both well-structured and ill-structured problems.

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