Problem 1
Use the Galerkin method to solve the 1D Laplace equation for ρ(x) =π 2ε 0 sin2 (π x) .
a) Use the appropriate Maple command to find the exact solution of this Laplace equation.
b) Using the same ui (x) as the class example, derive the formula for bi.
c) Calculate and plot the Galerkin solution and the exact solution for N = 15 and for N = 100.
restart
a)

$p := x \rightarrow \pi^2 \cdot e0 \cdot \sin^2(\mathrm{Pi} \cdot x) :$

$eq := y''(x) = -\frac{p(x)}{e0} :$

$ns := dsolve([eq, y(0) = 0, y(1) = 0])$

$$y(x) = -\frac{1}{8}\cos(2\pi x) + \frac{1}{8} - \frac{1}{4}\pi^2 x^2 + \frac{1}{4}\pi^2 x \qquad \textbf{(1)}$$

$bi := \frac{1}{e0}\left(int\left(p(x) \cdot \left(\frac{(x - x1)}{h}\right), x = x1..x2\right) + int\left(p(x) \cdot \left(\frac{(x3 - x)}{h}\right), x = x2..x3\right)\right) :$

$simplify(bi, size)$

$$\frac{1}{4}\frac{1}{h}\left(-2\cos(\pi x2)^2 + 2\sin(\pi x2)\,\pi\,(x1 - 2\,x2 + x3)\cos(\pi x2) + \cos(\pi x1)^2 + \cos(\pi x3)^2 + \pi^2\left(2\,x2^2 + (-2\,x1 - 2\,x3)\,x2 + x1^2 + x3^2\right)\right)$$

$bi := \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot (2 \cdot \cos^2(x2 \cdot \mathrm{Pi}) - \cos^2(x1 \cdot \mathrm{Pi}) - \cos^2(x3 \cdot \mathrm{Pi}))$

$$\frac{1}{2}\pi^2 h - \frac{1}{4}\,\frac{2\cos(x2\,\pi)^2 - \cos(x1\,\pi)^2 - \cos(x3\,\pi)^2}{h} \qquad \textbf{(3)}$$

I did some rearranging and some terms went away. I show my work on the paper I submit.

c)
First I make my A matrix function
$with(LinearAlgebra) :$
$fA := (i, j) \rightarrow piecewise(i = j, 2, i = j + 1, -1, i = j - 1, -1, 0) :$
Then I - initilize N and calculate h, function for b matrix. I create A and B matrix. Find inverse of A and multiply by B - that is my answer (Ms matrix). To plot I make X matrix where each value is greater than previous by h. I plot points where X matrix is my x axis and Ms is my y (my approximated solution).

$N := 15 :$

$h := \frac{1}{N - 1} :$

$A := Matrix\left(N, \frac{1}{h} \cdot fA\right) :$

$fb := i \rightarrow piecewise\Big(i = -1, \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot (2 \cdot \cos^2(((i - 1) \cdot h) \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i - 1) \cdot h - h))), i = N + 1, \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot (2 \cdot \cos^2(((i - 1) \cdot h) \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i - 1) \cdot h + h))), \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot (2 \cdot \cos^2((i - 1) \cdot h \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i - 1) \cdot h - h)) - \cos^2(\mathrm{Pi} \cdot ((i - 1) \cdot h + h)))\Big) :$

$B := evalf(Matrix(N, 1, fb)) :$
$MA := MatrixInverse(A) :$

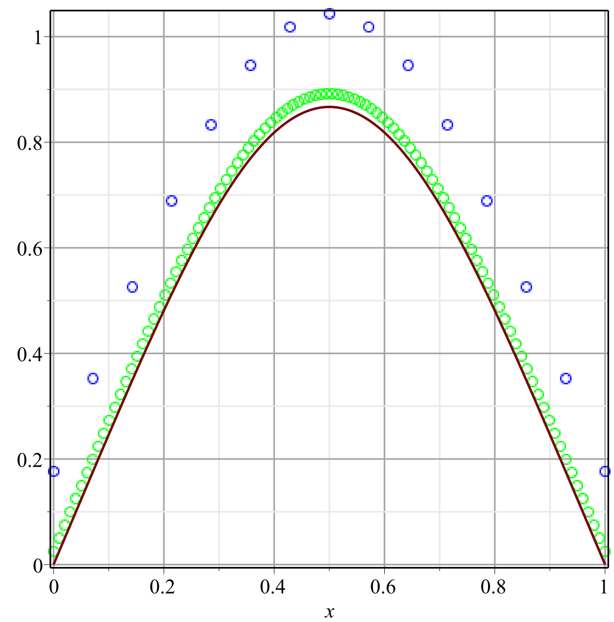
$Ms := Multiply(MA, B)$:
$fX := x \rightarrow h \cdot (x-1)$:
$X := Matrix(N, 1, fX)$:
$sol1 := [seq([X[i, 1], Ms[i, 1]], i = 1..N)]$:

Then I do the same but for different N.
$N := 100$:
$h := \dfrac{1}{N-1}$:
$A := Matrix\left(N, \dfrac{1}{h} \cdot fA\right)$:

$$fb := i \rightarrow piecewise\left(i = 0, \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot \left(2 \cdot \cos^2(((i-1) \cdot h) \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i-1) \cdot h - h))\right), i = N + 1, \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot \left(2 \cdot \cos^2(((i-1) \cdot h) \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i-1) \cdot h + h))\right), \frac{\pi^2 \cdot h}{2} - \frac{1}{4 \cdot h} \cdot \left(2 \cdot \cos^2((i-1) \cdot h \cdot \mathrm{Pi}) - \cos^2(\mathrm{Pi} \cdot ((i-1) \cdot h - h)) - \cos^2(\mathrm{Pi} \cdot ((i-1) \cdot h + h))\right)\right) :$$

$B := evalf(Matrix(N, 1, fb))$:
$MA := MatrixInverse(A)$:
$Ms := Multiply(MA, B)$:
$fX := x \rightarrow h \cdot (x-1)$:
$X := Matrix(N, 1, fX)$:
$sol2 := [seq([X[i, 1], Ms[i, 1]], i = 1..N)]$:

Display both plots on the same plot with the actual solution plot.
$with(plots)$:
display(pointplot(sol1, axes = boxed, gridlines = true, symbol = circle, color = blue, symbolsize = 15),
pointplot(sol2, axes = boxed, gridlines = true, symbol = circle, color = green, symbolsize = 15),
plot(eval(y(x), ns), x = 0 ..1))



It looks good. I had problems with end points. Comparing to the exmaple what we did in class I have to change end points to 0 and N+1 for i, because 1 and N would mess it up and it would not give me the right result. It also works fine with fb just being a regular function of i (no piecewise, just all terms always present).

Problem 2
Here I repeat exactly the same process with a few adjustments. 1) My solution to the integral obviously is different. Howere there is the same term that goes away as in problem one and in class example. I did not write it down explicitly on the paper. But its that same 2Pi(x1-2*x2+x3) term that goes to zero.

restart

a)

$p := x \to \pi^2 \cdot e0 \cdot \sin\left(\frac{\mathrm{Pi} \cdot x}{2}\right) :$

$eq := y''(x) = -\frac{p(x)}{e0} :$

$ns := dsolve([eq, y(0) = 0, y(1) = 0])$

$$y(x) = 4 \sin\left(\frac{1}{2} \pi x\right) - 4 x \tag{4}$$

b)

$$bi := \frac{1}{e0}\left(int\left(p(x)\cdot\left(\frac{(x-x1)}{h}\right), x=x1..x2\right) + int\left(p(x)\cdot\left(\frac{(x3-x)}{h}\right), x=x2..x3\right)\right):$$

$simplify(bi, size)$

$$\frac{2\,\pi\,(x1-2\,x2+x3)\cos\left(\frac{1}{2}\,\pi\,x2\right) - 4\sin\left(\frac{1}{2}\,\pi\,x3\right) - 4\sin\left(\frac{1}{2}\,\pi\,x1\right) + 8\sin\left(\frac{1}{2}\,\pi\,x2\right)}{h}$$

$$bi := \frac{4}{h}\cdot\left(2\cdot\sin\left(\frac{x2\cdot\mathrm{Pi}}{2}\right) - \sin\left(\frac{x1\cdot\mathrm{Pi}}{2}\right) - \sin\left(\frac{x3\cdot\mathrm{Pi}}{2}\right)\right)$$

$$\frac{4\left(2\sin\left(\frac{1}{2}\,\pi\,x2\right) - \sin\left(\frac{1}{2}\,\pi\,x1\right) - \sin\left(\frac{1}{2}\,\pi\,x3\right)\right)}{h} \qquad \textbf{(6)}$$

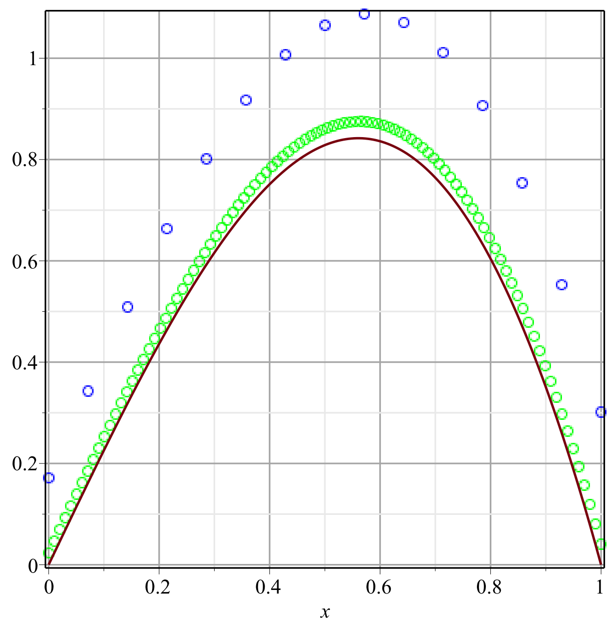
c)Here same process as in problem 1. Same issue with the boundaries as in problem 2, so I have same solution for that issue as I have in problem 1.

```
with(LinearAlgebra):
fA := (i, j) -> piecewise(i = j, 2, i = j + 1, -1, i = j - 1, -1, 0):
N := 15:
h := 1/(N - 1):
A := Matrix(N, (1/h)·fA):
fb := i -> piecewise(i = 0, (4/h)·(2·sin((π·(i - 1)·h)/2) - sin((π·((i - 1)·h + h))/2)), i = N + 1, (4/h)·(2
    ·sin((π·(i - 1)·h)/2) - sin((π·((i - 1)·h - h))/2)), (4/h)·(2·sin((π·(i - 1)·h)/2)
    - sin((π·((i - 1)·h - h))/2) - sin((π·((i - 1)·h + h))/2))):
B := evalf(Matrix(N, 1, fb)):
MA := MatrixInverse(A):
Ms := Multiply(MA, B):
fX := x -> h·(x - 1):
X := Matrix(N, 1, fX):
sol1 := [seq([X[i, 1], Ms[ i, 1]], i = 1..N)]:

N := 100:
h := 1/(N - 1):
A := Matrix(N, (1/h)·fA):
fb := i -> piecewise(i = 0, (4/h)·(2·sin((π·(i - 1)·h)/2) - sin((π·((i - 1)·h + h))/2)), i = N + 1, (4/h)·(2
    ·sin((π·(i - 1)·h)/2) - sin((π·((i - 1)·h - h))/2)), (4/h)·(2·sin((π·(i - 1)·h)/2)
    - sin((π·((i - 1)·h - h))/2) - sin((π·((i - 1)·h + h))/2))):
B := evalf(Matrix(N, 1, fb)):
MA := MatrixInverse(A):
Ms := Multiply(MA, B):
```

$fX := x \rightarrow h \cdot (x - 1) :$
$X := Matrix(N, 1, fX) :$
$sol2 := [seq([X[i, 1], Ms[\ i, 1]], i = 1 ..N)] :$

$with(plots) :$
$display(pointplot(\mathrm{sol1}, \mathrm{axes} = \mathrm{boxed}, \mathrm{gridlines} = \mathrm{true}, \mathrm{symbol} = \mathrm{circle}, \mathrm{color} = \mathrm{blue}, \mathrm{symbolsize} = 15),$
$pointplot(\mathrm{sol2}, \mathrm{axes} = \mathrm{boxed}, \mathrm{gridlines} = \mathrm{true}, \mathrm{symbol} = \mathrm{circle}, \mathrm{color} = \mathrm{green}, \mathrm{symbolsize} = 15),$
$plot(eval(y(x), ns), x = 0 ..1))$



Plot looks what I expext.

Problem 3
We mentioned that the Monte Carlo method, as applied in the calculation of π, has a standard deviation that falls slowly, as 1/sqrt(N) . Generate the data to prove this behavior, using the process for estimating the value
of π that we created in class:
a) Your data should include runs where N ranges from 100 to 5,000 with a step of 200. Each one of these runs should be done 200 times and its standard deviation calculated using the inbuilt StandardDeviation Maple command.
b) Fit a 1/sqrt(x) function to your data, to show that they indeed follow this trend. Display the data and the fit in
the same plot.

```
restart
with(LinearAlgebra) :
pMC := proc(iM) local i, N; local X, Y :
 N := 0 :
 X := RandomVector(iM, generator = -1 ..1.0) :
 Y := RandomVector(iM, generator = -1 ..1.0) :
 for i from 1 to iM do
 if (X[i]^2 + Y[i]^2) < 1 then
   N := N + 1 :
 end if
 end do:
 return N :
 end proc:
with(Statistics) :
N := { } :
SDLX := { } :
SDLY := { } :
SDL := { } :

 for i from 100 by 200 to 5000 do
   M := { };
   for k from 1 to 200 do
     a := evalf(pMC(i)·4 / i);
     M := [op(M), a];
     N := N ∪ {[i, a]};
   end do:
   s := StandardDeviation(M);
  SDLX := [op(SDLX), i];
  SDLY := [op(SDLY), s];
  SDL := SDL ∪ {[i, s]};
 end do:
X := Vector(SDLX, datatype = float);
Y := Vector(SDLY, datatype = float);
```

$$\left[\begin{array}{c} \textit{1 .. 25 Vector}_{\textit{column}} \\ \textit{Data Type: float}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right]$$

$$\left[\begin{array}{c} \textit{1 .. 25 Vector}_{\textit{column}} \\ \textit{Data Type: float}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right] \quad \textbf{(7)}$$

```
X[3]
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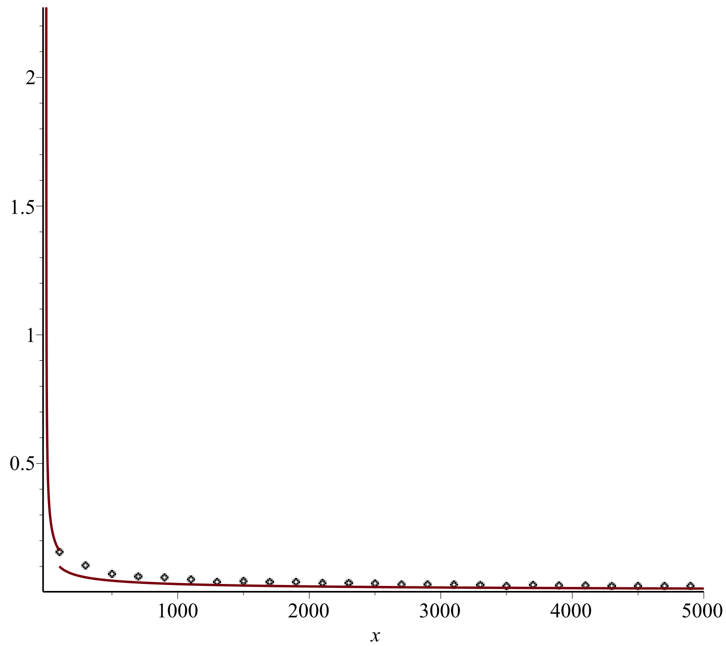
$$500. \quad \textbf{(8)}$$

$Fit\left(o+\frac{l}{\mathrm{sqrt}(x)}, X, Y, x\right)$

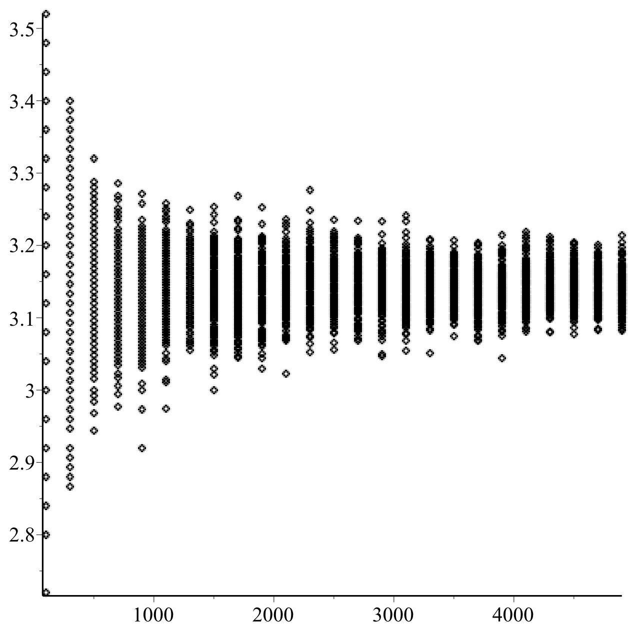
$$\frac{1.60630203130107}{\sqrt{x}}+0.000669804749276769 \tag{9}$$

$with(plots):$

$display\left(pointplot(SDL), plot\left(\frac{1}{\mathrm{sqrt}(x)}, x=100..5000\right), plot\left(Fit\left(o+\frac{l}{\mathrm{sqrt}(x)}, X, Y, x\right), x=100 ...5000\right)\right)$



$pointplot(N)$


3.5
3.4
3.3
3.2
3.1
3
2.9
2.8
1000
2000
3000
4000