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Hamiltonian Mechanics

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Introducing Hamiltonian formulation:

So far, we learned two formulations of mechanics: Newtonian and Lagrangian. Many problems can be solved much more easily by using Lagrangian formulation, because with Lagrangian we can use any set of generalized coordinates. This chapter introduces the third formulation of mechanics – Hamiltonian.

Hamiltonian approach has a similar idea to the Lagrangian, so it is easier to compare Hamiltonian to Lagrangian instead of Newtonian mechanics. In Lagrangian formulation, we were dealing with Lagrangian function. In Hamiltonian formulation, we have Hamiltonian function, which is defined as the sum of multiplication of generilized momenta and velosities minus Lagrangian. Hamiltonian function is represented in Eq.1.

$$H = -L + \sum_{m=1}^{M} p_m \dot{q}_m \tag{1}$$

If the general coordinates we use are "natural" (relation between generalized and Cartesian coordinates is time independent), then Hamiltonian function is just the total energy: T+U. Hamiltonian is the function of generalized coordinates, generalized momenta and time. Hamiltonian approach is defining a point in phase space (this is the name for the space when we talk about Hamiltonian approach).

Hamilton's equations:

To understand the Hamiltonian mechanics we start with the easiest case: one-dimensional systems. While by solving Lagrangian function, the equation of motion for one-dimensional system is a single second-order differential equation for the generalized coordinate q. In Hamiltonian, we get two fist-order differential equations:

$$\dot{q} = \frac{\partial H}{\partial p} \tag{2}$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{3}$$

We can combine those two equations to get a single second-order equation for q if we need to.

When we move to the n-dimensional system, we have n generalized coordinates and momenta. For the easier writing, we substitute all generalized coordinates by a single bold **q** and generalized momenta by **p**. Same as we did in Lagrangian formalism, we assume that any constrains are holonomic (number of generalized coordinates correspond to the number of degrees of freedom). When we set up Hamilton's equation we get 2n equations, where n is the number of generalized coordinates.

Hamiltonian time dependence:

It is worth looking at the time derivative of Hamiltonian function. Hamiltonian has an explicit time dependence, but it may also vary as **q** and **p** are changing with time. When we do partial derivative and substitute Hamilton's equations we find that total time derivative equals to partial time derivative. That means that if Hamiltonian changes with time then it is explicitly time dependent. If Hamiltonian does not depend explicitly on time (which is most of the time) then Hamiltonian is constant, therefore it is conserved. For this chapter we talk only about Hamiltonian that uses "natural" coordinates, therefore we always take that Hamiltonian is just the total energy and energy is indeed conserved.

Solving Hamiltonian:

The approach to solve Hamiltonian has more steps then Lagrangian but it is very similar. First, we have to choose generalized coordinates (position). Then we find kinetic (T) and potential (U) energies in terms of these coordinates and their derivatives (velocities). We also need to find generalized momenta (which is usually position derivative of T, unless U depends on velocity) and express generalized velocities in terms of generalized momenta and coordinates. Now we are ready to write down Hamiltonian as a function of position and momentum. Then we solve for Hamilton's equations to get the equations of motion.

Hamiltonian Advantages over Lagrangian:

When we were dealing with Lagrangian, we talked about ignorable coordinates, which means that Lagrangian does not depend on this coordinate. Since Hamiltonian and Lagrangian are linked, Hamiltonian would have the same ignorable coordinates as Lagrangian. When generalized coordinate is ignorable, the corresponding generalized moment is constant. For Hamiltonian formalism, it means that two-dimensional problem actually reduces to a one-dimensional problem, because Hamiltonian depends on position and momentum. Therefore, if we are using ignorable coordinates in Hamiltonian, it simplifies our problem a lot, while for Lagrangian not necessarily. This is one of the advantages of Hamiltonian over Lagrangian.

Another advantage of Hamiltonian over Lagrangian comes from the mathematical perspective. When solving Lagrangian we get n second-order equations, while for Hamiltonian formalism we get 2n first-order equations. First order equations are much easier to deal with. Also, we can eliminate p and get second-order equation if needed. In addition, same as Lagrange's, Hamilton's equations are invariant under any coordinate change in n-dimensional configuration space defined by the old coordinate. However, Hamiltonian formalism has more flexibility because we can change coordinates in 2n-dimentional phase space: we can change **q** and **p.** Therefore we get new coordinates: **Q=Q(q,p)** and **P=P(q,p).** If we change both **p** and **q** coordinates it is called – canonical transformation.

Phase-orbits:

As I mentioned before, Hamiltonian operates in the phase space. Any point in the phase space can be defined with the vector **z=(q,p)**. This vector is a 2n-dimentional vector that comprises all generalized coordinates and their corresponding momenta. Given an initial condition $\mathbf{z}_0$ at chosen time t_0, Hamilton's equation define a unique trajectory called phase-space orbit. From Hamilton's equations, it follows that only one distinct orbit goes through some point $\mathbf{z}_0$. That means that phase-space orbits never cross each other.

Since phase space has 2n-dimensions, the visualization of the phase-orbits of the system with more than 1 degree of freedom gets complicates because we cannot clearly visualize systems in four and more dimensions. Because of that, we constrain our discussion of the phase-space orbits to the one-dimension systems. The phase-orbit plot is the parametric plot of momentum and position with respect to time. If we launch an identical system with slightly different initial conditions, these systems remain close in phase space unless the motion is chaotic.

Liouville's Theorem:

Now let us consider we have enormous number of identical systems, so each system has its own specific velocity and a phase-orbit that it follows. We can treat this kind of system as a fluid with a density ρ. Each system can be considered as a dot in the phase space, therefore the density of the space that is occupied by these dots is in dots per volume of phase space. As this cloud of dots is moving in the phase space, the surface area where these dots are enclosed changes because each dot has a different velocity. However, as we proved earlier, no phase-orbit can cross each other, so none of these dots can escape from this enclosed area. Therefore, the number of dots in the enclosed surface area is constant in time. Liouville's theorem states that the volume enclosed by the any closed surface area is constant in time as the surface is moving in the phase space. Since volume and number of dots is not changing, density stays constant. To prove Liouville's statement we have to use some complicated math.

We start with investigating the change in volumes. I shall remind that each point in phase space has its own velocity, therefore for each position there is a unique velocity. Also, I introduce vector **n,** which is normal to the surface. Through geometry relations, the book proves that the change in volume is equal to the surface integral of the scalar product of **n** and **v**. If the scalar product is positive, the volume is increasing; if product is negative – the volume is decreasing. Next, we use the power of the divergence theorem, to rewrite the change of volume as a volume integral of the divergence of vector **v.**

The solutions of Hamiltonian equation define a point in phase space. Since the derivatives of the q and p are actually Hamilton's equations, the velocity of that point can be described in terms of these Hamilton's equations. The divergence of any velocity in phase space appears to be zero. Therefore, the change in volume that the points occupy in any closed surface is zero – Liouville's theorem proved.

Liouville's theorem is true for any system that obeys Hamilton's equation, even if the coordinates are not "natural" and forces are not conservative. There is no corresponding theorem for Lagrangian, which makes Liuville's theorem to be one of the most important advantages of the Hamiltonian over the Lagrangian formalism.