MATH3013 FINAL PROJECT

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Abstract

Heat distribution problems are very common in our daily life. By using physical law we can derive some partial differential equations to describe the distribution. However, we seldom have analytic solution for those PDE. In this report, I will introduce a way to solve Laplace equation concerning the heat steady-state distribution.

1 Introduction

In this section, we will derive the differential equation for steady-state distribution of heat in an area.

Assume there is a control area **R** in a plane with boundary **S** and assume no heat is generated or lost (see Figure 1).

By the law of conservation of thermal energy, we have the following

$$\textit{Rate of change of heat inside } \mathbf{R} = \textit{Heat flowing through } \mathbf{S} \textit{ at this moment}$$

In mathematical statement, using vector calculus

$$\frac{\partial}{\partial t}\iint_R c\rho u(\vec{x},t)\,dA = \oint_S \nabla u(\vec{x},t)\cdot\vec{n}\,ds$$

Where c, ρ, u are respectively the *specific heat capacity*, *density*, *temperature at a point.*

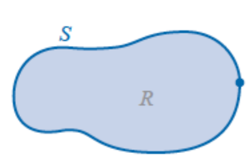


Figure 1: Area **R** with boundary **S**

By Gauss's Divergence Theorem

$$\oint_S \nabla u(\vec{x},t)\cdot \vec{n}\, ds = \iint_R \nabla\cdot\nabla\, u(\vec{x},t)\, dA$$

Thus

$$\iint_R c\rho \frac{\partial}{\partial t} u(\vec{x},t)\, dA = \iint_R \Delta\, u(\vec{x},t)\, dA$$

Because **R** is random, we have

$$c\rho \frac{\partial}{\partial t} u(\vec{x},t) = \Delta\, u(\vec{x},t)$$

In the discussion of steady-state distribution of heat where the temperature in every point would not change as time tends to infinite, the left hand equals 0.

$$\Delta u(\vec{x}) = 0$$

or

$$u_{xx} + u_{yy} = 0$$

which is also called **Laplace's equation**.

2 Numerical Scheme

2.1 Finite difference method

In this section, we will discuss the numerical method taken to solve the Laplace's equation in a rectangular area **R**=(a,b) × (c,d). Note that when g is continuous, the solution is unique.

$$u_{xx} + u_{yy} = 0 \tag{1}$$

$$\textit{Boundary condition}:\ u(x,y) = g(x,y), \quad \textit{with}\ (x,y)\in \mathbf{S}$$

Using Taylor's theorem, we have

$$\frac{\partial}{\partial x^2} u(x,y) = \frac{u(x-\Delta x,y) - 2u(x,y) + u(x+\Delta x,y)}{\Delta x^2} + \frac{\Delta x^2}{12}\frac{\partial}{\partial x^4} u(\zeta,y) \tag{2}$$

$$\frac{\partial}{\partial y^2} u(x,y) = \frac{u(x,y-\Delta y) - 2u(x,y) + u(x,y+\Delta y)}{\Delta y^2} + \frac{\Delta y^2}{12}\frac{\partial}{\partial y^4} u(x,\eta) \tag{3}$$

where $\zeta \in (x - \Delta x, x + \Delta x)$, $\eta \in (y - \Delta y, y + \Delta y)$.

Substitute (2), (3) into (1), and rearrange the equation

$$\frac{u(x-\Delta x, y) - 2u(x,y) + u(x+\Delta x, y)}{\Delta x^2} +$$

$$\frac{u(x, y-\Delta y) - 2u(x,y) + u(x, y+\Delta y)}{\Delta y^2}$$

$$= \frac{\partial}{\partial x^4} u(\zeta, y) + \frac{\partial}{\partial y^4} u(x, \eta)$$

If we divide $\mathbf{R}$ into $m \times n$ rectangular grids with equal size $\Delta x \times \Delta y$ and label each mesh point with index $\mathbf{i}$ and $\mathbf{j}$. Above difference equation suggests a finite-difference method as following (see Figure 2).

$$2[(\frac{\Delta x}{\Delta y})^2 + 1]\omega_{i,j} - (\omega_{i-1,j} + \omega_{i+1,j}) - (\frac{\Delta x}{\Delta y})^2(\omega_{i,j-1} + \omega_{i,j+1}) = O(\Delta x^2 + \Delta y^2) \quad (4)$$

for i=1,2,...,n-1 and j=1,2,...,m-1, and

$$\omega_{i,j} = g(x_i, y_j), \quad with\ (x,y) \in \mathbf{S}$$

where ω approximates u. Now we have to solve this linear system.

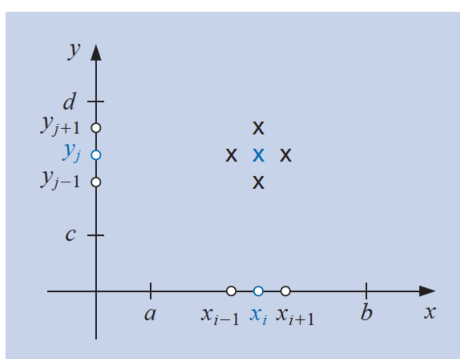


Figure 2: Mesh points involved in a single difference equation

2.2 Use Gauss-Seidel iterative method to solve the system

Let $\alpha = 2[(\frac{\Delta x}{\Delta y})^2 + 1]$, $\beta = (\frac{\Delta x}{\Delta y})^2$, by neglecting the local truncation error, (4) becomes

$$\alpha\omega_{i,j} - (\omega_{i-1,j} + \omega_{i+1,j}) - \beta(\omega_{i,j-1} + \omega_{i,j+1}) = 0$$

Putting $\omega_{i,j}$ on the left, we generate a Gauss-Seidel iterative method to solve above linear system.

$$\omega_{i,j}^{(k)} = \frac{\omega_{i-1,j}^{(k-1)} + \omega_{i+1,j}^{(k-1)} + \beta(\omega_{i,j-1}^{(k-1)} + \omega_{i,j+1}^{(k-1)})}{\alpha}$$

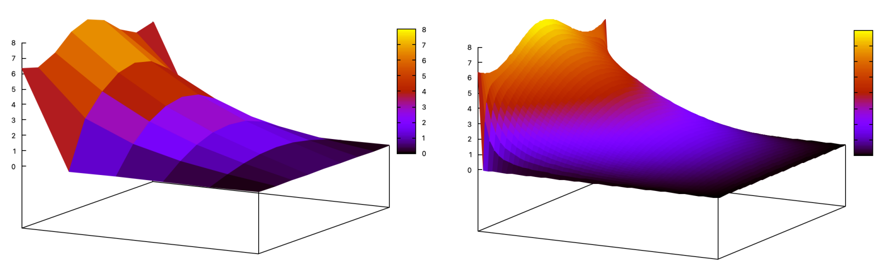
where k=1,2,... denotes the number of times of iteration and $\omega_{(i,j)}^{(0)}$ is set to be 0 for i=1,2,...,n-1 and j=1,2,...,m-1.

3 Example of implementation

Assume a plate with uniform material occupying a two-dimensional space $(-4, 4) \times (0, 5)$ has temperature u(x,0)=cos(x)+7 on its lower lateral and 0 temperature on the other three laterals. Find the steady-state distribution of heat for the plate.

We will stop when the estimated error is less than 0.001.

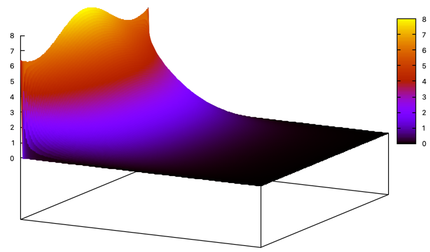
We first use a 5×8 grid, we get into 0.001 with 27 iterations.
With a 15×24 grid, we get into 0.001 within 163 iterations.
With a 50×80 grid, we get into 0.001 within 899 iterations.



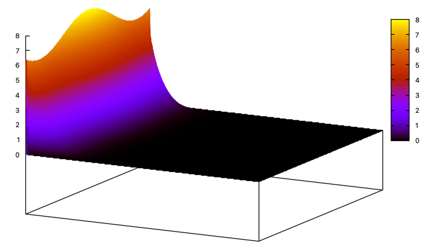
(a) 5×8 grid (b) 50×80 grid

Figure 3: More and more details

Nevertheless, as we continue to increase the mesh points, the temperature in the area away from the source of heat (the boundary) vanishes. This results



(a) 150×240 grid



(b) 500×800 grid

Figure 4: Temperature far away vanishes

from the round-off error in calculation (we are using the datatype **double**).

4 Conclusion

As we see, to get more detailed graph, the iterations needed is increasing which results in smaller step size and more calculation, finally we have nonnegligible round-off error. To reduce the number of iterations, we may refer to some more advanced technique like relaxation technique or iterative refinement. By the way, sparse matrix will significantly improve the efficiency when dealing with even denser grid partition.

5 Reference

[1]Richard L.Burden, J.Douglas Faires, Numerical analysis 9th, cha.12, p.716, 2010.
[2]James Stewart, Calculus 8th, cha.16, p.1181, 2016.