

[Link to Mathematica .nb file](#)

Definitions

Define the spherical harmonics

```
In[1]:= A[l_, m_] := Sqrt[(2 - KroneckerDelta[m, 0]) (2 l + 1) (l - m)! / (4 π (l + m)!)] ;

B[l_, m_, j_, k_] := (2^l m! (l+m+k-1)/2)! / (j! k! (m-j)! (l-m-k)! ((-l+m+k-1)/2)!);

Yxyz[l_, m_, x_, y_, z_] := { Sum[Sum[(-1)^(j/2) A[l, m] B[l, m, j, k] x^(m-j) y^j z^k, {k, 0, l-m}], {
  Sum[Sum[(-1)^(j-1)/2 A[l, Abs[m]] B[l, Abs[m], j, k] x^(Abs[m]-j) y^j z^k,
    {j, 1, Abs[m], 2}]}

Y[l_, m_, x_, y_] :=
  Expand[FullSimplify[Yxyz[l, m, x, y, Sqrt[1-x^2-y^2]]] /. Sqrt[1-x^2-y^2] -> z;
```

Compute our polynomial basis **bp**

```
In[5]:= bp[n_, x_, y_] := Module[{l, m, μ, ν},
  l = Floor[Sqrt[n]];
  m = n - l^2 - l;
  μ = l - m;
  ν = l + m;
  If[EvenQ[ν], x^(μ/2) y^(ν/2), x^(μ-1)/2 y^(ν-1)/2 Sqrt[1-x^2-y^2]]];
```

Compute the spherical harmonic vectors in the polynomial basis, **p**

```
In[8]:= p[l_, m_, lmax_] := Module[{Ylm},
  Ylm = Y[l, m, x, y] /. z -> Sqrt[1-x^2-y^2];
  Join[{Evaluate[Ylm /. {Sqrt[1-x^2-y^2] -> 0, x -> 0, y -> 0}]}],
  Table[Coefficient[Ylm, bp[n, x, y]] /.
    {Sqrt[1-x^2-y^2] -> 0, x -> 0, y -> 0}, {n, 1, (lmax+1)^2-1}]]];
```

The columns of the change of basis matrix **A_l** are just **p**

```
In[9]:= A1[lmax_] := Transpose[Flatten[Table[p[l, m, lmax], {l, 0, lmax}, {m, -l, l}], 1]];
```

Examples

Show bp up to $n=15$

```
In[10]:= {Table[bp[n, x, y] /.  $\sqrt{1-x^2-y^2} \rightarrow z$ , {n, 0, 15}]} // TableForm
```

Out[10]//TableForm=

1	x	z	y	x^2	xz	xy	yz	y^2	x^3	x^2z	x^2y	xyz	xy^2	y^2z
---	---	---	---	-------	------	------	------	-------	-------	--------	--------	-------	--------	--------

Show \mathbf{A}_l for $l_{\max} = 2$

```
In[11]:= A1[2] // MatrixForm
```

Out[11]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2\sqrt{\pi}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{5}{\pi}}}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\frac{3}{\pi}}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\frac{3}{\pi}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\frac{3}{\pi}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{5}{\pi}}}{4} & 0 & \frac{\sqrt{\frac{15}{\pi}}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{15}{\pi}}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{15}{\pi}}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{15}{\pi}}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{5}{\pi}}}{4} & 0 & -\frac{\sqrt{\frac{15}{\pi}}}{4} \end{pmatrix}$$

```
In[12]:=
```

LaTeXify

```
Out[12]= LaTeXify
```

Make \mathbf{A}_l L^AT_EX-friendly

```
In[12]:= A1TeX[lmax_] := TeXForm[ $\frac{1}{2\sqrt{\pi}}$ ] TeXForm[FullSimplify[ $2\sqrt{\pi} A1[lmax]$ ]];
```

Print A_l for $l_{\max} = 2$

In[13]:= A1TeX[2]

Out[13]= $\frac{1}{2\sqrt{\pi}} \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{5}}{2} & 0 & \frac{\sqrt{15}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\sqrt{5}}{2} & 0 & -\frac{\sqrt{15}}{2} \end{array} \right)$