Link to Mathematica .nb file

Definitions

Normalization constants

```
In[1]:= A[l_, m_] := Sqrt[((2 - KroneckerDelta[m, 0]) (2 l + 1) (l - m)!)/(4 π (l + m)!)];

B[l_, m_, j_, k_] := (2^l m! ((l+m+k-1)/2)!)/(j! k! (m - j)! (l - m - k)! ((-l+m+k-1)/2)!);
```

Spherical harmonics in $x^i y^j z^k$ form

```
In[3]:= Yxyz[l_, m_, x_, y_, z_] := {
  Sum[Sum[(-1)^(j/2) A[l, m] B[l, m, j, k] x^(m-j) y^j z^k, {k, 0, l - m}], {
  Sum[Sum[(-1)^((j-1)/2) A[l, Abs[m]] B[l, Abs[m], j, k] x^(Abs[m]-j) y^j z^k,
   {j, 1, Abs[m], 2}]
```

Spherical harmonics in contracted form

```
In[4]:= Y[l_, m_, x_, y_] :=
  Expand[FullSimplify[Yxyz[l, m, x, y, Sqrt[1 - x^2 - y^2]]]] /. Sqrt[1 - x^2 - y^2] → z;
```

As a check, compute them from Mathematica's complex SphericalHarmonicY function

```
In[5]:= Yθϕ[l_, m_, θ_, ϕ_] := If[m == 0,
    SphericalHarmonicY[l, 0, θ, ϕ],
    If[m < 0,
     ⅈ Sqrt[1/2] (SphericalHarmonicY[l, m, θ, ϕ] + (-1)^(m+1) SphericalHarmonicY[l, -m, θ, ϕ]),
     Sqrt[1/2] (SphericalHarmonicY[l, -m, θ, ϕ] + (-1)^m SphericalHarmonicY[l, m, θ, ϕ])]];
```

Benchmarks

Compare to the Mathematica version up to $l_{max} = 10$

```
In[6]:= With[{lmax = 10}, AllTrue[Assuming[θ ∈ Reals && Cos[θ] ≥ 0, Flatten[Table[
 FullSimplify[Y[l, m, Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ]] == Yθϕ[l, m, θ, ϕ]],
     {l, 0, 3}, {m, -l, l}]]]], TrueQ]]

Out[6]= True
```

Examples

Show the spherical harmonics up to $l_{max} = 4$

In[7]:=
```
With[{lmax = 4},
 Grid[Table[CenterArray[Table[Text[Style[Y[l, m, x, y], FontSize → 7]], {m, -l, l}],
    2 lmax + 1, ""], {l, 0, lmax}], Frame → All]]
```

Out[7]=

				$\frac{1}{2\sqrt{\pi}}$				
			$\frac{1}{2}\sqrt{\frac{3}{\pi}}\,y$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\,z$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\,x$			
		$\frac{1}{2}\sqrt{\frac{15}{\pi}}\,x y$	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\,y z$	$\frac{\sqrt{\frac{5}{\pi}}}{2} - \frac{3}{4}\sqrt{\frac{5}{\pi}}\,x^2 - \frac{3}{4}\sqrt{\frac{5}{\pi}}\,y^2$	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\,x z$	$\frac{1}{4}\sqrt{\left(\frac{15}{\pi}\right)}x^2 - \frac{1}{4}\sqrt{\left(\frac{15}{\pi}\right)}y^2$		
	$\frac{3}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x^2 y - \frac{1}{4}\sqrt{\left(\frac{35}{2\pi}\right)}y^3$	$\frac{1}{2}\sqrt{\frac{105}{\pi}}\,x y z$	$\sqrt{\frac{21}{2\pi}}\,y - \frac{5}{4}\sqrt{\left(\frac{21}{2\pi}\right)}x^2 y - \frac{5}{4}\sqrt{\left(\frac{21}{2\pi}\right)}y^3$	$\frac{1}{2}\sqrt{\frac{7}{\pi}}\,z - \frac{5}{4}\sqrt{\frac{7}{\pi}}\,x^2 z - \frac{5}{4}\sqrt{\frac{7}{\pi}}\,y^2 z$	$\sqrt{\frac{21}{2\pi}}\,x - \frac{5}{4}\sqrt{\left(\frac{21}{2\pi}\right)}x^3 - \frac{5}{4}\sqrt{\left(\frac{21}{2\pi}\right)}x y^2$	$\frac{1}{4}\sqrt{\left(\frac{105}{\pi}\right)}x^2 z - \frac{1}{4}\sqrt{\left(\frac{105}{\pi}\right)}y^2 z$	$\frac{1}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x^3 - \frac{3}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x y^2$	
$\frac{3}{4}\sqrt{\left(\frac{35}{\pi}\right)}x^3 y - \frac{3}{4}\sqrt{\left(\frac{35}{\pi}\right)}x y^3$	$\frac{9}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x^2 y z - \frac{3}{4}\sqrt{\left(\frac{35}{2\pi}\right)}y^3 z$	$\frac{9}{2}\sqrt{\frac{5}{\pi}}\,x y - \frac{21}{4}\sqrt{\frac{5}{\pi}}\,x^3 y - \frac{21}{4}\sqrt{\frac{5}{\pi}}\,x y^3$	$3\sqrt{\left(\frac{5}{2\pi}\right)}y z - \frac{21}{4}\sqrt{\left(\frac{5}{2\pi}\right)}x^2 y z - \frac{21}{4}\sqrt{\left(\frac{5}{2\pi}\right)}y^3 z$	$\frac{3}{2\sqrt{\pi}} - \frac{15x^2}{2\sqrt{\pi}} + (105x^4)/(16\sqrt{\pi}) - \frac{15y^2}{2\sqrt{\pi}} + (105x^2y^2)/(8\sqrt{\pi}) + (105y^4)/(16\sqrt{\pi})$	$3\sqrt{\left(\frac{5}{2\pi}\right)}x z - \frac{21}{4}\sqrt{\left(\frac{5}{2\pi}\right)}x^3 z - \frac{21}{4}\sqrt{\left(\frac{5}{2\pi}\right)}x y^2 z$	$\frac{9}{4}\sqrt{\frac{5}{\pi}}\,x^2 - \frac{21}{8}\sqrt{\frac{5}{\pi}}\,x^4 - \frac{9}{4}\sqrt{\frac{5}{\pi}}\,y^2 + \frac{21}{8}\sqrt{\frac{5}{\pi}}\,y^4$	$\frac{3}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x^3 z - \frac{9}{4}\sqrt{\left(\frac{35}{2\pi}\right)}x y^2 z$	$\frac{3}{16}\sqrt{\left(\frac{35}{\pi}\right)}x^4 - \frac{9}{8}\sqrt{\left(\frac{35}{\pi}\right)}x^2 y^2 + \frac{3}{16}\sqrt{\left(\frac{35}{\pi}\right)}y^4$