Link to Mathematica .nb file

Derivation

Let’s take the line integral of G along the quartic approximation to the occultor limb.

$$\vec{r} = x\,\hat{x} + y\,\hat{y}$$

$$y(x) = b - r + \frac{x^2}{2r} + \frac{x^4}{8r^3}$$

$$\left(\frac{dr}{dx}\right)_y = \frac{x}{r} + \frac{1}{2}\frac{x^3}{r^3}$$

$$G_y = x^{\frac{\mu+2}{2}} y^{\frac{\nu}{2}} = x^{\frac{\mu+2}{2}} \left(b - r + \frac{x^2}{2r} + \frac{x^4}{8r^3}\right)^{\frac{\nu}{2}}$$

$$P = \int G_y \left(\frac{dr}{dx}\right)_y dx$$

$$P = 2\,(b-r)^{\frac{\nu}{2}} \int_0^{\mathrm{Cos}[\lambda]} x^{\frac{\mu+2}{2}} \left(1 + \frac{x^2}{2r(b-r)} + \frac{x^4}{8r^3(b-r)}\right)^{\frac{\nu}{2}} \left(\frac{x}{r} + \frac{1}{2}\frac{x^3}{r^3}\right) dx$$

$$(1+Z)^{\frac{\nu}{2}} = \sum_{i=0}^{\frac{\nu}{2}} \binom{\frac{\nu}{2}}{i} Z^i$$

$$P = 2\,(b-r)^{\frac{\nu}{2}} \int_0^{\mathrm{Cos}[\lambda]} x^{\frac{\mu+2}{2}} \sum_{i=0}^{\frac{\nu}{2}} \binom{\frac{\nu}{2}}{i} \left(\frac{x^2}{2r(b-r)} + \frac{x^4}{8r^3(b-r)}\right)^i \left(\frac{x}{r} + \frac{1}{2}\frac{x^3}{r^3}\right) dx$$

$$P = 2\,(b-r)^{\frac{\nu}{2}} \int_0^{\mathrm{Cos}[\lambda]} x^{\frac{\mu+2}{2}} \sum_{i=0}^{\frac{\nu}{2}} \binom{\frac{\nu}{2}}{i} \left(\frac{x^2}{2r(b-r)}\right)^i \left(1 + \frac{x^2}{4r^2}\right)^i \left(\frac{x}{r} + \frac{1}{2}\frac{x^3}{r^3}\right) dx$$

$$P = \frac{2\,(b-r)^{\frac{\nu}{2}}}{r} \sum_{i=0}^{\frac{\nu}{2}} \binom{\frac{\nu}{2}}{i} \left(\frac{1}{2r(b-r)}\right)^i$$

$$\sum_{j=0}^{i} \binom{i}{j} \left(\frac{1}{4r^2}\right)^j \left(\int_0^{\mathrm{Cos}[\lambda]} x^{\frac{\mu+2}{2}+2i+2j+1} dx + \frac{1}{2r^2}\int_0^{\mathrm{Cos}[\lambda]} x^{\frac{\mu+2}{2}+2i+2j+3} dx\right)$$

$$P = \frac{(b-r)^{\frac{\nu}{2}}}{r} \sum_{i=0}^{\frac{\nu}{2}} \left(\mathrm{Binomial}\left[\frac{\nu}{2}, i\right] \left(\frac{1}{2r(b-r)}\right)^i \sum_{j=0}^{i} \left(\mathrm{Binomial}[i, j]\right.\right.$$

$$\left.\left.\left(\frac{1}{4r^2}\right)^j \mathrm{coslam}^{2\left(\frac{\mu+2}{4}+i+j+1\right)} \left(\frac{1}{\frac{\mu+2}{4}+i+j+1} + \frac{1}{2r^2}\frac{\mathrm{coslam}^2}{\frac{\mu+2}{4}+i+j+2}\right)\right)\right)$$

In[1428]:=
```
Exact[b_, r_, μ_, ν_] := Module[{sinphi, phi},
    sinphi = (1 - r^2 - b^2)/(2 b r);
    phi = ArcSin[sinphi];
    ∫_{π-phi}^{2 π+phi} (r Cos[ψ])^((μ+2)/2) (b + r Sin[ψ])^(ν/2) r Cos[ψ] dψ];
Quadratic[b_, r_, μ_, ν_] := Module[{sinlam, coslam},
    sinlam = (1 - r^2 + b^2)/(2 b);
    coslam = Sqrt[1 - sinlam^2];
    (4 (b - r)^(ν/2))/r Sum_{i=0}^{ν/2} (Binomial[ν/2, i] coslam^(3+μ/2+2 i))/(r^i (6 + 4 i + μ) (2 (b - r))^i)];
Quartic[b_, r_, μ_, ν_] := Module[{sinlam, coslam},
    sinlam = (1 - r^2 + b^2)/(2 b);
    coslam = Sqrt[1 - sinlam^2];
    ((b - r)^(ν/2))/r Sum_{i=0}^{ν/2} (Binomial[ν/2, i] (1/(2 r (b - r)))^i
        Sum_{j=0}^{i} (Binomial[i, j] (1/(4 r^2))^j coslam^(2 ((μ+2)/4+i+j+1)) (1/((μ+2)/4 + i + j + 1) + 1/(2 r^2) coslam^2/((μ+2)/4 + i + j + 2))))
  ];
```

$$\int_{\pi-\mathrm{phi}}^{2\pi+\mathrm{phi}} (r\cos\psi)^{\frac{\mu+2}{2}} (b + r\sin\psi)^{\frac{\nu}{2}}\, r\cos\psi \, d\psi$$

$$\frac{4(b-r)^{\frac{\nu}{2}}}{r}\sum_{i=0}^{\frac{\nu}{2}}\left(\frac{\mathrm{Binomial}\left[\frac{\nu}{2}, i\right]\mathrm{coslam}^{3+\frac{\mu}{2}+2i}}{r^i(6+4i+\mu)(2(b-r))^i}\right)$$

$$\frac{(b-r)^{\frac{\nu}{2}}}{r}\sum_{i=0}^{\frac{\nu}{2}}\left(\mathrm{Binomial}\left[\frac{\nu}{2}, i\right]\left(\frac{1}{2r(b-r)}\right)^i \sum_{j=0}^{i}\left(\mathrm{Binomial}[i,j]\left(\frac{1}{4r^2}\right)^j \mathrm{coslam}^{2\left(\frac{\mu+2}{4}+i+j+1\right)}\left(\frac{1}{\frac{\mu+2}{4}+i+j+1}+\frac{1}{2r^2}\frac{\mathrm{coslam}^2}{\frac{\mu+2}{4}+i+j+2}\right)\right)\right)$$

Evaluate the exact expression, the quadratic approximation, and the quartic approximation for an occultor of radius $r_o = 10$:

In[1431]:=
```
Exact[SetAccuracy[10.5, 30], SetAccuracy[11, 30], 0, 0]
Quadratic[10.5, 11, 0, 0]
Quartic[10.5, 11, 0, 0]
```

Out[1431]= 0.042189211634903094408205553

Out[1432]= 0.042107

Out[1433]= 0.0421889

The quartic expression is pretty good!