Link to Mathematica .nb file

Definitions

Define the polynomial basis

In[1]:=
$$A[l\_,\ m\_] := \sqrt{\frac{(2 - \mathrm{KroneckerDelta}[m,\ 0])\ (2\,l + 1)\ (l - m)\,!}{4\,\pi\ (l + m)\,!}};$$

$$B[l\_,\ m\_,\ j\_,\ k\_] := \frac{2^{l}\ m\,!\ \left(\frac{l+m+k-1}{2}\right)!}{j\,!\ k\,!\ (m - j)\,!\ (l - m - k)\,!\ \left(\frac{-l+m+k-1}{2}\right)!};$$

$$Yxyz[l\_,\ m\_,\ x\_,\ y\_,\ z\_] := \begin{cases} \mathrm{Sum}\left[\mathrm{Sum}\left[(-1)^{\frac{j}{2}}\ A[l,\ m]\ B[l,\ m,\ j,\ k]\ x^{m-j}\ y^{j}\ z^{k},\ \{k,\ 0,\ l - m\}\right],\ \{ \right. \\ \mathrm{Sum}\left[\mathrm{Sum}\left[(-1)^{\frac{j-1}{2}}\ A[l,\ \mathrm{Abs}[m]]\ B[l,\ \mathrm{Abs}[m],\ j,\ k]\ x^{\mathrm{Abs}[m]-j}\ y^{j}\ z^{k},\right.\right. \\ \quad \left.\{j,\ 1,\ \mathrm{Abs}[m],\ 2\}\right] \end{cases}$$

$$Y[l\_,\ m\_,\ x\_,\ y\_] :=$$
$$\quad \mathrm{Expand}\left[\mathrm{FullSimplify}\left[Yxyz\left[l,\ m,\ x,\ y,\ \sqrt{1 - x^{2} - y^{2}}\right]\right]\right]\ /.\ \sqrt{1 - x^{2} - y^{2}} \to z;$$

$$\begin{aligned} &bp[n\_,\ x\_,\ y\_] := \mathrm{Module}\left[\{l,\ m,\ \mu,\ \nu\},\right. \\ &\quad l = \mathrm{Floor}\left[\sqrt{n}\right]; \\ &\quad m = n - l^{2} - l; \\ &\quad \mu = l - m; \\ &\quad \nu = l + m; \\ &\quad \left.\mathrm{If}\left[\mathrm{EvenQ}[\nu],\ x^{\frac{\mu}{2}}\ y^{\frac{\nu}{2}},\ x^{\frac{\mu-1}{2}}\ y^{\frac{\nu-1}{2}}\ \sqrt{1 - x^{2} - y^{2}}\right]\right]; \end{aligned}$$

Compute the total flux for terms for which ν (and thus also μ) is even

In[6]:=
$$\mathrm{Assuming}\left[\mu \ge 0\ \&\&\ \nu \ge 0,\ \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^{\frac{\mu}{2}}\ y^{\frac{\nu}{2}}\ dy\ dx\right];$$

$$\mathrm{Simplify}[\%]$$

Out[7]=
$$\frac{\left(1 + i^{\mu}\right)\ \left(1 + i^{\nu}\right)\ \mathrm{Gamma}\left[\frac{2+\mu}{4}\right]\ \mathrm{Gamma}\left[\frac{6+\nu}{4}\right]}{(2 + \nu)\ \mathrm{Gamma}\left[\frac{1}{4}\ (8 + \mu + \nu)\right]}$$

Since μ and ν are both even, this simplifies to

In[8]:=
$$r_{n} = \begin{cases} \frac{\mathrm{Gamma}\left[\frac{\mu}{4}+\frac{1}{2}\right]\ \mathrm{Gamma}\left[\frac{\nu}{4}+\frac{1}{2}\right]}{\mathrm{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \frac{\mu}{2}\ \text{and}\ \frac{\nu}{2}\ \text{even} \\ 0 & \text{otherwise} \end{cases}$$

Out[8]=
$$\begin{cases} \frac{\mathrm{Gamma}\left[\frac{1}{2}+\frac{\mu}{4}\right]\ \mathrm{Gamma}\left[\frac{1}{2}+\frac{\nu}{4}\right]}{\mathrm{Gamma}\left[2+\frac{\mu+\nu}{4}\right]} & \frac{1}{4}\ \text{and even}\ \mu\ \nu \\ 0 & \text{True} \end{cases}$$

Compute the total flux for terms for which ν (and thus also μ) is odd

In[9]:=
$$\text{Assuming}\left[\mu \geq 0\ \&\&\ \nu \geq 0,\ \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^{\frac{\mu-1}{2}}\, y^{\frac{\nu-1}{2}} \sqrt{1-x^2-y^2}\ dy\, dx\right];$$

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Simplify[%]
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Out[10]=
$$-\frac{\left(i + i^{\mu}\right)\left(i + i^{\nu}\right)\sqrt{\pi}\ \text{Gamma}\left[\frac{1+\mu}{4}\right]\text{Gamma}\left[\frac{1+\nu}{4}\right]}{8\ \text{Gamma}\left[\frac{1}{4}\left(8+\mu+\nu\right)\right]}$$

Since μ and ν are both odd, this simplifies to

In[11]:=
$$r_n = \begin{cases} \frac{\sqrt{\pi}}{2}\frac{\text{Gamma}\left[\frac{\mu}{4}+\frac{1}{4}\right]\text{Gamma}\left[\frac{\nu}{4}+\frac{1}{4}\right]}{\text{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \frac{\mu-1}{2} \text{ and } \frac{\nu-1}{2} \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Combine the expressions

$$r_n = \begin{cases} \frac{\text{Gamma}\left[\frac{\mu}{4}+\frac{1}{2}\right]\text{Gamma}\left[\frac{\nu}{4}+\frac{1}{2}\right]}{\text{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \frac{\mu}{2} \text{ and } \frac{\nu}{2} \text{ even} \\ \frac{\sqrt{\pi}}{2}\frac{\text{Gamma}\left[\frac{\mu}{4}+\frac{1}{4}\right]\text{Gamma}\left[\frac{\nu}{4}+\frac{1}{4}\right]}{\text{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \frac{\mu-1}{2} \text{ and } \frac{\nu-1}{2} \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

In[12]:=
$$r[n\_] := \text{Module}\left[\left\{l = \text{Floor}\left[\sqrt{n}\right],\ m = n - \text{Floor}\left[\sqrt{n}\right]^2 - \text{Floor}\left[\sqrt{n}\right],\ \mu,\ \nu\right\},\right.$$
$$\mu = l - m;$$
$$\nu = l + m;$$
$$\left.\begin{cases} \frac{\text{Gamma}\left[\frac{\mu}{4}+\frac{1}{2}\right]\text{Gamma}\left[\frac{\nu}{4}+\frac{1}{2}\right]}{\text{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \text{Mod}\left[\frac{\mu}{2}, 2\right] == \text{Mod}\left[\frac{\nu}{2}, 2\right] == 0 \\ \frac{\sqrt{\pi}}{2}\frac{\text{Gamma}\left[\frac{\mu}{4}+\frac{1}{4}\right]\text{Gamma}\left[\frac{\nu}{4}+\frac{1}{4}\right]}{\text{Gamma}\left[\frac{\mu+\nu}{4}+2\right]} & \text{Mod}\left[\frac{\mu-1}{2}, 2\right] == \text{Mod}\left[\frac{\nu-1}{2}, 2\right] == 0 \\ 0 & \text{True} \end{cases}\right];$$

Benchmarks

Compare it to direct integration up to n=25

Our formula

In[13]:=
```
{Table[r[n], {n, 0, 25}]} // TableForm
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Out[13]//TableForm=

π	0	$\frac{2\pi}{3}$	0	$\frac{\pi}{4}$	0	0	0	$\frac{\pi}{4}$	0	$\frac{2\pi}{15}$	0	0	0	$\frac{2\pi}{15}$	0	$\frac{\pi}{8}$

Mathematica integration

In[14]:=
$$\left\{\text{Table}\left[\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \text{bp}[n, x, y]\ dy\, dx,\ \{n, 0, 25\}\right]\right\} \text{ // TableForm}$$

Out[14]//TableForm=

π	0	$\frac{2\pi}{3}$	0	$\frac{\pi}{4}$	0	0	0	$\frac{\pi}{4}$	0	$\frac{2\pi}{15}$	0	0	0	$\frac{2\pi}{15}$	0	$\frac{\pi}{8}$