**Tilde Approximation.** State and prove the Tilde-Approximations for the following functions of $n$. (*Hint: use the definition of the Tilde approximation.*)

1. $f(n) = n + 1$
2. $f(n) = 1 + \frac{1}{n}$
3. $f(n) = (1 + 1/n)(1 + 2/n)$
4. $f(n) = 2n^3 - 15n^2 + n$
5. $f(n) = (\log(2n)) / (\log(n))$
6. $f(n) = \left(\log\left(n^2 + 1\right)\right) / (\log(n))$
7. $f(n) = n^{100}/2^n + 1$

**Big-O Notation.** State and prove the order of growth using the Big-$O$ notation for the following functions of $n$.

8. $f(n) = 5n^3 + 5 - 1$
9. $f(n) = 3n^2 - 25n + 5$
10. $f(n) = (2n + 25)\log\left(6n^2 + 10\right)$
11. $f(n) = \left(n^2 + 5\log(n)\right) / (3n - 1)$

15) Show directly that $f(n) = n^2 + 3n^3 \in (n^3)$. That is, use the definitions of $O$ and $\Omega$ to show that $f(n)$ is in both $O(n^3)$ and $\Omega(n^3)$.

16) Using the definitions of $O$ and $\Omega$, show that

$$6n^2 + 20n \in O(n^3), \text{ but } 6n^2 + 20n \notin \Omega(n^3).$$

27) Show the correctness of the following statements.

(a) $\lg n \in O(n)$

(b) $n \in O(n \lg n)$

(c) $n \lg n \in O\left(n^2\right)$

(d) $2^n \in \Omega\left(5^{\ln n}\right)$

(e) $\lg^3 n \in o\left(n^{0.5}\right)$

29) Consider the following algorithm:

```
for ( i = 1 ; i <= 1 . 5 n ; i++)
        cout << i ;
for ( i = n ; i >= 1 ; i - - )
        cout << i ;
```

(a) What is the output when n = 2, n = 4, and n = 6?

(b) What is the time complexity T(n)? You may assume that the input n is divisible by 2.

**Tilde Approximation.** State and prove the Tilde-Approximations for the following functions of $n$. (*Hint: use the definition of the Tilde approximation.*)

Tilde-Approximations

$f(n) \sim g(n) \Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$

1a) $f(n) = n+1$

$g(n) = n$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \underbrace{\frac{1}{n}}_{\to 0} = 1$$

b) $f(n) = 1 + \frac{1}{n}$

$g(n) = 1$

$$\lim_{n\to\infty} \frac{1+\overbrace{\frac{1}{n}}^{\to 0}}{1} = \lim_{n\to\infty} 1 = 1$$

c) $f(n) = \left(1+\frac{1}{n}\right) \cdot \left(1+\frac{2}{n}\right) = 1 + \frac{2}{n} + \frac{1}{n} + \frac{2}{n^2}$

$= 1 + \frac{3}{n} + \frac{2}{n^2}$

$g(n) = 1$

$$\lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{2}{n^2}}{1} = \lim_{n\to\infty} \frac{1+0+0}{1} = \lim_{n\to\infty} 1 = 1$$

d) $f(n) = 2n^3 - 15n^2 + n$

$g(n) \, 2 \, n^3$

$$\lim_{n\to\infty} \frac{2n^3 - 15n^2 + n}{2n^3} = \lim_{n\to\infty} \frac{2n^3}{2n^3} - \lim_{n\to\infty} \frac{15n^2}{2n^3} + \lim_{n\to\infty} \frac{n}{2n^3}$$

$= 1$

e) $f(n) = \frac{\log(2n)}{\log n} = \frac{\log 2 + \log n}{\log n} = \frac{\log 2}{\log n} + 1$

$g(n) = 1$

$$\lim_{n\to\infty} \underbrace{\frac{\log 2}{\log n}}_{\to 0} + 1$$

f) $f(n) = \frac{\log(n^2+1)}{\log n} \approx \frac{\log(n^2)}{\log n} = \frac{2\log(n)}{\log n} = 2$

$g(n) = 2$

$$\lim_{n\to\infty} \frac{2}{2} = 1$$

g) $f(n) = \frac{n^{100}}{2^n} + 1 = 1$

$g(n) = 1$

$$\lim_{n\to\infty} \frac{1}{1} = 1$$

**Big-O Notation.** State and prove the order of growth using the Big-$O$ notation for the following functions of $n$.

8. $f(n) = 5n^3 + 5 - 1$
9. $f(n) = 3n^2 - 25n + 5$
10. $f(n) = (2n + 25)\log\left(6n^2 + 10\right)$
11. $f(n) = \left(n^2 + 5\log(n)\right) / (3n - 1)$

a) $f(n) = 5n^3+5-1 = 5n^3+4$

$f(n) \in O(n^3)$

$|f(n)| \leq c \cdot |g(n)|\ ;\ \forall n \geq n_0\ ;\ n_0 \geq 1\ ;\ c > 0$

$5n^3+4 \leq c \cdot n^3$

$5n^3+4 \leq 5n^3+4n^3$

$5n^3+4 \leq \underbrace{9}_{c}\underbrace{n^3}_{g(n)}$

pre $n+1$

$5(n+1)^3+4 \leq 9(n+1)^3$

$5n^3+4+15n^2+15n+5 \leq 9n^3+27n^2+27n+9$

$c=9$

$n_0=1$

$\Rightarrow f(n) \in O(n^3)$

$f(n) \leq c\,g(n)$

b) $f(n) = 3n^2-25n+5$

$f(n) \in O(n^2)$

$|f(n)| \leq c \cdot |g(n)|$

$|3n^2-25n+5| \leq c \cdot |n^2|$

$|3n^2-25n+5| \leq |3n^2-25n^2+5n^2|$

$|3n^2-25n+5| \leq |-17n^2|$

$3n^2-25n+5 \leq 17n^2$

$c = 17$

$n_0 = 1$

c) $f(n) = \underbrace{(2n+25)}_{n} \log \underbrace{(6n^2+10)}$

$\log(6n^2+10)$

$\log(6n^2)$

$\log 6 + \log n^2$

$\approx 2\log n$

$\Rightarrow O(n \cdot \log(n))$

$(2n+25)\log(6n^2+10) \leq c\, g(n)$

$(2n+25)\log(6n^2+10) \leq c \cdot n\log n$

$$\frac{(2n+25)\log(6n^2+10)}{n\log n} \leq c$$

15) Show directly that f(n) = n² + 3n³ ∈ (n³). That is, use the definitions of O and Ω to show that f(n) is in both O(n³) and Ω(n³).

$O(n^3)$

$f(n) \leq c g(n)$

$n^2+3n^3 \leq n^3+3n^3$

$n^2+3n^3 \leq 4n^3$

$c=4$

$n=1$

$\Omega(n^3)$

$f(n) \geq c \cdot g(n)$

$n^2+3n^3 \geq c \cdot n^3$

$n^2+3n^3 \geq 3n^3$

$c=3$

$n=1$

16) Using the definitions of O and Ω, show that

6n² + 20n ∈ O(n³), but 6n² + 20n ∉ Ω(n³).

$f(n) \in O(n^3)$

$f(n) \notin \Omega(n^3)$

$f(n) \leq c \cdot g(n)$

$6n^2+20n \leq c \cdot n^3$

$6n^2+20n \leq 6n^3+20n^2$

$6n^2+20n \leq 26n^3$

$f(n) \notin \Omega(n^3)$

$$\lim_{n\to\infty} \frac{6n^2+20n}{cn^3} = \lim_{n\to\infty} c \; \frac{6}{n} + \frac{20}{n^2}$$

$= 0 \Rightarrow f(n) \notin \Omega(n^3)$

27) Show the correctness of the following statements.

(a) $\lg n \in O(n)$

(b) $n \in O(n \lg n)$

(c) $n \lg n \in O(n^2)$

(d) $2^n \in \Omega(5^{\ln n})$

(e) $\lg^3 n \in o(n^{0.5})$

a) $\lg n \in O(n)$

$$\lim_{n\to\infty} \frac{\lg n}{n} = \lim_{n\to\infty} \frac{(\lg(n))'}{n'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

b) $\underbrace{n}_{\to f(n)} \in O(\underbrace{n \lg n}_{\to g(n)})$

$$\lim_{n\to\infty} \frac{n}{n \lg n} = \lim_{n\to\infty} \frac{1}{\lg n} = 0$$

c) $n \lg n \in O(n^2)$

$$\lim_{n\to\infty} \frac{n \lg n}{n^2} = \lim_{n\to\infty} \frac{1 \lg n}{n} = \frac{1}{n} = 0$$

d) $2^n \in \Omega(5^{\ln n})$

$$\lim_{n\to\infty} \frac{2^n}{5^{\ln n}} = \lim_{n\to\infty} \frac{2^n}{n^{\ln(5)}} = \lim_{n\to\infty} \frac{2^n}{n^{1.61}} = \lim_{n\to\infty} \frac{(2^n)'}{n^{1.61}} =$$

$$\frac{2^n \ln(2)}{1.61\, n^{0.61}} = \frac{\ln(2)}{1.61} \cdot \lim_{n\to\infty} \frac{2^n}{n^{0.61}} \approx \lim_{n\to\infty} 2^n = \infty$$

$$5^{\ln(n)} = e^{\ln(5^{\ln(n)})} = e^{\ln(n)\ln(5)} = n^{\ln(5)}$$

(e) $\lg^3 n \in o(n^{0.5})$

$$\lim_{n\to\infty} \frac{\lg^3 n}{n^{0.5}} = \lim_{n\to\infty} \frac{(\lg^3 n)'}{(n^{0.5})'} = \lim_{n\to\infty} \frac{1}{n^{0.5}} = 0$$

29) Consider the following algorithm:

```
for ( i = 1 ; i <= 1 . 5 n ; i++)
        cout << i ;
for ( i = n ; i >= 1 ; i - - )
        cout << i ;
```

(a) What is the output when n = 2, n = 4, and n = 6?

(b) What is the time complexity T(n)? You may assume that the input n is divisible by 2.

a,

↳ $n=2$ 1;2;3;2;1

$n=4$ 1;2;3;4;5;6;4;3;2;1

$n=6$ 1;2;3;4;5;6;7;8;9;6;5;4;3;2;1

b, $T(n) = ?$

$T(n) = 1.5n + n$

$\approx O(n)$

$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i-1}(i+j)\right)=\sum_{i=0}^{n-1}\left[\sum_{j=0}^{i-1} i+\sum_{j=0}^{i-1} j\right]$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} i+\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} j \qquad \frac{(i-1)\cdot(i-1+1)}{2}=\frac{(i-1)\cdot i}{2}$$

$$=\frac{i^2-i}{2}$$

$$=\sum_{i=0}^{n-1} i^2+\sum_{i=0}^{n-1}\frac{i^2-i}{2}=$$

$$\sum_{i=0}^{n-1} i^2+\frac{1}{2}\sum_{i=0}^{n-1} i^2-i$$

$$=\sum_{i=0}^{n-1} i^2+\frac{1}{2}\left(\sum_{i=0}^{n-1} i^2-\sum_{i=0}^{n-1} i\right)\Rightarrow\Theta(n^3)+$$

$$\Theta(n^3)-\Theta(n^2)\Rightarrow\Theta(n^3)$$

$$\sum_{i}^{n} i^k\Rightarrow\Theta(n^{k+1})$$