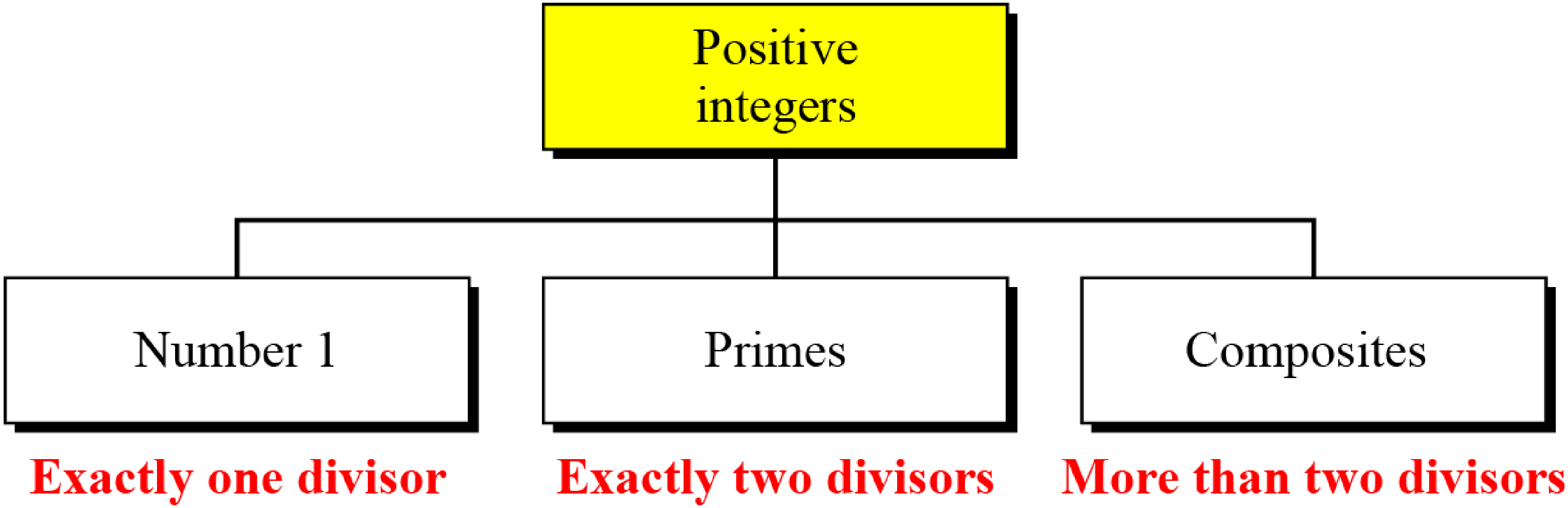
UNIT II
NUMBER THEORY AND PUBLIC KEY CRYPTOGRAPHY

INTRODUCTION TO NUMBER THEORY

Ref:Cryptography and Network Security by Behrouz Forouzan

Three groups of positive integers



A prime is divisible only by itself and 1.

Checking for Primeness

Given a number n, how can we determine if n is a prime? The answer is that we need to see if the number is divisible by all primes less than $\sqrt{n}$

Is 97 a prime?

Solution

The floor of $\sqrt{97} = 9$. The primes less than 9 are 2, 3, 5, and 7. We need to see if 97 is divisible by any of these numbers. It is not, so 97 is a prime.

Is 301 a prime?

Solution

The floor of $\sqrt{301} = 17$. We need to check 2, 3, 5, 7, 11, 13, and 17. The numbers 2, 3, and 5 do not divide 301, but 7 does. Therefore 301 is not a prime.

Sieve of Eratosthenes

https://commons.wikimedia.org/wiki/File:Sieve_of_Eratosthenes_animation.gif#/media/File:Sieve_of_Eratosthenes_animation.gif

Sieve of Eratosthenes

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Euler's Phi-Function

- *Euler's phi-function, ϕ (n), which is sometimes called the Euler's totient function plays a very important role in cryptography. The function finds the number of integers that are both smaller than n and relatively prime to n*

1. $\phi(1) = 0$.
2. $\phi(p) = p - 1$ if p is a prime.
3. $\phi(m \times n) = \phi(m) \times \phi(n)$ if m and n are relatively prime.
4. $\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

Find the GCD(HCF) of

1)12&13

2)12&15

3)36&60

Example

1.What is the value of $\Phi(13)$?

Solution

Because 13 is a prime, $\Phi(13) = (13 - 1) = 12$.

2.What is the value of $\Phi(10)$?

Solution

We can use the third rule:

$\phi(10) = \phi(2) \times \phi(5) = 1 \times 4 = 4$, because 2 and 5 are primes.

3.What is the value of $\phi(240)$?

$\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

Solution

4. Can we say that $\phi(49) = \phi(7) \times \phi(7) = 6 \times 6 = 36$?

Solution

3.We can write $240 = 2^4 \times 3^1 \times 5^1$.

Then $\phi(240) = (2^4 - 2^3) \times (3^1 - 3^0) \times (5^1 - 5^0) = 64$

4. No. The third rule applies when m and n are relatively

prime. Here $49 = 7^2$. We need to use the fourth rule:

$\phi(49) = 7^2 - 7^1 = 42$.

5.What is the number of elements in Z_{14}^*

Solution:

$\Phi(14) = \phi(2) \times \phi(7) = 1 \times 6 = 6$

The members are 1,3,5,7,11,13

UNIT II
NUMBER THEORY AND PUBLIC KEY CRYPTOGRAPHY

INTRODUCTION TO NUMBER THEORY-LECTURE 2

Ref:Cryptography and Network Security by Behrouz Forouzan

Fermat's Little Theorem

- First Version

$$\boxed{a^{p-1} \equiv 1 \bmod p}$$

- Second Version

$$\boxed{a^{p} \equiv a \bmod p}$$

Examples

1.Find the result of 6^{10} mod 11.

Solution

We have 6^{10} mod 11 = 1. This is the first version of Fermat's little theorem where $p = 11$.

$a^{p-1} \equiv 1 \bmod p$

2. Find the result of 3^{12} mod 11.

Solution

Here the exponent (12) and the modulus (11) are not the same. With substitution this can be solved using Fermat's little theorem.

$a^p \equiv a \bmod p$

$$3^{12} \bmod 11 = (3^{11} \times 3) \bmod 11 = (3^{11} \bmod 11)\,(3 \bmod 11) = (3 \times 3) \bmod 11 = 9$$

Multiplicative Inverses

- Find the multiplicative inverse of 8 mod 11, using the Euclidean Algorithm.

gcd(8, 11) = 1

11 = 8(1) + 3

8 = 3(2) + 2

3 =2(1) + 1

Now reverse the process using the equations

1 = 3 − 2(1)

1 = 3 − (8 − 3(2))(1) = 3 − (8 − (3(2)) = 3(3) − 8

1 = (11 − 8(1))(3) − 8 = 11(3) − 8(4) = 11(3) + 8(−4)

Therefore 1 ≡ 8(−4) mod 11, or if we prefer a residue value for the multiplicative inverse, 1 ≡ 8(7) mod 11.

Work at home

- Find the multiplicative inverses of the following:

1) 50 mod 71

2) 43 mod 64

Euler's Theorem

- *First Version: If a and n are coprime then*

$$\boxed{a^{\phi(n)} \equiv 1 \pmod{n}}$$

Second Version: similar to first version but removes the condition that a and n should be prime

$$\boxed{a^{k \times \phi(n) + 1} \equiv a \pmod{n}}$$

The second version of Euler's theorem is used in the RSA cryptosystem

Euler's Phi-Function

To Remember:

1. $\phi(1) = 0$.
2. $\phi(p) = p - 1$ if p is a prime.
3. $\phi(m \times n) = \phi(m) \times \phi(n)$ if m and n are relatively prime.
4. $\phi(p^e) = p^e - p^{e-1}$ if p is a prime.

Examples

1.Find the result of 6^{24} mod 35.

Solution

We have $6^{24} \bmod 35 = 6^{\phi(35)} \bmod 35 = 1$.

2. Find the result of 20^{62} mod 77

Solution

If we let $k = 1$ on the second version, we have

$$20^{62} \bmod 77 = (20 \bmod 77)\,(20^{\phi(77)\,+\,1} \bmod 77) \bmod 77$$

$$= (20)(20) \bmod 77 = 15.$$

PRIMALITY TESTING

- Deterministic Algorithms
- Probabilistic Algorithms
- Recommended Primality Test

Deterministic Algorithms

Divisibility Algorithm

Algorithm *Pseudocode for the divisibility test*

```
Divisibility_Test (n)                    // n is the number to test for primality
{
  r ← 2
  while (r < √n )
  {
   if (r | n) return "a composite"
   r ← r + 1
  }
  return "a prime"
}
```

AKS algorithm

The AKS primality test (also known as Agrawal–Kayal–Saxena primality test and cyclotomic AKS test) is a deterministic algorithm created and published by , computer scientists at the Indian Institute of Technology Kanpur, on August 6, 2002

Probabilistic Algorithms

- *Fermat Test*

$$\boxed{\textbf{If } n \textbf{ is a prime, then } a^{n-1} \equiv 1 \bmod n.}$$

If n is a prime, $a^{n-1} \equiv 1 \bmod n$

If n is a composite, it is possible that $a^{n-1} \equiv 1 \bmod n$

Does the number 561 pass the Fermat test?

Solution

Use base 2

$$2^{561-1} = 1 \bmod 561$$

The number passes the Fermat test, but it is not a prime, because 561 = 33 × 17.

Square Root Test

If n is a prime, $\sqrt{1} \bmod n = \pm 1$.
If n is a composite, $\sqrt{1} \bmod n = \pm 1$ and possibly other values.

What are the square roots of 1 mod n if n is 7 (a prime)?

Solution

The only square roots are 1 and −1. We can see that

$$1^2 = 1 \bmod 7 \qquad (-1)^2 = 1 \bmod 7$$
$$2^2 = 4 \bmod 7 \qquad (-2)^2 = 4 \bmod 7$$
$$3^2 = 2 \bmod 7 \qquad (-3)^2 = 2 \bmod 7$$

Note that we don't have to test 4, 5 and 6 because 4 = –3 mod 7, 5 = –2 mod 7 and 6 = –1 mod 7.

What are the square roots of 1 mod n if n is 22 (a composite)?

Solution

Surprisingly, there are only two solutions, +1 and −1, although 22 is a composite.

$$1^2 = 1 \bmod 22$$
$$(-1)^2 = 1 \bmod 22$$

- *Miller-Rabin Test*

$$n - 1 = m \times 2^k$$

Idea behind Fermat primality test

$$a^{n-1} = a^{m \times 2^k} = [a^m]^{2^k} = [a^m]^{\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k \text{ times}}}$$

Algorithm *Pseudocode for Miller-Rabin test*

```
Miller_Rabin_Test (n, a)                          // n is the number; a is the base.
{
    Find m and k such that n − 1 = m × 2^k
    T ← a^m mod n
    if (T = ± 1)   return "a prime"
    for (i ← 1 to k − 1)                          // k − 1 is the maximum number of steps.
    {
        T ← T^2 mod n
        if (T = +1) return "a composite"
        if (T = −1) return "a prime"
    }
    return "a composite"
}
```

Does the number 561 pass the Miller-Rabin test?

Solution

Using base 2, let $561 - 1 = 35 \times 2^4$, which means

$m = 35$, $k = 4$, and $a = 2$.

Initialization:	$T = 2^{35} \bmod 561 = 263 \bmod 561$	
$k = 1$:	$T = 263^2 \bmod 561 = 166 \bmod 561$	
$k = 2$:	$T = 166^2 \bmod 561 = 67 \bmod 561$	
$k = 3$:	$T = 67^2 \bmod 561 = +1 \bmod 561$	$\rightarrow$ **a composite**

Work at home

1.What are the square roots of 1mod n ,if n is 8?

2.What are the square roots of 1 mod n,if n is 17?

3.Check whether 61 passes Miller Rabin test

Recommended Primality Test

- *Today, one of the most popular primality test is a combination of the divisibility test and the Miller-Rabin test.*

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INTRODUCTION TO NUMBER THEORY-CRT

Ref:Cryptography and Network Security by Behrouz Forouzan

CHINESE REMAINDER THEOREM

The Chinese remainder theorem (CRT) is used to solve a set of congruent equations with one variable but different moduli, which are relatively prime, as shown below:

$$
\begin{aligned}
x &\equiv a_1 \pmod{m_1} \\
x &\equiv a_2 \pmod{m_2} \\
&\cdots \\
x &\equiv a_k \pmod{m_k}
\end{aligned}
$$

- Solution To Chinese Remainder Theorem

1. Find $M = m_1 \times m_2 \times \dots \times m_k$. This is the common modulus.

2. Find $M_1 = M/m_1$, $M_2 = M/m_2$, …, $M_k = M/m_k$.

3. Find the multiplicative inverse of $M_1, M_2, \dots, M_k$ using corresponding moduli $(m_1, m_2, \dots, m_k)$.Call the inverses

$$M_1^{-1}, M_2^{-1}, \dots, M_k^{-1}.$$

4. The solution to the simultaneous equations is

$$\boxed{x = (a_1 \times M_1 \times M_1^{-1} + a_2 \times M_2 \times M_2^{-1} + \cdots + a_k \times M_k \times M_k^{-1}) \bmod M}$$

Example

Find the solution to the simultaneous equations:

$$x \equiv 2 \pmod{3}$$
$$x \equiv 3 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

Solution

We follow the four steps.

1. $M = 3 \times 5 \times 7 = 105$

2. $M_1 = 105 / 3 = 35$, $M_2 = 105 / 5 = 21$, $M_3 = 105 / 7 = 15$

3. The inverses are $M_1^{-1} = 2$, $M_2^{-1} = 1$, $M_3^{-1} = 1$

4. $x = (2 \times 35 \times 2 + 3 \times 21 \times 1 + 2 \times 15 \times 1) \bmod 105$

   $= 23 \bmod 105$

Work at home

1. $x \equiv 3 \bmod 5$

   $x \equiv 1 \bmod 7$

   $x \equiv 6 \bmod 8$

UNIT II
NUMBER THEORY AND PUBLIC KEY CRYPTOGRAPHY

INTRODUCTION TO NUMBER THEORY-Quadratic Congruence_LECTURE 7

Ref:Cryptography and Network Security by Behrouz Forouzan

QUADRATIC CONGRUENCE

In cryptography, we also need to discuss quadratic congruence—that is, equations of the Form $a_2x^2 + a_1x + a_0 \equiv 0 \pmod{n}$. *We limit our discussion to quadratic equations in which* $a_2 = 1$ *and* $a_1 = 0$, *that is equations of the form*

$$\boxed{x^2 \equiv a \pmod{n}.}$$

How do you solve congruences of the form $x^2 \equiv a \pmod{m}$? *Said another way, how do you find square roots in modular arithmetic?*

Simple Example

We start off with a simple example.

Calculate x^2 modulo $m = 11$ for $x = 0, 1, 2, \cdots, 10$.

$$0^2 \equiv 0 \pmod{11}$$
$$1^2 \equiv 1 \pmod{11}$$
$$2^2 \equiv 4 \pmod{11}$$
$$3^2 \equiv 9 \pmod{11}$$
$$4^2 \equiv 5 \pmod{11}$$
$$5^2 \equiv 3 \pmod{11}$$
$$6^2 \equiv 3 \pmod{11}$$
$$7^2 \equiv 5 \pmod{11}$$
$$8^2 \equiv 9 \pmod{11}$$
$$9^2 \equiv 4 \pmod{11}$$
$$10^2 \equiv 1 \pmod{11}$$

The above calculation shows that the values of x^2 modulo $m = 11$ can only be $1, 3, 4, 5, 9$. So equations such as $x^2 \equiv a \pmod{11}$ for $a = 1, 3, 4, 5, 9$ have solutions. For example, the solutions for the equation $x^2 \equiv 5 \pmod{11}$ are $x = 4$ and $x = 7$.

Quadratic Congruence Modulo a Prime

- *We first consider the case in which the modulus is a prime*
- The equation $x^2 \equiv 3 \pmod{11}$ has two solutions, 5 and 6.
- The equation $x^2 \equiv 2 \pmod{11}$ has no solution. No integer x can be found such that its square is 2 mod 11.

- *Quadratic Residues and Nonresidue*
- If $x^2 \equiv a \pmod{n}$ has a solution, we say a is a "quadratic residue mod n." If this congruence has no solution, we say x is a "quadratic non-residue mod p.

- There are 10 elements in $Z_{11}*$. Exactly five of them are quadratic residues and five of them are nonresidues. In other words, $Z_{11}*$ is divided into two separate sets, QR and QNR

Division of $Z_{11}*$ elements into QRs and QNRs

$Z_{11}* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

QR set = $\{1, 3, 4, 5, 9\}$

Each element has a square root

QNR set = $\{2, 6, 7, 8, 10\}$

No element has a square root

Work at home

- Solve the following quadratic equations:

a. $x^2 \equiv 3 \pmod{23}$

b. $x^2 \equiv 2 \pmod{11}$

c. $x^2 \equiv 7 \pmod{19}$

- *Euler's Criterion*

a. *If* $a^{(p-1)/2} \equiv 1 \pmod{p}$, *a is a quadratic residue modulo p.*

b. *If* $a^{(p-1)/2} \equiv -1 \pmod{p}$, *a is a quadratic nonresidue modulo p.*

- To find out if 14 or 16 is a QR in Z_{23}^*, we calculate:

$14^{(23-1)/2} \bmod 23 \rightarrow 22 \bmod 23 \rightarrow -1 \bmod 23$ nonresidue

$16^{(23-1)/2} \bmod 23 \rightarrow 16^{11} \bmod 23 \rightarrow 1 \bmod 23$ residue

Quadratic Congruence Modulo a Composite

$$x^2 \equiv a \bmod (n)$$
$$n = p_1 \times p_2 \times \ . \ \ . \ \ . \times p_k$$

$$\longrightarrow$$

$$x^2 \equiv a_1 \pmod{p_1}$$
$$x^2 \equiv a_2 \pmod{p_1}$$
$$\bullet\bullet\bullet$$
$$x^2 \equiv a_k \pmod{p_k}$$

$$\longrightarrow$$

$$x_1 \equiv \pm\, b_1 \pmod{p_1}$$
$$x_2 \equiv \pm\, b_2 \pmod{p_1}$$
$$\bullet\bullet\bullet$$
$$x_\mathrm{k} \equiv \pm\, b_\mathrm{k} \pmod{p_k}$$

Quadratic Congruence Modulo a Composite

Assume that $x^2 \equiv 36 \pmod{77}$. We know that $77 = 7 \times 11$. We can write

$$x^2 \equiv 36 \pmod{7} \equiv 1 \pmod{7} \quad \text{and} \quad x^2 \equiv 36 \pmod{11} \equiv 3 \pmod{11}$$

The answers are $x \equiv +1 \pmod{7}$, $x \equiv -1 \pmod{7}$, $x \equiv +5 \pmod{11}$, and $x \equiv -5 \pmod{11}$. Now we can make four sets of equations out of these:

Set 1:	$x \equiv +1 \pmod{7}$	$x \equiv +5 \pmod{11}$
Set 2:	$x \equiv +1 \pmod{7}$	$x \equiv -5 \pmod{11}$
Set 3:	$x \equiv -1 \pmod{7}$	$x \equiv +5 \pmod{11}$
Set 4:	$x \equiv -1 \pmod{7}$	$x \equiv -5 \pmod{11}$

- How hard is it to solve a quadratic congruence modulo a composite? The main task is the factorization of the modulus. In other words, the complexity of solving a quadratic congruence modulo a composite is the same as factorizing a composite integer. As we have seen, if n is very large, factorization is infeasible.

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NUMBER THEORY AND PUBLIC KEY CRYPTOGRAPHY

INTRODUCTION TO NUMBER THEORY-

Exponentiation and Logarithm_LECTURE 9

Ref:Cryptography and Network Security by Behrouz Forouzan

EXPONENTIATION AND LOGARITHM

Exponentiation: $y = a^x$ $\rightarrow$ **Logarithm:** $x = \log_a y$

Exponentiation

- *Fast Exponentiation:square-and-multiply method*

Algorithm *Pseudocode for square-and-multiply algorithm*

Square_and_Multiply (a, x, n)
{
  $y \leftarrow 1$
  for (i $\leftarrow$ 0 to $n_b - 1$) // n_b is the number of bits in x
  {
    if ($x_i = 1$) $y \leftarrow a \times y \mod n$ // multiply only if the bit is 1

    $a \leftarrow a^2 \mod n$ // squaring is not needed in the last iteration
  }
  return y
}

The process for calculating $y = a^x$ using the Algorithm is shown. (for simplicity, the modulus is not shown). In this case, $x = 22 = (10110)_2$ in binary. The exponent has five bits.

Demonstration of calculation of a^{22} using square-and-multiply method

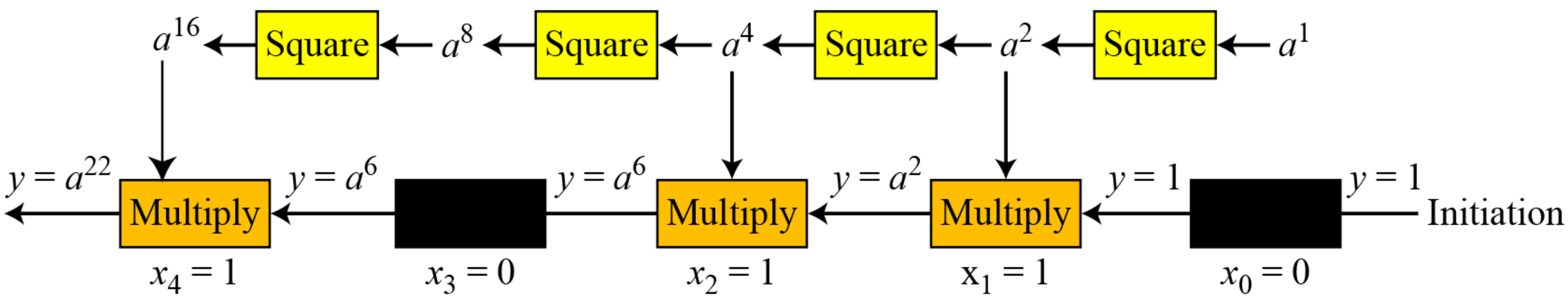

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Table *Calculation of 17^{22} mod 21*

i	x_i	*Multiplication* *(Initialization: $y = 1$)*	*Squaring* *(Initialization: $a = 17$)*
0	0	$\rightarrow$	$a = 17^2 \bmod 21 = 16$
1	1	$y = 1 \times 16 \bmod 21 = 16$ $\rightarrow$	$a = 16^2 \bmod 21 = 4$
2	1	$y = 16 \times 4 \bmod 21 = 1$ $\rightarrow$	$a = 4^2 \bmod 21 = 16$
3	0	$\rightarrow$	$a = 16^2 \bmod 21 = 4$
4	1	$y = 1 \times 4 \bmod 21 = 4$ $\rightarrow$	

```
}
```

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Logarithm

- *Exhaustive Search*

Algorithm *Exhaustive search for modular logarithm*

```
Modular_Logarithm (a, y, n)
{
    for (x = 1 to n − 1)                    // k is the number of bits in x
    {
        if (y ≡ a^x mod n) return x
    }
    return failure
}
```

```
}
```

This is an inefficient algorithm as the complexity is exponential

- *Discrete Logarithm is the second approach*
- *Order of the Group*.
- What is the order of group $G = <Z_{21}*, \times>$? $|G| = \phi(21) = \phi(3) \times \phi(7) = 2 \times 6 = 12$. There are 12 elements in this group: 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, and 20. All are relatively prime with 21.

Order of an Element

- Find the order of all elements in $G = \langle Z_{10}^*, \times \rangle$.

Solution

This group has only $\phi(10) = 4$ elements: 1, 3, 7, 9. We can find the order of each element by trial and error.

a. $1^1 \equiv 1 \bmod (10) \rightarrow \text{ord}(1) = 1$.
b. $3^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(3) = 4$.
c. $7^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(7) = 4$.
d. $9^2 \equiv 1 \bmod (10) \rightarrow \text{ord}(9) = 2$.

a. $1^1 \equiv 1 \bmod (10) \rightarrow \text{ord}(1) = 1$.
b. $3^1 \equiv 3 \bmod (10)$; $3^2 \equiv 9 \bmod (10)$; $3^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(3) = 4$.
c. $7^1 \equiv 7 \bmod (10)$; $7^2 \equiv 9 \bmod (10)$; $7^4 \equiv 1 \bmod (10) \rightarrow \text{ord}(7) = 4$.
d. $9^1 \equiv 9 \bmod (10)$; $9^2 \equiv 1 \bmod (10) \rightarrow \text{ord}(9) = 2$.

- *Primitive Roots* *In the group* $G = <Z_n{*}, \times>$, *when the order of an element is the same as* $\phi(n)$, *that element is called the primitive root of the group.*
- *Cyclic Group* *If g is a primitive root in the group, we can generate the set* $Z_n{*}$ *as* $Z_n{*} = \{g^1, g^2, g^3, \ldots, g^{\phi(n)}\}$

- The group $G = \langle Z_{10}^*, \times \rangle$ has two primitive roots. It can be found that the primitive roots are 3 and 7. The following shows how we can create the whole set Z_{10}^* using each primitive root.

$g = 3 \rightarrow$	$g^1 \bmod 10 = 3$	$g^2 \bmod 10 = 9$	$g^3 \bmod 10 = 7$	$g^4 \bmod 10 = 1$
$g = 7 \rightarrow$	$g^1 \bmod 10 = 7$	$g^2 \bmod 10 = 9$	$g^3 \bmod 10 = 3$	$g^4 \bmod 10 = 1$

The group $G = \langle Z_n^*, \times \rangle$ is a cyclic group if it has primitive roots.
The group $G = \langle Z_p^*, \times \rangle$ is always cyclic.

The idea of Discrete Logarithm

Properties of $G = \langle Z_p^*, \times \rangle$:

1.Its elements include all integers from 1 to $p - 1$.

2. It always has primitive roots.

3. It is cyclic. The elements can be created using g^x Where x is an integer from 1 to $\phi(n) = p - 1$.

4.The primitive roots can be thought as the base of logarithm.

Solution to Modular Logarithm Using Discrete Logs

- *Tabulation of Discrete Logarithms*
- *To solve problem of type y=a ^{x} mod n when y is given and x need to be found*

Table 9.6 *Discrete logarithm for* $\mathbf{G} = \langle \mathbf{Z}_7{}^*, \times \rangle$

y	1	2	3	4	5	6
$x = L_3\, y$	6	2	1	4	5	3
$x = L_5\, y$	6	4	5	2	1	3

1.There are 6 elements in this group:1, 2,3, 4, 5, and 6.
2.Find primitive roots

Work at home

- Find x in each of the following cases:

a. $4 \equiv 3^x \pmod 7$.

b. $6 \equiv 5^x \pmod 7$.

Exponentiation: $y = a^x$ $\rightarrow$ **Logarithm:** $x = \log_a y$

Public Key Cryptography

Cryptography and Network Security
by William Stallings

Private-Key Cryptography

- traditional **private/secret/single key** cryptography uses **one** key
- shared by both sender and receiver
- if this key is disclosed communications are compromised
- also is **symmetric**, parties are equal
- hence does not protect sender from receiver forging a message & claiming is sent by sender

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses **two** keys – a public & a private key
- **asymmetric** since parties are **not** equal
- uses clever application of number theoretic concepts to function
- complements **rather than** replaces private key cryptography

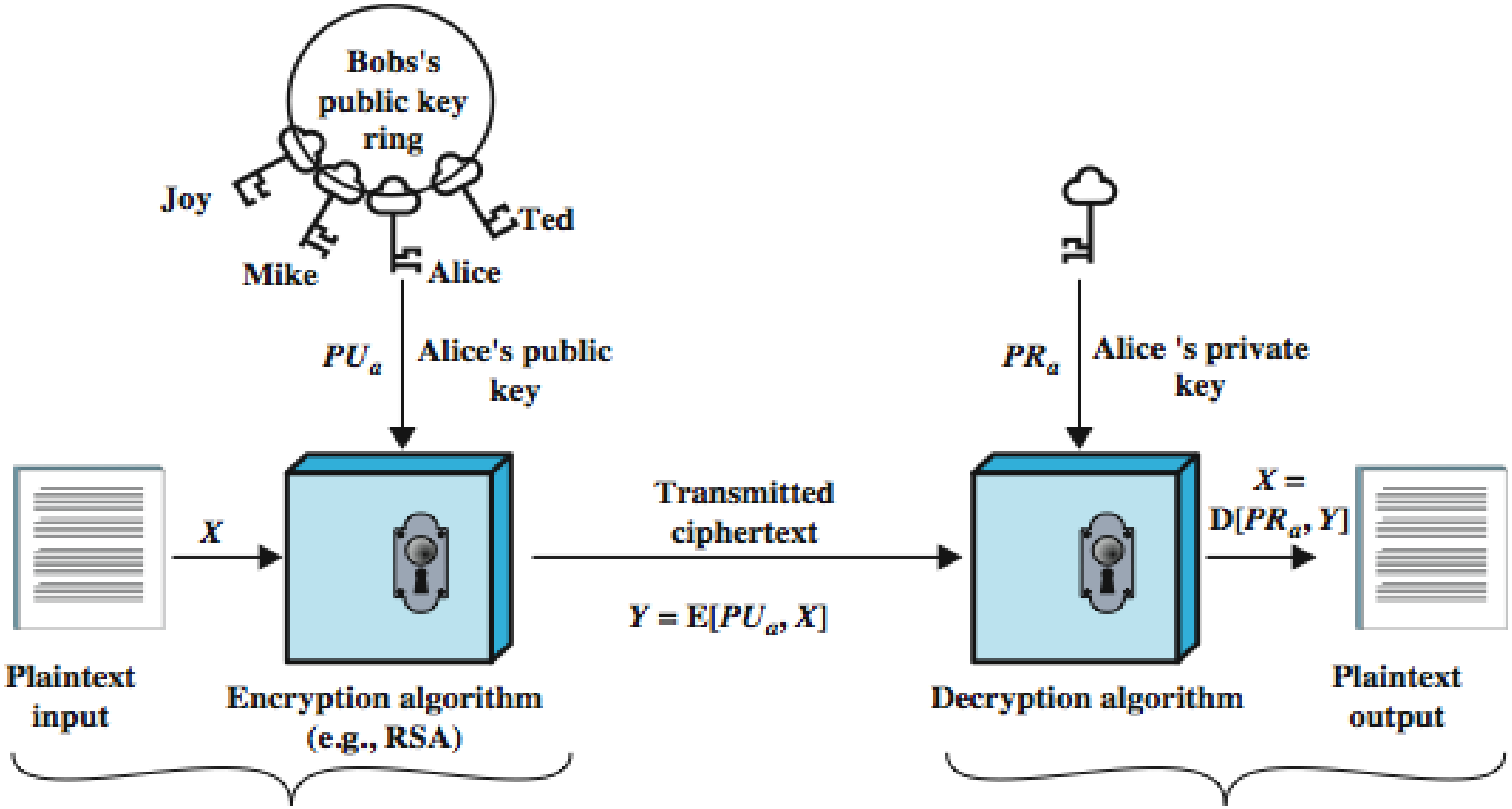
Why Public-Key Cryptography?

- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford University in 1976

Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a related **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- **infeasible to determine private key from public**
- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

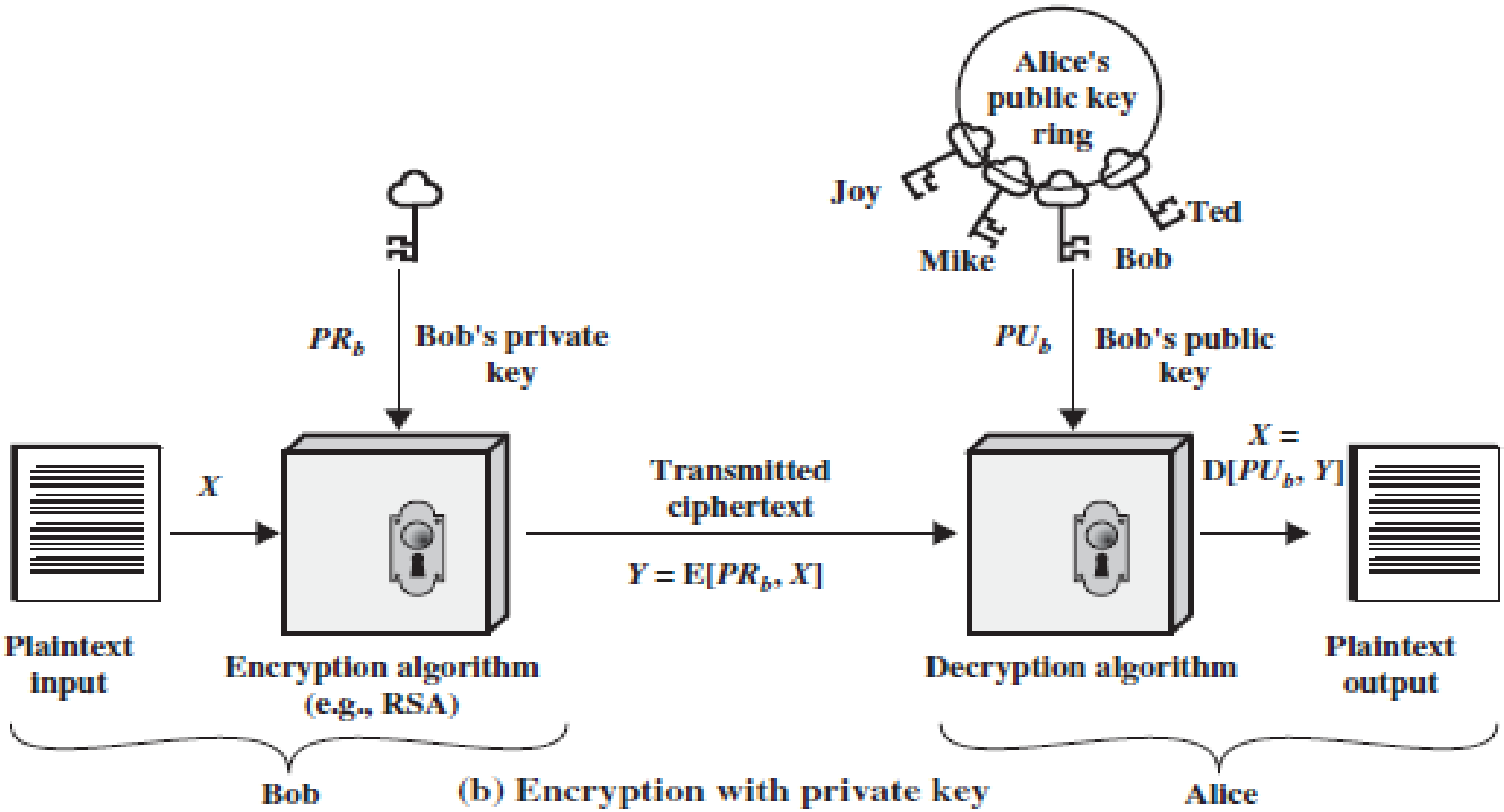
Public-Key Cryptography



(a) Encryption with public key

Bob sends a confidential message to Alice

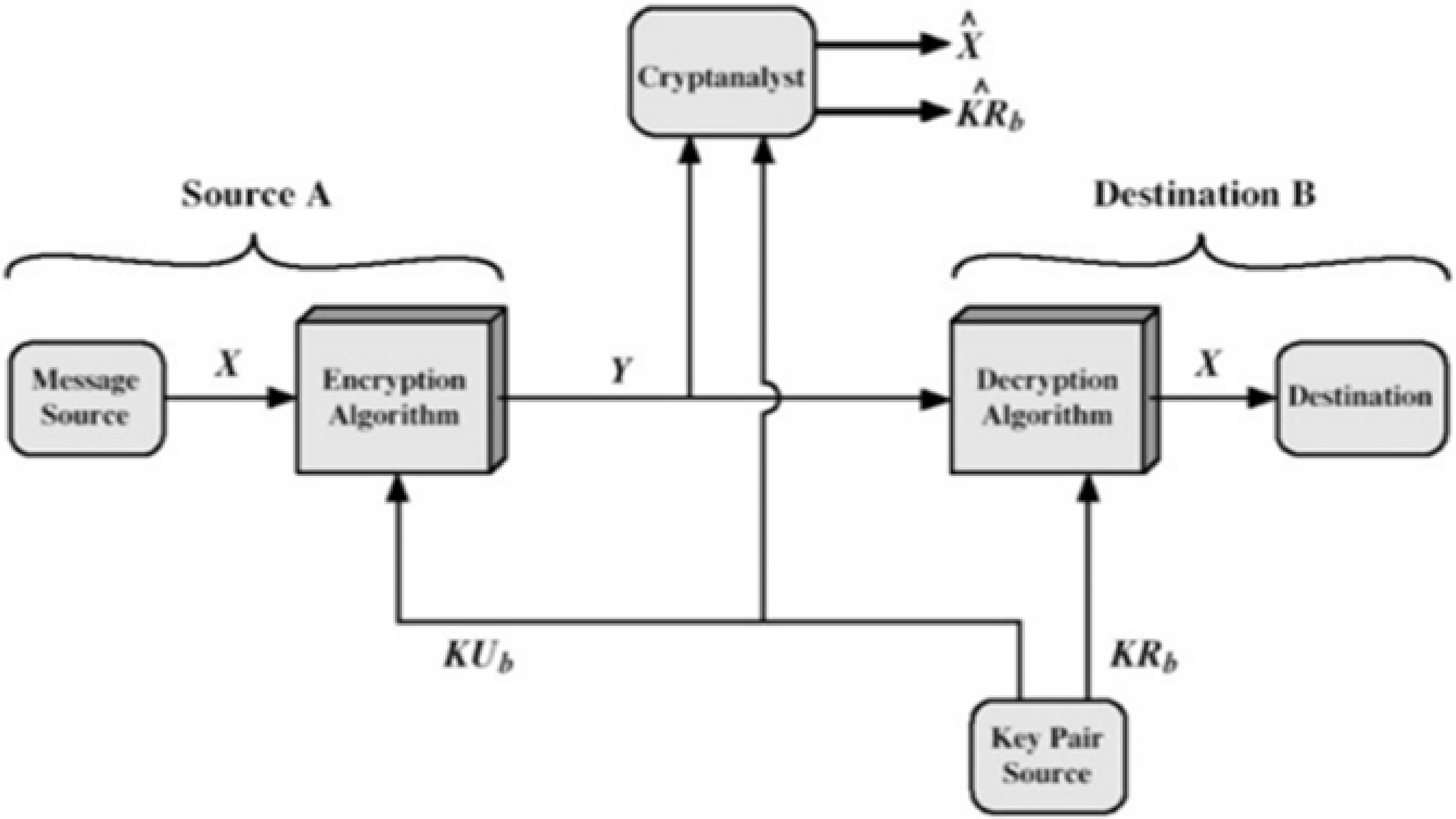
Public-Key Cryptography



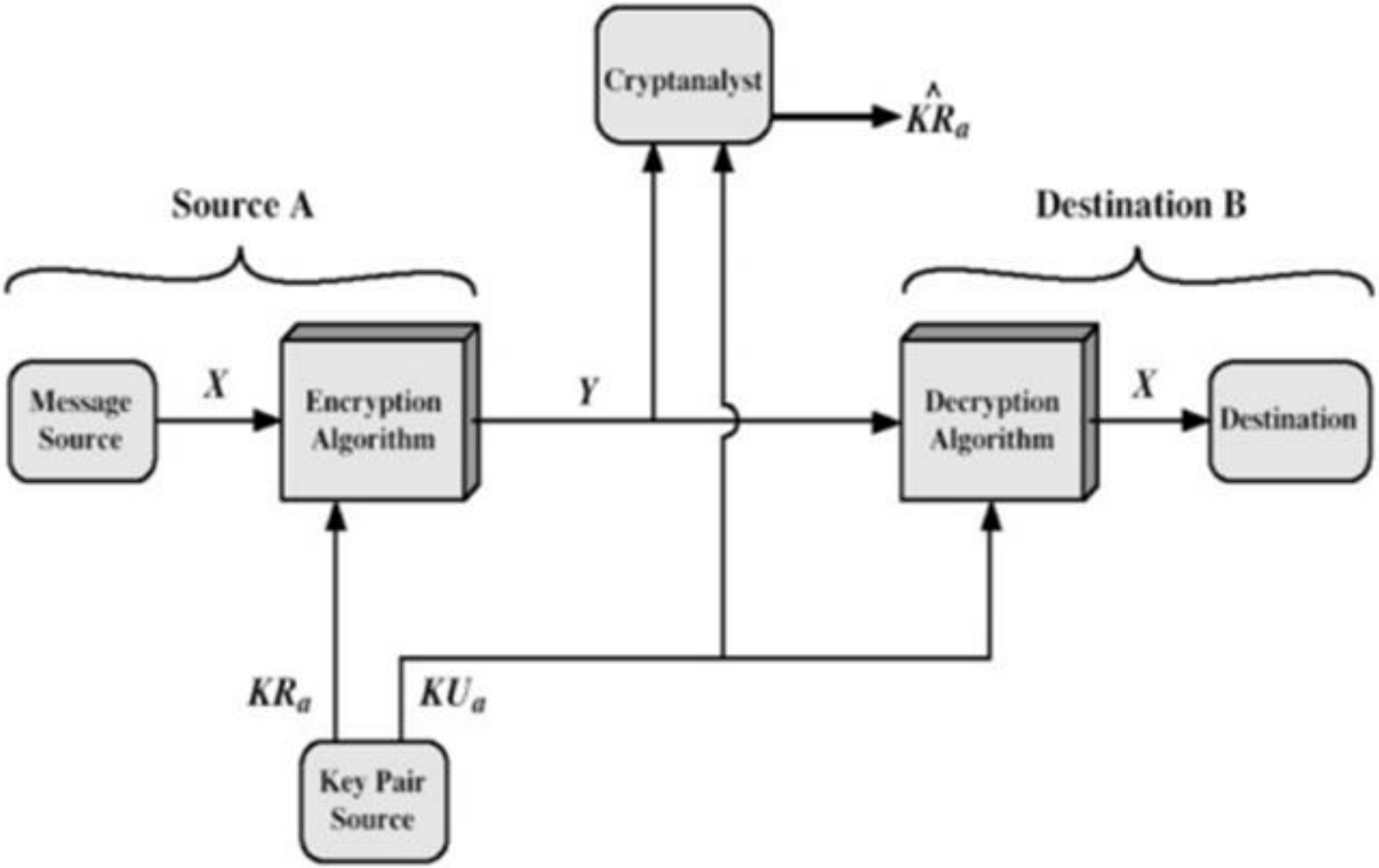
Authentication is achieved here

Symmetric vs Public-Key

Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key. *Needed for Security:* 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed to Work:* 1. One algorithm is used for encryption and decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one). *Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

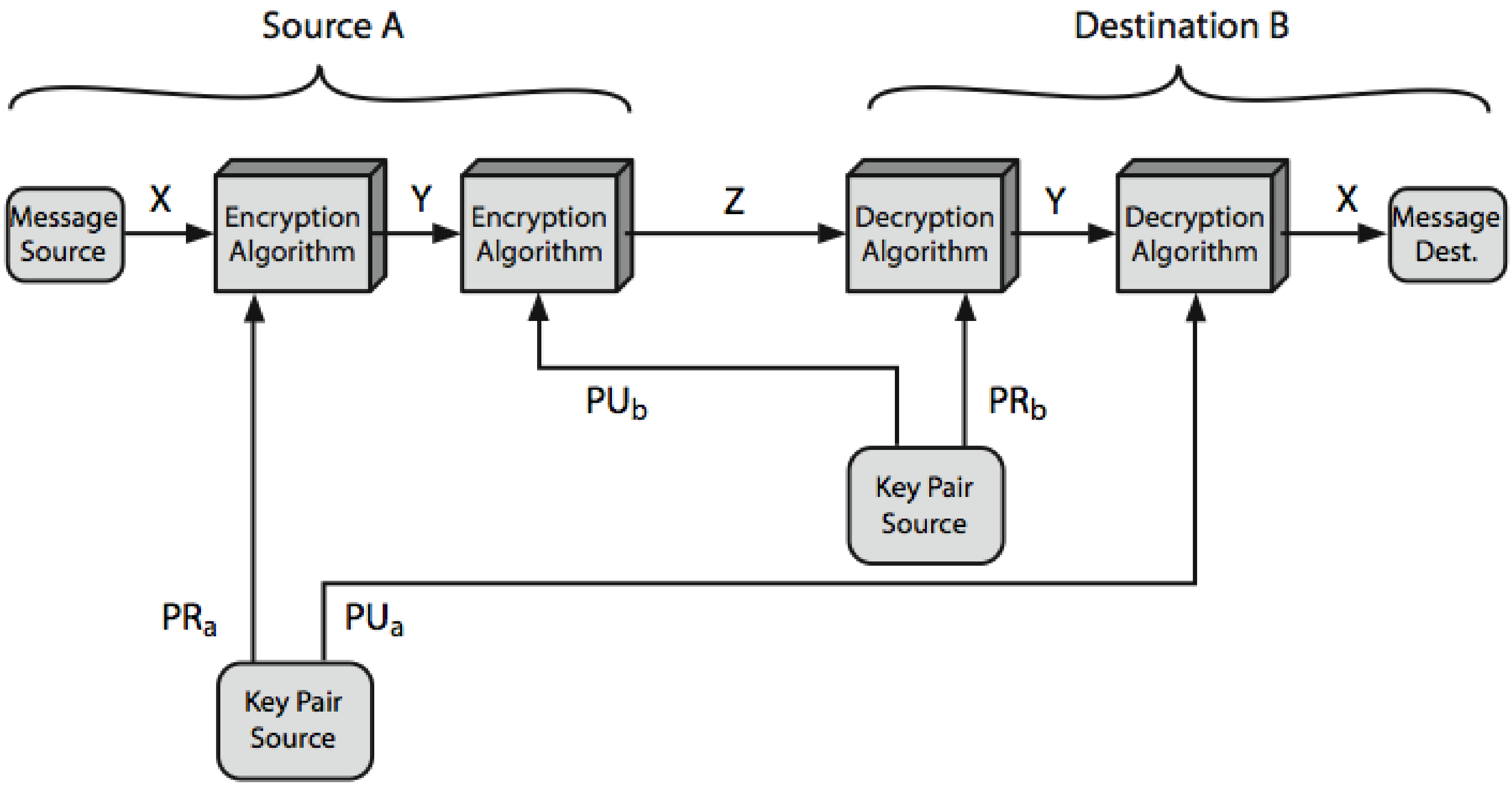


Public-Key Cryptosystem:Secrecy



Public key encryption to provide authentication

Public-Key Cryptosystems



Confidentiality and Authentication

$$Z = \mathrm{E}(PU_b, \mathrm{E}(PR_a, X))$$

$$X = \mathrm{D}(PU_a, \mathrm{D}(PR_b, Z))$$

- Encrypt the message using senders private key. Provides digital signature (authentication).
- Encrypt again using receivers public key. The message can be decrypted by only by the intended receiver (confidentiality).

Public-Key Applications

- can classify uses into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

Public-Key Requirements-Deffie and Hellman

1. It is computationally easy for party B to generate a key pair (public (PU) and private (PR)).
2. It is computationally easy for sender A knowing PUb and the message to be encrypted to generate the corresponding ciphertext C = E(PUb (M)).

- 3. It is computationally easy for the receiver B to decrypt the resulting ciphertext using his private key (PRb) to recover the original message. M = DPRb (C) = DPRb [E(PUb (M))].
- 4. It is computationally infeasible for an opponent, knowing the public key PUb, to determine the private key PRb

5. It is computationally infeasible for an opponent, knowing PUb and C to recover the plaintext message M.

6. A sixth requirement that, although useful, is not necessary for all public-key applications - the encryption and decryption can be applied in either order: M = EPUb [D(PRb (M))] = DPUb [E(PRb (M))].

These are formidable requirements which only a few algorithms have satisfied

Public-Key Requirements

- need a trapdoor one-way function
- one-way function(A one-way function is a function that maps a domain into a range such that every function value has a unique inverse, with the condition that the calculation of the function is easy whereas the calculation of the inverse is infeasible: has
  - $Y = f(X)$ easy
  - $X = f^{-1}(Y)$ infeasible

- a trap-door one-way function has
  - $Y = f_k(X)$ easy, if k and X are known
  - $X = f_k^{-1}(Y)$ easy, if k and Y are known
  - $X = f_k^{-1}(Y)$ infeasible, if Y known but k not known
- a practical public-key scheme depends on a suitable trap-door one-way function
- A trap door function is a family of invertible functions f_k

Security of Public Key Schemes-Cryptanalysis

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are too large (>512bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems

- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

Asymmetric Key Algorithm
RSA Algorithm

Ref:Cryptography and Network Security-Atul Kahate

RSA Algorithm

- Ron Rivest , Adi Shamir and Len Adleman at MIT developed the asymmetric key cryptography system.
- RSA solves the problem of key agreements and distribution.
- In this approach each communicating party possesses a key pair, made up of one public key and one private key.
- RSA is the most popular asymmetric key cryptographic algorithm that employ prime numbers.

Data Encryption Standard (RSA)

RSA Pseudo code:

- Select P,Q where P and Q are prime numbers, $P \neq Q$
- Calculate n, where $n = P * Q$
- Calculate Φ (n), where $\Phi(n) = (P-1)(Q-1)$
- Select Random **'e'** such as : $\gcd(\Phi(n), e) = 1$ and $1 < e < \Phi(n)$,
- Calculate **'d'** , where $d.e \equiv 1 \pmod{\varphi(n)}$
- Public key {e,n}
- private key {d,n}

Example 1

Use RSA algorithm to encrypt the following text number: Plaintext: 88

- Select P = 17, Q = 11
- Calculate n , n= P * Q =17 *11 = 187
- Calculate Φ (n),Φ (n) = (P-1)(Q-1)=16 * 10 = 160
- Select Random E such as: gcd (e, Φ (n),) = 1 and 1<e< Φ (n),

  e = 7 because gcd (7,160) = 1
- Calculate d , where d.e ≡ 1 (mod φ(n))

  d * 7 ≡ 1 mod 160 [d = 7^{-1} mod 160]

  d = 23

Cont…

- Public key { 187, 7} {n,e}
- Private key {187, 23} {n,d}

Encryption: $c = m^e \bmod n$

$88^7 \bmod 187 = 11 = \text{Cipher}$

Decryption: $m = c^d \bmod n$

$11^{23} \bmod 187 = 88$

Try this as home work

1. Select primes p=11, q=3 and find out the public and private key. Perform encryption and decryption on the message m=7

2. Consider an RSA cryptosystem with n=pq, where p=7 and q=13. Let the public key be (n,35) and private key $(\varphi(n), p, q, a)$. Then the decrypted value of 10 is ______

Diffie Hellman key exchange

Ref:Cryptography and Network Security- Atul Kahate

Diffie Hellman key exchange

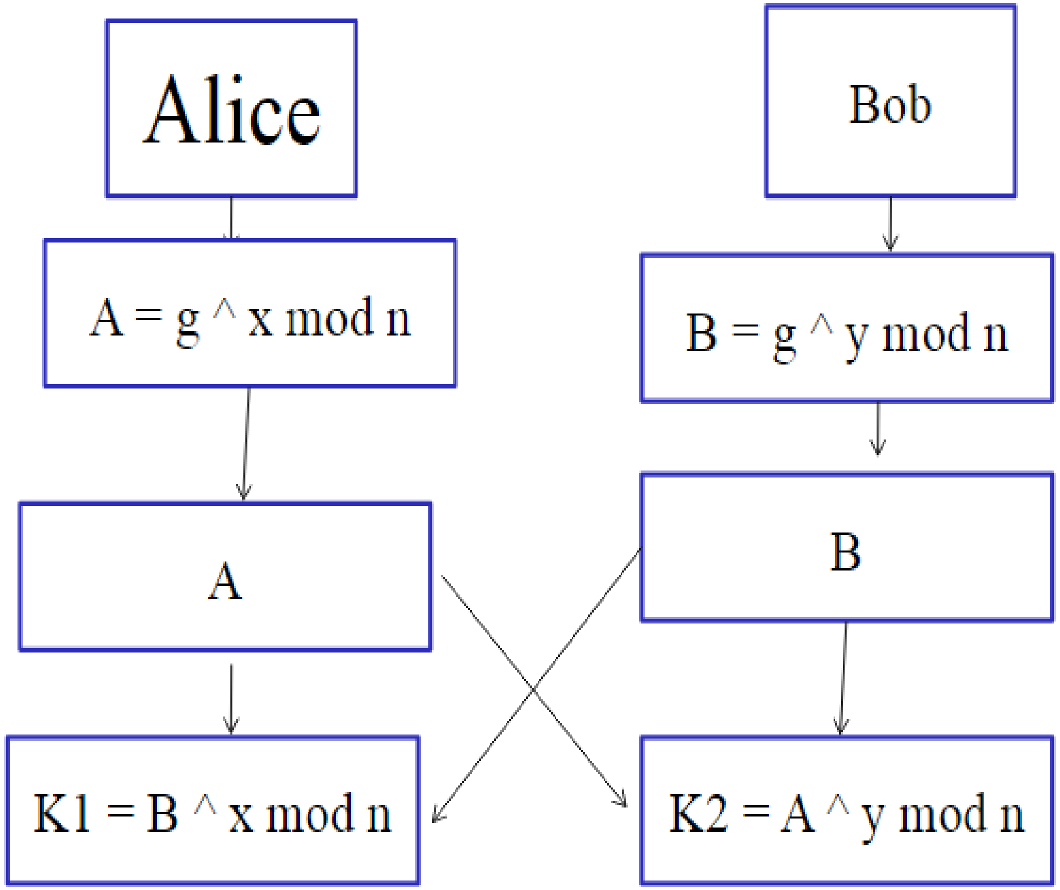
- Devised by Whitefield Diffie and Mertin Hellman as a solution to the problem of key agreement.
- This permits parties for key exchange but not for encryption/Decryption of messages.
- In this technique both the parties interested to communicate can agree on a symmetric key.

Working of the algorithm

Alice and Bob agree on two large prime numbers **'n'** and **'g'**, need not be kept secret.

x,y – random numbers

K1=K2=K that becomes the shared secret key between Alice and Bob



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Example

1. Firstly, Alice and Bob agree on two large prime numbers, n and g. These two integers need not be kept secret. Alice and Bob can use an insecure channel to agree on them.

   Let $n = 11$, $g = 7$.

2. Alice chooses another large random number x, and calculates A such that:
   $A = g^x \bmod n$

   Let $x = 3$. Then, we have, $A = 7^3 \bmod 11 = 343 \bmod 11 = 2$.

3. Alice sends the number A to Bob.

   Alice sends 2 to Bob.

4. Bob independently chooses another large random integer y and calculates B such that
   $B = g^y \bmod n$

   Let $y = 6$. Then, we have, $B = 7^6 \bmod 11 = 117649 \bmod 11 = 4$.

5. Bob sends the number B to Alice.

Bob sends 4 to Alice.

6. A now computes the secret key $K1$ as follows:
$K1 = B^x \bmod n$

We have, $K1 = 4^3 \bmod 11 = 64 \bmod 11 = 9.$

7. B now computes the secret key $K2$ as follows:
$K2 = A^y \bmod n$

We have, $K2 = 2^6 \bmod 11 = 64 \bmod 11 = 9.$

Mathematical theory

$$K1 = B^x \bmod n.$$

$$B = g^y \bmod n.$$

$$K1 = (g^y)^x \bmod \mathrm{n} = g^{yx} \bmod n$$

$$K2 = A^y \bmod n.$$

$$A = g^x \bmod n.$$

$$K2 = (g^x)^y \bmod n = g^{xy} \bmod n$$

$$K^{yx} = K^{xy}$$

Therefore, in this case, we have $K1 = K2 = K$.

Problems with the algorithm

- Man in the middle attack also called as Bucket Brigade attack.
- Makes the actual communicators believe that they are talking to each other but they are actually talking to the man-in-the-middle.

Alice	Tom	Bob
$n = 11, g = 7$	$n = 11, g = 7$	$n = 11, g = 7$

Alice	Tom	Bob
$x = 3$	$x = 8, y = 6$	$y = 9$

Alice	Tom	Bob
$A = g^x \bmod n$ $= 7^3 \bmod 11$ $= 343 \bmod 11$ $= 2$	$A = g^x \bmod n$ $= 7^8 \bmod 11$ $= 5764801 \bmod 11$ $= 9$	$B = g^y \bmod n$ $= 7^9 \bmod 11$ $= 40353607 \bmod 11$ $= 8$
	$B = g^y \bmod n$ $= 7^6 \bmod 11$ $= 117649 \bmod 11$ $= 4$	

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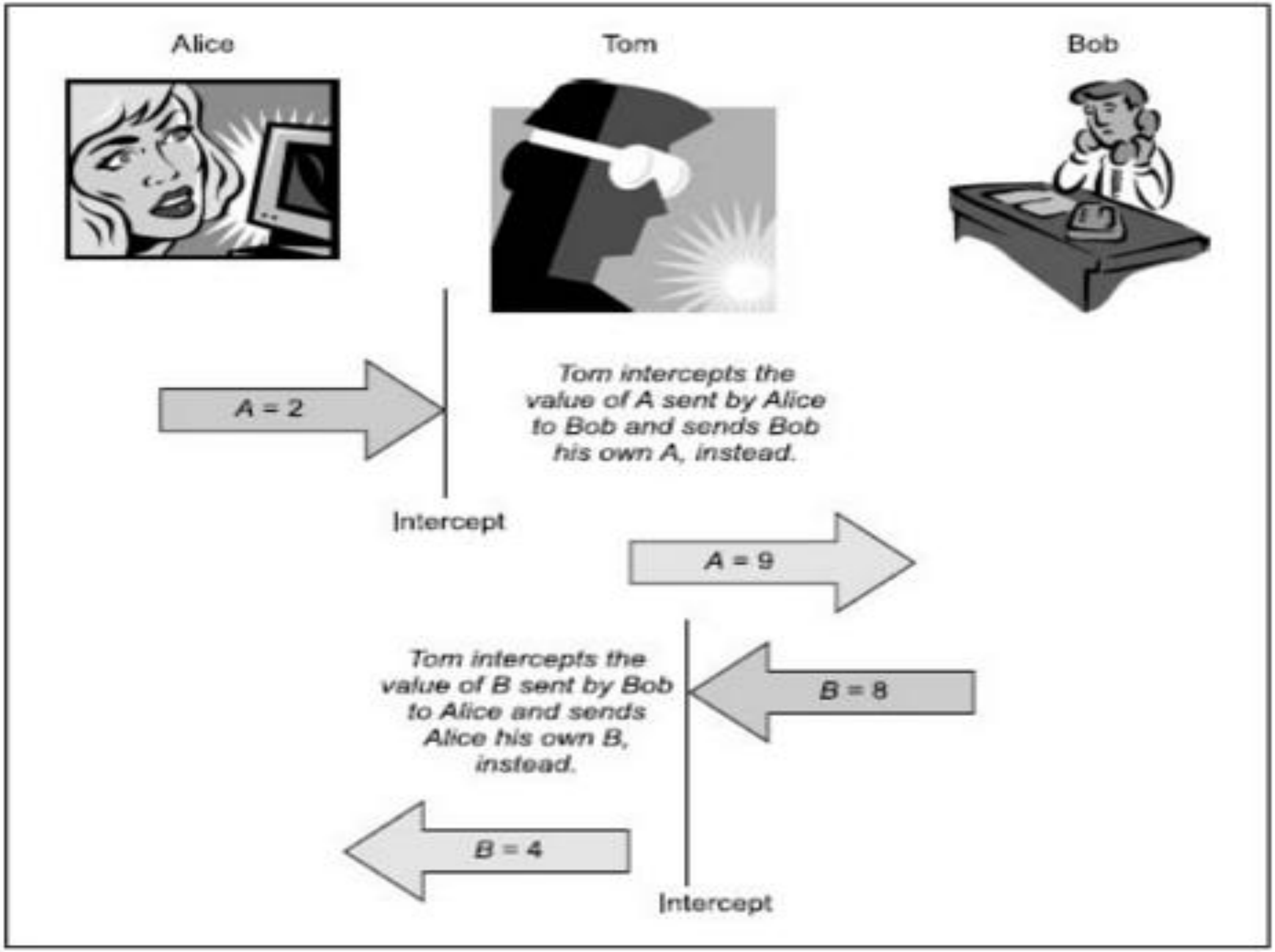
Alice
Tom
Bob
A = 2
Intercept
Tom intercepts the value of A sent by Alice to Bob and sends Bob his own A, instead.
A = 9
B = 8
Tom intercepts the value of B sent by Bob to Alice and sends Alice his own B, instead.
B = 4
Intercept

Alice	Tom	Bob
$A = 2, B = 4^*$	$A = 2, B = 8$	$A = 9^*, B = 8$

(Note: * indicates that these are the values after Tom hijacked and changed them.)

Alice		Tom		Bob	
$K1$	$= B^x \bmod n$	K1	$= B^x \bmod n$	$K2$	$= A^y \bmod n$
	$= 4^3 \bmod 11$		$= 8^8 \bmod 11$		$= 9^9 \bmod 11$
	$= 64 \bmod 11$		$= 16777216 \bmod 11$		$= 387420489 \bmod 11$
	$= 9$		$= 5$		$= 5$
		K2	$= A^y \bmod n$		
			$= 2^6 \bmod 11$		
			$= 64 \bmod 11$		
			$= 9$		

Example:Try it

- Calculate the Diffie hellman key for the public values n=10 and g=3

  Consider x=5 ;y=11

- Suppose p(here n) is a prime of around 300 digits, and a(x) and b(y) at least 100 digits each.
- Discovering the shared secret given g, p,

  g^a mod p and g^b mod p would take longer than the lifetime of the universe, using the best known algorithm. This is called the discrete logarithm problem.

How can two parties agree on a secret value when all of their messages might be overheard by an eavesdropper?

- The Diffie-Hellman algorithm accomplishes this, and is still widely used.
- With sufficiently large inputs, Diffie-Hellman is very secure.