The Determinant

- **Introduction:** The following material about determinants has been collated from Chapter 3 of the textbook by Lay. It contains all the information about determinants that you are expected to be familiar with. From now onwards, you would be free to use any of the definitions and results presented here. Probably, you are already familiar with most of these results. They will not be presented in class. However, you are welcome to review Chapter 3 of Lay and try out some practice exercises if desired.

The Determinant

- **Remark:** Propositions about determinants will be numbered independently as Prop D1, Prop D2, etc.
- **Definition of the Determinant:** If $A \in F^{2\times 2}$ where $A = [a_{ij}]$, then det A is defined to be the scalar $a_{11} a_{22} - a_{12} a_{21}$. Thus det is a function from $F^{2\times 2}$ to F.
- We extend this definition recursively to $F^{n\times n}$. as follows:
- **Notation:** If $A \in F^{n\times n}$, let $A_{i,j}$ denote the $(n-1) \times (n-1)$ matrix obtained from A by omission of the i-th row and j-th column.
- **Column expansion formula:** A formula for the determinant is given by:

  $\det A = \Sigma (-1)^{i+j} a_{ij} \det A_{i,j}$, where the summation is taken for i = 1 to n.
- **Row expansion formula:** Another formula for the determinant is given by:

  $\det A = \Sigma (-1)^{i+j} a_{ij} \det A_{i,j}$, where the summation is taken for j = 1 to n.

The Determinant - 1

- **Proposition D1:** The following hold for the determinant of a square matrix A:
  i. If the matrix A’ is obtained from A by interchanging two rows, then det A’ = – det A
  ii. If the matrix A’ is obtained from A by multiplying some row by $\lambda \in F$, then det A’ = λ det A
  iii. If the matrix A’ is obtained from A by adding a multiple of one row to another row, then det A’ = det A
- **Remark 1**: The above indicates what happens to the determinant when an elementary row operation – interchange, scaling, or replacement – is applied.
- **Remark 2:** The above holds if row is replaced by column.
- **Remark 3**: It follows directly from the above that if the rows (or columns) of A are linearly dependent, then det A = 0.

Procedure for Computing the Determinant

- **Proposition D2:** If an $n \times n$ matrix A is upper triangular, then det A = $a_{11}a_{22}\ldots\ldots a_{nn}$
- **Corollary D2.1:** In order to determine the determinant of an $n \times n$ matrix, use elementary row operations of interchange and replacement type only to reduce A to an upper triangular matrix A'. If r is the number of row interchanges carried out, then det A = $(-1)^r$ det A'.
- **Remark 1**: This follows directly from Proposition D1 and the definition (using the column expansion).
- **Remark 2**: The above method is far less computationally intensive than using either row or column expansion. **NB**: Most advanced textbooks use a different definition (formula) for the determinant; however, it is equally inefficient computationally.

Further Properties of the Determinant - 1

- **Proposition D3:** An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.
- **Remark:** The above gives another useful property equivalent to invertibility for square matrices. Consequently, we need to extend our theorem on invertibility of matrices (see next slide).

Very Important Theorem – Ver 1.1

- **Theorem 1**: The following are equivalent for an $m \times m$ square matrix A:

a. A is invertible

b. A is row equivalent to the identity matrix

c. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

d. The system of equations $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

e. Det $A \neq 0$

Further Properties of the Determinant - 2

- **Proposition D4:** for all $A, B \in F^{n\times n}$, det (AB) = (det A) (det B)
- **Corollary D4.1:** If A is invertible, then det A^{-1} = $(\det A)^{-1}$
- **Remark: W**hile det (AB) = (detA)(detB), in general $\det(A + B) \neq \det A + \det B$. *As we shall later, the determinant is not a linear function or linear transformation.*
- **Proposition D5:** For all $A \in F^{n\times n}$, det A^T = det A.

Cramer’s Rule

- **Remark:** If you have not studied this topic before, it is nicely presented in the book by Lay: Section 3.3
- **Definition:** For any $n \times n$ matrix A and any vector $\mathbf{b}$ in R^n, define $A_i(\mathbf{b})$ to be the matrix obtained by replacing the i-th column of A by $\mathbf{b}$.
- **Proposition D6 (Cramer’s Rule):** Let A be any invertible $n \times n$ matrix. For any vector $\mathbf{b}$ in R^n, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by:

  $x_i = (\det A_i(\mathbf{b}))/(\det A)$ for $i = 1,2,..,n$

- Cramer’s Rule is (usually) not a practical method for solving systems of linear equations since it requires computation of $(n + 1)$ determinants.

Application of Cramer's Rule

- **Terminology and Notation:** For any n×n matrix A, we define the cofactor $C_{ij} = (-1)^{i+j} \det(A_{ij})$
- Definition: the classical adjoint of A (written adj A) is the matrix whose entries are the cofactors of A transposed. In other words, adj A is the matrix:

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

- **Proposition D7: Inverse Formula:** Let A be any invertible n×n matrix. Then:

$$A^{-1} = (1/\det A)(\operatorname{adj} A)$$

Application of Determinants to Areas and Volumes - 1

- **Proposition D8:** (a) If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

  (b) If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Application of Determinants to Areas and Volumes - 2

- **Remark:** *We will shortly introduce the topic of linear transformations. You should revisit the following propositions after that.*
- **Proposition D9:** (a) Let $T: R^2 \to R^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in R^2, then {area of T(S)} = |det A| × {area of S}.

  (b) Let $T: R^3 \to R^3$ be the linear transformation determined by a 3×3 matrix A. If S is a parallelepiped in R^3, then {volume of T(S)} = |det A| × {volume of S}.

Application of Determinants to Areas and Volumes - 3

- **Proposition D10:** The conclusions of Proposition D9 hold whenever S is a region in R^2 with finite area or a region in R^3 with finite volume. In other words:

  {area or volume of T(S)} = |det A| × {area or volume of S}.