MTH 100 : Lecture 29

Algebra of Linear Transformations

Let V and W be vector spaces over a field F.

Notation: $W^V \equiv$ The set of all functions from V to W

$L(V,W) \equiv$ The set of all linear transformations from V to W

Proposition:

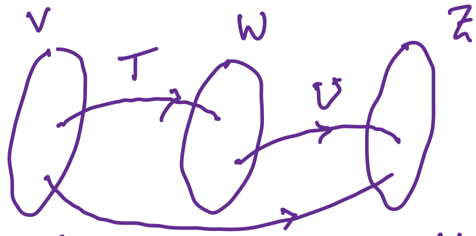
(a) The set W^V of all functions from V to W is a vector space over F.

(b) The set $L(V,W)$ of all linear transformations from V to W is a subspace of W^V

Proof: Exercise (For (b) use test for a subspace)

Proposition: Let V, W and Z be vector spaces over a field F. Let T be a linear transformation from V to W and U be a linear transformation from W to Z. Then the composed function UT from V to Z defined by $(UT)(v) = U(T(v))$ for all $v \in V$ is a linear transformation from V into Z.

Proof: Exercise



(Composition UT or $U \circ T$)

Linear Operators:

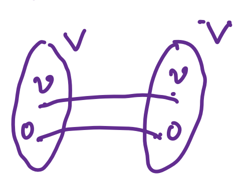
- A linear operator on a vector space V is a linear transformation from V to V.
- $L(V, V) \equiv$ The space of all linear operators on V

  (i.e. the space of all linear transformations from V into V)

- $L(V, V)$ is of primary importance because we can define a 'multiplication', i.e. composition of linear operators.

  (Note that we can't do this on $L(V, W)$ when W is different from V.)

- By the previous proposition, the composition of two linear operators on V is a linear operator on V.
- Note Define the identity operator $I : V \to V$ by $I(v) = v \ \forall \ v \in V$



- Composition of Linear operators satisfies the following properties:

(a) $IU = UI = U$ for all linear operator U where I is the identity operator on V

(b) $(T_1 T_2)T_3 = T_1(T_2 T_3)$ (Associative Law)

(c) $U(T_1+T_2) = UT_1 + UT_2$

(d) $(T_1+T_2)U = T_1 U + T_2 U$

(e) $c(UT_1) = (cU)T_1 = U(cT_1)$.

* (f) However, this multiplication is not commutative.

- Verification of the above properties is left as an exercise.

- A vector space with a multiplication which satisfies properties (a) through (e) above is called an $\boxed{\text{ALGEBRA}}$.

Thus $L(V\ V)$ is an Algebra.

A Fundamental Isomorphism

Proposition: Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then there is an isomorphism between $L(V,W)$ and $F^{m\times n}$

Outline of a Proof:
We take a fixed ordered basis $\alpha=\{v_1,v_2,\dots,v_n\}$ for V and a fixed ordered basis $\beta=\{w_1,w_2,\dots,w_m\}$ for W.
For any linear transformation $T\in L(V,W)$, we can find the matrix of T with respect to the bases α and β. Let us denote it by $[T]_{\alpha\to\beta}$
Clearly $[T]_{\alpha\to\beta}\in F^{m\times n}$

Define the mapping $\phi: L(V,W)\longrightarrow F^{m\times n}$ by $\phi(T)=[T]_{\alpha\to\beta}$

- Now show that ϕ is linear, 1-1 and onto. (need to show!!)
  Hence ϕ is an isomorphism.

Note: The isomorphism ϕ above is defined in terms of the bases α and β and is therefore dependent on the choice of α and β.

Proposition: If $\dim V=n$, $\dim W=m$ then $\dim L(V,W)=mn$

Proof: can be proved in two different ways.

First way: Using the previous proposition, we can say that $L(V,W)$ is isomorphic to $F^{m\times n}$.

Since $\dim(F^{m\times n}) = mn$, we conclude that $\boxed{\dim L(V,W) = mn}$

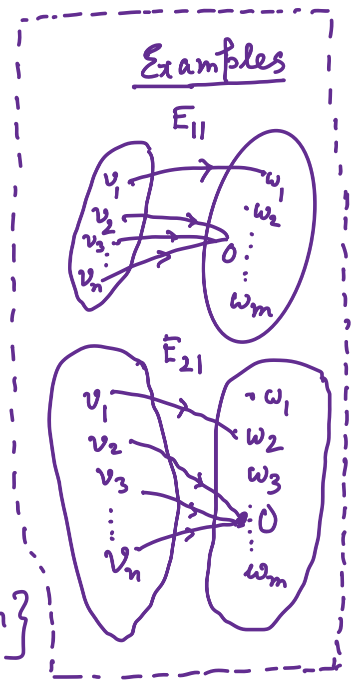
Second way: Let us take a fixed ordered basis $\alpha = \{v_1, \dots, v_n\}$ for V and a fixed ordered basis $\beta = \{w_1, \dots, w_m\}$ for W.

Define a linear transformation

$$E_{ij} : V \longrightarrow W$$

by $E_{ij}(v_j) = w_i$ and

$$E_{ij}(v_k) = 0 \quad \text{for } k \neq j$$



- It can be shown that the family $S = \{E_{ij} : i = 1, \dots m;\ j = 1, \dots, n\}$ forms a basis for $L(V,W)$.

Since the number of elements in S is mn, we conclude that $\dim L(V,W) = \boxed{mn}$

Note: The matrix of the linear transformation E_{ij} with regard to the basis α and β is the matrix $[E_{ij}]$ defined earlier.

(Recall the $m \times n$ matrix $[E_{ij}]_{m\times n}$ is defined as the matrix whose (i,j)th entry is 1 and rest of the entries are zero.)

i.e. $[E_{ij}]_{\alpha\to\beta} = [E_{ij}]$

Recall that the matrices $[E_{ij}]$ form a basis for $F^{m\times n}$.

Proposition ⊛:

Suppose T and U are linear operators on a finite dimensional vector space V and β is a fixed ordered basis for V.

Then $[UT]_\beta = [U]_\beta [T]_\beta$

(So, the matrix of the product of two linear operators is the product of their matrices. Thus we have a seamless transition from operators to matrices and vice-versa.

Note: If $\left.\begin{array}{l} T: V \to V \\ U: V \to V \end{array}\right\}$ then $UT: V \to V$ and all the three operators have corresponding matrices with respect to a fixed ordered basis of V.)

Alternative Statement of previous proposition ⊛ :

The mapping $\phi : L(V,V) \longrightarrow F^{n\times n}$ given by

$\phi(T) = [T]_\beta$ is a vector space isomorphism which also preserves products,

i.e. $\phi(UT) = \phi(U)\phi(T)$

Generalization of Proposition ⊛ :

The Proposition ⊛ can be extended to the composition of linear transformations $T : V \to W$ and $U : W \longrightarrow Z$ in the following way:

Suppose that $\dim V = n$, $\dim W = m$, $\dim Z = k$

Then $UT : V \longrightarrow Z$ will be a linear transformation from a space of dimension n to a space of dimension k

i.e. its matrix would be an $k \times n$ matrix.

Let α, β, γ be bases of V, W, Z respectively,

then $\underset{k\times n}{[UT]_{\alpha\to\gamma}} = \underset{k\times m}{[U]_{\beta\to\gamma}} \; \underset{m\times n}{[T]_{\alpha\to\beta}}$