MTH 100 : Lecture 31

How to Determine Eigen Values and Eigen Vectors :

Note: It is easy to verify whether a particular vector is an eigen vector of a given matrix A or not.

Similarly, given some scalar, we can verify whether it is an eigen value or not.

- However we need to find a systematic method to find eigen values.

Ex: Let $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

Let $v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$

$v_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

To check if v is an eigen vector of A :

Calculate: $Av = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 20 \\ -11 \\ 38 \end{bmatrix}$$

But $\begin{bmatrix} 20 \\ -11 \\ 38 \end{bmatrix} \neq \lambda \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ for any $\lambda \in F$.

So, $Av \neq \lambda v$ for any λ.

So, v is not an eigen vector of A.

Now $Av_1 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 1 . \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

So, v_1 is an eigen vector corresponding to the eigen value 1.

Again, $Av_2 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} = 1 . \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$

So, v_2 is an eigen vector corresponding to the eigen value 1.

Now $Av_3 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

So, v_3 is an eigen vector corresponding to the eigen value 0.

Now $v_4 = v_1 + v_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$

and $Av_4 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$

So, v_4 is (as expected) an eigen vector corresponding to the eigen value 1.

<u>Ex:</u> $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

Let $\lambda = 3$: can we find out if 3 is an eigen value of A or not.

If λ is an eigen value of A, then there exists a vector $v \in \mathbb{R}^3$, $v \neq 0$ such that

$$Av = \lambda v \Rightarrow Av - \lambda v = 0$$
$$\Rightarrow Av - \lambda I v = 0$$
$$\Rightarrow (A - \lambda I)v = 0$$

i.e. the homogeneous system $(A - \lambda I)X = 0$ has a nontrivial solution.

Now $A - \lambda I$
$$= A - 3I = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 1 \\ 6 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -2 \\ 0 & -8 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -8 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 8R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\xrightarrow[R_3 \to -\frac{1}{6}R_3]{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

= RREF matrix

So, $(A - 3I)X = 0$ has only trivial solution.

Therefore $\lambda = 3$ is not an eigen value of A.

Proposition: A scalar λ is an eigen value of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$

(Note: Characteristic Equation of A: $\det(A - \lambda I) = 0$
Characteristic Polynomial of A: $\det(A - \lambda I)$)

Proof: λ is an eigen vector of A

$\Leftrightarrow$ There is a non zero vector v such that $Av = \lambda v$

$\Leftrightarrow$ The system $(A - \lambda I)x = 0$ has a non-trivial solution

$\Leftrightarrow$ The matrix $(A - \lambda I)$ is not invertible (By the first theorem of the course)

$\Leftrightarrow$ $\det(A - \lambda I) = 0$ (By the extended version of the first theorem)

$\Leftrightarrow$ λ is a root of the characteristic equation.

Ex: Given $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

Find the characteristic polynomial of A and eigen values of A.

Characteristic Polynomial of A

$$= \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & -1 \\ -3 & -1-\lambda & 1 \\ 6 & 4 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left[(-1-\lambda)(-1-\lambda) - 4\right] + 2\left[1 \times 6 - (-3)(-1-\lambda)\right] + (-1)\left[(-3)(4) - 6(-1-\lambda)\right]$$

$= (4-\lambda)(1+2\lambda+\lambda^2-4) + 2(3-3\lambda) - (-6+6\lambda)$

$= (4-\lambda)(\lambda^2+2\lambda-3) + 2(3-3\lambda) - (-6+6\lambda)$

$= 4\lambda^2 + 8\lambda - 1\cancel{2} - \lambda^3 - 2\lambda^2 + 3\lambda + \cancel{6} - 6\lambda + \cancel{6} - 6\lambda$

$= \boxed{-\lambda^3 + 2\lambda^2 - \lambda}$

Now the characteristic polynomial of A

$= -\lambda^3 + 2\lambda^2 - \lambda \quad = \quad -\lambda(\lambda^2 - 2\lambda + 1)$

$= -\lambda(\lambda-1)^2$

Hence the eigen values of A are 0 and 1.

<u>Note:</u> • 0 is an eigen value of A with <u>multiplicity 1</u> and 1 is an eigen value of A with <u>multiplicity 2</u>.

<u>Note:</u> • det $(A-\lambda I)$ is a polynomial of degree n and it is called the characteristic polynomial of A.

• It has atmost n roots, counting multiplicities. Hence an $n \times n$ matrix can have atmost n eigen values (counting multiplicities)

• It is possible for a matrix with real entries to have no real eigen values.

<u>Ex:</u> Given $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Then det $(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-1)(1)$

$= \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2$

So, the characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$

$$\Rightarrow \quad \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(1)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$$

$$\Rightarrow \quad \lambda = 1 \pm i$$

<u>Note</u>: If complex roots are allowed, an $n \times n$ matrix has exactly n eigen values (counting multiplicities): This follows from the so-called Fundamental Theorem of Algebra. Therefore, we must clearly specify which field is being considered when we talk about the eigen values of a matrix.

Eigenvalues of Similar Matrices

- Recall that an $n \times n$ matrix B is similar to an $n \times n$ matrix A if there exists an invertible matrix P such that $B = PAP^{-1}$ (or $A = P^{-1}BP$).

- Note that if A and B are similar matrices then $B = PAP^{-1}$ for some invertible matrix P and so $\det B = \det(PAP^{-1})$

$$= (\det P)(\det A)(\det P^{-1})$$

$$= \det A$$

- **Proposition:** If the $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.

Proof:

$$\det(B - \lambda I) = \det(PAP^{-1} - \lambda I)$$
$$= \det(PAP^{-1} - P(\lambda I)P^{-1})$$
$$= \det(P(A - \lambda I)P^{-1})$$
$$= (\det P)\det(A - \lambda I)\det(P^{-1})$$
$$= \det(A - \lambda I)(\det P)\det(P^{-1})$$
$$= \det(A - \lambda I) \cdot 1$$
$$= \det(A - \lambda I)$$

(QED)

Note: The eigenvectors of A and B are not necessarily the same.