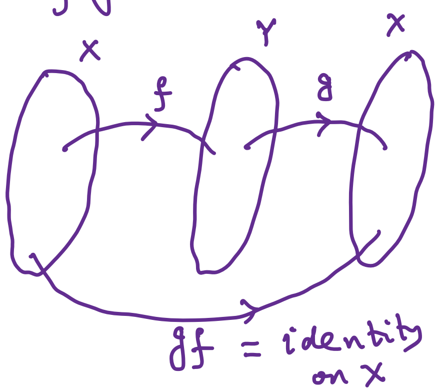
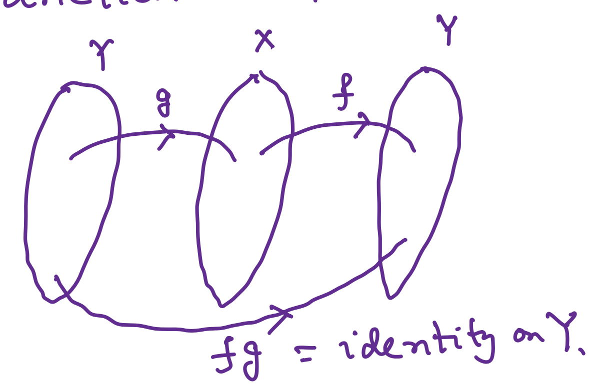
MTH 100: Lecture 30

Invertible functions

Definition: A function $f: X \to Y$ is called invertible if there exists a function $g: Y \to X$ such that gf is the identity function on X and fg is the identity function on Y

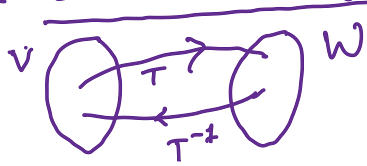




- If f is invertible, then the function g is unique and is called the inverse of f, denoted by f^{-1}.
- A function f is invertible if and only if f is 1-1 and onto (i.e. bijective)

Invertibility of Linear Transformations

Proposition: If $T: V \to W$ is an invertible linear transformation, its inverse function $T^{-1}: W \to V$ is also a linear transformation.



Proof: Exercise

Note: We had earlier referred to invertible linear transformations as <u>isomorphisms</u>. We may use either term.

Corollary: Isomorphism is <u>an equivalence relation</u> on the set of all vector spaces over a given field F.

Outline of Proof: Reflexive property is obvious. Prove symmetric and transitive property using earlier propositions.

Eigen Vectors and Eigen Values:

- A scalar λ is called an <u>eigen value</u> of a $n \times n$ matrix A if there is a <u>non trivial solution</u> of $Ax = \lambda x$. Such a vector x is called an <u>eigen vector corresponding</u> to the <u>eigen value λ</u>.

- Thus an <u>eigen vector</u> of an $n \times n$ matrix A is a non-zero vector x such that $Ax = \lambda x$ for some scalar λ.

- If v is an eigen vector corresponding to an eigen value λ_1, then it cannot be an eigen vector corresponding to some different eigen value λ_2.

For if $Av = \lambda_1 v$ and $Av = \lambda_2 v \Rightarrow \lambda_1 v = \lambda_2 v \Rightarrow (\lambda_1 - \lambda_2)v = 0$

If $\lambda_1 \neq \lambda_2$, $\lambda_1 - \lambda_2 \neq 0$ and so $v = 0$, not possible for an eigen vector.

- Eigen values are sometimes called characteristic values or latent roots.

Eigen vectors are sometimes called characteristic vectors.

Note: • The "zero vector" is not considered as an eigen vector since $A0 = \lambda 0$ for all matrices A and all scalars λ.

- However 0 is allowed to be an eigen value for a matrix A. (Note that $Ax = 0.x \Rightarrow Ax = 0$)

Proposition: An $n \times n$ matrix A is invertible if and only if 0 is not an eigen value for A.

Proof: Let 0 be an eigen value for A.

Then the equation $Ax = 0.x$ has a nontrivial solution.

But $Ax = 0$ has a nontrivial solution if and only if A is not invertible.

Therefore an $n \times n$ matrix A is invertible if and only if 0 is not an eigen value of A.

- Thus we have another condition to add to our first theorem (of the course).

<u>Note:</u> An eigen vector is <u>not unique</u> since all scalar multiples of an eigen vectors are also eigen vectors (corresponding to the same eigen value)

$Ax = \lambda x \Rightarrow A(cx) = c(Ax)$
$= c(\lambda x) = \lambda(cx)$

<u>Proposition:</u> Let A be an $n \times n$ matrix and $V = F^n$

Then the set $X = \{ v \in V : v \text{ is an eigen vector of } A \text{ corresponding to } \lambda \} \cup \{0\}$

$= \{ v \in V : Av = \lambda v \}$

is a subspace of V.

<u>Proof:</u> • Let $v_1, v_2 \in X$. Then $Av_1 = \lambda v_1$ and $Av_2 = \lambda v_2$

$\Rightarrow A(v_1 + v_2) = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$

$\Rightarrow v_1 + v_2 \in X$

Similarly if $v \in X$ and $c \in F$, then $Av = \lambda v \Rightarrow A(cv) = cAv = c(\lambda v)$

$\Rightarrow A(cv) = \lambda(cv)$

$\Rightarrow cv \in X$

So, $\boxed{X \text{ is a subspace of } V}$

- <u>Another Proof:</u>

Note that $v \in X \Leftrightarrow Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow (A - \lambda I)v = 0$

$\Leftrightarrow v \in \text{Nul}(A - \lambda I)$

Hence $X = \text{Nul}(A - \lambda I)$ and therefore X is a subspace of V

<u>Note:</u> The subspace X defined above is called the <u>eigen space</u> of A <u>corresponding to λ</u>.

Fundamental Result about Eigenvectors and Eigen Values

Proposition: If $v_1, v_2, \ldots, v_p$ are eigen vectors corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the matrix A, then the set $\{v_1, v_2, \ldots, v_p\}$ is linearly independent.

Corollary: An $n \times n$ matrix A can have atmost n distinct eigen values.

Proof of the Proposition:

Proof will be by contradiction:

Assume that $v_1, v_2, \ldots, v_p$ are linearly dependent.

Let m be the smallest number such that $v_1, v_2, \ldots, v_m$ are linearly independent and v_{m+1} is a linear combination of the preceeding vectors.

Then there exist scalars $c_1, c_2, \ldots c_m \in F$ such that $c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = v_{m+1}$ ......(1)

Then $A(c_1 v_1 + c_2 v_2 + \cdots + c_m v_m) = A v_{m+1}$

$\Rightarrow c_1 A v_1 + c_2 A v_2 + \cdots + c_m A v_m = A v_{m+1}$

$\Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_m \lambda_m v_m = \lambda_{m+1} v_{m+1}$ ....(2)

Now multiplying ① by λ_{m+1}, we get

$$c_1 \lambda_{m+1} v_1 + c_2 \lambda_{m+1} v_2 + \cdots + c_m \lambda_{m+1} v_m = \lambda_{m+1} v_{m+1} \quad \cdots\cdots ③$$

Now ② − ③ ⇒

$$c_1(\lambda_1 - \lambda_{m+1}) v_1 + c_2(\lambda_2 - \lambda_{m+1}) v_2 + \cdots + c_m(\lambda_m - \lambda_{m+1}) v_m = 0$$

Since $v_1, v_2, \ldots, v_m$ are <u>linearly independent</u>,

$$c_1(\lambda_1 - \lambda_{m+1}) = c_2(\lambda_2 - \lambda_{m+1}) = \cdots = c_m(\lambda_m - \lambda_{m+1}) = 0$$

But $\lambda_1 - \lambda_{m+1} \neq 0$, $\lambda_2 - \lambda_{m+1} \neq 0, \ldots, \lambda_m - \lambda_{m+1} \neq 0$

Hence $c_1 = c_2 = \cdots = c_m = 0$

and so from ① we conclude $v_{m+1} = 0$

But v_{m+1} is an eigen vector of A, corresponding to the eigen value λ_{m+1} and so $v_{m+1} \neq 0$, a contradiction.

Therefore $\boxed{\{v_1, v_2, \ldots, v_p\} \text{ is linearly independent}}$