

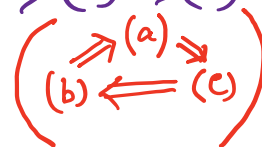
MTH 100: Lecture 7

Theorem ①: The following are equivalent for a $m \times m$ square matrix A :

- (a) A is invertible
- (b) A is row equivalent to the identity matrix.
- (c) The homogeneous system $AX = \bar{0}$ has only the trivial solution.
- (d) The system of equation $AX = \bar{b}$ has at least one solution for every $\bar{b} \in \mathbb{R}^m$

Proof: First we will prove $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

($(a) \Leftrightarrow (d)$: Later)



• $(a) \Rightarrow (c)$:

Given: A is invertible

To show: $AX = \bar{0}$ has only the trivial solution.

• Let u be a solution of $AX = \bar{0}$

$$\Rightarrow Au = \bar{0}$$

$$\Rightarrow A^{-1}(Au) = A^{-1}(\bar{0}) \quad \left(\begin{array}{l} \text{Since } A \text{ invertible} \\ A^{-1} \text{ exists} \end{array} \right)$$

$$\Rightarrow A^{-1}(Au) = \bar{0}$$

$$\Rightarrow (A^{-1}A)u = \bar{0}$$

$$\Rightarrow I.u = \bar{0} \Rightarrow \boxed{u = \bar{0}}$$

So, $AX = \bar{0}$ has only the trivial solution.

(c) \Rightarrow (b): Given: $AX = \bar{0}$ has only the trivial solution.

To show: A is row equivalent to the identity matrix.

- Let R be the RREF matrix of A .
Then $RX = \bar{0}$ also has only the trivial solution.
 - $\Rightarrow R$ has no free variable
 - $\Rightarrow R$ has only basic variables
 - $\Rightarrow R$ has leading entry as 1 in each row
(There are m rows)
 - $\Rightarrow R$ has exactly one 1 in each column
(There are m columns)
 - $\Rightarrow R$ is $I \Rightarrow A$ is row equivalent to I

(b) \Rightarrow (a): Given: A is row equivalent to the identity matrix I

To show: A is invertible

- A is row equivalent to I
 - \Rightarrow There are elementary row operations $e_1, e_2, \dots, e_{p-1}, e_p$ such that
$$e_p(e_{p-1} \dots (e_2(e_1(A))) \dots) = I$$
- Let E_i be the elementary matrix corresponding to e_i for $i = 1, 2, \dots, p$
(ie. $E_i = e_i(I)$)
- Then $E_p(E_{p-1} \dots (E_2(E_1 A)) \dots) = I$ (By Proposition 5)

$$\Rightarrow (E_p E_{p-1} \dots E_2 E_1) A = I$$

$$\text{Let } B = E_p E_{p-1} \dots E_2 E_1$$

Then B is invertible (By observation ④ and Proposition ⑥)
and we have $BA = I$

Multiplying both sides by B^{-1} from the left, we obtain

$$B^{-1}(BA) = B^{-1} \cdot I$$

$$\Rightarrow (B^{-1}B)A = B^{-1}$$

$$\Rightarrow I \cdot A = B^{-1} \Rightarrow \boxed{A = B^{-1}}$$

So, A is the inverse of an invertible matrix
 $\Rightarrow \boxed{A \text{ is invertible}}$ (By Observation ②)

Calculation of Inverse matrix:

Corollary(1.1): An invertible matrix A is a product of elementary matrices.

(Note: Any sequence of row operations that reduces A to I also transforms I to A^{-1}
(We are using Theorem 1(b) here))

Proof: If A is invertible, then by Theorem 1(b), A is row equivalent to I .

So, there are some elementary row operations e_1, e_2, \dots, e_p such that

$$e_p(e_{p-1} \dots (e_2(e_1(A))) \dots) = I$$

Let E_1, E_2, \dots, E_p be the corresponding elementary matrices (i.e. $E_i = e_i(I)$),
 then $E_p(E_{p-1} \dots (E_2(E_1 A)) \dots) = I$

$$\Rightarrow (E_p E_{p-1} \dots E_2 E_1) A = I$$

$$\Rightarrow A = (E_p E_{p-1} \dots E_2 E_1)^{-1} I$$

$$\Rightarrow A = (E_1^{-1} E_2^{-1} \dots E_{p-1}^{-1} E_p^{-1}) I$$

$$\Rightarrow \boxed{A = E_1^{-1} E_2^{-1} \dots E_{p-1}^{-1} E_p^{-1}}$$

So, A is a product of some elementary matrices

Note: We can say $A^{-1} = (E_1^{-1} E_2^{-1} \dots E_p^{-1})^{-1}$

$$\Rightarrow A^{-1} = (E_p^{-1})^{-1} \dots (E_2^{-1})^{-1} (E_1^{-1})^{-1}$$

$$\Rightarrow A^{-1} = E_p \dots E_2 E_1$$

$$\Rightarrow A^{-1} = (E_p \dots E_2 E_1) I$$

$$\Rightarrow A^{-1} = E_p (\dots E_2 (E_1 I) \dots)$$

$$\Rightarrow \boxed{A^{-1} = e_p (\dots e_2 (e_1(I)) \dots)}$$

Thus the same sequence of row operations that reduces A to I also reduces I to A^{-1} .

Method of obtaining A^{-1} :

Form the enlarged matrix $[A:I]$ and carry out elementary row operations till 'A' part becomes I. The final result has the form $[I:A^{-1}]$.

- Corollary(1.2): If A has a left inverse or a right inverse, then it has an inverse.

Note: (1) B is a left inverse of A if $BA=I$
(2) D is a right inverse of A if $AD=I$