MTH 100: Lecture 26

Definition: A linear transformation $T: V \to W$ is called an isomorphism if it is injective and surjective (i.e. if Range $T = W$)

Proposition: Let V and W be finite dimensional spaces.

(a) An isomorphism $T: V \to W$ takes any arbitrary basis of V to a basis of W.

(b) Conversely, if a linear transformation $T: V \to W$ takes some basis of V to a basis of W, then it is an isomorphism.

Sketch of a Proof:

(a): Given: $T: V \to W$ is an isomorphism

Now, T is onto $\Rightarrow$ Range $T = W \Rightarrow$ Rank $T = \dim W$

T is 1-1 $\Rightarrow$ Ker $T = \{0\} \Rightarrow$ nullity $T = 0$

By the Rank Theorem,

$$\text{Rank } T + \text{nullity } T = \dim V = n \text{ (say)}$$
$$\Rightarrow \dim W + 0 = n$$
$$= \dim W = \dim V = n$$

Let $\{v_1, \ldots, v_n\}$ be a Basis of V.

Then $\{Tv_1, \ldots, Tv_n\}$ is a spanning set of Range $T = W$. Since $\dim W = n$, $\{Tv_1, \ldots Tv_n\}$ forms a basis of W.

(b) $\Leftarrow$:

Assume that $\dim V = n$ and T takes some basis $\{v_1, \ldots, v_n\}$ of V to a basis $\{Tv_1, \ldots, Tv_n\}$ of W

Therefore $\dim W = n$ and Rank $T = n$

Hence Range $T = W$ and T is onto.

Now using nullity Theorem :

$$\text{Rank } T + \text{nullity } T = \dim V = n$$
$$\Rightarrow n + \text{nullity } T = n$$
$$\Rightarrow \text{nullity } T = 0$$
$$\Rightarrow \text{Ker } T = \{0\}$$
$$\Rightarrow T \text{ is } 1\text{-}1.$$

Therefore T is an isomorphism.

Proposition: Two finite dimensional vector spaces V and W (over the same field F) are isomorphic if and only if $\dim V = \dim W$.

Proof: $\Rightarrow$: Assume $T: V \longrightarrow W$ is an isomorphism.
Want to show $\dim V = \dim W$.

Suppose $\dim V = n$ and let $\{v_1, \ldots, v_n\}$ be a basis of V. Then by the previous proposition $\{Tv_1, \ldots, Tv_n\}$ is a basis of W. Hence $\dim W = n$

So, $\boxed{\dim V = \dim W}$

$\Leftarrow$: Assume that $\dim V = \dim W$.
Want to show that V and W are isomorphic.

Let $\{v_1, \ldots, v_n\}$ be a basis of V
and $\{w_1, \ldots, w_n\}$ be a basis of W.
Consider the unique linear transformation $T: V \longrightarrow W$ such that $Tv_i = w_i$ for $i = 1, 2, \ldots, n$.

Since T takes a basis of V to a basis of W, by the previous proposition, T is an isomorphism and hence V and W are isomorphic.

Remark: Every vector space of dimension n over a field F is isomorphic to F^n. In particular, every vector space of dimension n over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$.

An Important Linear Transformation:

Left multiplication by a Matrix:

Let A be a $m \times n$ matrix.

Define $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

by $T_A(x) = Ax$

[Note that $\underbrace{A_{m\times n}\, x_{n\times 1}}_{m\times 1}$ is a $m\times 1$ matrix]

- T_A is a linear transformation.

- For $x, y \in \mathbb{R}^n$,

$$T_A(x+y) = A(x+y) = Ax + Ay$$
$$= T_A(x) + T_A(y)$$

- For $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$T_A(cx) = A(cx) = cAx = cT_A(x)$$

Hence T_A is a linear transformation.

Consider the Reverse Problem:

Suppose V and W are finite dimensional vector spaces over the field F.

Suppose $T : V \to W$ is a linear transformation
We will associate a matrix with this linear transformation.

Coordinate Systems:

Suppose V is a finite dimensional vector space.

An ordered basis for a finite dimensional vector space V is a finite sequence of vectors which is linearly independent and spans V.

In other words,

an ordered basis is a basis with the vectors taken in a specified fixed order.

Thus,
given an order basis of V

$$B = \{u_1, \cdots, u_n\},$$

we can express any vector $u \in V$ uniquely in the form $u = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$

The scalars x_i are called the coordinates of u relative to the (ordered) basis B

Remark: given a fixed ordered basis B for a finite dimensional vector space V, we can find an n-tuple in F^n

(usually F is $\mathbb{R}$ or $\mathbb{C}$) corresponding to any vector u in V as follows: $u \longrightarrow (x_1, x_2, \ldots, x_n)$ where x_i are the coordinates of u relative to B

Rather than the n-tuple $(x_1, x_2, \ldots, x_n)$ we express it as a column vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- This vector is called the coordinate vector of u (relative to B) and is written $[u]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Ex: Let $V = R_n(t)$ be the vector space of polynomials of degree $\leq n$

Then $B = \{1, t, t^2, \ldots, t^n\}$ is an ordered basis of $R_n(t)$

Let $v = 2t^3 \in R_n(t)$

Then v can be written as:

$$v = 0.1 + 0.t + 0.t^2 + 2.t^3 + 0.t^4 + \cdots + 0.t^n$$

Hence the coordinate vector of v relative to B is $[v]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$