Quiz 8

Nov 8th, 2024

Time: 15 minutes Max marks = 5

Name: ____________ Roll no.: ____________ Group: ____________

Instructions: Notes, books, computers, cell phones and other electronic devices are not allowed.

Problem 1. (a) Find the matrix of the linear operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where $T(x, y, z) = (3x + 2y - 3z, -2x + y + z, x + 2y + z)$ with respect to the standard basis.

(b) Find the matrix of T with regard to the basis $\beta = \{(1, 2, 4), (0, -1, 1), (2, 3, 8)\}$

(c) Find $[Tv]_\beta$ where $v = (1, 2, 3)$.

Rubrics of Quiz 8 | Total marks = 5

1 (a) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ where

$$T(x,y,z) = (3x + 2y - 3z,\ -2x + y + z,\ x + 2y + z)$$

Let $\alpha = \{ e_1 = (1,0,0),\ e_2 = (0,1,0) \text{ and } e_3 = (0,0,1) \}$ be the standard basis of $\mathbb{R}^3$

$$Te_1 = (3, -2, 1) = 3e_1 - 2e_2 + e_3$$
$$Te_2 = (2, 1, 2) = 2e_1 + 1e_2 + 2e_3$$
$$Te_3 = (-3, 1, 1) = -3e_1 + 1e_2 + 1e_3$$

+.5

Hence $[T]_\alpha = \begin{bmatrix} 3 & 2 & -3 \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

+.5

+1

(b) Let $\beta = \{ (1,2,4),\ (0,-1,1),\ (2,3,8) \}$

Then $[T]_\beta = P\,[T]_\alpha\, P^{-1}$

+.5 They can write here or later

Writing the formula has .5 marks

where $P = P_{\alpha \to \beta} = Q^{-1}$ where $Q = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

+.5

Now we calculate P

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

So,

$$P = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

+1

Note: They can use any other method to compute Q^{-1} to get P. As long as the expression of Q and P are correct and their work is correct, it will be acceptable.

③

$= P[T]_\alpha P^{-1}$ (They can write the formula here instead of at the beginning & it has .5 marks)

Therefore

$$[T]_\beta = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -35 & -16 & 37 \\ -11 & -6 & 13 \\ 19 & 9 & -20 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

+1

$$= \boxed{\begin{bmatrix} 81 & 53 & 178 \\ 29 & 19 & 64 \\ -43 & -29 & -95 \end{bmatrix}}$$

Note: They can multiply in different order and so they don't need to show the intermediate matrices as long as the expression for $[T]_\beta$ (The last matrix) is correct.

c) $v = (1,2,3)$

+.5

$$[Tv]_\alpha = T\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3\times1+2\times2-3\times3\\-2\times1+2+3\\1+4+3\end{pmatrix} = \begin{bmatrix}-2\\3\\8\end{bmatrix}_\alpha$$

Then

+.5

$$[Tv]_\beta = P[Tv]_\alpha = \begin{bmatrix}-11 & 2 & 2\\-4 & 0 & 1\\6 & -1 & -1\end{bmatrix}\begin{bmatrix}-2\\3\\8\end{bmatrix}$$

$$= \begin{bmatrix}22+6+16\\8+0+8\\-12 \quad 3-8\end{bmatrix}_\beta = \boxed{\begin{bmatrix}44\\16\\-23\end{bmatrix}_\beta}$$

Another way:

$$[Tv]_\beta = [T]_\beta\,[v]_\beta$$

Now $$[v]_\beta = P[v]_\alpha = \begin{bmatrix}-11 & 2 & 2\\-4 & 0 & 1\\6 & -1 & -1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

+.5

$$= \begin{bmatrix}-11+4+6\\-4+0+3\\6-2-3\end{bmatrix} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}_\beta$$

Then $$[Tv]_\beta = \begin{bmatrix}81 & 53 & 178\\29 & 19 & 64\\-43 & -29 & -95\end{bmatrix}\begin{bmatrix}-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-81-53+178\\-29-19+64\\43+29-95\end{bmatrix} = \begin{bmatrix}44\\16\\-23\end{bmatrix}_\beta$$

+.5

Note: 1 In all parts, if they make calculation mistake but their method is correct, give proportional marks.

2 In (b), some students may try to find $T(1,2,4)$, $T(0,-1,1)$ and $T(2,3,8)$ and write them as a linear combination of $(1,2,4)$, $(0,-1,1)$ and $(2,3,8)$.

This way, they can calculate $[T]_\beta$ but ~~however~~ the calculation is not easy.

If however, some of the students get the correct matrix, they get full credit & in case proportional marks is justified, please award proportional marks.

3 In part (c) also, without explicitly using P they can calculate the coordinate $[T v]_\beta$ by solving system of equations. If their method & answer is correct, they get full credit.

For correct method but calculation mistake, they should get proportional credit.