MTH 100: Lecture 35

Inner Products:

Definition: An inner product on a (real) vector space V is a function, that to each pair of vectors u and v in V associates a scalar (real number) $\langle u, v \rangle$ and satisfies the following axioms:

1. $\langle u, v \rangle = \langle v, u \rangle \quad \forall u, v \in V$
2. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \forall u, v, w \in V$
3. $\langle cu, v \rangle = c\langle u, v \rangle \quad \forall u, v \in V$ and $\forall c \in \mathbb{R}$
4. $\langle u, u \rangle \geq 0 \quad \forall u \in V$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

- A vector space with an inner product is called an inner product space.

Note: The above definition holds for real inner products. For complex inner products, the first axiom above becomes:

$$\langle u, v \rangle = \overline{\langle v, u \rangle} \quad (\text{the complex Conjugate})$$

Definition: If we regard u, v in $\mathbb{R}^n$ as $n \times 1$ matrices (column vectors), then the transpose u^T is a $1 \times n$ matrix (row vector). Then the matrix product $u^T v$ is a 1×1 matrix which is a real number. This real number is called the inner product or dot product and is written as $u \cdot v$

Examples of Inner Product space:

(1) Let $V = \mathbb{R}^n$

For $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

define $\langle u, v \rangle = u \cdot v = u^T v$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Then the above is an inner product.

(1) $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

$$= [v_1 \; v_2 \; \cdots \; v_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \langle v, u \rangle$$

$\forall \; u, v \in \mathbb{R}^n$

(2) If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ are in $\mathbb{R}^n$,

then
$$\langle u+v, w\rangle = \left\langle \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \right\rangle$$
$$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + \cdots + (u_n+v_n)w_n$$
$$= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n)$$
$$= \langle u, w\rangle + \langle v, w\rangle$$

(3) For $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$

and $c \in \mathbb{R}$,
$$\langle cu, v\rangle = \left\langle c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\rangle$$
$$= \left\langle \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\rangle = (cu_1)v_1 + (cu_2)v_2 + \cdots + (cu_n)v_n$$
$$= c(u_1v_1) + c(u_2v_2) + \cdots + c(u_nv_n)$$
$$= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$$
$$= c\langle u, v\rangle$$

(4) For any $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$,

$$\langle u, u \rangle = u_1 u_1 + u_2 u_2 + \cdots + u_n u_n$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$$

Furthermore

$$\langle u, u \rangle = 0 \iff u_1^2 + u_2^2 + \cdots + u_n^2 = 0$$
$$\iff u_1 = 0, u_2 = 0, \ldots, u_n = 0$$
$$\iff u = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \bar{0} \text{ (zero vector)}$$

Therefore the product defined is an inner product.

<u>Example:</u>

The space $\mathbb{R}_n[t]$ of all polynomials of degree less than or equal to n can be made into an inner product space in the following way:

Let $t_0, t_1, t_2, \cdots, t_n$ be distinct real numbers (<u>note:</u> There are $(n+1)$ numbers).

For any two polynomials p and q in $\mathbb{R}_n[t]$, define $\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)$

It can be shown that the four axioms for an inner product hold with the above definition. (exercise)

Why are $(n+1)$ points taken?

Note: The above inner product for polynomials is used when the values at specific points are important. (Interpolation problems)

Example: The space $C[a,b]$ of all continuous functions on the closed interval $[a,b]$ can be made into an inner product space with the following definition

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt$$

It can be verified that the four axioms for an inner product hold with the above definition (exercise).

Note: The above inner product plays a very important role in the study of continuous functions and their applications in signals and systems.

Length and Distance in Inner Product Spaces

Definition: The length or norm of any vector u in an inner product space is the non-negative number $\|u\| = \sqrt{\langle u, u\rangle}$

Note: In the case of $\mathbb{R}^n$, we get the length or norm as the nonnegative number

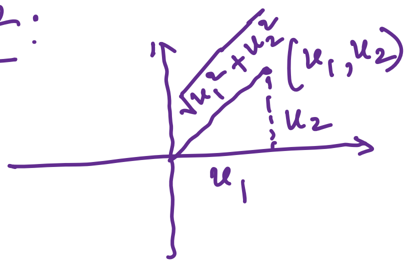
$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

This coincides with the usual notion of length as the distance from the origin to the point (u_1, u_2) or (u_1, u_2, u_3) in $\mathbb{R}^2$ or $\mathbb{R}^3$.

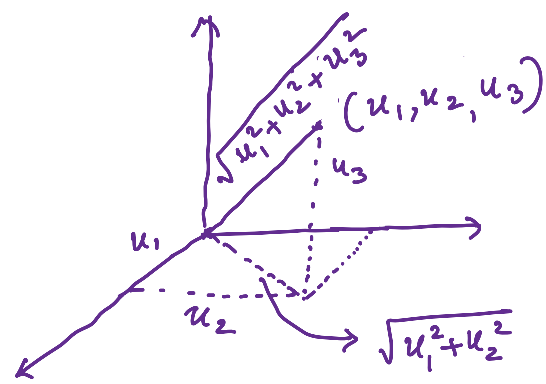
We can easily see that

for any scalar c, $\|cu\| = |c|\,\|u\|$

$\mathbb{R}^2$:



$\mathbb{R}^3$:



$$\|u\| \neq 0 \quad \left\| \frac{1}{\|u\|} \cdot u \right\| = \frac{1}{\cancel{\|u\|}} \cancel{\|u\|} = 1$$

- A vector whose length is one is called unit vector.

Given any non-zero vector u, the vector $\frac{u}{\|u\|}$ has norm one : This is called normalizing

- The distance between any two vectors u and v in V is defined as

$$\text{dist}(u,v) = \|u-v\| = \sqrt{\langle u-v,\, u-v\rangle}$$

(check in $\mathbb{R}^2$ and $\mathbb{R}^3$)

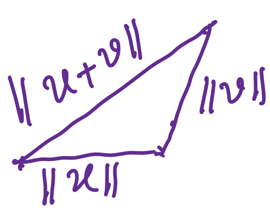
Proposition: (The Cauchy-Schwarz inequality)

If V is an inner product space, for all $u, v \in V$, $|\langle u,v\rangle| \leq \|u\|\,\|v\|$

(check in $\mathbb{R}^2$ and $\mathbb{R}^3$)

Proposition (The triangle inequality)

If V is an inner-product space, then for all $u, v \in V$, $\|u+v\| \leq \|u\| + \|v\|$



- The triangle inequality can be proved using Cauchy Schwarz inequality.

If $u, v \in V$, then

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + \langle u,v\rangle + \langle u,v\rangle + \|v\|^2$$

$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{(By Cauchy-Schwarz inequality)}$$

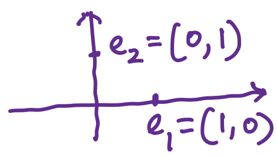
$$= (\|u\| + \|v\|)^2$$

$$\Rightarrow \boxed{\|u+v\| \leq \|u\| + \|v\|}$$

Orthogonality: Two vectors u, v in V are called orthogonal to each other if $\langle u, v \rangle = 0$

Notation for Orthogonality : $u \perp v$

Ex: In $\mathbb{R}^2$



$\langle e_1, e_2 \rangle = 1\times 0 + 0 \times 1 = 0+0 = 0$

Hence $e_1 \perp e_2$

Note: The zero vector is orthogonal to every vector in V.

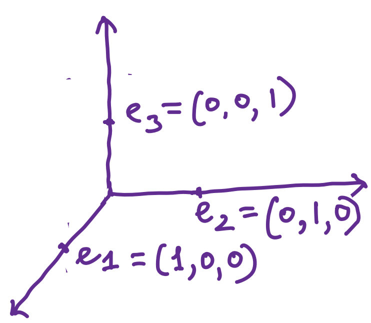
For every $u \in V$, $\langle \underset{\substack{\downarrow \\ \text{The zero vector}}}{0}, u \rangle = \langle \underset{\substack{\downarrow \\ \text{scalar zero}}}{0} . u , u \rangle = 0 \langle u, u \rangle = 0$

$\Rightarrow \boxed{0 \perp u \ \ \forall u \in V}$

Definition: A set of vectors $\{u_1, u_2, \dots, u_p\}$ is said to be an Orthogonal set if any two distinct vectors in the set are orthogonal to each other

i.e. if $\langle u_i, u_j \rangle = 0$ whenever $i \neq j$

Ex: In $\mathbb{R}^3$



In $\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ is an orthogonal set.

Proposition: An orthogonal set of <u>nonzero</u> vectors in V is linearly independent.

Proof: Suppose $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of **nonzero** vectors and suppose $c_1 u_1 + c_2 u_2 + \cdots + c_p u_p = 0$

Then $\langle c_1 u_1 + c_2 u_2 + \cdots + c_p u_p, u_1 \rangle = \langle 0, u_1 \rangle = 0$

$$\Rightarrow c_1 \langle u_1, u_1 \rangle + c_2 \langle u_2, u_1 \rangle + \cdots + c_p \langle u_p, u_1 \rangle = 0$$

$$\Rightarrow c_1 \langle u_1, u_1 \rangle + c_2 \times 0 + \cdots + c_p \times 0 = 0$$

(Since $\langle u_i, u_1 \rangle = 0$ for $i = 2, \ldots p$)

$$\Rightarrow c_1 \langle u_1, u_1 \rangle = 0$$

Since $\langle u_1, u_1 \rangle > 0$, we get $c_1 = 0$

Similarly $c_2 = c_3 = \cdots = c_p = 0$

Therefore the set $S = \{u_1, u_2, \ldots, u_p\}$ is linearly independent.