MTH100: Lecture 25

Proposition:

(a) A linear transformation $T: V \longrightarrow W$ is completely determined by its action on a basis of V.

(b) Conversely, given a basis $B = \{v_1, \ldots, v_n\}$ of V, and a list of n vectors $w_1, \ldots, w_n$ (not necessarily distinct) in the co-domain space W, there is a unique linear transformation T such that $T(v_1) = w_1$, $T(v_2) = w_2, \ldots, T(v_n) = w_n$.

Proof: Exercise

Outline of a proof:

(a) If $\{v_1, \ldots, v_n\}$ is a Basis of V, then any arbitrary vector $v \in V$ can be written as a linear combination of $v_1, \ldots, v_n$.

Thus there exist $c_1, \ldots, c_n \in V$ s.t. $v = c_1 v_1 + \cdots + c_n v_n$

$\Rightarrow T(v) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n)$

Thus the image of any arbitrary vector under T is a linear Combination of $Tv_1, \ldots, Tv_n$

i.e. T is completely determined by $Tv_1, \ldots, Tv_n$

(b) First note that the transformation T defined by $Tv_i = w_i$ for $i = 1, 2, \ldots, n$ is a linear transformation (check!!)

Now if there are two linear transformations T_1 & T_2 with the same properties i.e. $T_1(v_i) = w_i$, $T_2(v_i) = w_i$ for $i = 1, 2, \ldots, n$

then for any arbitrary vector $v = c_1 v_1 + \cdots + c_n v_n$, $c_1, \ldots, c_n \in F$

$T_1(v) = T_1(c_1 v_1 + \cdots + c_n v_n) = c_1 T_1 v_1 + \cdots + c_n T_1 v_n = c_1 T_2 v_1 + \cdots + c_n T_2 v_n = T_2(c_1 v_1 + \cdots + c_n v_n)$

Thus $T_1 = T_2$ and so such T is unique.

Rank of a Linear Transformation:

- For the time being, we will assume V to be finite-dimensional.

Definition: Let $T: V \longrightarrow W$ be a linear transformation. Then Rank of T is defined to be the $\boxed{\text{dimension of the Range of } T}$

Remark:
Range(T) is finite dimensional and

$$\dim(\text{Range}(T)) \leq \dim V$$

Thus definition of Rank(T) is valid.

Proof: Use the previous proposition:
Given that V is finite dimensional.

Let $B = \{v_1, \ldots, v_n\}$ is a Basis for V.

Let $Tv_1 = w_1, \ldots, Tv_n = w_n$

Then $w_1, w_2, \ldots, w_n$ Span Range T:

Let $w \in$ Range T, Then there exists an $v \in V$ s.t. $Tv = w$, Since B is a basis of V, there exist scalars $c_1, \ldots, c_n$ s.t. $v = c_1v_1 + \cdots + c_nv_n$

Now $w = Tv = T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)$
$= c_1w_1 + \cdots + c_nw_n$

Therefore $\boxed{\dim(\text{Range}(T)) \leq \dim(V) = n}$

Recall from Last time:

- For a linear transformation $T: V \to W$, we defined $\ker T = \text{Null}\, T = \{ v \in V : T(v) = 0 \}$ and showed that it is a subspace of V.
- If $\ker T$ is finite-dimensional, then $\dim(\ker T)$ is called the $\boxed{\text{nullity of } T}$

Theorem (Rank Theorem for Linear Transformations):

Suppose that $T: V \to W$ is a linear transformation and V is finite dimensional.

Then $\boxed{\text{Rank}(T) + \text{nullity}(T) = \dim V}$.

Note: We have already seen that if $T: V \to W$ is a linear transformation and V is finite-dimensional, then range T is also finite dimensional and

$\dim(\text{range}\, T) \leq \dim V$.

i.e. $\text{Rank}(T) \leq \dim V$.

Proof of the Rank Theorem:

- Assume that $\dim V = n$ and $\text{nullity}(T) = k$.
  Let $v_1, v_2, \dots, v_k$ be a basis of $\ker T$.
  Expand this to a basis B of V by inserting the additional vectors $v_{k+1}, \dots, v_n$.
- We will show that
  $T(v_{k+1}), \dots, T(v_n)$ form a basis for Range(T).

Firstly all the vectors $T(v_1), \dots, T(v_n)$ span Range (T)

Any element of Range(T) is of the form $T(v)$ for some $v \in V$.

Since $v_1, \dots, v_n$ form a basis of V, there exist scalars $c_1, \dots, c_n \in F$ such that $v = c_1 v_1 + \dots + c_n v_n$

$$\Rightarrow \quad T(v) = T(c_1 v_1 + \dots + c_n v_n)$$
$$= c_1 T(v_1) + \dots + c_n T(v_n)$$

Since, $T(v_1) = T(v_2) = \dots = T(v_k) = 0$,

actually $T(v_{k+1}), \dots, T(v_k)$ span Range T.

Now suppose that

$$c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \cdots + c_nT(v_n) = 0$$

$$\Rightarrow T(c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_nv_n) = 0$$

$$\Rightarrow c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_nv_n \in \ker T$$

So, there exist scalars $b_1, \ldots, b_k$ such that $c_{k+1}v_{k+1} + \cdots + c_nv_n = b_1v_1 + \cdots + b_kv_k$

$$\Rightarrow b_1v_1 + \cdots + b_kv_k - c_{k+1}v_{k+1} - \cdots - c_nv_n = 0$$

Since $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ form a basis of V, they are linearly independent and hence $c_{k+1} = \cdots = c_n = 0$

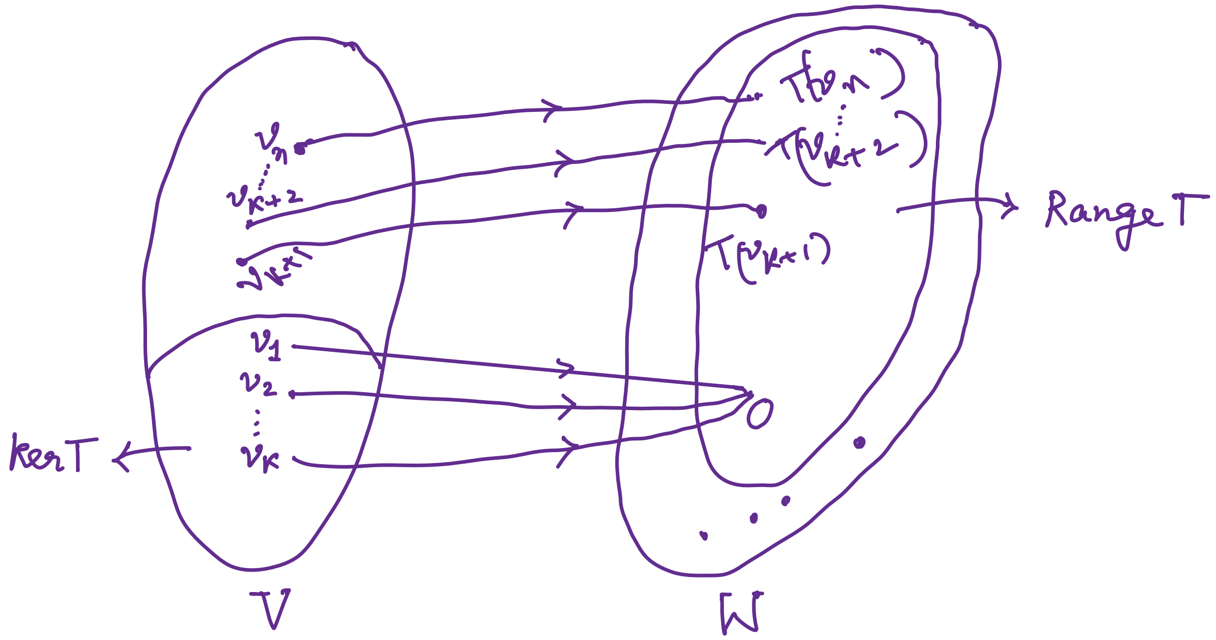
Thus $T(v_{k+1}), \ldots, T(v_n)$ form a basis of Range T.

Now $\operatorname{Rank} T = \dim(\operatorname{Range} T) = n-k$

Hence $\operatorname{Rank} T + \operatorname{nullity} T = n-k+k$

$$= n = \dim V$$

(QED)



RangeT
kerT
V
W