13 Let V be an inner product space and S be a finite subset of V.

Let $S^\perp = \{v \in V : \langle v, u\rangle = 0 \ \forall u \in S\}$

Then $S^\perp$ is a subspace of V

- Firstly, $\langle \bar{0}, u\rangle = 0 \ \forall u \in S$

$\Rightarrow \bar{0} \in S^\perp$

- If $v_1, v_2 \in S^\perp$ then $\left.\begin{array}{l}\langle v_1, u\rangle = 0 \\ \langle v_2, u\rangle = 0\end{array}\right\} \forall u \in S$

$\Rightarrow \langle v_1 + v_2, u\rangle = \langle v_1, u\rangle + \langle v_2, u\rangle$

$\Rightarrow v_1 + v_2 \in S^\perp \qquad = 0 + 0 = 0 \quad \forall u \in S$

- If $v \in S^\perp$ and $c \in F$ (scalar)

then $\langle cv, u\rangle = c\langle v, u\rangle = 0 \ \forall u \in S$

$\Rightarrow cv \in S^\perp$

Hence $S^\perp$ is a subspace of V

Now let $W = \text{Span}(S)$

Then we will show that $\boxed{S^\perp = W^\perp}$

Now $S \subset W$

If $v \in W^\perp \Rightarrow \langle v, u\rangle = 0 \ \forall u \in W \Rightarrow \langle v, u\rangle = 0$
$\forall u \in S$
(since $S \subset W$)

$\Rightarrow v \in S^\perp$

Thus $\boxed{W^\perp \subset S^\perp}$

Conversely let $v \in S^{\perp}$

Let $u \in W = \text{Span}(S)$

Then there exist $u_1, u_2, \ldots, u_n \in S$

s.t. $u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$

$$\Rightarrow \langle v, u \rangle = \langle v, c_1 u_1 + \cdots + c_n u_n \rangle$$
$$= c_1 \langle v, u_1 \rangle + \cdots + c_n \langle v, u_n \rangle$$
$$= 0 + \cdots + 0 = 0 \quad \forall u \in W$$

$\Rightarrow v \in W^{\perp}$

So, $\boxed{S^{\perp} \subset W^{\perp}}$

Combining $\boxed{S^{\perp} = W^{\perp}}$

(14) $A \in \mathbb{R}^{m\times n}$ ($m\times n$ matrix with real entries) (27)

Let $A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$

i.e. $v_1^T, v_2^T, \ldots, v_m^T$ are the rows of A.

Now if $\bar{y} \in \mathbb{R}^n$ be such that $\bar{y} \in (\text{Row} A)^\perp$

then $v_1^T \bar{y} = 0,\ v_2^T \bar{y} = 0, \ldots, v_m^T \bar{y} = 0$

$$\Rightarrow A\bar{y} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \cdot \bar{y} = \bar{0}$$

$\Rightarrow \bar{y} \in \text{Nul} A \quad \Rightarrow (\text{Row} A)^\perp \subset \text{Nul} A$

Conversely if $\bar{y} \in \text{Nul} A$, then $A\bar{y} = \bar{0}$

$$\Rightarrow \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \cdot \bar{y} = 0 \Rightarrow v_1^T \bar{y} = 0,\ v_2^T \cdot \bar{y} = 0, \ldots \cdots y_m^T \bar{y} = 0$$

Now if $\bar{x} \in (\text{Row} A)$, then $\exists\ c_1, c_2, \ldots c_m \in \mathbb{R}$

s.t. $\bar{x} = c_1 v_1^T + c_2 v_2^T + \cdots + c_m v_m^T$

Then $\langle \bar{x}, \bar{y} \rangle = (c_1 v_1^T + c_2 v_2^T + \cdots + c_m v_m^T) \cdot \bar{y}$

$= c_1 v_1^T \bar{y} + c_2 v_2^T \cdot \bar{y} + \cdots + c_m v_m^T \cdot \bar{y} = \bar{0}$

Hence $\bar{y} \perp \bar{x} \Rightarrow \bar{y} \in (\text{row } A)^{\perp}$

So, $\text{Nul } A \subset (\text{Row } A)^{\perp}$

Hence $\boxed{\text{Nul } A = (\text{Row } A)^{\perp}}$

15 (a) Let V be a complex inner product space.

Then $\langle u, v\rangle = \overline{\langle v, u\rangle}$

Now $\langle u, cv\rangle = \overline{\langle cv, u\rangle}$

$\Rightarrow \langle u, cv\rangle = \overline{c\langle v, u\rangle}$

$\Rightarrow \langle u, cv\rangle = \bar{c}\,\overline{\langle v, u\rangle}$

$\Rightarrow \boxed{\langle u, cv\rangle = \bar{c}\langle u, v\rangle}$

(Since $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \quad \forall z_1, z_2 \in \mathbb{C}$)

(b) For $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$

We define $\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$

Now $\langle x, y\rangle = \sum_{i=1}^{n} x_i \bar{y}_i = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$

$= \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n$

$= \bar{y}_1 \bar{\bar{x}}_1 + \cdots + \bar{y}_n \bar{\bar{x}}_n = \overline{y_1 \bar{x}_1 + \cdots + y_n \bar{x}_n}$

$= \overline{\sum_{i=1}^{n} y_i \bar{x}_i} = \overline{\langle y, x\rangle}$

So, $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall\, x, y \in \mathbb{C}^n$

- Now $\langle x+y, z \rangle = \sum_{i=1}^{n} (x+y)_i \overline{z_i}$

$$= \sum_{i=1}^{n} (x_i + y_i) \overline{z_i} = \sum_{i=1}^{n} (x_i \overline{z_i} + y_i \overline{z_i})$$

$$= \sum_{i=1}^{n} x_i \overline{z_i} + \sum_{i=1}^{n} y_i \overline{z_i}$$

$$= \langle x, z \rangle + \langle y, z \rangle \quad \forall\, x, y, z \in \mathbb{C}^n$$

(can also write explicitly without summation notation)

- $\langle cx, y \rangle = \sum_{i=1}^{n} (cx)_i \overline{y_i} = \sum_{i=1}^{n} c\, x_i \overline{y_i}$

$$= c \sum_{i=1}^{n} x_i \overline{y_i} = c \langle x, y \rangle \quad \forall\, x, y \in \mathbb{C}^n$$

$\quad$ & $\forall\, c \in \mathbb{C}$ (scalar)

- $\langle x, x \rangle = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2$

$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$$

and $\langle x, x \rangle = 0 \Rightarrow |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 0$

$$\Rightarrow |x_1| = 0,\ |x_2| = 0,\ \ldots,\ |x_n| = 0$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$