Solution for Tutorial 1

1. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 1 \end{bmatrix}$

$\xrightarrow{R_2 \to \frac{1}{4} R_4} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 5 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix}$

$\xrightarrow{R_3 \to \left(-\frac{2}{3} R_3\right)} \begin{bmatrix} 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow[R_1 \to R_1 - \frac{3}{2} R_3]{R_2 \to R_2 - \frac{1}{2} R_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow$ RREF matrix

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$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$
$R_3 \to R_3 - 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & \frac{7}{3} & -\frac{10}{3} \end{bmatrix} \xrightarrow{R_3 \to \frac{3}{7}R_3} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix}$$

$R_2 \to R_2 + \frac{4}{3}R_3$
$R_1 \to R_1 - \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \to \text{RREF matrix}$$

(3) 2×2 RREF matrices will be of the form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} \; x \in \mathbb{R}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Similarly for 2×3 and 3×3 RREF matrices.

(4) A relation T on $\mathbb{R}$ is defined by:

$x \, T \, y$ if $y - x \in \mathbb{Z}$.

(a) Reflexivity Since $a - a = 0 \in \mathbb{Z} \;\; \forall \, a \in \mathbb{R}$,

$a \, T \, a \;\; \forall \, a \in \mathbb{R}$

(b) Symmetry: Let $a, b \in \mathbb{R}$ s.t. $a \, T \, b$

So, $b - a \in \mathbb{Z} \Rightarrow -(b-a) \in \mathbb{Z}$

$\Rightarrow a - b \in \mathbb{Z} \Rightarrow b \, T \, a$

(c) Transitivity: Let $a, b, c \in \mathbb{R}$ be such that $a \, T \, b$ and $b \, T \, c$.

$\Rightarrow b - a \in \mathbb{Z}, \; c - b \in \mathbb{Z} \Rightarrow (c-b) + (b-a) \in \mathbb{Z}$

$\Rightarrow c - a \in \mathbb{Z} \Rightarrow a \, T \, c$

So, T is an equivalence relation.

Since any real number can be written as the sum of an integer and a real number in $[0, 1)$,
Each equivalence class contains exactly one real number $r \in [0, 1)$

This is the special representative of the equivalence class.

5 Let $\mathbb{R}^{m \times n}$ be the set of all $m \times n$ matrices with real entries.
For any $A, B \in \mathbb{R}^{m \times n}$, We define ARB if A is row equivalent to B

(a) Reflexivity:
If $A \in \mathbb{R}^{m \times n}$, then A is row equivalent to itself

$\Rightarrow ARA$

(b) Symmetry:
Let $A, B \in \mathbb{R}^{m \times n}$ and ARB.
$\Rightarrow$ A is row equivalent to B
$\Rightarrow$ there exist elementary row operations $e_1, \dots, e_p$ s.t. $B = (e_p \cdots e_2 e_1) A$

$\Rightarrow$ there exists inverse elementary row operations $f_1, f_2, \dots, f_p$ s.t.
$A = (f_1 \cdots f_{p-1} f_p) B \Rightarrow BRA$

(c) Transitivity:

Let $A, B, C \in \mathbb{R}^{m \times n}$ s.t ARB and BRC.

So, there exists elementary row operations

$$e_1, \dots e_p \text{ s.t. } B = (e_p \dots e_1) A$$

and $e_1', \dots, e_m'$ s.t. $C = (e_m', \dots e_1') B$

Then $C = (e_m' \dots e_1' e_p \dots e_1) A$

$\Rightarrow ARC$

So, R is an equivalence relation.

(6) Let E be an equivalence relation on a set X.

Let [a] and [b] are two distinct equivalence classes under E.

If [a] and [b] are not disjoint,

there exists an element $c \in X$ s.t.

$$c \in [a] \cap [b]$$

So, aEc and bEc

$\Rightarrow aEc$ and cEb (By symmetry)

$\Rightarrow aEb$ (By transitivity)

$\Rightarrow bEa$ (By Symmetry)

Now if $x \in [b]$, then xEb and bEa

$\Rightarrow xEa \Rightarrow x \in [a]$

So, $[b] \subseteq [a]$.

Similarly $[a] \subseteq [b]$

Hence $[a] = [b]$, a contradiction to the fact that $[a]$ and $[b]$ are distinct equivalence classes.

Thus $[a]$ and $[b]$ are disjoint.

Also note that every element a of X belongs to the equivalence class $[a]$ (Since $a E a$ as E is reflexive)

7 Let $\mathcal{P} = \{ P_i : i \in I \}$ where I is some index set be a partition of the set X.

i.e. $X = \bigcup_{i \in I} P_i$; $P_i \cap P_j = \phi$ for $i \neq j$

Define a relation E on X by

$x E y$ if and only if $x, y \in P_i$ for some i.

Now E is an equivalence relation.

(a) Reflexivity: For any $x \in X$, there exists an $i \in I$ s.t. $x \in P_i$ So, $x E x$.

(b) Symmetry:

Suppose $x, y \in X$ be s.t. $x E y$

Then $\exists$ some fixed $i \in I$ s.t.

$x, y \in E_i \Rightarrow y E x$

(c) Transitivity:

Suppose $x, y, z \in X$ be s.t. $x E y$ and $y E z$

Then $x, y \in P_i$ for some $i \in I$

and $y, z \in P_j$ for some $j \in I$

But then $y \in P_i \cap P_j \Rightarrow i = j$

(as the ~~...~~ parts of the partitions are disjoint)

$\Rightarrow x, z \in P_i$

$\Rightarrow x E z$

Now suppose $x \in X$, and consider $[x]$. Let $y \in [x]$

Then $x \in P_i$ for some $i \in I$

$y \in [x] \Rightarrow x E y \Rightarrow y \in P_i$

$\Rightarrow [x] \subset P_i$

Conversely if P_i is a part of P,

then $x \in P_i$ for some $x \in X$.

Now if $y \in P_i$, then $y E x \Rightarrow y \in [x]$

$\Rightarrow P_i \subset [x]$

So, $P_i = [x]$. Thus $[x]$ corresponds to a part of the partition P.

So, The equivalence classes correspond to the parts of the partition P.

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$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 7 & -7 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 7R_2]{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 7/3 & -\frac{10}{3} \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{3}{7}R_3} \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix} \xrightarrow[R_2 \to R_2 + \frac{4}{3}R_3]{R_1 \to R_1 - \frac{1}{3}R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix}}_{\text{(RREF matrix)}}$$

So, the corresponding system of equation is:

$$\left.\begin{aligned} x_1 + \frac{15}{7}x_4 &= 0 \\ x_2 - \frac{4}{7}x_4 &= 0 \\ x_3 - \frac{10}{7}x_4 &= 0 \\ x_4 &= x_4 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -\frac{15}{7}x_4 \\ x_2 &= +\frac{4}{7}x_4 \\ x_3 &= \frac{10}{7}x_4 \\ x_4 &= x_4 \end{aligned}$$

So, the set of solution

$$= \left\{ x_4 \begin{bmatrix} -15/7 \\ 4/7 \\ 10/7 \\ 1 \end{bmatrix} : x_4 \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{bmatrix} -\frac{15}{7} \\ 4/7 \\ 10/7 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ tu : u \in \mathbb{R} \right\}$$

where $u = \begin{bmatrix} -15/7 \\ 4/7 \\ 10/7 \\ 1 \end{bmatrix}$