MTH 100 : Lecture 19

Last time:

We defined Basis for a vector space and finite & infinite dimensional vector space.

Recall that: A basis for a vector space V is a linearly independent set S of vectors such that $\text{span}(S) = V$

Alternative Definition for Basis:

Proposition: $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of the vector space V if and only if every vector $v \in V$ is uniquely expressible as a linear combination of the elements of B.

Note: In some books, the above is used as the definition of a Basis, and then it is shown that a Basis is a linearly independent spanning set.

Proof: exercise

Proposition (Steinitz Exchange Lemma):

Suppose $v_1, v_2, \ldots, v_n$ are linearly independent vectors in a vector space V and suppose $V = \text{Span}\{w_1, w_2, \ldots, w_m\}$

Then (a) $n \leq m$

(b) $\{v_1, v_2, \ldots, v_n, w_{n+1}, w_{n+2}, \ldots, w_m\}$ span V after reordering the w's if necessary.

Proposition: If V is a finite dimensional vector space, then any two bases of V have the same number of elements.

Proof: Let B_1 and B_2 be two bases of V with k_1 and k_2 vectors respectively.

Want to show: $k_1 = k_2$

Now B_1 is a linearly independent set of vectors in V and B_2 is a spanning set of V.

So, by Steinitz exchange lemma, $k_1 \leq k_2$

Now B_2 is a linearly independent set of vectors in V and B_1 is a spanning set of V.

So, by Steinitz exchange lemma, $k_1 \leq k_2$. Hence $\boxed{k_1 = k_2}$

Definition: The dimension of a finite dimensional vector space is the number of elements in a basis for V. This is written as $\boxed{\dim(V)}$

Note: The above proposition ensures that this is a proper definition.

Example: $\boxed{\dim(\mathbb{R}^n) = n}$ (Recall: $e_1, e_2, \ldots, e_n$ is a basis of $\mathbb{R}^n$)

Special case: The dimension of the zero subspace of any vector space is taken as zero.
(It doesn't have a basis)

Proof of Steinitz Exchange Lemma

Given: V is a vector space
$v_1, v_2, \ldots, v_n$ is a linearly independent set of vectors
and $V = \mathrm{Span}\{w_1, w_2, \ldots, w_m\}$

Step I: Since $v_1 \in V$, we can write

$$v_1 = c_1 w_1 + c_2 w_2 + \cdots + c_m w_m \quad \cdots\cdots (1)$$

where $c_1, c_2, \ldots, c_m \in F$

If $c_i = 0 \ \forall i$, then $v_1 = 0$ which is not possible since $v_1, v_2, \ldots, v_n$ are linearly independent.

So, $c_i \neq 0$ for atleast one i

Renumbering if necessary, we can assume $c_1 \neq 0$.

Then (1) $\Rightarrow c_1 w_1 = v_1 - c_2 w_2 - \cdots - c_m w_m$

$$\Rightarrow c_1^{-1} c_1 w_1 = c_1^{-1} v_1 - c_1^{-1} c_2 w_2 - \cdots - c_1^{-1} c_m w_m$$

$$\Rightarrow w_1 = d_1 v_1 + d_2 w_2 + \cdots + d_m w_m \quad \cdots\cdots (2)$$

where $d_1, d_2, \ldots, d_m$ are scalars.

From here we can conclude

$$\text{Span}\{v_1, w_2, w_3, \ldots, w_m\} = \text{span}\{w_1, w_2, \ldots, w_m\} = V$$

Let $x \in V$.

Then $x = f_1 w_1 + f_2 w_2 + \cdots + f_m w_m$ for scalars $f_1, f_2, \ldots, f_m \in F$.

$$= f_1 (d_1 v_1 + d_2 w_2 + \cdots + d_m w_m) + f_2 w_2 + \cdots + f_m w_m$$

$$= f_1 d_1 v_1 + (f_1 d_2 + f_2) w_2 + \cdots + (f_1 d_m + f_m) w_m$$

$$= h_1 v_1 + h_2 w_2 + \cdots + h_m w_m$$

$$\in \text{span}\{v_1, w_2, \ldots, w_m\}$$

$$\text{span}\{v_1, w_2, \ldots, w_m\} = V$$

Step II: Since $v_2 \in V = \text{Span}\{v_1, w_2, \ldots, w_m\}$, we can write $v_2 = l_1 v_1 + l_2 w_2 + \cdots + l_m w_m$

where $l_1, l_2, \ldots, l_m \in F$

Now atleast one of $l_2, l_3, \ldots, l_m$ is nonzero.

Otherwise $v_2 = l_1 v_1$ that contradicts the fact that $v_1, v_2, \ldots, v_n$ are linearly independent.

Renumbering if necessary, we can assume that $l_2 \neq 0$

Then $l_2 w_2 = -l_1 v_1 + v_2 - l_3 w_3 - \cdots - l_m w_m$

$$\Rightarrow w_2 = -l_2^{-1} l_1 v_1 + l_2^{-1} v_2 - l_2^{-1} l_3 w_3 - \cdots l_2^{-1} l_m w_m$$

Proceeding as before we can conclude

$$\text{Span}\{v_1, v_2, w_3, \ldots, w_m\} = \text{Span}\{v_1, w_2, w_3, \ldots, w_m\}$$
$$= \text{Span}\{w_1, w_2, w_3, \ldots, w_m\} = V$$

This process will stop after the nth step atmost (since there are only n vectors $v_1, v_2, \ldots, v_n$)

Now we can think of two situations.

<u>Case 1:</u> $n \leq m$

Then we are in the following situation:

$$\{\overset{\overset{v_1}{\downarrow}}{w_1}, \overset{\overset{v_2}{\downarrow}}{w_2}, \ldots, \overset{\overset{v_n}{\downarrow}}{w_n}, w_{n+1}, \ldots w_m\}$$

We have replaced n of the w-vectors and renumbering if necessary we get

$$\text{span}\{v_1, v_2, \ldots, v_n, w_{n+1}, \ldots, w_m\} = V$$

In this case we proved the lemma.

<u>Note:</u> If $n = m$, then the vectors $w_{n+1}, \ldots$ etc. are not there in the original spanning set.

Case 2: $n > m$

Then we are in the following situation:

$$\begin{array}{ccccccc} v_1 & v_2 & \cdots & v_m, & v_{m+1}, & \cdots, & v_n \\ \downarrow & \downarrow & & \downarrow & & & \\ \{w_1, & w_2, & \cdots, & w_m\} & & & \end{array}$$

Now, $\{v_1, v_2, \cdots, v_m\}$ is a spanning set for V.

Then $v_{m+1} \in \text{Span}\{v_1, v_2, \cdots, v_m\}$

i.e. $v_{m+1} = p_1 v_1 + p_2 v_2 + \cdots + p_m v_m$

where $p_1, p_2, \cdots, p_m$ are scalars.

But this contradicts the linear independence of $\{v_1, v_2, \cdots, v_m, v_{m+1}, \cdots, v_n\}$.

Thus Case 2 cannot happen and in this case $n \le m$ and the lemma is proved.