MTH 100: Lecture 8

Corollary (1.2): If A has a left inverse or a right inverse then it has an inverse.

Note: ① B is a left inverse of A if $BA = I$
② D is a right inverse of A if $AD = I$

Proof: **Case ①:** Suppose A has a left inverse.

Then there exists a matrix B such that $BA = I$

Consider the homogeneous system $A\bar{x} = \bar{0}$

$$\Rightarrow B(A\bar{x}) = B.\bar{0}$$
$$\Rightarrow (BA)\bar{x} = \bar{0}$$
$$\Rightarrow I.\bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$$

So, $A\bar{x} = \bar{0}$ has only the trivial solution

By Theorem (1), A is invertible.

Furthermore, $BA = I$ (given)

$$\Rightarrow (BA)A^{-1} = I.A^{-1}$$ (we have shown that A^{-1} exists)

$$\Rightarrow B(AA^{-1}) = A^{-1}$$
$$\Rightarrow B.I = A^{-1}$$
$$\Rightarrow \boxed{B = A^{-1}}$$

Case (2): Suppose A has a right inverse.

Then there exists a matrix D such that

$$AD = I$$

So, A is a left inverse of D

Therefore by case (1), D is invertible

Now $(AD)D^{-1} = I . D^{-1}$

$$\Rightarrow A(DD^{-1}) = D^{-1}$$

$$\Rightarrow A . I = D^{-1}$$

$$\Rightarrow A = D^{-1}$$

Therefore A is the inverse of an invertible matrix D

So, A is invertible

and $A^{-1} = (D^{-1})^{-1}$

$$\Rightarrow \boxed{A^{-1} = D}$$

Corollary (1.3): Suppose a square matrix A can be factored as a product of square matrices i.e. $A = A_1 A_2 \cdots A_n$ with $n \geq 2$ (A_i's are all square matrices)

Then A is invertible if and only if each A_i is invertible.

Proof: '$\Leftarrow$':

If A_i's are all invertible then $A = A_1 A_2 \cdots A_n$ is also invertible

and $A^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ (By Observation 4)

'$\Rightarrow$': Given: A is invertible

To show: Each A_i is invertible

First we will show that A_n is invertible

Consider the homogeneous system

$$A_n \bar{x} = \bar{0}$$

$$\Rightarrow (A_1 A_2 \cdots A_{n-1}) A_n \bar{x} \quad (A_1 A_2 \cdots A_{n-1}) \bar{0}$$

$$\Rightarrow (A_1 A_2 \cdots A_{n-1} A_n) \bar{x} = \bar{0}$$

$$\Rightarrow \quad A\bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$$

(By Theorem 1 since A is invertible)

So, the homogeneous system $A_n \bar{x} = \bar{0}$ has only the trivial solution.

Therefore by Theorem 1, A_n is invertible.

Now $A_1 A_2 \cdots A_{n-1} A_n = A$

$\Rightarrow (A_1 A_2 \cdots A_{n-1} A_n) A_n^{-1} = A A_n^{-1}$ (since A_n^{-1} exists)

$\Rightarrow (A_1 A_2 \cdots A_{n-1}) A_n A_n^{-1} = A A_n^{-1}$

$\Rightarrow (A_1 A_2 \cdots A_{n-1}) I = A A_n^{-1}$

$\Rightarrow A_1 A_2 \cdots A_{n-1} = A A_n^{-1}$

Let $B = A A_n^{-1}$

Then B is an invertible matrix and

$$B = A_1 A_2 \cdots A_{n-1}$$

Repeating the same argument, we conclude that A_{n-1} is invertible.

Continuing the same process we can show that each A_i is invertible.

Final Part of Theorem 1 : (a) $\Leftrightarrow$ (d)

(a) $A_{m\times m}$ is invertible

(d) The non-homogeneous system $A\bar{x}=\bar{b}$ has atleast one solution for any choice of $\bar{b}\in\mathbb{R}^m$.

(a) $\Rightarrow$ (d) Given : A is invertible

To show: $A\bar{x}=\bar{b}$ has atleast one solution for any choice of $\bar{b}\in\mathbb{R}^m$.

Let $\bar{b}\in\mathbb{R}^m$ be any arbitrary but fixed vector.

Let $\bar{v}=A^{-1}\bar{b}$ (A^{-1} exists since A is invertible)

Since A^{-1} is a $m\times m$ matrix and $\bar{b}$ is a $m\times 1$ vector, $\bar{v}$ is a $m\times 1$ vector.

Now $A\bar{v}=A(A^{-1}\bar{b})$

$\Rightarrow A\bar{v}=(AA^{-1})\bar{b}$

$\Rightarrow A\bar{v}=I.\bar{b}$

$\Rightarrow A\bar{v}=\bar{b}$

So, $\bar{v}$ is a solution of the non-homogeneous system $A\bar{x}=\bar{b}$ as required.

(d) $\Rightarrow$ (a): Given: $A\bar{x} = \bar{b}$ has atleast one solution for any choice of $\bar{b} \in \mathbb{R}^m$

To show: A is invertible.

Let $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \in \mathbb{R}^m$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \in \mathbb{R}^m$, ... $\bar{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^m$

By the given condition
$A\bar{x} = \bar{e}_i$ has atleast one solution for $i = 1, 2, \ldots, m$

Let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m$ be the solutions.
i.e. $A\bar{v}_i = \bar{e}_i$ for $i = 1, 2, \ldots, n$

Let $B = [\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m]$

Then B is a $m \times m$ matrix
and

$$AB = A[\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m]$$
$$= [A\bar{v}_1, A\bar{v}_2, \ldots, A\bar{v}_m]$$
$$= [\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_m]$$

$$\Rightarrow AB = I_{m\times m}$$

So, B is a right inverse of A.
Now by corollary (1.2), A has an inverse and so, A is invertible. (QED)

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Note (Explanation): Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}_{m \times m}$

and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}_{m \times m}$

Now $AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1m}b_{m1} & a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1m}b_{m2} & \cdots & a_{11}b_{1m}+\cdots+a_{1m}b_{mm} \\ a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2m}b_{m1} & a_{21}b_{12}+a_{22}b_{22}+\cdots+a_{2m}b_{m2} & \cdots & a_{21}b_{1m}+\cdots+a_{2m}b_{mm} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{mm}b_{m1} & a_{m1}b_{12}+a_{m2}b_{22}+\cdots+a_{mm}b_{m2} & \cdots & a_{m1}b_{1m}+\cdots+a_{mm}b_{mm} \end{bmatrix}$$

$$= \left[\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} b_{1m} \\ b_{2m} \\ \vdots \\ b_{mm} \end{bmatrix} \right]$$

$$= \left[A \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \quad A \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{m2} \end{bmatrix} \cdots \cdots A \begin{bmatrix} b_{1m} \\ b_{2m} \\ \vdots \\ b_{mm} \end{bmatrix} \right]$$

$$= \left[A\overline{v}_1 \quad A\overline{v}_2 \quad \cdots \quad A\overline{v}_m \right]$$

where

$$\overline{v}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix} = \text{The } i\text{th column vector of the matrix } B$$

for $i = 1, 2, \ldots, m$

$\left(\text{i.e. } B = \left[\overline{v}_1 \quad \overline{v}_2 \quad \cdots \quad \overline{v}_m \right] \right)$

So, $AB = \left[A\overline{v}_1 \quad A\overline{v}_2 \quad \cdots \quad A\overline{v}_m \right]$

Corollary (1.4) (Alternative version of Last equivalence of Theorem 1)

A matrix A is invertible if and only if the system $A\bar{x}=\bar{b}$ has a unique solution for any choice of vector $\bar{b} \in \mathbb{R}^m$

Note: The proof of the implication:

"The system $A\bar{x}=\bar{b}$ has a unique solution for any choice of vector $\bar{b} \in \mathbb{R}^m$
$\Rightarrow$ A is invertible"

is exactly same as the proof of (d) $\Rightarrow$ (a) in Theorem 1.

Now in the converse part,
to prove that A is invertible $\Rightarrow$ $A\bar{x}=\bar{b}$ has a unique solution for any choice of vector $\bar{b} \in \mathbb{R}^m$,

first we need to prove existence of a solution of $A\bar{x}=\bar{b}$ for any choice of $\bar{b}\in\mathbb{R}^m$

in exactly the same way as $(a)\Rightarrow(d)$ in Theorem 1.

To prove the uniqueness of the solution

assume $A\bar{v}_1=\bar{b}$ and $A\bar{v}_2=\bar{b}$ be two such solutions.

Then $A(\bar{v}_1-\bar{v}_2)=A\bar{v}_1-A\bar{v}_2=\bar{b}-\bar{b}=\bar{0}$

$\Rightarrow A(\bar{v}_1-\bar{v}_2)=\bar{0}$

i.e. $\bar{v}_1-\bar{v}_2$ is a solution of the homogeneous system $A\bar{x}=\bar{0}$

Since A is invertible, by Theorem 1, the homogeneous system $A\bar{x}=\bar{0}$ has only the trivial solution.

Hence $\bar{v}_1-\bar{v}_2=\bar{0}\Rightarrow\bar{v}_1=\bar{v}_2$

i.e. the solution is unique.