Solution of [redacted] Worksheet 12

(1) $$A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[\lambda^2+\lambda+10\right] + (-1)\left[20-12\lambda-12\right]$$
$$+ (-1)\left[24+4\lambda\right]$$
$$= (3-\lambda)(\lambda^2+\lambda+10) - (8-12\lambda) - (24+4\lambda)$$
$$= 3\lambda^2+3\lambda+30-\lambda^3-\lambda^2-10\lambda-8+12\lambda-24-4\lambda$$
$$= -\lambda^3+2\lambda^2+\lambda-2$$
$$= -\left[\lambda^3-2\lambda^2-\lambda+2\right] = -\left[\lambda^2(\lambda-2)-1(\lambda-2)\right]$$
$$= -(\lambda-2)(\lambda^2-1) = -(\lambda-2)(\lambda+1)(\lambda-1)$$

So, the eigenvalues are $2, -1, 1$.

For $\lambda = 2$:

$$A-\lambda I = \begin{bmatrix} 3-2 & -1 & -1 \\ -12 & -2 & 5 \\ 4 & -2 & -1-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -12 & -2 & 5 \\ 4 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3-4R_1]{R_2 \to R_2+12R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -14 & -7 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{14}R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 2R_2}]{} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

So, the system $(A - 2I)X = 0 \Rightarrow \left.\begin{aligned} x_1 - \tfrac{1}{2}x_3 &= 0 \\ x_2 + \tfrac{1}{2}x_3 &= 0 \\ x_3 &= x_3 \end{aligned}\right\}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Taking $x_3 = 2$, we get an eigen vector as $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\underline{\lambda = 1}$

$$(A - \lambda I) = \begin{bmatrix} 3-1 & -1 & -1 \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + 12R_1 \\ R_3 \to R_3 - 4R_1}]{} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -7 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{7}R_2} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & +1 & \frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -\frac{3}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

So, the system becomes

$$\left.\begin{aligned} x_1 - \tfrac{3}{7}x_3 &= 0 \\ x_2 + \tfrac{1}{7}x_3 &= 0 \\ x_3 &= x_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} +\frac{3}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}$$

Taking $x_3 = 7$,

we get an eigenvector as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$

$\lambda = \cancel{8}\ -1$

$$(A - \lambda I) = \begin{bmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to -\frac{1}{2}R_2]{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & -\frac{1}{4} & -1/4 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 + \frac{1}{4}R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the system becomes

$$\left.\begin{aligned} x_1 - \tfrac{1}{2}x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_3 &= x_3 \end{aligned}\right\}$$

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

Now choosing $x_3 = 2$ we get an eigen vector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Now for each eigenvalues $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$, (4)

the geometric multiplicity = algebraic multiplicity $= 1$

So, A is diagonalizable.

(2) $$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

The characteristic Polynomial is

$$\det[A - \lambda I] = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(4-\lambda) + 18\right] + (-3)\left[18 - 3(4-\lambda)\right] + 3\left[-18 + 6(5+\lambda)\right]$$

$$= (1-\lambda)(-20 - 4\lambda + 5\lambda + \lambda^2 + 18) - 3(6+3\lambda) + 3(12+6\lambda)$$

$$= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda$$

$$= -\lambda^3 + 12\lambda + 16 = -\lambda^3 + 4\lambda^2 - 4\lambda^2 + 16\lambda - 4\lambda + 16$$

$$= -(\lambda - 4)(\lambda + 2)^2$$

Now for $\lambda_1 = 4$,

$$(A - \lambda I) = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow[\substack{R_2 \to \frac{1}{3}R_2 \\ R_3 \to \frac{1}{6}R_3}]{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & +1 & -1 \\ 1 & -3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{4}R_4} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Solving we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Taking $x_3 = 2$, we get an eigen vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Now for $\lambda_2 = -2$,

$$(A - \lambda_2 I) = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \xrightarrow[\substack{R_2 \to \frac{1}{3}R_2 \\ R_3 \to \frac{1}{6}R_3}]{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, we get the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, the two eigen vectors are (Taking $x_2 = 1$, $x_3 = 0$ & $x_2 = 0$, $x_3 = 1$)

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, the geometric multiplicity of λ_1 $=1$ = algebraic multiplicity of λ_1 is 1

the geometric multiplicity of λ_2 $=1$ = algebraic multiplicity of λ_2 is 2

So, A is diagonalizable

Infact $A = PDP^{-1}$ where $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

and $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(2)(b)

$$B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(B-\lambda I) = \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix}$$

$$= (-3-\lambda)\big[(5-\lambda)(-2-\lambda) - (6)(-1)\big]$$

$$+ 1\big[(-1)(-6) - (-7)(-2-\lambda)\big]$$

$$+ (-1)\big[(-7)(6) - (-6)(5-\lambda)\big]$$

$$= (-3-\lambda)\big[-10-5\lambda+2\lambda+\lambda^2+6\big]$$

$$+ \big[6-14-7\lambda\big] - \big[-42+30-6\lambda\big]$$

$$= -\lambda^3 + 12\lambda + 16$$

$$= -(\lambda+2)^2(\lambda - 4)$$

So, the eigen values are

$\lambda_1 = 4$ and $\lambda_2 = -2$

For $\lambda_1 = 4$:

$$(A-\lambda_1 I) = \begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{R_3 \to -\frac{1}{6}R_3} \begin{bmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ -7 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & 1 \\ -7 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + 7R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{6}R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\left.\begin{aligned} x_1 &= 0 \\ x_2 &= x_3 \\ x_3 &= x_3 \end{aligned}\right\}$ So, there is one eigen vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now $\lambda_2 = -2$

$$(A - \lambda_2 I) = \begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 7R_1} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow[R_2 \to \frac{1}{6}R_2]{R_1 \to (-1)R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - 6R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$
$x_3 = 0$

$$\left.\begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \\ x_3 = 0 \end{array}\right\} \text{There is only one free variable } x_2$$

So, corresponding eigen vector is

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence geometric multiplicity of λ_2 (which is 1) < algebraic multiplicity of λ_2 (which is 2)

Hence B is <u>not diagonalizable</u>.

(3) It is possible for A to be not diagonalizable.

For diagonalizability, Dimension of the eigen space corresponding to the third eigen value should be 2.

But it may turn out to be 1

In that case A will not be diagonalizable.

In other cases also A may not be diagonalizable.

<u>Case 2</u>: λ_1: geometric multiplicity $= 2$
algebraic multiplicity $= 3$

λ_2: geometric multiplicity
= algebraic multicity = 3

λ_3 : geometric multiplicity
= algebraic multicity = 1

(Total algebraic multiplicity = 7)

Case 3:

λ_1: algebraic multiplicity
= geometric multiplicity = 2

λ_2: geometric multiplicity = 3
$\neq$ algebraic multiplicity = 4

λ_3: geometric multiplicity
= algebraic multiplicity = 1

4 False:

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\det A = 1 \neq 0$

So, A is invertible and so row equivalent to the identity matrix

Then $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$

So, $\lambda = 1$ is the only eigen value of A with algebraic multiplicity 2.

For eigen vector, we consider

$(A-\lambda I)X = 0 \Rightarrow [A - 1.I]X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} x_1 = x_1 \\ x_2 = 0 \end{matrix}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(x_1 is the free variable)

So, dimension of the eigen space is 1 and the (basis) eigen vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now geometric multiplicity $= 1 <$ algebraic multiplicity $= 2$

Hence A is not diagonalizable.

5 $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 10$$

$$= \lambda^2 - 4\lambda + 13$$

So, eigen values are roots of

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

Let us take $\lambda = 2 + 3i$ so that $a = 2$, $b = -3$

Hence the matrix $B = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$

Now $A - \lambda I = \begin{bmatrix} 1-(2+3i) & 5 \\ -2 & 3-(2+3i) \end{bmatrix}$

$$= \begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix}$$

Now $(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \left.\begin{array}{l} (-1-3i)x + 5y = 0 \\ -2x + (1-3i)y = 0 \end{array}\right\}$

Both the equations represent the same relationship and so we get the second equation and putting $y = 2$ we get $x = (1-3i)$

So, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

Hence $P = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$

and $A = PBP^{-1}$

Now to express B as a rotation followed by scaling, $\left(|\lambda| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13} \right)$

$$B = \sqrt{13} \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}$$

Rotation is through an angle ϕ in the positive direction, where ϕ is the angle between the x-axis & the ray joining $(0,0)$ and $(2,-3)$

Infact $\phi = \sin^{-1}\left(\frac{-3}{\sqrt{13}}\right)$

(6) Eigen Values of the matrix

$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ are $2+3i$ and $2-3i$

The corresponding eigen vectors are

$\begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1-3i & 1+3i \\ 2 & 2 \end{bmatrix}$

and $D = P^{-1}AP = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$

or $\boxed{A = PDP^{-1}}$ where $D = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$

Check: $AP = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1-3i & 1+3i \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 11-3i & 11+3i \\ 4+6i & 4-6i \end{bmatrix}$

and $PD = \begin{bmatrix} 1-3i & 1+3i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix} = \begin{bmatrix} 11-3i & 11+3i \\ 4+6i & 4-6i \end{bmatrix}$

7. $V = C^{\infty}[\mathbb{R}]$, Then V is a vector space over $\mathbb{R}$,
$D: V \longrightarrow V$ is defined by $Df = f'$

If λ is any eigen value of D, then there exists a non zero function $f \in C^{\infty}[\mathbb{R}]$ Such that $Df = \lambda f \Rightarrow f'(x) = \lambda f(x)$

$$\Rightarrow \frac{df}{f} = \lambda \, dx$$

$$\Rightarrow \ln f(x) = \lambda x + c \Rightarrow f(x) = Ae^{\lambda x}$$
(where $A = e^{c}$)

Thus any real number $\lambda \in \mathbb{R}$ is an eigen value of D and the Corresponding eigen vector is $Ae^{\lambda x}$ (for any constant $A \in \mathbb{R}$)

(Note that $Ae^{\lambda x} \in C^{\infty}[\mathbb{R}]$ since it is infinitely differentiable)

8. Let $t_0, t_1, t_2, \ldots, t_n \in \mathbb{R}$ be distinct $(n+1)$ real numbers

$$\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)$$
$\forall \, p, q \in \mathbb{R}_n[t]$

- Now $\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)$

$$= q(t_0)p(t_0) + q(t_1)p(t_1) + \cdots + q(t_n)p(t_n)$$

$$= \langle q, p \rangle \quad \forall \, p, q \in \mathbb{R}_n[t]$$

- $$\langle p+q, r\rangle = (p+q)(t_0)r(t_0) + (p+q)(t_1)r(t_1) + \cdots\cdots + (p+q)(t_n)r(t_n)$$

$$= [p(t_0)+q(t_0)]r(t_0) + [p(t_1)+q(t_1)]r(t_1) + \cdots\cdots + [p(t_n)+q(t_n)]r(t_n)$$

$$= [p(t_0)r(t_0) + p(t_1)r(t_1) + \cdots\cdots + p(t_n)r(t_n)] + [q(t_0)r(t_0) + q(t_1)r(t_1) + \cdots + q(t_n)r(t_n)]$$

$$= \langle p, r\rangle + \langle q + r\rangle \quad \forall\ p, q, r \in \mathbb{R}_n[t]$$

- If $c \in \mathbb{R}$,

$$\langle cp, q\rangle = (cp)(t_0)q(t_0) + (cp)(t_1)q(t_1) + \cdots\cdots + (cp)(t_n)q(t_n)$$

$$= cp(t_0)q(t_0) + cp(t_1)q(t_1) + \cdots\cdots + cp(t_n)q(t_n)$$

$$= c[p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots\cdots + p(t_n)q(t_n)]$$

$$= c\langle p, q\rangle \quad \forall\ p, q \in R_n[t]$$

- Now $\langle p, p\rangle = p(t_0)p(t_0) + p(t_1)p(t_1) + \cdots\cdots + p(t_n)p(t_n)$

$$= [p(t_0)]^2 + [p(t_1]^2 + \cdots + [p(t_n)]^2 \geq 0$$

$$\forall\ p \in R_n[t]$$

Furthermore,

$$\langle p, p \rangle = 0 \Rightarrow [p(t_0)]^2 + [p(t_1)]^2 + \cdots + [p(t_n)]^2 = 0$$

$$\Rightarrow p(t_i) = 0 \quad \text{for } i = 0, 1, \ldots, n$$

$$\Rightarrow p(t) \equiv 0$$

A polynomial of degree $\leq n$ can have atmost n distinct zeros ~~and the~~ unless it is the zero polynomial.

Therefore to obtain the ~~...~~ last property of the inner product we need $(n+1)$ distinct points.

(9) Let $V = C[a, b]$

and $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt$

- Now $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt = \int_a^b g(t) f(t)\, dt$

$$= \langle g, f \rangle \quad \forall f, g \in C[a, b]$$

- $\langle f + g, h \rangle = \int_a^b (f + g)(t) h(t)\, dt$

$$= \int_a^b [f(t) + g(t)] h(t)\, dt = \int_a^b [f(t) h(t) + g(t) h(t)]\, dt$$

$$= \int_a^b f(t) h(t)\, dt + \int_a^b g(t) h(t)\, dt$$

$$= \langle f, h \rangle + \langle g, h \rangle \quad \forall f, g, h \in C[a, b]$$

- For any $c \in \mathbb{R}$ (or $\mathbb{F}$ (field))

$$\langle cf, g\rangle = \int_a^b (cf)(t)\, g(t)\, dt = \int_a^b c f(t) g(t)\, dt$$

$$= c \int_a^b f(t) g(t)\, dt = c\langle f, g\rangle \quad \forall\ f, g \in C[a,b]$$

- Now $\langle f, f\rangle = \int_a^b f(t) f(t)\, dt = \int_a^b [f(t)]^2\, dt \geq 0$

and $\langle f, f\rangle = 0 \Rightarrow \int_a^b [f(t)]^2\, dt = 0$

$$\Rightarrow f(t) \equiv 0$$

(By continuity f is zero everywhere on $[a, b]$)

10 $x_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

Using Gram-Schmidt orthogonalization process

let $v_1 = x_1 = \boxed{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \frac{4\times2 + 1\times1}{2\times2+1\times1+2\times2} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \frac{9}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}}$$

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \frac{6+1-2}{2\times2+1\times1+2\times2}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - \frac{6+2}{2\times2+(-2)\times(-2)}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \frac{5}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix}}$$

So, an orthonormal basis will be

$$\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$$

ie $$\left\{ \frac{1}{\sqrt{2^2+1^2+2^2}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2^2+(-2)^2}} \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{\left(-\frac{1}{9}\right)^2+\left(\frac{4}{9}\right)^2+\left(-\frac{1}{9}\right)^2}} \begin{pmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \frac{3}{\sqrt{2}} \begin{pmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{pmatrix} \right\}$$

11 $V = R_2[t]$

a) $\langle p, q\rangle = p(0)q(0) + p(-2)q(-2) + p(2)q(2)$

Let us denote the three polynomials as

$x_1 = 1, \quad x_2 = t, \quad x_3 = t^2$

Using Gram-Schmidt orthogonalization process

$v_1 = x_1 = \boxed{1}$

$$v_2 = x_2 - \frac{\langle x_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} 1$$

$$= t - \frac{0\times 1 + (-2)\times 1 + (2)\times 1}{1\times 1 + 1\times 1 + 1\times 1} 1 = t - \frac{0}{3}\times 1$$

$\Rightarrow \boxed{v_2 = t}$

$$v_3 = x_3 - \frac{\langle x_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t$$

$$= t^2 - \frac{0 + (-2)^2\times 1 + (2)^2\times 1}{1\times 1 + 1\times 1 + 1\times 1} \times 1$$

$$- \frac{0\times 0 + (-2)^2(-2) + 2^2\times 2}{0\times 0 + (-2)(-2) + (2)(2)} \cdot t$$

$$= t^2 - \frac{8}{3} - 0 = \boxed{t^2 - \frac{8}{3}}$$

So, the orthogonal basis is

$$\left\{ 1,\ t,\ t^2 - \frac{8}{3} \right\}$$

Check: $\langle 1, t \rangle = 1 \times 0 + 1(-2) + 1(2) = -2+2 = \boxed{0}$

$$\left\langle 1, t^2 - \frac{8}{3} \right\rangle = 1 \times \left(-\frac{8}{3}\right) + 1\left((-2)^2 - \frac{8}{3}\right) + 1\left(2^2 - \frac{8}{3}\right)$$
$$= -\frac{8}{3} + 4 - \frac{8}{3} + 4 - \frac{8}{3} = 8 - \cancel{3} \times \frac{8}{\cancel{3}}$$
$$= 8 - 8 = \boxed{0}$$

$$\left\langle t, t^2 - \frac{8}{3} \right\rangle = 0 \times \left(-\frac{8}{3}\right) + (-2)\left((-2)^2 - \frac{8}{3}\right) + (2)\left(2^2 - \frac{8}{3}\right)$$
$$= 0 - 2\left(4 - \frac{8}{3}\right) + 2\left(4 - \frac{8}{3}\right) = \boxed{0}.$$

(b) Now let $p(t) = (1 + 2t + 3t^2)$

$$= c_1 \times 1 + c_2 \times t + c_3 \left(t^2 - \frac{8}{3}\right)$$

$\Rightarrow c_1 - \frac{8}{3} c_3 = 1, \quad c_2 = \boxed{2}, \quad c_3 = \boxed{3}$

$\Rightarrow c_1 = 1 + \frac{8}{3} c_3 = 1 + \frac{8}{3} \times 3 = \boxed{9}$

Then $p(t) = 9(1) + 2(t) + 3\left(t^2 - \frac{8}{3}\right)$

Hence the coordinates of $p(t)$ with respect to this orthogonal basis is $\begin{bmatrix} 9 \\ 2 \\ 3 \end{bmatrix}$

12) Given $W = \text{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\} \subset \mathbb{R}^3$

Let $x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3$ be such that

$$\left\langle \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle = 0 \Rightarrow x_1 + 2x_2 + 3x_3 = 0$$

Thus we need to solve
$$\left.\begin{aligned} x_1 &= -2x_2 - 3x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}\right\}$$

So, the solution is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\0\\1\end{bmatrix}$

Taking $x_2 = 1$, $x_3 = 0$ we get $u_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

Taking $x_3 = 1$, $x_2 = 0$ we get $u_2 = \begin{bmatrix}-3\\0\\1\end{bmatrix}$

clearly $\langle x, u_1\rangle = 0$ and $\langle x, u_2\rangle = 0$

but $\langle u_1, u_2\rangle = 6 \neq 0$

We use Gram-Schmidt normalization process on $\{u_1, u_2\}$

Let $v_1 = u_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = \begin{bmatrix}-3\\0\\1\end{bmatrix} - \frac{6}{5}\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{3}{5}\\-\frac{6}{5}\\1\end{bmatrix}$$

- W is a one dimensional subspace of $\mathbb{R}^3$ and its orthogonal basis is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

$W^{\perp}$ has orthogonal basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{5} \\ -6/5 \\ 1 \end{pmatrix} \right\}$

Since $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/5 \\ -6/5 \\ 1 \end{pmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$,

it is linearly independent

and since $\dim(\mathbb{R}^3) = 3$, the above set forms a basis of $\mathbb{R}^3$.