Solution of Worksheet 11

1. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 4\right] + 2\left[-4-2(1-\lambda)\right] + 2\left[-4-2(1-\lambda)\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda + 1 - 4\right] + 2\left[2\lambda - 6\right] + 2\left[2\lambda - 6\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right] + 4\left[\lambda - 3\right] + 4\left[\lambda - 3\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right] + 4\left[\lambda - 3\right]$$

$$= (1-\lambda)\left[(\lambda-3)(\lambda+1)\right] + 4\left[\lambda - 3\right]$$

$$= (\lambda - 3)\left\{1 - \lambda^2 + 8\right\}$$

$$= (\lambda - 3)(9 - \lambda^2) = (\lambda - 3)(3 - \lambda)(3 + \lambda)$$

So, the eigen values are

$\lambda_1 = 3$ (algebraic multiplicity 2)

$\lambda_2 = -3$ (algebraic multiplicity 1)

For $\lambda_1 = 3$, $A - \lambda I = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -\frac{1}{2} R_1}$$

RREF matrix $= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So, the solution of the system is: $x_1 = x_2 + x_3$
$x_2 = x_2$
$x_3 = x_3$

$$x = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Taking $x_2 = 1$ and $x_3 = 0$, we get $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Taking $x_2 = 0$ and $x_3 = 1$, we get $u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

However $\langle u_1, u_2 \rangle = 1 \neq 0$

So, we apply Gram-Schmidt process to get orthogonal eigenvectors for $\lambda_1 = 3$: Let $u_1' = u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$u_2' = u_2 - \frac{\langle u_2, u_1 \rangle}{\langle u_1', u_1' \rangle} \cdot u_1'$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalising we get

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Now for $\lambda_2 = -3$, $A - \lambda I = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

$$\xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_1 - 2R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2}$$

$$\text{RREF matrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving we get $\left.\begin{matrix} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{matrix}\right\}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Taking $x_3 = 1$ we get $u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Let ~~us~~ us normalize to get $v_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Therefore $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

and $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

and $A = PDP^T$

(2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = B$ (say)

The characteristic polynomial is $(2-\lambda)(3-\lambda)$

and so the eigen values in descending order are $\lambda_1 = 3$, $\lambda_2 = 2$

~~Therefore~~ For $\lambda_1 = 3$, $B - \lambda_1 I = \begin{bmatrix} 2-3 & 0 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

$\xleftarrow{R_1 \to (-1)R_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

So, the ~~basic~~ solution is

$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, the associated ~~solution~~ eigen vector corresponding to $\lambda_1 = 3$ can be taken as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (taking $x_2 = 1$)

For $\lambda_2 = 2$, $B - \lambda_2 I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

So, the system is $\left.\begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Taking $x_1 = 1$ we get an ~~orthonor~~ eigen vector corresponding to $\lambda_2 = 2$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Therefore $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$

<u>To compute U:</u>

$$u_1 = \frac{Av_1}{\sigma_1}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

But $\{u_1, u_2\}$ is not a basis for $\mathbb{R}^3$

So, we will find a u_3 orthogonal to both u_1 and u_2.

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0 \qquad \Rightarrow 2x_1 + x_2 = 0$$

$$x_1 - x_3 = 0 \qquad x_1 = x_3$$

$$\Rightarrow x_1 = -\frac{1}{2}x_2$$

$$x_2 = x_2$$

$$x_3 = x_1 = -\frac{1}{2}x_2$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

Taking $x_2 = -2$, we get $u_3' = \begin{bmatrix} +1 \\ -2 \\ +1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Normalizing, we get $u_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ 1/\sqrt{6} \end{bmatrix}$

Therefore $A = U \Sigma V^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5) Let $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$

First we find the singular value decomposition of $B = A^T$.

$$B = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times 2}$$

Then $B^T B = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix} = C \text{ (say)}$$

$$\det(C - \lambda I) = \begin{vmatrix} 333-\lambda & 81 \\ 81 & 117-\lambda \end{vmatrix}$$

$$= (333-\lambda)(117-\lambda) - (81)(81)$$
$$= 38961 - 333\lambda - 117\lambda + \lambda^2 - 6561$$
$$= \lambda^2 - 450\lambda + 32400$$
$$= (\lambda - 360)(\lambda - 90)$$

So, the eigen values are $\lambda_1 = 360$
$\lambda_2 = 90$

Then $\sigma_1 = \sqrt{360} = 6\sqrt{10}$
$\sigma_2 = \sqrt{90} = 3\sqrt{10}$

Eigenvectors corresponding to $\lambda_1 = 360$

$$(C - \lambda_1 I) = \begin{bmatrix} 333-360 & 81 \\ 81 & 117-360 \end{bmatrix} = \begin{bmatrix} -27 & 81 \\ 81 & -243 \end{bmatrix}$$

$$\begin{bmatrix} -27 & 81 \\ 81 & -243 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} -27 & 81 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -\frac{1}{27}R_1} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

So, the system as
$$\left.\begin{aligned} x_1 - 3x_2 &= 0 \\ x_2 &= x_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= 3x_2 \\ x_2 &= x_2 \end{aligned}\right\}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Taking $x_2 = 1$, we get $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as an eigenvector

For $\lambda_2 = 90$,

$$(C - \lambda_2 I) = \begin{bmatrix} 333-90 & 81 \\ 81 & 117-90 \end{bmatrix} = \begin{bmatrix} 243 & 81 \\ 81 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 243 & 81 \\ 81 & 27 \end{bmatrix} \xrightarrow{R_1 \to R_2} \begin{bmatrix} 81 & 27 \\ 243 & 81 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 81 & 27 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{81}R_1} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

So, the solution is
$$\left.\begin{aligned} x_1 + \tfrac{1}{3}x_2 &= 0 \\ x_2 &= x_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= -\tfrac{1}{3}x_2 \\ x_2 &= x_2 \end{aligned}\right\}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Taking $x_2 = -3$, we get $v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

we can normalize all the vectors in the end.

Now to find u_2' we take -

$$u_1 = Bv_1 = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 40 \end{bmatrix}$$

We can take $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

and $u_2 = Bv_2 = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -20 \\ -10 \\ 20 \end{bmatrix}$

we can take $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

Note that $\langle u_1, u_2 \rangle = 0$ i.e. u_1 and u_2 are orthogonal.

To get an orthogonal basis for $\mathbb{R}^3$, we have to find a vector orthogonal to both u_1 and u_2.

Thus we need to solve

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$

$$x_1 + 2x_2 + 2x_3 = 0$$
$$-2x_1 - x_2 + 2x_3 = 0$$

The Coefficient matrix is $\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix}$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

= RREF matrix

So, the system is

$$\left.\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}\right\}$$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Taking $x_3 = 1$, we get $u_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ as a suitable vector

Now we will normalize all the vectors v_1, v_2 and u_i's.

So, we get $v_1' = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$

and $v_2' = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}$

and $u_1' = \frac{1}{\sqrt{9}}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

$u_2' = \frac{1}{\sqrt{9}}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

and $u_3' = \frac{1}{\sqrt{9}}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

Let $\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \\ 0 & 0 \end{bmatrix}$ $V = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$

and $U = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Then $B = U\Sigma V^T$ is the SVD of B

Then $A = B^T = (U\Sigma V^T)^T = (V^T)^T \Sigma^T U^T$

$= V\Sigma U^T$ is the SVD of A

Thus $A = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \times$

$$\times \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

is the SVD of A.

1(a)

4) (a) Given that A is an $n \times n$ invertible matrix

Let λ be any eigen value of A.
Since A is invertible, $\lambda \neq 0$.
If $x \neq 0$ be an eigenvector corresponding to λ, then

$$Ax = \lambda x \Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x) \Rightarrow x = \lambda(A^{-1}x)$$
$$\Rightarrow A^{-1}x = \lambda^{-1}x$$

So, x is an eigen-vector of A^{-1} corresponding to λ^{-1}.
Thus λ^{-1} is an eigenvalue of A^{-1}
Therefore eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

(b) Since A is invertible, $\operatorname{rank}(\operatorname{Col} A) = \operatorname{rank} A = n$
and so it has n nonzero singular values
$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$
Then singular value decomposition of A will be

$$A = U \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix} V^T$$

Then
$$A^{-1} = (V^T)^{-1} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}^{-1} U^{-1}$$

$$= V \begin{bmatrix} \sigma_1^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_2^{-1} & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_n^{-1} \end{bmatrix} U^T$$

(Since $V^T = V^{-1}$, $U^T = U^{-1}$ By orthogonality)

This will be a singular value decomposition of A^{-1}.

5. Given that U is an $m \times n$ matrix with orthonormal columns and $x, y \in \mathbb{R}^n$

(a) Now $Ux \cdot Uy = (Ux)^T (Uy) = (x^T U^T)(Uy)$

(Here same as $\langle Ux, Uy \rangle$)

$$= x^T (U^T U) y$$

$$= x^T y \quad \left(\text{since } U^T U = I_n \text{ (given)}\right)$$

$$= x \cdot y$$

(b) $\|Ux\|^2 = \langle Ux, Ux \rangle = (Ux)\cdot(Ux)$

$$= x \cdot x \quad (\text{By (a)})$$

$$= \|x\|^2$$

$\Rightarrow \|Ux\| = \|x\|$

(c) Since $Ux \cdot Uy = x \cdot y$ (By (a))

we get $Ux \cdot Uy = 0$ if and only if $x \cdot y = 0$