MTH 100 : Lecture 2

Two Special type of Matrices :

Ex:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 6} \longrightarrow \text{Echelon form}$$

Ex:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 6} \longrightarrow \text{Reduced Row Echelon form (RREF)}$$

Note:

The word Echelon came from the Latin word $\boxed{\text{Scala}}$ which means Ladder.

It came to French as "échelle" and then to English which now means Level or Step.

Definition:
An $m \times n$ matrix is said to be in **echelon form** if:

(1) All nonzero rows are above all zero rows.

(2) Each leading entry of a row (the first nonzero entry in a row) is to the right of the leading entry of the row above it.

(3) All entries in a column below a leading entry are zero.

Note: (3) is a consequence of (2). Still we have written it explicitly here in the interest of clarity.

Definition:
A $m \times n$ matrix is said to be in **Reduced Row Echelon form (RREF)** if

(1) All non zero rows are above all zero rows.

(2) Each leading (i.e. the first nonzero entry)

entry of a row is to the right of the leading entry of a row above it.

(3) The leading entry (i.e. the first nonzero entry) in each non zero row is 1.

(4) Each column which contains such a leading entry (necessarily 1) has all its other entries as zero

<u>Note:</u> RREF matrix is in Echelon form and has two further requirements (3) and (4).

<u>Example:</u> <u>An Example of Row-Reduction</u> (Gauss-Jordan Elimination)

$$A = \begin{bmatrix} 0 & 5 & 10 & 8 \\ 1 & 2 & 6 & 7 \\ 2 & 4 & 12 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 6 & 7 \\ 0 & 5 & 10 & 8 \\ 2 & 4 & 12 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + (-2)R_1} \begin{bmatrix} 1 & 2 & 6 & 7 \\ 0 & 5 & 10 & 8 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

(Echelon form)

(Forward Phase)

$$\xrightarrow[R_3 \to -\frac{1}{8}R_3]{R_2 \to \frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 6 & 7 \\ 0 & 1 & 2 & \frac{8}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 + (-7)R_3]{R_2 \to R_2 + (-\frac{8}{5})R_3} \begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + (-2)R_2} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \to \text{RREF matrix}$$

(Backward Phase)

(Pivot columns)

Ex: Find the RREF form of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \left(-\frac{1}{3}\right) R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 5R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(RREF matrix)

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$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \left(-\frac{1}{4}\right) R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 8R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \left(-\frac{1}{10}\right) R_3} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 3R_2]{R_1 \to R_1 - 7R_3} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(RREF matrix)

Row Reduction Algorithm (Gauss-Jordan elimination):

- The input is an $m \times n$ matrix.
- We carry out elementary row operations on the input matrix.
- **Pivot Position:** A position corresponding to a leading position in an echelon form (a leading 1 in the RREF of the matrix)
  Its column is called a $\boxed{\text{Pivot column.}}$

- At the start move all zero rows to the bottom using interchange operations.

- **Step 1:**
  Start with Left most nonzero column.
  It will be the pivot column.

- **Step 2**
  Using interchange operation make the top element of the pivot column nonzero.
  (This will be the pivot position)

- **Step 3**
  Use replacement operations to make all entries in the pivot column below the pivot position as 0's.

- **Step 4**

  Cover (ignore) the row containing pivot position and all rows above it (if any).

  Repeat steps 1 to 4 for all rows below until all the non zero rows have been processed.

Note: Steps 1 – 4 is called $\boxed{\text{Forward Phase}}$ which produces a matrix in echelon form. (This portion of the algorithm is referred to as Gaussian Reduction or Gaussian elimination)

- **Step 5 :**

  Use scaling operation to make all the pivot elements 1.

- **Step 6 :**

  Starting with the right-most pivot, creat zeros in the entire column above it, by using replacement operations.

  Repeat this step moving leftward and upward.

Steps 5–6 is called the Backward Phase of the algorithm which produces an RREF matrix.

Note : The algorithm will stop after a finite number of steps and we will get a RREF matrix.

Conclusion:

Definition: If A and B are $m \times n$ matrices, we say that B is row equivalent to A if B can be obtained from A by a finite sequence of row operations.

Proposition 1: Given any $m \times n$ matrix A, there exists an RREF matrix which is row equivalent to A.

Proof: The proof is given by the above algorithm.

(i.e. we have given a constructive proof, rather than a pure existence proof.)

Note: In the example done in class, all the matrices in the intermediate steps were row equivalent to the original matrix A. However we obtained an RREF matrix only at the final step.

Proposition 2:
Row Equivalence is an equivalence relation on the set $R^{m\times n}$ of $m\times n$ matrices with entries from the field $\mathbb{R}$ (set) of real numbers. (m, n are fixed)

Proof: Exercise.

Note: If '$\sim$' denotes the relation of row equivalence, then
- $A \sim A$ (scaling first row by 1) for every $m\times n$ matrix A,
- $A \sim B \Rightarrow B \sim A$ (reversing the operation)
- $A \sim B$, $B \sim C \Rightarrow$ Combining the operations $A \sim C$

So, row equivalence is an equivalence relation.

Remark:
- Later on we will also work with matrices with complex entries. The proposition will continue to hold with $\mathbb{R}$ replaced by $\mathbb{C}$.
- Every equivalence relation induces a partition of the underlying set.

(The parts of the partition are called equivalence classes)

The equivalence classes are pairwise disjoint subsets whose union is the whole set.

Conversely given any partition of a set, there exists a corresponding equivalence relation. (Justify yourself!)

Note: Define $x \sim y$ if and only if x, y belong to the same set of the partition.

This is an equivalence relation (Show!).

Remark: The RREF matrix of any given matrix is unique.

i.e. A matrix can't be row equivalent to two distinct RREF matrices.

Alternatively, two distinct RREF matrices can't be row-equivalent to each other.

Thus, inside each equivalence class for this equivalence relation, there is a distinctive member i.e. the one and only RREF matrix in it.

- This fact can be used to determine whether two matrices are row-equivalent to each other.