MTH 100 : Lecture 9

LU factorization of a matrix

Motivation: $Ax = b_1$, $Ax = b_2$, ..., $Ax = b_m$

(So, b_i's change but A remains fixed)

- One way of solving is to find A^{-1} and then find $A^{-1} b_i$ for $i = 1, 2, \ldots, m$

- However a more efficient way is to factor $A = LU$ which requires reducing to an echelon form only. Then the equation can be solved.

Definition: Suppose A is a $m \times n$ matrix which can be reduced to an echelon form (upper triangular) matrix without using row

interchange operations.
(So, only replacement operations are used in the forward phase of the row reduction algorithm)

Then A can be factorized as

$$A = LU$$

where L is an $m \times m$ Lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form matrix obtained from A by row reduction.

Any such factorization is called LU factorization of A.

- The matrix L is invertible and is called a Unit Lower triangular matrix.

- <u>Application to Solving Linear System</u>

Consider $A\bar{x} = \bar{b}$

Let $A = LU$

$A\bar{x} = \bar{b} \Rightarrow (LU)\bar{x} = \bar{b}$

$\Rightarrow L(U\bar{x}) = b \Rightarrow \boxed{U\bar{x} = L^{-1}\bar{b}}$

<u>Note that</u>

$L(L^{-1}\bar{b}) = (LL^{-1})\bar{b} = \bar{b}$

So, $\boxed{L^{-1}\bar{b} \text{ is a solution of } L\bar{y} = \bar{b}}$

- The solution of the $A\bar{x} = \bar{b}$ is replaced by

the solution of $\left.\begin{array}{l} L\bar{y} = \bar{b} \\ \text{and } U\bar{x} = \bar{y} \end{array}\right\}$

These systems are triangular and therefore easy to solve.

Ex: Let $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$

- We row reduce A to echelon form without interchanges or scaling and by adding multiples of a row to a lower row at every step:

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 5 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \text{ (say)}$$

Note of Caution: In general U doesn't always have 1's in the diagonal.

Let us write the row operations e_i and their inverses f_i:

$e_1 : R_2 \to R_2 - R_1$ $\qquad$ $f_1 : R_2 \to R_2 + R_1$

$e_2 : R_3 \to R_3 - 2R_1$ $\qquad$ $f_2 : R_3 \to R_3 + 2R_1$

$e_3 : R_3 \to R_3 - 5R_2$ $\qquad$ $f_3 : R_3 \to R_3 + 5R_2$

To get L from I, we find $(f_1 f_2 f_3) I$

Explanation:

Note: The same steps which take A to U will take L to I.

So, $I = e_3(e_2(e_1(L))) = E_3(E_2(E_1 L))$

$= (E_3 E_2 E_1) L$ where $E_i = e_i(I)$

$\Rightarrow L = (E_3 E_2 E_1)^{-1} I = (E_1^{-1} E_2^{-1} E_3^{-1}) I$

$\Rightarrow L = f_1(f_2(f_3(I)))$

So,
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[f_3 : R_3 \to R_3 + 5R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow[f_2 : R_3 \to R_3 + 2R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \xrightarrow[f_1 : R_2 \to R_2 + R_1]{} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 1 & \boxed{1} & 0 \\ 2 & 5 & \boxed{1} \end{bmatrix}_{3\times 3}$$

$$= L \text{ (say)}$$

Note that L is a lower triangular matrix with 1's on the diagonal i.e. L is a unit lower triangular matrix.

check: $A = LU$ (??)

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$= A$ (as desired)

Ex: Solve $A\bar{x} = \bar{b}$ where $\bar{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

and $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$

Note that $A\bar{x} = \bar{b}$ is solved by solving $L\bar{y} = \bar{b}$ and then solving $U\bar{x} = \bar{y}$

We have $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$

and $U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (By the previous example)

Let $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and consider $L\bar{y} = \bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} y_1 &= 2 & \Rightarrow \quad & y_1 = \boxed{2} \\ y_1 + y_2 &= -1 & \Rightarrow \quad & y_2 = -1 - y_1 = -1-2 = \boxed{-3} \\ 2y_1 + 5y_2 + y_3 &= 1 & \Rightarrow \quad & y_3 = 1 - 2y_1 - 5y_2 \\ & & & \quad = 1 - 2(2) - 5(-3) = \boxed{12} \end{aligned}$$

(Forward substitution)

Now we solve $U\bar{x} = \bar{y}$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 12 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 - 2x_3 &= 2 \\ x_2 + x_3 &= -3 \\ x_3 &= 12 \end{aligned}\right\}$$

Then $x_3 = \boxed{12}$

$x_2 = -3 - x_3 = -3 - 12 = \boxed{-15}$

and $x_1 = 2 + x_2 + 2x_3 = 2 + (-15) + 2(12)$

$= \boxed{11}$

(Backward substitution)

<u>Check:</u> $\underline{A\bar{x} = \bar{b}}$ (??)

$$A\bar{x} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -15 \\ 12 \end{bmatrix} = \begin{bmatrix} 11 + 15 - 24 = 2 \\ 11 - 12 = -1 \\ 22 - 45 + 24 = 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$= \bar{b}$

(as desired)

Ex: Find the solution of $A\bar{x} = \bar{b}$ for general $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with this given A.

Here after solving $L\bar{y} = \bar{b}$, we will get:

$$\left.\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - b_1 \\ y_3 &= b_3 - 2b_1 - 5(b_2 - b_1) \end{aligned}\right\}$$

Then $U\bar{x} = \bar{y} \Rightarrow x_3 = \boxed{y_3}$

and $x_2 = y_2 - x_3 = \boxed{y_2 - y_3}$

and $x_1 = y_1 + x_2 + 2x_3$

$\Rightarrow x_1 = y_1 + y_2 - y_3 + 2y_3$

$\Rightarrow x_1 = \boxed{y_1 + y_2 + y_3}$

So, from the entries of L and U, the system can be solved easily for any $\bar{b}$.