Addendum to Lecture 15

Proposition 3: If A is a square matrix, then A is row equivalent to the identity matrix iff the homogeneous system $A\bar{x}=\bar{0}$ has only the trivial solution.

Proof: $\Rightarrow$: If $A_{n\times n}$ is a square matrix and A is row equivalent to the identity matrix I_n, then after row operations, A will be row reduced to the RREF matrix I_n.

Hence the system of equation will be $I_n \bar{x}_{n\times 1} = \bar{0}$

$$\Rightarrow \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. the system $A\bar{x}=\bar{0}$ has only the trivial solution.

$\Leftarrow$: Conversely assume that $A\bar{x}=\bar{0}$ has only the trivial solution.

Then if we row reduce A to its RREF matrix, there will be only basic variables and no free variable (because free variable will give non-trivial solution).

Thus RREF matrix will have only pivot columns. Since it is a square matrix there will be no zero rows and it will be the identity matrix I_n.

i.e. A is row equivalent to I_n

Proposition 4: The system $A\bar{x}=\bar{b}$ is consistent iff the rightmost column of R (where R is the RREF matrix corresponding to A) is not a pivot column.
i.e. there is no row of the form
$[0, 0, \ldots, 0, p]$ with p nonzero.

Proof: $\Rightarrow$:

If there is a row of the form
$[0, 0, \ldots, 0, p]$ with $p \neq 0$,
then if we write the reduced set of equations explicitly, one of the equation will be
$$0.x_1 + 0.x_2 + \cdots + 0.x_n = p$$
$$\Rightarrow \quad 0 = p \text{ where } p \neq 0$$
a contradiction
and $A\bar{x}=\bar{b}$ is inconsistent.

Thus $A\bar{x}=\bar{b}$ consistent $\Rightarrow$ There is no row of the form $[0, 0, \ldots, p]$ with $p \neq 0$

$\Leftarrow$ If there is no row of the form
$[0, 0, \ldots, p]$ with $p \neq 0$,
then if we write the reduced set of equations explicitly, there will be basic variables and possibly some free variables. The basic variables can be solved in terms of free variables and so the system $A\bar{x}=\bar{b}$ is consistent.

Observation 4:

If $A_1, A_2, \dots A_n \; (n \geq 2)$ are invertible matrices then $C = A_1 A_2 \cdots A_n$ is invertible and

$$C^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$$

Proof: For $n = 2$,

$$(A_2^{-1} A_1^{-1})(A_1 A_2) = A_2^{-1}(A_1^{-1} A_1) A_2 = A_2^{-1} I A_2$$
$$= A_2^{-1} A_2 = I$$

and
$$(A_1 A_2)(A_2^{-1} A_1^{-1}) = A_1 (A_2 A_2^{-1}) A_1^{-1} = A_1 I A_1^{-1}$$
$$= A_1 A_1^{-1} = I$$

Thus for $n = 2$, $(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$

Now assume that
$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$$

Now
$$(A_{k+1}^{-1} A_k^{-1} \cdots A_1^{-1})(A_1 \cdots A_k A_{k+1})$$
$$= A_{k+1}^{-1} (A_k^{-1} \cdots A_1^{-1})(A_1 \cdots A_k) A_{k+1}$$
$$= A_{k+1}^{-1} I A_{k+1} = A_{k+1}^{-1} A_{k+1} = I$$

and
$$(A_1 \cdots A_k A_{k+1})(A_{k+1}^{-1} A_k^{-1} \cdots A_1^{-1})$$
$$= (A_1 \cdots A_k)(A_{k+1} A_{k+1}^{-1})(A_k^{-1} \cdots A_1^{-1})$$
$$= (A_1 \cdots A_k) I (A_k^{-1} \cdots A_1^{-1}) = (A_1 \cdots A_k)(A_k^{-1} \cdots A_1^{-1})$$
$$= I$$

Thus the formula is true for $n = k+1$ if it is true for $n = k$.

Since the formula is proved for $n=2$, by the principle of mathematical induction it is true for all positive integer n.

<u>Proposition 5:</u> If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A,

$$e(A) = EA$$

<u>Proof:</u> Please see solution of Worksheet 3. (problem 13)

<u>Properties:</u> Let V be a vector space over a field F.

Then (c) $0.u = \bar{0} \quad \forall\, u \in V$

(d) $c.\bar{0} = \bar{0} \quad \forall\, c \in F$

(e) $-u = (-1)u \quad \forall\, u \in V$

<u>Proof:</u> (c) $\bar{0} + 0.u = 0.u = (0+0).u = 0.u + 0.u$

$\Rightarrow \bar{0} + 0.u + (-0.u) = 0.u + 0.u + (-0.u)$

$\Rightarrow \bar{0} + (0.u + (-0.u)) = 0.u + (0.u + (-0.u))$

$\Rightarrow \bar{0} + \bar{0} = 0.u + \bar{0} \Rightarrow \bar{0} = 0.u \Rightarrow \boxed{0.u = \bar{0}}$

(d) $\bar{0} + c.\bar{0} = c\bar{0} = c(\bar{0}+\bar{0}) = c\bar{0} + c\bar{0}$

$\Rightarrow \bar{0} + c.\bar{0} + (-c\bar{0}) = c\bar{0} + c\bar{0} + (-c\bar{0})$

$\Rightarrow \bar{0} + (c.\bar{0} + (-c\bar{0})) = c\bar{0} + (c\bar{0} + (-c\bar{0}))$

$$\Rightarrow \quad \bar{0} + \bar{0} = c\bar{0} + \bar{0} \Rightarrow \bar{0} = c\bar{0} \Rightarrow \boxed{c\bar{0} = \bar{0}}$$

(e) $\bar{0} = 0.u = ((-1) + 1)u = (-1)u + 1u = (-1)u + u$

$$\Rightarrow \quad \bar{0} + (-u) = (-1)u + u + (-u)$$

$$\Rightarrow \quad \bar{0} + (-u) = (-1)u + (u + (-u))$$

$$\Rightarrow \quad -u = (-1)u + \bar{0} \Rightarrow \boxed{-u = (-1)u}$$

Proposition: Test 1: Let V be a vector space over a field F. Then a nonempty subset W is a subspace of V

$\Leftrightarrow$ (1) $\bar{0} \in W$

(2) $u + v \in W \ \forall \ u, v \in W$

(3) $cu \in W \ \forall \ c \in F$ and $\forall \ u \in W$

Proof: $\Rightarrow$ If W is a subspace of V, then W is a vector space over the field F.

Then closure properties of addition & scalar multiplication are satisfied in W

$\Rightarrow \quad u + v \in W \ \forall \ u, v \in W$ and $cu \in W \ \forall c \in F$ and $\forall u \in W$

Also W must have the zero element and so $\bar{0} \in W$

$\Leftarrow$: Assume that W satisfies (1), (2) and (3)

(2) and (3) $\Rightarrow$ closure properties of addition and scalar multiplication are satisfied in W.

(1) $\Rightarrow \bar{0} \in W$

Now $c = -1$ in (3) $\Rightarrow (-1)u = -u \in W \quad \forall\, u \in W$

i.e. additive inverse exists in W $\forall\, u \in W$.

Now associative property of addition, commutative property of addition and all other properties of scalar multiplication are hereditary and hence are satisfied in W (since they are satisfied in V).

Therefore W is a subspace of V.

<u>Note:</u> (1) can be replaced by (1'): $W \neq \phi$

i.e. (1), (2), (3) $\Leftrightarrow$ (1'), (2), (3).

<u>$\Rightarrow$:</u> Since $\bar{0} \in W$, $W \neq \phi$ and (1') holds.

<u>$\Leftarrow$:</u> Since $W \neq \phi$, there exists $u \in W$

By (3) $\quad (-1)u = -u \in W$ (By taking $c = -1$)

and By (2) $\quad u + (-u) \in W \Rightarrow \bar{0} \in W$ and so (1) holds.

<u>Proposition: Test (2):</u> Let V be a vector space over a field F.

Then a nonempty subset W is a subspace of V

$$\Leftrightarrow \quad cu + v \in W \quad \forall\, u, v \in W \text{ and } \forall\, c \in F$$

The two tests are equivalent:

Test (1) $\Rightarrow$ Test (2)

If $u, v \in W$ and $c \in F$, then

$cu \in W$ (By 3 of Test(1))

$\Rightarrow$ $cu + v \in W$ (By (2) of Test(1))

Test (2) $\Rightarrow$ Test(1)

Since $W \neq \phi$, there exists an element $u \in W$

Taking $c = -1$ and $v = u$

we have $(-1)u + u = -u + u = 0 \in W$

$\boxed{\text{i.e. (1) of Test (1) holds.}}$

Now taking $c = 1$, we get $(1)u + v = u + v \in W$
$\forall\, u, v \in W$

$\boxed{\text{i.e. (2) of Test (1) holds.}}$

Taking $v = 0$ we get $cu + 0 = cu \in W \ \forall\, u \in V$
and $\forall\, c \in F$

$\boxed{\text{i.e. (3) of Test (1) holds}}$