Worksheet 2

1. The Augmented matrix of the system:

$$A = \left[\begin{array}{cccc|c} 3 & 2 & 7 & 9 & 7 \\ 6 & 14 & 22 & 15 & 13 \\ 1 & 4 & 5 & 2 & 2 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 2 \\ 6 & 14 & 22 & 15 & 13 \\ 3 & 2 & 7 & 9 & 7 \end{array}\right]$$

$R_2 \to R_2 - 6R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 2 \\ 0 & -10 & -8 & 3 & 1 \\ 0 & -10 & -8 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 2 \\ 0 & -10 & -8 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_2 \to \left(-\frac{1}{10}\right) R_2$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & 2 & 2 \\ 0 & 1 & \frac{4}{5} & -\frac{3}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 - 4R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{9}{5} & \frac{16}{5} & 12/5 \\ 0 & 1 & \frac{4}{5} & -\frac{3}{10} & -\frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \text{RREF matrix.}$$

- Since the right most column of the RREF matrix of the Augmented matrix is not a Pivot column, the System is Consistent.

. The reduced system is:

$$x + \frac{9}{5}z + \frac{16}{5}\omega = \frac{12}{5} \Rightarrow x = \frac{12}{5} - \frac{9}{5}z - \frac{16}{5}\omega$$

$$y + \frac{4}{5}z - \frac{3}{10}\omega = -\frac{1}{10} \Rightarrow y = -\frac{1}{10} - \frac{4}{5}z + \frac{3}{10}\omega$$

$$z = 0 + z + 0$$

$$\omega = 0 + 0 + \omega.$$

So,
$$\begin{bmatrix} x \\ y \\ z \\ \omega \end{bmatrix} = \underbrace{\begin{bmatrix} 12/5 \\ -1/10 \\ 0 \\ 0 \end{bmatrix}}_{u} + z\underbrace{\begin{bmatrix} -\frac{9}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{bmatrix}}_{v} + \omega\underbrace{\begin{bmatrix} -16/5 \\ 3/10 \\ 0 \\ 1 \end{bmatrix}}_{d}$$ where $z, \omega \in \mathbb{R}$

The solution set is

$$\Rightarrow \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = u + tv + sd \;,\; t, s \in \mathbb{R} \right\}$$

where $u = \begin{bmatrix} 12/5 \\ -1/10 \\ 0 \\ 0 \end{bmatrix}$ is a solution of $Ax = b$ (can check!)

and

$v = \begin{bmatrix} -9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix}$ and $d = \begin{bmatrix} -\frac{16}{5} \\ \frac{3}{10} \\ 0 \\ 1 \end{bmatrix}$ (can check!)

associated homogeneous system

are solutions of $Ax = 0$

2

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & a \end{array}\right] \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -8+a \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -5+a \end{array}\right] \xrightarrow{R_1 \to R_1 - 5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -5+a \end{array}\right]$$

RREF matrix

So, the system is consistent if $a-5=0$

ie if $\boxed{a=5}$

In that case

the system reduces to

$$\left.\begin{aligned} x+7z&=1 \\ y-2z&=-1 \\ 0&=0 \end{aligned}\right. \quad \Rightarrow \quad \left.\begin{aligned} x&=1-7z \\ y&=-1+2z \\ z&=z \end{aligned}\right\}$$

So, the solution set is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

The system is inconsistent if $a \neq 5$

In that case the system becomes:

$$\left.\begin{aligned} x+7z&=1 \\ y-2z&=-1 \\ 0&=a-5 (\neq 0) \end{aligned}\right\}$$

and so it is inconsistent.

(3) [redacted]

$$\left.\begin{aligned} x+y+z&=3\\ 2x+3y+4z&=9\\ x-y+az&=b \end{aligned}\right\}$$

The Augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 2 & 3 & +4 & 9\\ 1 & -1 & a & b \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 2 & 3\\ 0 & -2 & a-1 & b-3 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & 2 & 3\\ 0 & 0 & a+3 & b+3 \end{array}\right] \xleftarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 2 & 3\\ 0 & 0 & a+3 & b+3 \end{array}\right]$$

So, the system has

(1) a unique solution if $a+3 \neq 0$ ie $\boxed{\text{if } a \neq -3}$
(b can be any real number)

(2) Infinitely many solution if $\left.\begin{aligned} a+3&=0\\ \text{and } b+3&=0 \end{aligned}\right\}$

ie $\boxed{\text{if } a=-3,\ b=-3}$

(3) No solution if $a+3=0$ and $b+3 \neq 0$

ie $\boxed{\text{if } a=-3 \text{ and } b \neq -3}$

1

4 Yes, It is possible for a homogeneous system $Ax=b$ $(b \neq 0)$ to be inconsistent, when the corresponding associated homogeneous system $Ax=0$ has a unique solution (i.e. the trivial solution).

Example:

$$2x+3y=13$$
$$3x+5y=21$$
$$x+y=6$$

Augmented Matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 13 \\ 3 & 5 & 21 \\ 1 & 1 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 3 & 5 & 21 \\ 2 & 3 & 13 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$
$R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 - R_3}]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The corresponding system reduces to:
$$x = 0$$
$$y = 0$$
$$\underbrace{0 = 1}_{\text{(not possible)}}$$

The matrix has a row of the form $[0, 0, \dots, 0, b]$ $b \neq 0$

$[0, 0, 1]$

(The last column is a pivot column)

and so the system $Ax = b$ is inconsistent.

However, the corresponding homogeneous system has a unique (~ trivial solution) solution.

(Note that the number of equations > ~~no~~ number of unknowns here)

5 (a) Find the values of x for which the following matrix is an augmented matrix corresponding to a consistent system.

$$A = \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

(b) Find the RREF of the matrix formed by replacing x in A by π.

(a)

$$\begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & -15 & 6 & x^3 - 4x \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & 0 & 0 & x^3 - 4x + 3x^2 \end{bmatrix}$$

So, the system is consistent

if and only if $x^3 + 3x^2 - 4x = 0$

i.e. if and only if $x(x^2 + 3x - 4) = 0$

$\Rightarrow x(x+4)(x-1) = 0$

$$\boxed{\text{So, the matrix is an augmented matrix of a consistent system when } x = 0, -4, 1}$$

(b) First we find the RREF form of the matrix A.

Now

$$A \longrightarrow \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & 0 & 0 & x^3+3x^2-4x \end{bmatrix} \quad \text{(done in part (a))}$$

$$= \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{bmatrix} 1 & -2 & +1 & \pi \\ 0 & 1 & -2/5 & \pi^2/5 \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1+2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5} \\ 0 & 1 & -\frac{2}{5} & \pi^2/5 \\ 0 & 0 & 0 & \pi^3+3\pi^2-4\pi \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{\pi^3+3\pi^2-4\pi}} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi+\frac{2\pi^2}{5} \\ 0 & 1 & -2/5 & \pi^2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Continued)

$$\xrightarrow[R_2 \to R_2 - \frac{\pi^3}{5} R_3]{} \begin{bmatrix} 1 & 0 & \frac{1}{5} & \pi + \frac{2\pi^2}{5} \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - \left(\pi + \frac{2\pi^2}{5}\right) R_3]{} \begin{bmatrix} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(6) If the matrix B has been obtained from the matrix A by an elementary row operation, then the vector v is a solution of $Av = 0$ if and only if v is a solution of $Bv = 0$

Proof: Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$

Let e be an elementary row operation and Let $B = e(A) = [b_{ij}]$

and Let $v \in \mathbb{R}^n$ $\quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$\Rightarrow$: Assume that $Av = 0$

Will show that $Bv = 0$.

Since e affects only one or two rows of A, we need to consider only the affected entries of Bv (note that Bv is a column vector).

Case 1: Let e be a scale operation

$R_i \longrightarrow cR_i$ (c is a nonzero scalar).

Then the ith entry of Bv is

$$b_{i1}v_1 + b_{i2}v_2 + \cdots + b_{in}v_n$$

$$= ca_{i_1}v_1 + ca_{i_2}v_2 + \cdots + ca_{in}v_n$$

$$= c(a_{i_1}v_1 + a_{i_2}v_2 + \cdots + a_{in}v_n) = c.0 . \quad (\text{Since } Av = 0)$$

$$= 0$$

Case 2 Let e be an interchange

say $R_i \longleftrightarrow R_k$.

We need to consider only the ith and kth entries of Bv.

The ith entry is

$$b_{i1}v_1 + \cdots + b_{in}v_n = \underbrace{a_{k1}v_1 + \cdots + a_{kn}v_n}_{\text{(interchange)}} = 0 \quad \left(\text{Since } Av = 0\right)$$

Similarly kth entry of Bv is also zero.

Case 3: Let e be a replacement operation say $R_i \to R_i + cR_k$ where c is a nonzero scalar.

We need to consider only the ith entry of Bv.

ith entry of Bv

$= b_{i1}v_1 + \cdots + b_{in}v_n$

$= (a_{i1} + c\, a_{k1})v_1 + \cdots + (a_{in} + c\, a_{kn})v_n$

$= (a_{i1}v_1 + \cdots + a_{in}v_n) + c(a_{k1}v_1 + \cdots + a_{kn}v_n)$

$= 0 + 0 = 0$ (since $Av = 0$)

So, $Bv = 0$ in all cases.

$\Leftarrow$ given $Bv = 0$

Will show $Av = 0$.

Now we know that if $B = e(A)$, there exists an ~~inverse~~ elementary row operation of the same type (denoted by e^{-1}) such that $A = e^{-1}(B)$

Now applying the first part to the matrix B and $e^{-1}(B)$, we get the required result.