$$\boxed{\text{MTH 100 : Lecture 24}}$$

Ex: Fix $1 \le i \le n$

Let $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined by

$$P_i(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots x_n) = (0, 0, \dots, 0, x_i, 0, \dots 0)$$

Then P_i is a Linear transformation:

- Let $u = (x_1, x_2, \dots, x_n)$ and $v = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

Now $P_i(u+v) = P_i\big((x_1, \dots, x_n) + (y_1, \dots, y_n)\big)$

$$= P_i\big((x_1 + x_2, \dots, x_n + y_n)\big)$$

$$= (0, \dots, 0, x_i + y_i, 0, \dots, 0)$$

$$= (0, \dots, 0, x_i, 0, \dots, 0) + (0, \dots, 0, y_i, 0, \dots, 0)$$

$$= P_i(u) + P_i(v)$$

- Next let $u = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let $c \in \mathbb{R}$

Then $P_i(cu) = P_i\big(c(x_1, \dots, x_n)\big) = P_i\big((cx_1, \dots, cx_n)\big)$

$$= (0, \dots, 0, cx_i, 0, \dots, 0) = c(0, \dots, 0, x_i, 0, \dots, 0)$$

$$= cP_i(u)$$

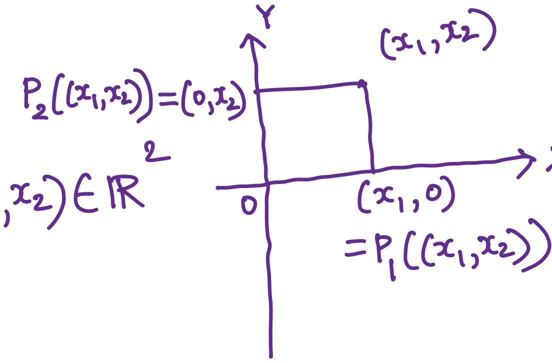
Therefore P_i is a linear transformation.

Ex: Let us take $n=2$.

Then we will get two linear transformations

$$P_1, P_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\left.\begin{aligned} P_1((x_1,x_2)) &= (x_1,0) \\ \text{and } P_2((x_1,x_2)) &= (0,x_2) \end{aligned}\right\} \forall (x_1,x_2) \in \mathbb{R}^2$$



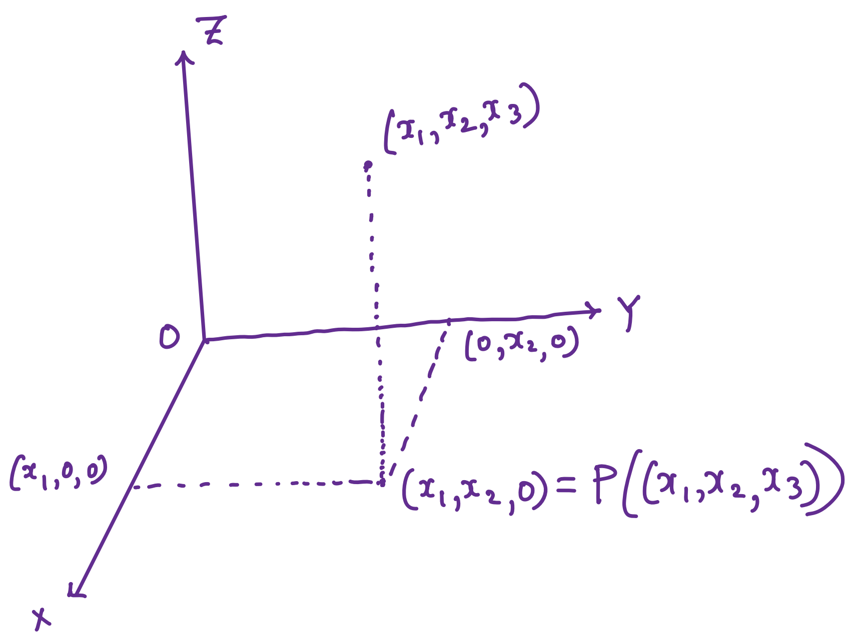
Ex: Look at the transformation for $\mathbb{R}^3$

Ex: Define $P_{12} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$P_{12}((x_1,x_2,x_3)) = (x_1,x_2,0)$$

Show that P_{12} is a linear transformation

Similarly define transformations P_{23} and P_{31}.



Ex: Fix i, j such that $1 \le i < j \le n$

Define $P_{ij} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$P_{ij}((x_1, \dots, x_i, \dots, x_j, \dots, x_n))$$
$$= (0, \dots, 0, x_i, 0, \dots, 0, x_j, 0, \dots, 0)$$

Show that P_{ij} is a linear transformation.

Remarks:

(1) If $T : V \to W$ is linear, then

(a) $T(0) = 0$ (b) $T(-v) = -T(v)$

Proof:

(a) $T(v) = T(v+0) = T(v) + T(0)$ (Since T is Linear)

$\Rightarrow (-T(v)) + T(v) = (-T(v)) + T(v) + T(0)$

$\Rightarrow 0 = 0 + T(0) \Rightarrow \boxed{T(0) = 0}$

(b) $T((-v) + v) = T(0) = 0$ (By (a))

$\Rightarrow T(-v) + T(v) = 0$ (Since T is Linear)

$\Rightarrow T(-v) + T(v) + (-T(v)) = 0 + (-T(v))$

$\Rightarrow T(-v) + 0 = -T(v) \Rightarrow \boxed{T(-v) = -T(v)}$

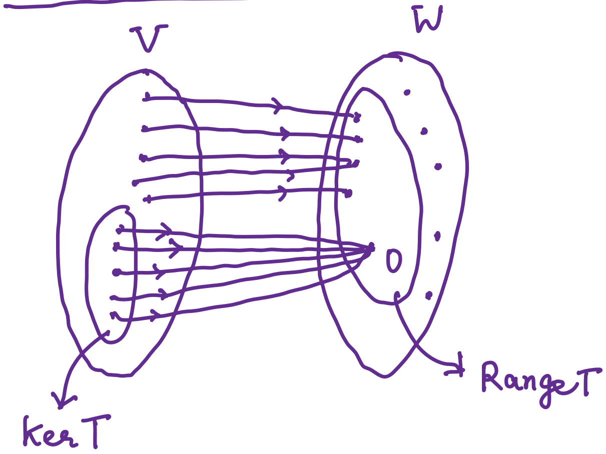
(2) If T is linear, T preserves linear combinations:

i.e. $T(c_1 v_1 + c_2 v_2 + \dots + c_k v_k)$

$$= c_1 T(v_1) + c_2 T(v_2) + \dots + c_k T(v_k)$$

Proof: Exercise

Two important Subspaces associated with a Linear Transformation:



(1) Let $T : V \longrightarrow W$ be a linear transformation.

- Then the kernel of T, $\ker T = \{ v \in V : T(v) = 0 \in W \}$ is a subspace of V.
  $\ker T$ is also called the null space of T, denoted by $\operatorname{Nul} T$.
- The range of T,
  $\operatorname{Range} T = \{ \omega \in W : \omega = T(v) \text{ for some } v \in V \}$
  is a subspace of W.

Proof:

- If $u, v \in \ker T$, then $T(u) = 0$, $T(v) = 0$

Now, $T(u+v) = T(u) + T(v) = 0 + 0 = 0$

$\Rightarrow u + v \in \ker T$

If $u \in \ker T$ and $c \in F$ then $T(u) = 0$

Now $T(cu) = c\,T(u) = c.0 = 0 \Rightarrow cu \in \ker T$

Hence $\ker T$ is a subspace of V.

- If $w_1, w_2 \in \text{Range}\,T$, then there exist $u, v \in V$ such that $T(u) = w_1$ and $T(v) = w_2$

Now, $w_1 + w_2 = T(u) + T(v) = T(u+v)$ (since T is linear)
and $u + v \in V$ (since V is a vector space)

So, $w_1 + w_2 \in \text{Range}\,T$

If $w \in \text{Range}\,T$ and $c \in F$, then there exists $v \in V$ such that $T(v) = w$

Now, $cw = cT(v) = T(cv)$ (since T is linear)
and $cv \in V$ (since V is a vector space)

So, $cw \in \text{Range}\,T$

Hence $\text{Range}\,T$ is a subspace of W.

Note: • For any linear transformation T,
$\ker T \neq \phi$ since $0 \in \ker T$ (since $T(0) = 0$)

Definition: A linear transformation $T: V \to W$ is called injective (1-1) if $Tu = Tv \Rightarrow u = v \;\forall\, u, v \in W$ (or equivalently $u \neq v \Rightarrow Tu \neq Tv \;\forall\, u, v \in V$)

Important Remark:
If $T: V \to W$ is a linear transformation, then T is injective (1-1) if and only if $\ker T = \{0\}$.

Proof:

$\Rightarrow$: Assume that T is 1-1. We have seen that $0 \in \ker(T)$ (since $T(0) = 0$)

Now let $v \in V$, $v \neq 0 \Rightarrow T(v) \neq T(0)$ (since T is 1-1)

$$\Rightarrow T(v) \neq 0$$

$$\Rightarrow v \notin \ker T$$

So, $\ker T = \{0\}$

$\Leftarrow$: Assume that $\ker T = \{0\}$

Now $T(u) = T(v)$

$$\Rightarrow T(u) - T(v) = 0$$

$$\Rightarrow T(u - v) = 0 \quad (\text{since } T \text{ is linear})$$

$$\Rightarrow u - v \in \ker T$$

$$\Rightarrow u - v = 0 \quad (\text{since } \ker T = \{0\})$$

$$\Rightarrow u = v \Rightarrow T \text{ is } 1\text{-}1. \quad (\text{QED})$$