MTH 100 : Lecture 20

How to create Bases:

Proposition: Suppose $S = \{v_1, v_2, \dots, v_n\}$ is a linearly independent set in a vector space V. Suppose v is a vector which is not in the span S. Then the set obtained by adjoining v to S is linearly independent.

Proof: Need to show that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + c v = \bar{0} \quad \cdots\cdots (1)$$

$$\Rightarrow c_1 = c_2 = \cdots\cdots = c_n = c = 0$$

Suppose $c \neq 0$. Then $c^{-1} \in F$ (where F is the field of scalars.)

Then (1) $\Rightarrow cv = -c_1 v_1 - c_2 v_2 - \cdots\cdots - c_n v_n$

$$\Rightarrow v = -c^{-1} c_1 v_1 - c^{-1} c_2 v_2 - \cdots\cdots - c^{-1} c_n v_n$$

$$\Rightarrow v = (-c^{-1} c_1) v_1 + (-c^{-1} c_2) v_2 + \cdots\cdots + (-c^{-1} c_n) v_n$$

$\Rightarrow v \in \text{Span}\{v_1, v_2, \dots, v_n\}$ which contradicts the assumption that $v \notin \text{Span}\{v_1, v_2, \dots, v_n\}$

So, $c = 0$. Then (1) $\Rightarrow c_1 v_1 + c_2 v_2 + \cdots\cdots + c_n v_n = \bar{0}$

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ (Since $v_1, v_2, \dots, v_n$ are linearly independent)

Hence $v_1, v_2, \dots, v_n, v$ are linearly independent.

Proposition: Any linearly independent set S in a finite dimensional vector space can be expanded to a basis.

Proof: Exercise

Hint: Use the previous proposition repeatedly. By Steinitz Exchange Lemma, the process has to stop and at that stage, a basis is obtained.

Proposition: Any finite spanning set S in a nonzero vector space can be contracted to a basis.

Proof: Exercise

Note: If a non-zero vector space V has a finite spanning set S, then it must be finite dimensional.

Summary:

Proposition: Let V be a non-zero finite dimensional vector space with dimension n.

Then,

- Any linear independent set of vectors must have $\leq n$ vectors.
  If a linearly independent set has n-vectors, then it must be a basis.
  i.e. it must also be a spanning set for V.
- Any spanning set for V must have $\geq n$ vectors. If a spanning set has n vectors, it must be a basis.
  i.e. it must also be linearly independent.

Note: Thus we can regard a basis as either a $\boxed{\text{maximal linearly independent set}}$ or as a $\boxed{\text{minimal spanning set.}}$

Ex: Let $V = \mathbb{R}^4$, $W = \mathrm{span}(S)$, where $S = \{w_1, w_2, w_3\}$

Insert v_1 and v_2 into S replacing suitable w's to get a new spanning set for W applying the method of Steinitz Exchange Lemma.

$$v_1 = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 9 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 8 \\ 12 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 9 \\ 11 \\ 19 \\ 33 \end{bmatrix},$$

$$w_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -3 \end{bmatrix}$$

Need to Solve:

$$x_1 w_1 + x_2 w_2 + x_3 w_3 = v_1$$

$$\Rightarrow \quad x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 9 \\ 11 \\ 19 \\ 33 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 9 \end{bmatrix}$$

The Augmented matrix $[w_1, w_2, w_3 \mid v_1]$

$$= \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 1 & 11 & -1 & 3 \\ 1 & 19 & -1 & 7 \\ 1 & 33 & -3 & 9 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{} \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 10 & 0 & 5 \\ 0 & 24 & -2 & 7 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 10 & 0 & 5 \\ 0 & 24 & -2 & 7 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 24R_2]{R_3 \to R_3 - 10R_2} \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to \left(-\frac{1}{2}R_3\right)} \left[\begin{array}{ccc|c} 1 & 9 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 9 & 0 & \frac{9}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 9R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the solution is $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = \frac{5}{2}$

Hence $0 . w_1 + \frac{1}{2} w_2 + \frac{5}{2} w_3 = v_1$

We can replace either w_2 or w_3 to get a new spanning set (w_1 can't be replaced).

Let $S_1 = \{w_1, v_1, w_3\}$ be our new spanning set.

Let us solve : $x_1 \omega_1 + x_2 v_1 + x_3 \omega_3 = v_2$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 7 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \\ 12 \end{bmatrix}$$

The augmented matrix $[\omega_1, v_1, \omega_3 \,\vdots\, v_2]$

$$= \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 1 & 3 & -1 & 4 \\ 1 & 7 & -1 & 8 \\ 1 & 9 & -3 & 12 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{} \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 5 \\ 0 & 7 & -2 & 9 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - 5R_2 \\ R_4 \to R_4 - 7R_2}]{} \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] \xrightarrow[R_3 \leftrightarrow R_4]{} \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to (-\frac{1}{2} R_3)]{} \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 + R_3]{} \left[\begin{array}{ccc:c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_1 = 0$, $x_2 = 1$, $x_3 = -1$

$\Rightarrow \quad 0.w_1 + 1.v_1 + (-1)\,w_3 = v_2$

Thus w_3 can be replaced by v_2
(w_1 can't be replaced)

Hence the new spanning set will be

$$\boxed{\{w_1, v_1, v_2\}}$$

Dimension of Subspaces:

Definition: A proper subspace of a vector space is a subspace different from the zero subspace and the entire space.

Proposition: If W is a proper subspace of a finite-dimensional space V, then W is also finite dimensional and $0 < \dim W < \dim V$.

Proof: Since W is a proper subspace of V,
$W \neq \{0\}$

So, there exists $w_1 \in W$ $(w_1 \neq 0)$

If $\operatorname{span}\{w_1\} = W$, then W is finite dimensional.
$(\dim W = 1)$

If $\operatorname{span}\{w_1\} \neq W$, there exists $w_2 \in W$ $(w_2 \neq 0)$
such that $w_2 \notin \operatorname{span}\{w_1\}$

By adjoining w_2 to w_1, we get a linearly independent set $\{w_1, w_2\}$.

Continuing in this way, we get a basis of W with atmost $\dim V$ elements
(By Steinitz Exchange lemma)

Hence W is finite dimensional
and $0 < \dim W \leq \dim V$

Since W is a proper subspace of V, there exists $v \in V$ $(v \neq 0)$ such that $v \notin W$. Adjoining v to any basis of W, we will have a linearly independent set in V. Hence $\dim W$ is strictly less than $\dim V$.

i.e. $\boxed{\dim W < \dim V}$