MTH 100: Lecture 21

Example of another infinite dimensional vector space:

Ex: Let $C[a,b]$ be the vector space of all real valued continuous functions defined on $[a,b]$.

Question: Is $C[a,b]$ finite dimensional?

Answer: No. $C[a,b]$ is infinite dimensional.

Assume BWOC that $C[a,b]$ is finite dimensional.

Let $P[a,b]$ be the set of all (real valued) polynomials with domain $[a,b]$.

Now $P[a,b] \subset C[a,b]$

Furthermore $P[a,b]$ is a subspace of $C[a,b]$.
(check!)

Now the space $P[a,b]$ is infinite dimensional.

(The proof essentially uses the same argument we used to prove that $R[t]$ is infinite dimensional)

Now $P[a,b]$ is a subspace of $C[a,b]$.
Thus if $C[a,b]$ is finite dimensional,
then $P[a,b]$ will also be finite dimensional.
— a contradiction.
Hence $\dim(C[a,b]) = \infty$.

<u>Proof of the fact that $P[a,b]$ is infinite dimensional:</u>

Suppose BWOC that $P[a,b]$ is finite dimensional.
Then it has a finite Basis,
say $\{p_1(x), p_2(x), \dots, p_k(x)\}$
Let $N = \max\{\deg p_1, \deg p_2, \dots, \deg p_k\}$
and let $p(x) = x^{N+1}$
Then $p(x)$ can't be written as a linear combination of $p_1, p_2, \dots, p_k$ because any linear combination of $p_1, p_2, \dots, p_k$ will be a polynomial of degree $\leq N$
and $\deg p(x) = N+1$,
a contradiction
Hence $P[a,b]$ is infinite dimensional.

Note: For the space $R[t]$, $1, t, t^2, \ldots, t^n, \ldots$ is a basis because these are linearly independent and any polynomial can be written as a finite linear combination of these polynomials.

If $p(t) \in R[t]$ and $\deg p(t) = N \, (< \infty)$ then there exist scalars $c_0, c_1, \ldots, c_N$ such that $p(t) = \sum_{i=0}^{N} c_i t^i$

Sum of Subspaces:

Definition: Let U and W be subspaces of the vector space V.

Then the sum of U and W is defined by

$$U + W = \{ u + w : u \in U, \, w \in W \}$$

Furthermore,

- $U + W$ is a subspace of V.
- In fact $U + W$ is the smallest subspace of V containing U and W.

Proposition: If U and W are finite-dimensional subspaces of the vector space V, then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Proof: • If either U or $W = \{0\}$, the result is obvious.

• Now let $B = \{k_1, k_2, \ldots, k_m\}$ be a basis of $U \cap W$

(If $U \cap W = \{0\}$, this step is not needed)

Since $U \cap W \subseteq U$, we can expand B to a basis B_1 of U, by adjoining vectors $u_1, u_2, \ldots, u_n$

i.e. $B_1 = \{k_1, k_2, \ldots, k_m, u_1, \ldots, u_n\}$, $m \geq 0$, $n \geq 0$

Similarly since $U \cap W \subseteq W$, we can expand B to a Basis B_2 of W, by adjoining the vectors $w_1, \ldots, w_p$

i.e. $B_2 = \{k_1, k_2, \ldots, k_m, w_1, \ldots, w_p\}$ $m \geq 0$, $p \geq 0$

Let $C = B \cup B_1 \cup B_2 = \{k_1, k_2, \ldots, k_m, u_1, \ldots, u_n, w_1, \ldots, w_p\}$

We claim that C is a basis for $U+W$

So, we need to prove that

(1) $\operatorname{Span} C = U+W$ (2) C is linearly independent.

① Let $v \in U+W$. Then $v = u+w$ where $u \in U$ $w \in W$

Then there exist scalars $c_1, \dots, c_m, d_1, \dots, d_n, f_1, \dots, f_m, g_1, \dots, g_p \in F$

such that $u = c_1 k_1 + \dots + c_m k_m + d_1 u_1 + \dots + d_n u_n$

$w = f_1 k_1 + \dots + f_m k_m + g_1 w_1 + \dots + g_p w_p$

So, $v = u + w = (c_1 + f_1) k_1 + \dots + (c_m + f_m) k_m$
$+ d_1 u_1 + \dots + d_n u_n + g_1 w_1 + \dots + g_p w_p$

Thus v is a linear combination of the elements of C.

Hence $U + W = \operatorname{Span}(C)$

② Now suppose

$c_1 k_1 + \dots + c_m k_m + d_1 u_1 + \dots + d_n u_n + g_1 w_1 + \dots + g_p w_p = \overline{0}$ .... ①

Then $c_1 k_1 + \dots + c_m k_m + d_1 u_1 + \dots + d_n u_n$

$= -g_1 w_1 - \dots - g_p w_p$ .... ②

Now the L.H.S. of ② is a vector in U and the R.H.S. of ② is a vector in W and so it is in $U \cap W$.

Hence we can write

$c_1 k_1 + \dots + c_m k_m + d_1 u_1 + \dots + d_n u_n$

$= f_1 k_1 + \dots + f_m k_m$ where $f_1, \dots, f_m \in F$

$\Rightarrow (c_1 - f_1) k_1 + \dots + (c_m - f_m) k_m + d_1 u_1 + \dots + d_n u_n = \overline{0}$

Since $\{k_1, \dots, k_m, u_1, \dots, u_n\}$ is a basis for U,

it is linearly independent.

So, $d_1 = d_2 = \dots = d_n = 0$

Then (1) becomes

$$c_1 k_1 + \cdots + c_m k_m + g_1 w_1 + \cdots + g_p w_p = \bar{0}$$

Since $\{k_1, \ldots, k_m, w_1, \ldots, w_p\}$ is a basis for W,

it is linearly independent and

therefore $c_1 = \cdots = c_m = g_1 = \cdots = g_p = 0$

Hence C is linearly independent and

So, C is a basis for $U+W$.

Now $\dim U + \dim W - \dim(U \cap W)$

$$= (m+n) + (m+p) - m$$

$$= \cancel{m} + n + m + p - \cancel{m}$$

$$= m + n + p$$

$$= \dim(U+W)$$

Therefore $\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$