Lecture : 5

Observation (R S):

- If a vector u is a given solution of $A\mathbf{x}=\bar{b}$, then another vector is a solution of $A\mathbf{x}=\bar{b}$ if and only if it is of the form $u+v$ where v is a solution of the associated homogeneous system.

- In case $A\mathbf{x}=0$ has only trivial solution (ie $v=0$), then there is a unique solution u. (otherwise we have infinitely many solutions.)

Proof: $\Rightarrow$: given : Let u be a solution of $A\mathbf{x}=\bar{b}$ and let w be another solution of $A\mathbf{x}=\bar{b}$

want to show : w is of the form $u+v$ where v is a solution of $A\mathbf{x}=\bar{0}$

Let $v = w-u$, then $w = u+v$

and $Av = A(w-u) = Aw - Au = \bar{b}-\bar{b} = \bar{0}$

So, v is a solution of the associated homogeneous equation and $w = u+v$

$\Leftarrow$:

Given: Let u be a solution of $Ax=\bar{b}$
and let v be a solution of $Ax=\bar{0}$

Want to show: $w=u+v$ is a solution of $Ax=\bar{b}$

We have $Aw = A(u+v) = Au + Av = \bar{b} + \bar{0} = \bar{b}$

So, w is a solution of $Ax=\bar{b}$

(Q.E.D.)

Example:

Consider the system of equation

$$x_1+x_2+x_3=1$$
$$2x_1-x_2+x_3=2$$

The Augmented Matrix [A : b]:

$$\left[\underbrace{\begin{matrix}1&1&1\\2&-1&1\end{matrix}}_{A}\;\vdots\;\underbrace{\begin{matrix}1\\2\end{matrix}}_{\bar{b}}\right] \xrightarrow{R_2\to R_2-2R_1} \left[\begin{array}{ccc|c}1&1&1&1\\0&-3&-1&0\end{array}\right]$$

$$\xrightarrow{R_2\to\left(-\frac{1}{3}\right)R_2} \left[\begin{array}{ccc|c}1&1&1&1\\0&1&\frac{1}{3}&0\end{array}\right] \xrightarrow{R_1\to R_1-R_2} \underbrace{\left[\begin{array}{ccc|c}1&0&\frac{2}{3}&1\\0&1&\frac{1}{3}&0\end{array}\right]}_{\text{(RREF matrix)}}$$

The corresponding System is:

$$\left.\begin{array}{l} x_1 + \frac{2}{3}x_3 = 1 \\ x_2 + \frac{1}{3}x_3 = 0 \end{array}\right\} \Rightarrow \underbrace{\begin{array}{l} x_1 = 1 - \frac{2}{3}x_3 \\ x_2 = -\frac{1}{3}x_3 \\ x_3 = x_3 \text{ (dummy equation)} \end{array}}$$

$$\Downarrow$$

So, the solution is of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{u} + t\underbrace{\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}}_{v} = u + tv$$

where $t \in \mathbb{R}$ (scalar)

where u is a solution of the given non-homogeneous system and v is a solution of the associated homogeneous system.

Check: $Au = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \bar{b}$

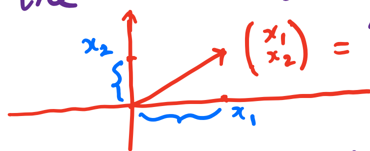
$$Av = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} - \frac{1}{3} + 1 \\ -\frac{4}{3} + \frac{1}{3} + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$:

- A vector in $\mathbb{R}^2$ is an ordered pair of real numbers (written either as column or row) (e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$ or (a, b))

(In case of $\mathbb{R}^3$, it is a 3-tuple)

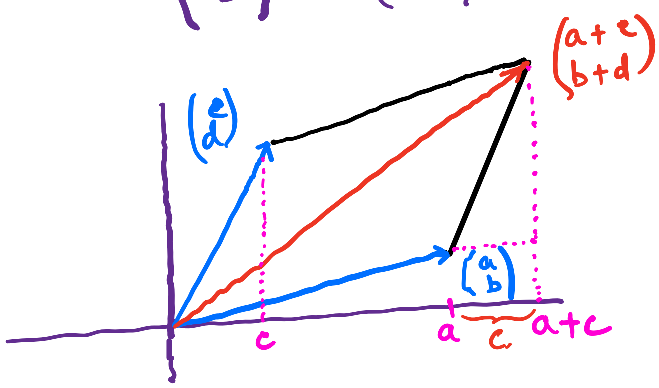
It gives us the geometric interpretation of the vector as the arrow pointing from $(0,0)$ to $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



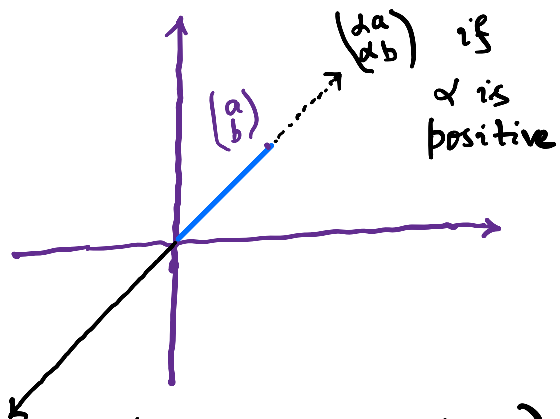
- We can add two vectors (coordinate wise) and multiply any vector by a real number and this is consistent with the geometric interpretation.

Thus $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$ and $\alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$

for any $\alpha \in \mathbb{R}$



(Addition of Vectors)



$\begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$ (if λ is negative)

(Scalar Multiplication of a vector)

Note that addition and scalar multiplication satisfies the following properties.

- $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} c+a \\ d+b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

(commutative property)

Similarly

- $\left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right] + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \left[\begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}\right]$

(Associative property)

- $\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

- $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Also $1\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1.a \\ 1.b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\alpha\left(\beta\begin{pmatrix} a \\ b \end{pmatrix}\right) = (\alpha\beta)\begin{pmatrix} a \\ b \end{pmatrix}$ for any $\alpha, \beta \in \mathbb{R}$

$$(\alpha+\beta)\begin{pmatrix}a\\b\end{pmatrix} = \alpha\begin{pmatrix}a\\b\end{pmatrix} + \beta\begin{pmatrix}a\\b\end{pmatrix}$$

for any $\alpha, \beta \in \mathbb{R}$

$$\alpha\left[\begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}c\\d\end{pmatrix}\right] = \alpha\begin{pmatrix}a\\b\end{pmatrix} + \alpha\begin{pmatrix}c\\d\end{pmatrix}$$

for any $\alpha \in \mathbb{R}$

- Note that all these properties follow from the corresponding properties of real numbers
  (Please verify them at home)

- Similarly we can define vector addition and scalar multiplication in $\mathbb{R}^3$ and also in $\mathbb{R}^n$ (for any positive integer n)

- <u>Geometrical interpretation of solutions:</u>

- In case we are looking with 2-tuples or 3 tuples, we can have a geometrical interpretation.
- Each vector correspond to a point either <u>in plane</u> (2-space) or <u>in space</u> (3-space)

- Then the solution of a homogeneous system is either the origin only or all the points on a line or, a plane through the origin.

- If a non-homogeneous system has even a single solution (i.e. a point in plane or space) then its entire solution set consists of either only that point or the line or plane through that point which is parallel to the solution of the associated homogeneous system.

From geometry:

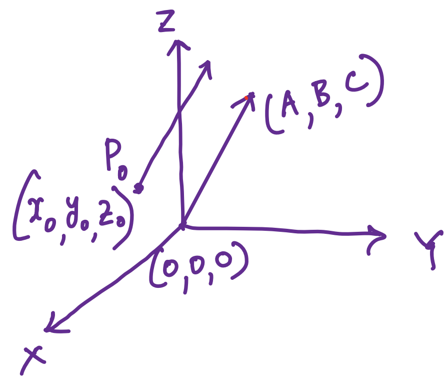
- Equation of the line through $P_0(x_0, y_0, z_0)$ parallel to a given vector

$$v = Ai + Bj + Ck$$

(i.e. the line segment from $(0,0,0)$ to (A,B,C))

is given by

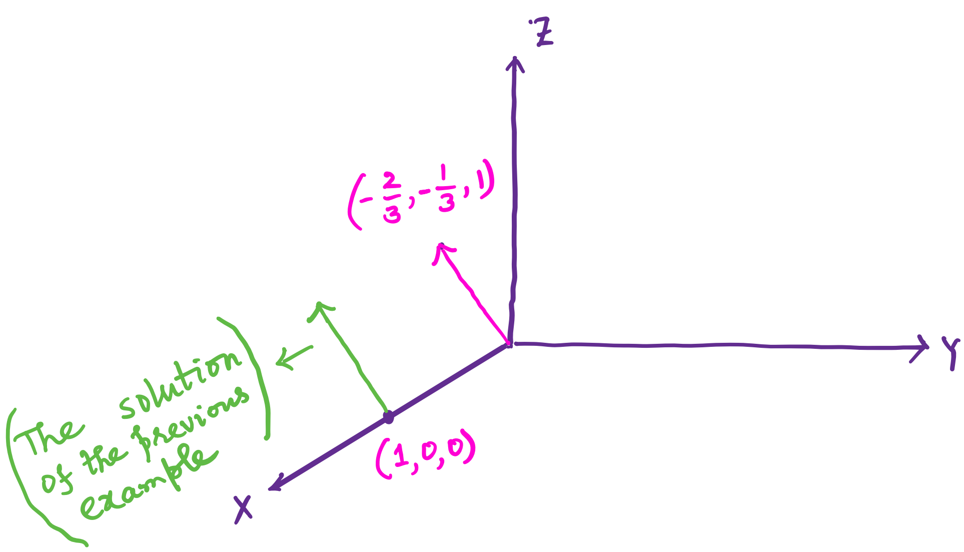
$$\left.\begin{aligned} x &= x_0 + tA \\ y &= y_0 + tB \\ z &= z_0 + tC \end{aligned}\right\} \text{ where } t \in \mathbb{R}$$



So, $(x, y, z) = (x_0, y_0, z_0) + t(A, B, C)$

$= u + tv$

- The solution we have obtained corresponds to the geometrical equation of the line through $(1,0,0)$ which is parallel to the vector determined by $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$



Summary For Non-homogeneous System:

Associated Homogeneous System $Ax=\bar{0}$		Non-homogeneous System $Ax=\bar{b}$
case 1:	Unique Solution (trivial) $\updownarrow$ (No free Variable) $\longrightarrow$	Inconsistent **or** Unique solution (non-zero)
case 2:	Infinitely many solutions $\updownarrow$ Atleast one free Variables $\longrightarrow$	Inconsistent **or** Infinitely many solutions.

Note: $A x=\bar{b}$ can be inconsistent in both cases. However, if it is consistent, nature of solutions corresponds to nature of solution of $Ax=\bar{0}$