1 $\Rightarrow$:

Given that $B = \{v_1, v_2, \cdots, v_n\}$ is a basis of the vector space V over the field F.

Want to show: Every vector $v \in V$ is uniquely expressible as a linear combination of elements of B.

Since $\text{Span}(B) = V$ and $v \in V$ is any vector, v can be expressed as a linear combination of vectors of B.

i.e., there exists $c_1, c_2, \cdots, c_n \in F$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Assume that v has another representation of the form $v = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$

Then $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$
$= d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$

$\Rightarrow (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \cdots + (c_n - d_n) v_n = 0$

Since $v_1, v_2, \ldots, v_n$ are linearly independent,

$c_1 - d_1 = 0, \quad c_2 - d_2 = 0, \quad \ldots, \quad c_n - d_n = 0$

$\Rightarrow c_1 = d_1, \; c_2 = d_2, \; \ldots, \; c_n = d_n.$

Thus ~~the repres~~ representation of v is unique.

$\Leftarrow$: Given: Every vector $v \in V$ is uniquely expressible as a linear combination of $\{v_1, v_2, \ldots, v_n\} = B$

To prove: B is a Basis of V

Clearly, since every vector $v \in V$ is uniquely expressible as a linear combination of $\{v_1, \ldots, v_n\} = B$, $\boxed{\text{Span } B = V}$

Also 0 (since every vector is uniquely expressible) is uniquely expressable as:

$0 = 0.v_1 + 0.v_2 + \cdots + 0.v_n$

Since this is the only possible way, B is $\boxed{\text{linearly independent.}}$

Hence B is a basis of V.

(4) For $s, t = 1, 2$, let $E_{s,t}$ be the 2×2 matrix whose (s,t)th element is 1 and all other elements are zero.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$,

then $A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$

So, $\mathbb{R}^{2\times 2} = \text{span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$

Also if $c_1 E_{11} + c_2 E_{12} + c_3 E_{21} + c_4 E_{22} = [0]$

$\Rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0$

So, E_{11}, E_{12}, E_{21} and E_{22} are linearly independent.

So, $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $\mathbb{R}^{2\times 2}$

In general for $\mathbb{R}^{m\times n}$,

$B = \{E_{st}, \ s = 1, 2, --, m, \ t = 1, 2, --, n\}$

is a basis for $\mathbb{R}^{m\times n}$. This is known as the standard basis.

1) $\{v_1, v_2\}$ is a linearly independent set

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$B = \{e_1, e_2, e_3\}$ is a Basis (and hence spanning set) of $\mathbb{R}^3$

Let us solve

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = v_1$$

The Augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So, $c_1 = 0, \; c_2 = 1, \; c_3 = 1$

$\Rightarrow \quad v_1 = 0.e_1 + 1.e_2 + 1.e_3$

So, a new spanning set is $B_1 = \{e_1, v_1, e_3\}$ (e_1 can't be replaced)

Now let us solve

$$c_1 e_1 + c_2 v_1 + c_3 e_3 = v_2$$

The Augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, $c_1 = 1, \; c_2 = 1, \; c_3 = 0$

$\Rightarrow \quad v_2 = 1 e_1 + 1.v_1 + 0.e_3$

So, we will have to replace e_1 by v_2 (e_3 can't be replaced)

So, the new spanning set is $\boxed{B_2 = \{v_2, v_1, e_3\}}$

Since B_2 has three vectors and $\dim(\mathbb{R}^3) = 3$, B_2 is a Basis

4 Given $u = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 10 \\ 9 \\ 21 \end{bmatrix} \in \mathbb{R}^3$ are linearly independent vectors.

We need to find a vector v not in the span $\{u, w\}$

Then $\{u, w, v\}$ will be a basis for $\mathbb{R}^3$

We solve

$c_1 u + c_2 w = b$ for any general vector $b \in \mathbb{R}^3$

Let $b = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

The Augmented matrix

$$\left[\begin{array}{cc|c} 3 & 10 & p \\ 3 & 9 & q \\ 7 & 21 & r \end{array}\right] \xrightarrow[R_3 \to R_3/7]{R_2 \to R_2/3} \left[\begin{array}{cc|c} 3 & 10 & p \\ 1 & 3 & q/3 \\ 1 & 3 & r/7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cc|c} 3 & 10 & p \\ 1 & 3 & q/3 \\ 0 & 0 & \frac{r}{7} - \frac{q}{3} \end{array}\right]$$

We don't need to reduce it to RREF matrix.

We can conclude that the above system is inconsistent if $\frac{r}{7} - \frac{q}{3} \neq 0$

We can make many choices of v.

e.g. $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\left(\frac{1}{7} - \frac{1}{3} \neq 0 \right)$ $\left(\text{or } \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \frac{2}{7} - \frac{1}{3} \neq 0 \right)$

So, $\left\{ \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 10 \\ 9 \\ 21 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \{u, w, v\}$

forms a Basis of $\mathbb{R}^3$.

5) Given: $A \in \mathbb{R}^{m \times m}$ Let $A = [v_1, \dots, v_m]$

$v_1, \dots, v_m \in \mathbb{R}^m$

$\Rightarrow$: Assume A to be invertible:

Want to show: $\{v_1, \dots, v_m\}$ forms a basis of $\mathbb{R}^m$

Consider the system of homogeneous equation

$$Ax = 0 \quad \text{or} \quad x_1 v_1 + \dots + x_n v_n = 0$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Since A is invertible, the above system have only the trivial solution

ie $x_1 v_1 + \dots + x_n v_n = 0 \Rightarrow x_1 = x_2 = \dots = x_n = 0$

$\Rightarrow v_1, v_2, \dots, v_n$ are linearly independent.

Since $\dim(\mathbb{R}^m) = m$,
$\{v_1, \dots v_m\}$ form a Basis.

$\Leftarrow$: Assume that $\{v_1, \dots, v_m\}$ is a Basis of $\mathbb{R}^m$.
Want to show that $A = [v_1, \dots, v_m]$

Since $\{v_1, \dots v_m\}$ is a Basis of $\mathbb{R}^m$,
it is linearly independent and so
$x_1 v_1 + \dots + x_n v_n = 0 \Rightarrow x_1 = \dots = x_n = 0$

$\Rightarrow$ The system of homogeneous equation
$Ax = 0$ has only the trivial solution.

$\Rightarrow$ A is invertible.

5) Let $A \in \mathbb{R}^{m \times m}$, $A \neq 0$.
Note that $\dim(\mathbb{R}^{m \times m}) = m^2$
Consider the matrices $I, A, A^2, \dots, A^{m^2}$

If $A^k = 0$ for any k, $1 < k \leq m^2$,
then A satisfies the polynomial $p(x) = x^k$
Similarly if $A^i = A^j$ for $i \neq j$ $(A^0 = I)$,
then A satisfies the polynomial
$q(x) = x^i - x^j$

Finally, assume that all the matrices $I, A, A^2, \dots, A^{m^2}$ are distinct.

Since $\dim(\mathbb{R}^{m\times m}) = m^2$, any list of more that m^2 matrices is linearly dependent in $\mathbb{R}^{m\times m}$.

Hence the $(1+m^2)$ matrices $I, A, A^2, \dots, A^{m^2}$ are linearly dependent.

So, there exists scalars $c_0, c_1, \dots, c_{m^2}$ not all zero such that

$$c_0 I + c_1 A + \dots + c_{m^2} A^{m^2} = [0]$$

Thus A satisfies the non-zero polynomial,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_{m^2} x^{m^2}$$

of degree $\le m^2$

7 ① Yes: $\mathbb{C}$ is a finite dimensional vector space over $\mathbb{R}$

$$\dim(\mathbb{C}) = 2$$

In fact the set $\{1, i\}$ forms a Basis of $\mathbb{C}$

$x_1 \times 1 + x_2 \times i = 0$ where $x_1, x_2 \in \mathbb{R}$

$\Rightarrow x_1 = 0, x_2 = 0$ (Equating real & imaginary parts)

Hence $\{1, i\}$ is linearly independent.

Also any complex number $a+bi \in \mathbb{C}$ can be written as $a+bi = a\times 1 + b\times i$

Hence $\{1, i\}$ is a spanning set of $\mathbb{C}$. So, $\boxed{\dim \mathbb{C} = 2}$

9

(b) There ~~is a~~ does n't exist a field F strictly lying between $\mathbb{R}$ and $\mathbb{C}$

ie $\mathbb{R} \subsetneq F \subsetneq \mathbb{C}$ is not possible

Any field F is a vector space over itself

but F is also a field over $\mathbb{R}$ (since $\mathbb{R} \subsetneq F$)

Hence if $\mathbb{R} \subsetneq F \subsetneq \mathbb{C}$

$\dim \mathbb{R} < \dim F < \dim \mathbb{C}$

$\Rightarrow 1 < \dim F < 2$ a Contradiction

8) (a) Given that the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.

Want to show that

$v_1 - v_2,\ v_2 - v_3, \ldots\ v_{n-1} - v_n,\ v_n$ are linearly independent.

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + \cdots + c_{n-1}(v_{n-1} - v_n) + c_n v_n = 0$$

Then $c_1 v_1 + (c_2 - c_1)v_2 + (c_3 - c_2)v_3$

$$+ \cdots + (c_{n-1} - c_{n-2})v_{n-1}$$

$$+ (c_n - c_{n-1})v_n = 0$$

$\Rightarrow c_1 = 0,\ c_2 - c_1 = 0,\ c_3 - c_2 = 0$

$\ldots,\ c_{n-1} - c_{n-2} = 0,\ c_n - c_{n-1} = 0$

(since $v_1, v_2, \ldots, v_n$ are l.i.)

$\Rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0, \ldots, c_{n-1} = 0,\ c_n = 0$

Hence $v_1 - v_2,\ v_2 - v_3, \ldots, v_{n-1} - v_n$ and v_n are linearly independent.

(b) Given:

$S = \{v_1, \ldots, v_n\}$ is linearly independent

and $w \in V$

Furthermore,

$v_1 + w, v_2 + w, \ldots, v_n + w$ are linearly dependent.

Want to show: $w \in \text{Span}(S)$

Since $v_1 + w, v_2 + w, \ldots, v_n + w$ are linearly dependent, there exist scalars $c_1, c_2, \ldots, c_n$ not all zero, such that $c_1(v_1 + w) + c_2(v_2 + w) + \cdots + c_n(v_n + w) = 0$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n + (c_1 + c_2 + \cdots + c_n) w = 0$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n + d w = 0$ where $d = c_1 + c_2 + \cdots + c_n$

If $d = 0$, then $c_1 v_1 + \cdots + c_n v_n = 0$

$\Rightarrow c_1 = \cdots = c_n = 0$ (since $v_1, v_2, \ldots, v_n$ are linearly independent)

But that is not possible (contradiction)

Hence $d \neq 0$

So, $c_1 v_1 + \cdots + c_n v_n + d w = 0$

$\Rightarrow d w = -c_1 v_1 - \cdots - c_n v_n$

$\Rightarrow w = -d^{-1} c_1 v_1 - \cdots - d^{-1} c_n v_n$ (Since $d \neq 0$)

$\Rightarrow \boxed{w \in \text{Span}\{v_1, v_2, \ldots, v_n\}}$

9. $V = C[a,b]$

No. V is infinite dimensional.

Assume BWOC that $C[a,b]$ is finite dimensional.

Let $P[a,b]$ be the set of all (real valued) polynomials with domain $[a,b]$.

Now $P[a,b] \subset C[a,b]$

Furthermore

$P[a,b]$ is a subspace of $C[a,b]$ (Check!)

Now the space $P[a,b]$ is infinite dimensional. (The proof essentially uses the same argument we used to prove that $\mathbb{R}[t]$ is infinite dimensional).

Also since, $P[a,b]$ is a subspace of $C[a,b]$ and if $C[a,b]$ is finite dimensional, then $P[a,b]$ will also be finite dimensional — a contradiction.

Hence $\dim(C[a,b]) = \infty$.

Proof of the fact that $P[a,b]$ is infinite dimensional:

Suppose BWOC that $P[a,b]$ is finite dimensional. Then it has a finite Basis, say $\{p_1(x), \ldots, p_k(x)\}$.

Let $N = \max\{\deg p_1, \deg p_2, \ldots, \deg p_k\}$

and let $p(x) = x^{N+1}$.

Then $p(x)$ can't be written as a linear combination of $p_1, p_2, \ldots, p_k$ because any linear combination of $p_1, p_2, \ldots, p_k$ will be a polynomial of degree $\leq N$ and $\deg p(x) = N+1$, a Contradiction

Hence $P[a,b]$ is infinite dimensional.

(10) Let $V = \mathbb{R}^\infty$

$W = \{\{a_n\} :$ only finitely many of the terms are non zero $\}$.

(a) Want to show that W is subspace of V

Let us define

$\text{tail}\{a_n\} = k$ where k is the least positive integer s.t. $a_i = 0 \ \forall i \geq k$.

(i.e. it is the index from which that tail Consisting of only zeros commence in $\{a_n\}$)

(1) Clearly the zero sequence $\{0\} = 0, 0, 0, \ldots$ belongs to W.

(2) Let $\{a_n\}, \{b_n\} \in W$

and $\text{tail}\{a_n\} = k_1$, $\text{tail}\{b_n\} = k_2$

and $k = \max\{k_1, k_2\}$

Then $\{c_n\} = \{a_n\} + \{b_n\} = \{a_n + b_n\}$

and $c_n = 0 \ \forall\ n \geq k$.

Hence $\{c_n\} \in W$.

(3) If $c \in \mathbb{R}$ and $\{a_n\} \in W$ and $\text{tail}\{a_n\} = k_1$

then $c\{a_n\} = \{c a_n\}$ and $c a_n = 0 \ \forall\ n \geq k_1$

Hence $\{c a_n\} \in W$.

So, W is a subspace of V.

(b) Is W finite dimensional?

Ans: No:

Suppose BWOC that W is finite dimensional.

Then it has a Basis B consisting of sequences $s_1, s_2, \ldots s_k$ for some positive integer k.

Let $N = \max\{\text{tail}(s_1), \text{tail}(s_2), \ldots, \text{tail}(s_K)\}$.

Consider the sequence $S = \{a_n\}$

where $a_n = 0$ for $n < N$

$= 1$ for $n = N$

$= 0$ for $n > N$.

Then $S = \{a_n\} \in W$ but $S \notin \text{Span}(B)$ because any ~~sequence~~ element of $\text{Span}(B)$ is a linear combination of $s_1, s_2, \ldots, s_K$ and so its tail will be less than or equal to N whereas tail $S = N+1$.

Thus $\text{Span}(B) \neq W$, a contradiction to the fact that B is a basis of W.

Hence W is infinite dimensional.

(c) Is V finite dimensional?

Answer: No. If V is finite dimensional, then since $W \subseteq V$ is a subspace of W, W will also be finite dimensional. But in (b) we showed that W is infinite dimensional. Hence V is also infinite dimensional.

(d) Let C be the space of all convergent sequences in $\mathbb{R}^\infty$.

Answer: No   Note that $W \subseteq C$

Every sequence in W converges to zero.

Since W is infinite dimensional, C can not be finite dimensional.

Remarks :

(1) If c_0 be the set of all sequences converging to zero,

then c_0 is a subspace of $\mathbb{R}^\infty$

and $W \subseteq c_0$

Therefore c_0 is infinite-dimensional.

(2) Let l_∞ be the set of all bounded sequences.

then l_∞ is a subspace of $\mathbb{R}^\infty$

Now $W \subsetneq c_0 \subsetneq C \subsetneq l_\infty \subsetneq \mathbb{R}^\infty$

and all of them are infinite dimensional.

- C and l_∞ are useful in signal processing.

11 Let $F = \mathbb{Z}_2 = \{0, 1\}$

and $V = F^n$

Then $\dim V = n$ (V is finite dimensional).

Now if W ~~is~~ a subspace of V,

W is also finite dimensional.

If $\dim W = k$, then $1 \le k \le n$

Thus W has a basis of k vectors

say $\{w_1, w_2, \dots, w_k\}$

So, any vector $w \in W$ can be written

uniquely as $w = c_1 w_1 + c_2 w_2 + \dots + c_k w_k$

where $c_i \in F = \mathbb{Z}_2$

i.e. $c_i = 0$ or 1.

Hence there are only 2^k possibilities

for w. Thus order of ~~W~~, $\boxed{|W| = 2^k}$

for $1 \le k \le n$

12 $S = \{u, w\}$ is a linearly independent set in $\mathbb{R}^3$

$u = (1, 2, 3)$, $w = (2, 4, 5)$

We need to find a vector v not in the span $\{u, w\}$

We solve $c_1 u + c_2 w = b$ for any general vector $b \in \mathbb{R}^3$

Let $b = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

The Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & p \\ 2 & 4 & q \\ 3 & 5 & r \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & p \\ 0 & 0 & q - 2p \\ 0 & -1 & r - 3p \end{array}\right]$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{cc|c} 1 & 2 & p \\ 0 & -1 & r - 3p \\ 0 & 0 & q - 2p \end{array}\right]$$

We don't need to reduce it to RREF matrix.
We can conclude that the above system is inconsistent if $q - 2p \neq 0$

We can many choices of v.

e.g. let $v = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ $(6 - 2 \times 2 \neq 0)$ $\left(\text{or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) (1 - 2 \times 1 \neq 0)$

So, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} \right\} = \{u, w, v\}$ forms a Basis of $\mathbb{R}^3$.