MTH 100 : Lecture 11

- We would like to study matrices further.

Instead of looking matrices as a rectangular array of numbers, we will look at a matrix as

<u>a collection of rows</u> (row vectors)

<u>a collection of columns</u> (column vectors)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \equiv \left(\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right)$$

or

$$\begin{pmatrix} \boxed{a_{11} \quad a_{12} \cdots a_{1n}} \\ \boxed{a_{21} \quad a_{22} \cdots a_{2n}} \\ \vdots \qquad \vdots \qquad \vdots \\ \boxed{a_{m1} \quad a_{m2} \cdots a_{mn}} \end{pmatrix}$$

- Also, if we multiply a matrix with a $(n \times 1)$ column we get a $(m \times 1)$ column

i.e.
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} - \\ - \\ \vdots \\ - \end{pmatrix}_{n \times 1} = \begin{pmatrix} - \\ - \\ \vdots \\ - \end{pmatrix}_{m \times 1}$$

Thus a <u>transformation</u> is taking place here.

- <u>Earlier</u> (<u>Ref:</u> Lecture 5) we have looked at rows or columns with 2 or 3 entries as vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and looked at their properties like <u>addition</u> and <u>scalar multiplication</u>.

- The important aspect is <u>these properties</u> and <u>not the set it self</u>.

- We can take <u>some other set</u> and observe the <u>same properties</u>.

The set can be : a collection of numbers
a Collection of row vectors
a collection of column vectors
a collection of some matrices
a collection of some sequence of numbers
a collection of some functions.

- This leads to Axiomatic approach to Mathematics.

Axioms : Some Rules to be followed by the elements of a set.

Thus we now have a Set and Axioms / Rules.

(Exact nature of the elements of the set is not important)

Any proof about the set uses only the axioms and logical reasoning.

This leads to Abstract system. or structure.

- Different sets following the same set of axioms give the same system.
- Different axioms leads to different systems.

- Advantages of Axiomatic approach:
- Proofs are more rigorous.
- It unifies different Mathematical objects.
- Ease of proof in abstract setting. (compared to concrete or specific Mathematical objects.)

(Compared to concrete or specific mathematical objects)

- Novelty

This approach leads to different branches of Mathematics.

Examples:

- Discrete Mathematics: Graphs, Lattices, Posets.
- Algebra: Groups, Rings, Fields.

- Measure Theory and Probability: Measure space, Measurable set.

- Topology: Metric Spaces and Topological Spaces.

- Functional Analysis: Banach Spaces or Hilbert Spaces.

- Linear Algebra: Vector Space over a field.

- Elements of the vector space are called vectors and elements of the field are called scalars.

- Let us look at the set of real numbers and observe its properties such as addition, subtraction, multiplication, division etc
- We can add two real numbers and get another real number,

We can multiply two real numbers and get another real number. i.e. $a+b \in \mathbb{R} \ \forall a,b \in \mathbb{R}$ and $a.b \in \mathbb{R} \ \forall a,b \in \mathbb{R}$

(Note: $\forall \equiv$ for all, $\in \equiv$ belongs to)

- Addition and multiplication satisfies associative and commutative properties.

$$\left.\begin{aligned} a+(b+c) &= (a+b)+c \\ a(bc) &= (ab)c \end{aligned}\right\} \forall a,b,c \in \mathbb{R}$$

$$\left.\begin{aligned} a+b &= b+a \\ ab &= ba \end{aligned}\right\} \forall a,b \in \mathbb{R}$$

- We have real number 0 (additive identity) such that $a+0 = 0+a = a \ \forall a \in \mathbb{R}$

and real number 1 (multiplicative identity) such that $1.a = a.1 = a \ \forall a \in \mathbb{R}$

- For every real number, we have its negative (additive inverse). For every $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$

For every nonzero real number, we have its reciprocal (multiplicative inverse). For every $a \in \mathbb{R}$ $(a \neq 0)$, there exists $a^{-1} \in \mathbb{R}$ such that $a^{-1}a = aa^{-1} = 1$

- Real numbers satisfies Distributive properties: $a(b+c) = ab+ac \ \forall a,b,c \in \mathbb{R}$

This leads to the concept of a Field.

Informal Definition: An algebraic system with addition and multiplication of elements in which universal addition, subtraction, multiplication and division except that division by zero element (denoted by 0) is not possible.

Formal Definition:

A field is a nonempty set F with two binary operations called addition and multiplication

(Usual symbols: '+' for addition and • or * for multiplication sometimes omitted altogether for multiplication.
Thus: $a+b$; $a \cdot b$ or $a*b$ or ab will be written)

which satisfies the following properties.

(A) Closure Under the operation:

For every element $a, b \in F$,
$a+b \in F$ and $a \cdot b \in F$

(B) The following properties hold:

(a) Associative Property:

$$\left.\begin{array}{l}(a+b)+c = a+(b+c) \\ (a \cdot b) \cdot c = a \cdot (b \cdot c)\end{array}\right\} \text{ for every } a, b, c \in F$$

(b) Commutative Property:

$$\left.\begin{array}{l}a+b = b+a \\ a \cdot b = b \cdot a\end{array}\right\} \text{ for every } a, b \in F$$

(c) Zero property and identity (unit) property:

There exists a zero element '0' and an identity or unit element '1' (not the same as zero element which satisfies

$$\left.\begin{aligned} a+0 &= 0+a = a \\ \text{and}\quad 1.a &= a.1 = a \end{aligned}\right\} \text{ for every } a \in F$$

(d) For any $a \in F$, it will have an additive inverse b and multiplicative inverse c (provided $a \neq 0$) such that

$$a+b=0$$

and $\quad a.c = 1 \quad$ (provided $a \neq 0$)

(e) Distributive Property:

$$a.(b+c) = a.b + a.c \quad \text{for every } a, b, c \in F$$

Examples:

(1) We have shown that set of real numbers $\mathbb{R}$ is a field with respect to usual addition and multiplication

(2) Set of rational numbers $\mathbb{Q}$ is a field with respect to usual addition and multiplication.

(3) Set of Complex numbers $\mathbb{C}$ is also a field with respect to usual addition and multiplication.

Examples:

(1) Let $\mathbb{Z}$ be the set of integers.

There is no multiplicative inverse for any non-zero integer.

So, $\mathbb{Z}$ is not a field with respect to usual addition and multiplication.

(2) Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 matrices with real entries.

Now commutative property of multiplication doesnot hold.

Therefore $\mathbb{R}^{2\times 2}$ is not a field.

(3) Let us consider the set of all 2×2 invertible matrices with real entries.

Now sum of two invertible matrices may not be an invertible matrix.

e.g. both $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are invertible but

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is not invertible.

So, the above set is not closed under matrix addition and therefore it is not a field.

<u>Properties:</u> The following properties hold for every field F.

- Since $1 \neq 0$, F must have atleast two elements. (This is an axiom)
- The zero element is unique and the unit element is unique.
- The additive inverse of every $c \in F$ is unique and is usually denoted by $-c$.
- The multiplicative inverse of every $c \in F$ $(c \neq 0)$ is unique and is usually denoted by c^{-1}.
- $0.c = 0$ for every $c \in F$

<u>Proof:</u> $0 + 0.c = 0.c = (0+0).c = 0.c + 0.c$ (By Distributive Property)

$\Rightarrow$ adding additive inverse of $0.c$ to both sides (from the right) we obtain $\boxed{0 = 0.c}$

- F has no <u>zero divisors.</u>

(Definition: An element $c \in F$ $(c \neq 0)$ is said to be a zero divisor if there exists an element $d \in F$ $(d \neq 0)$ such that $c.d = 0$)

<u>Proof:</u> If F has a zero divisor $c \in F$ $(c \neq 0)$, then there exists $d \in F$ $(d \neq 0)$ such that $c.d = 0$

$\Rightarrow$ Multiplying both sides by d^{-1} from the right we get $(c.d).d^{-1} = 0.d^{-1} \Rightarrow c.(d.d^{-1}) = 0$ (By the previous property)

$\Rightarrow c.1 = 0 \Rightarrow c = 0$, a contradiction.

Vector Space:

A vector space is a non-empty set V of objects called vectors together with an associated field F of scalars with two operations called addition and scalar multiplication which satisfies the following properties:

(A) Closure under operations:

$u + v \in V$ for every $u, v \in V$

and $cu \in V$ for every $u \in V$ and every $c \in F$

(B) The following properties hold for addition:

(a) associative property:

$(u+v) + w = u + (v+w)$ for every $u, v, w \in V$

(b) Identity Property:

There exists a 'zero vector' 0 such that

$0 + u = u + 0 = u$ for every vector $u \in V$

(c) Every vector $u \in V$ has an additive inverse $v \in V$ such that

$$u + v = v + u = 0$$

(d) Commutative property:

$u + v = v + u$ for every $u, v \in V$

Moreover:

(C) (a) $c(u+v) = cu + cv$ for every $u, v \in V$ and every $c \in F$

(b) $(c+d)u = cu + du$ for every $u \in V$ and every $c, d \in F$

(c) $c(du) = (cd)u$ for every $u \in V$ and every $c, d \in F$

(d) $1.u = u$ for every $u \in V$ where 1 is the unit element of F.

Note: (1) In lecture 5, we have shown that $\mathbb{R}^2$ is a vector space over the field $\mathbb{R}$.

(Although the definition of vector space was not mentioned there, all the properties were shown)

(2) In the same way we can show that

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

is a vector space over the base field $\mathbb{R}$.

Note that addition and scalar multiplication in $\mathbb{R}^n$ is defined by:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} \quad \text{and} \quad c\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix} \text{ for every } c \in \mathbb{R}$$