Worksheet 5

(1) Let U and W be two subspaces of a vector space V over a field F.

Let us consider $U \cap W$.

Clearly $U \cap W \subset V$

Now $0 \in U$, $0 \in W$ (since U and W are subspaces of V)

(1) So, $\boxed{0 \in U \cap W}$

(2) Now let $\omega_1, \omega_2 \in U \cap W$

then $\omega_1, \omega_2 \in U$ and $\omega_1, \omega_2 \in W$

So, $w_1 + w_2 \in U$ (since U is a subspace of V)
and $w_1 + w_2 \in W$ (since W is a subspace of V)

$\Rightarrow \boxed{w_1 + w_2 \in U \cap W}$

(3) Let $c \in F$ and $w \in U \cap W$
then $w \in U$ and $w \in W$
$\Rightarrow$ $cw \in U$ (since U is a subspace of V)
and $cw \in W$ (since W is a subspace of V)

$\Rightarrow \boxed{cw \in U \cap W}$

Therefore $U \cap W$ is a subspace of V

2 (a) $V = \mathbb{R}_n(t)$ = Vector space of polynomials of degree $\leq n$

$W = \{ p(t) \in W : \text{degree}(p(t)) = n \} \cup \{0(t)\}$

Now $p(t) = t^n \in W$, $q(t) = -t^n + t^{n-1} \in W$ $(n \geq 2)$

But $p(t) + q(t) = t^n - t^n + t^{n-1} = t^{n-1} \notin W$

So, $\boxed{\text{W is not a subspace of V}}$

(b) $V = \mathbb{R}^3$

$W = \{ (x, y, z) : x, y, z \in \mathbb{Q} \}$

Let $\omega = (x, y, z) \in W$ So, $x, y, z \in \mathbb{Q}$

Now let us take $\sqrt{2} \in \mathbb{R}$. (it is a scalar)

Now $\sqrt{2}\,\omega = (\sqrt{2}x, \sqrt{2}y, \sqrt{2}z) \notin W$

as $\sqrt{2}x, \sqrt{2}y, \sqrt{2}z \notin \mathbb{Q}$

So, W is not a subspace of V

(c) $\underline{V = \mathbb{R}^3}$:

$W = \{ (x, y, z) : xy = 0 \}$

So,

$(1, 0, 1) \in W$, $(0, 1, 0) \in W$

Now $(1, 0, 1) + (0, 1, 0) = (1, 1, 1) \notin W$

So, W is not a subspace of V

(d) $V = \mathbb{R}^3$

$W = \{ (x, y, z) : x^2 + y^4 + z^6 = 0 \}$

If $(x, y, z) \in W$ then $x^2 + y^4 + z^6 = 0$

$\Leftrightarrow x = 0, y = 0, z = 0$

$\Leftrightarrow (x, y, z) = (0, 0, 0)$

So $W = \{(0,0,0)\}$

Hence W is a subspace of V.

(4) (a) $V = \mathbb{R}^{2\times 2}$

$W = \{ A \in \mathbb{R}^{2\times 2} : A^T = A \}$

= Set of all symmetric (2×2) matrices over $\mathbb{R}$

(1) clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(2) If $A, B \in W$ then $A^T = A$, $B^T = B$

Now $(A+B)^T = A^T + B^T$ (can check explicitly)

$= A + B$ (since $A, B \in W$)

So, $A + B \in W$.

(3) If $c \in \mathbb{R}$ and $A \in W$ then $A^T = A$

Now $(cA)^T = c A^T$ (can check explicitly)

$= cA$ (since $A \in W$)

$\Rightarrow cA \in W$. Hence W is a subspace of V

(b) Let $B \in \mathbb{R}^{2\times 2}$ be a fixed matrix.

Now $W = \{ A \in \mathbb{R}^{2\times 2} : AB = BA$, B is a fixed matrix in $\mathbb{R}^{2\times 2} \}$

(1) Now $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} B = B \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(2) If $A_1, A_2 \in W$ then $\left.\begin{matrix} A_1 B = B A_1 \\ \text{and } A_2 B = B A_2 \end{matrix}\right\}$

So, $(A_1 + A_2) B = A_1 B + A_2 B$

$= BA_1 + BA_2$ (since $A_1, A_2 \in W$)

$= B(A_1 + A_2)$

$\Rightarrow A_1 + A_2 \in W$

(3) If $c \in \mathbb{R}$ and $A \in W$ then $AB = BA$.

Then, $(cA)B = c(AB) = c(BA)$

$= (cB)A =$

$= (Bc)A = B(cA)$

$\Rightarrow cA \in W$

So, W is a subspace of V

(c) $W = \{ A \in \mathbb{R}^{2\times 2} : BA = 0$ for some fixed matrix $B \in \mathbb{R}^{2\times 2} \}$

(1) clearly $B \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(2) Now if $A_1, A_2 \in W$, then
$BA_1 = 0, \; BA_2 = 0$

Then $B(A_1 + A_2) = BA_1 + BA_2$ (Since $A_1, A_2 \in W$)
$= 0 + 0$

$\Rightarrow A_1 + A_2 \in W$

(3) If $c \in \mathbb{R}$ and $A \in W$,
then $BA = 0$

Then $B(cA) = c(BA) = c \cdot 0$ (Zero matrix)
$= 0$

$\Rightarrow cA \in W$.

Thus W is a subspace of V.

(d) The above results are true for general matrices also.
In the above solution, general matrices are used and general properties of matrix addition, multiplication, transpose and scalar multiplication is used.

(4) Let $V = \mathbb{R}^{n\times n}$

(a) Let W be the set of upper triangular matrices

So, $W = \left\{ [a_{ij}]_{n\times n} : a_{ij} = 0 \text{ if } i > j, \ i, j = 1, 2, \dots n \right\}$

(1) clearly $[0]_{n\times n} \in W$

(2) If $A, B \in W$, then $A = [a_{ij}]_{n\times n}$
$B = [b_{ij}]_{n\times n}$

and $a_{ij} = 0$ for $i > j$
$b_{ij} = 0$ for $i > j$

Now $A + B = [a_{ij} + b_{ij}]_{n\times n}$

where $a_{ij} + b_{ij} = 0 + 0 = 0$ for $i > j$

$\Rightarrow A + B \in W$

(3) If $c \in \mathbb{R}$ and $A \in W$
then $A = [a_{ij}]_{n\times n}$, $a_{ij} = 0$ for $i > j$

Now $[cA]_{n\times n} = [c a_{ij}]_{n\times n}$ and $c a_{ij} = 0$ for $i > j$

So, $cA \in W$

Thus W is a subspace of V

(b) If $A, B \in W$,

let $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0$ for $i > j$

$B = [b_{ij}]_{n \times n}$, $b_{ij} = 0$ for $i > j$

Then $AB = [c_{ik}]_{n \times n}$

where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

Now let $i > k$ (fixed)

Then $\boxed{a_{ij} = 0 \text{ for } i > j}$ and $b_{jk} = 0$ for $j > k$

and so $\boxed{b_{jk} = 0 \text{ for } j \geq i}$

Hence $\sum_{j=1}^{n} a_{ij} b_{jk} = 0 \Rightarrow c_{ik} = 0$ for $i > k$

Hence $AB \in W$

Now [redacted] if A is invertible, then A can be reduced to identity [redacted] matrix by elementary row operations.

Thus there exists elementary matrices $E_1, E_2, \ldots, E_p$ such that $E_p \cdots E_2 E_1 A = I$

Now since A is upper triangular, $E_1, \ldots E_p$ are all upper triangular [redacted] and $A^{-1} = E_p \cdots E_2 E_1$

Since product of upper triangular matrix is upper triangular, A^{-1} is upper triangular

(5) (a) Suppose u_1 and u_2 are two additive inverse of $u \in V$

Then $u_1 + u = 0$ (zero vector)

$\Rightarrow (u_1 + u) + u_2 = 0 + u_2$

$\Rightarrow u_1 + (u + u_2) = u_2$

$\Rightarrow u_1 + 0 = u_2$

$\Rightarrow u_1 = u_2$

Hence additive inverse of any vector is unique.

(b) $0.u = (0+0)u$

$= 0u + 0.u \quad \forall u \in V$

Let v be the additive inverse of $0.u$

Then $v + 0.u = v + 0.u + 0.u$

$\Rightarrow \underset{\text{zero vector}}{0} = \underset{\text{zero vector}}{0} + 0.u$

$\Rightarrow \boxed{0 = 0.u}$

(c) $c(\underset{\text{(zero vector)}}{0}) = c(\underset{\text{zero vector}}{0 + 0}) = \underset{\text{zero vector}}{c0 + c0}$

Adding inverse of $\underset{\text{(zero vector)}}{c0}$ (say v)

$v + c(\underset{\text{(zero vector)}}{0}) = v + \underset{\text{zero vector}}{c0 + c0} \Rightarrow 0 = \underset{\text{(zero vector)}}{0} + c0$

$= 0 = \underset{\text{(zero vector)}}{c0}$

(d) $u + v = u + w$ for $u, v, w \in V$

then additing additive inverse of u (say $-u$) on the left of both sides

$$-u + (u+v) = -u + (u+w)$$

$$\Rightarrow (-u+u) + v = (-u+u) + w$$

$$\Rightarrow \underbrace{0}_{\text{zero vector}} + v = \underbrace{0}_{\text{zero vector}} + w$$

$$\Rightarrow \boxed{v = w}$$

(f) Let $X = \mathbb{R}^{2\times 2}$ (Set of all 2×2 matrices with real entries)

and the operation be matrix multiplication.

Now there is a non-invertible matrix A such that $AB = AC \not\Rightarrow B = C$

Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Now $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $AC = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So, $AB = AC$ but $B \neq C$

Thus Cancellation Law is not satisfied here.

7 Let $Q(\sqrt{2}) = \{ a + b\sqrt{2} : a, b \in Q \}$

Want to show $Q(\sqrt{2})$ is a field.

Note that $Q(\sqrt{2}) \subset \mathbb{R}$ ($Q(\sqrt{2})$ is a subset of the field $\mathbb{R}$)

(1) Additive closure: If $a + b\sqrt{2}$, $c + d\sqrt{2} \in Q(\sqrt{2})$,

then $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in Q(\sqrt{2})$

(2) Additive identity $0 = 0 + 0\sqrt{2} \in Q(\sqrt{2})$

(3) Additive inverse If $a + b\sqrt{2} \in Q(\sqrt{2})$,

then $(-a) + (-b)\sqrt{2} \in Q(\sqrt{2})$

and $(a + b\sqrt{2}) + (-a - b\sqrt{2}) = 0$

(4) Multiplicative closure $a + b\sqrt{2}$, $c + d\sqrt{2} \in Q(\sqrt{2})$

$\Rightarrow (a + b\sqrt{2})(c + d\sqrt{2})$

$= (ac + 2bd) + (ad + bc)\sqrt{2} \in Q(\sqrt{2})$

(5) Multiplicative identity $1 = 1 + 0\sqrt{2} \in Q(\sqrt{2})$

(6) Multiplicative inverse

If $a + b\sqrt{2} \in Q(\sqrt{2})$ and $a + b\sqrt{2} \neq 0$

then $x = \dfrac{1}{a + b\sqrt{2}} = \dfrac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})}$

$= \dfrac{a}{a^2 - 2b^2} - \dfrac{b}{a^2 - 2b^2}\sqrt{2}$

Also $\quad a^2 - 2b^2 \neq 0$

because $\quad a^2 - 2b^2 = 0 \quad \Rightarrow \quad \dfrac{a^2}{b^2} = 2$

$\Rightarrow \quad \sqrt{2} \in Q$ $\quad$ a contradiction

So, $\quad x = \dfrac{a}{a^2 - 2b^2} + \left(\dfrac{-b}{a^2 - 2b^2}\right)\sqrt{2} \in Q(\sqrt{2})$

and $\quad (a + b\sqrt{2}).x = x(a + b\sqrt{2}) = 1$

Thus every nonzero element in $Q(\sqrt{2})$ has a multiplicative inverse in $Q(\sqrt{2})$

- All other properties follow since they are inherited from $\mathbb{R}$.

So, $Q(\sqrt{2})$ is a subfield of the field $\mathbb{R}$ and so it is a field.

8 (a) Is $\mathbb{R}$ a vector space over $\mathbb{Q}$?

Yes: • If $x, y \in \mathbb{R}$, $x+y$ is well defined & $x+y \in \mathbb{R}$

• If $q \in \mathbb{Q}$ and $x \in \mathbb{R}$, then $q.x \in \mathbb{R}$ (since $\mathbb{Q} \subseteq \mathbb{R}$)

Furthermore all the vector space axioms are satisfied since $\mathbb{R}$ satisfies all the field axioms.

(b) Is $\mathbb{C}$ a vector space over $\mathbb{R}$?

Yes:

• If $z_1 = a+bi$ and $z_2 = c+di \in \mathbb{C}$

then $z_1 + z_2 = (a+c) + (b+d)i$ is well defined & $z_1 + z_2 \in \mathbb{C}$

• If $c \in \mathbb{R}$ and $z_1 = a+ib \in \mathbb{C}$ then $cz_1 = (ca) + i(cb) \in \mathbb{C}$ is well defined since $\mathbb{R} \subset \mathbb{C}$.

• All other vector space axioms are satisfied.

(c) Generalization: If F and K are two fields with $F \subseteq K$ (i.e. F is a subfield of K), then K is a vector space over F with the natural definition of addition and scalar multiplication as above.

(9)

Modular Arithmetic :

Note: Here we make use of remainder theorem.

If $z \in \mathbb{Z}$ and n is a positive integer,

then $z = qn + r$ where either $r = 0$
or $0 < r < n$

Then $z \pmod{n} = r$ which is uniquely defined

and since $0 \leq r < n$, $r \in \mathbb{Z}_n$.

(a) Let $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$

and the operations on $\mathbb{Z}_n$ are modular addition and modular multiplication

$$x \oplus y = (x+y) \pmod{n}$$
$$x \otimes y = (x \cdot y) \pmod{n}$$

Now let $x \in \mathbb{Z}_n$. then $0 \leq x \leq n-1$

If $x=0$, let $y=0$

If $1 \leq x \leq n-1$, let $y = n - x$

Then $1 \leq y = n-x \leq n-1$

and $x + y = x + n - x = n$

So, $x \oplus y = n \pmod{n} = 0$

If $x = 0$, $y = 0$, then $x \oplus y = 0 \oplus 0 = 0$

Thus, Every element of $\mathbb{Z}_n$ has an additive inverse in $\mathbb{Z}_n$.

(b) To show that $\mathbb{Z}_3$ and $\mathbb{Z}_5$ are fields.

$\mathbb{Z}_3 = \{0, 1, 2\}$ $\qquad$ $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

We assume (as shown in class) that $\oplus$ and $\otimes$ satisfy closure, associativity, commutativity and distributivity in $\mathbb{Z}_n$.

- In $\mathbb{Z}_3$, 0 is the additive identity.

and
$$\left.\begin{array}{l} 0 \oplus 0 = 0 \\ 1 \oplus 2 = 3 \pmod 3 = 0 \end{array}\right\}$$
so, additive inverse exists for every element of $\mathbb{Z}_3$

1 is the multiplicative identity.

and
$$\left.\begin{array}{l} 1 \otimes 1 = 1 \\ 2 \otimes 2 = 4 \pmod 3 = 1 \end{array}\right\}$$
So $\left.\begin{array}{l} 1^{-1} = 1 \\ \text{and } 2^{-1} = 2 \end{array}\right\}$ in $\mathbb{Z}_3$

i.e. every nonzero element in $\mathbb{Z}_3$ has multiplicative inverse.

$\boxed{\text{So, } \mathbb{Z}_3 \text{ is a field.}}$

- In $\mathbb{Z}_5$, 0 is the additive identity.

and
$$\left.\begin{array}{l} 0 \oplus 0 = 0 \\ 1 \oplus 4 = 5 \pmod 5 = 0 \\ 2 \oplus 3 = 5 \pmod 5 = 0 \end{array}\right\}$$
So, additive inverse exists for every element of $\mathbb{Z}_5$

1 is the multiplicative identity

and
$$\left.\begin{array}{l} 1 \otimes 1 = 1 \\ 2 \otimes 3 = 6 \pmod 5 = 1 \\ 4 \otimes 4 = 16 \pmod 5 = 1 \end{array}\right\}$$
So, $\left.\begin{array}{l} 1^{-1} = 1 \\ 2^{-1} = 3 \\ 3^{-1} = 2 \\ 4^{-1} = 4 \end{array}\right\}$ in $\mathbb{Z}_5$

So, every non zero element in $\mathbb{Z}_5$ has multiplicative inverse and so $\boxed{\mathbb{Z}_5 \text{ is a field.}}$

(c) Z_4 and Z_6 are not fields.

- In Z_4, $\quad 2 \otimes 2 = 4 \pmod 4 = 0$

  but $2 \pmod 2 \neq 0$.

So, 2 is a zero divisor in Z_4.

But we know that a field cannot have zero divisor. Hence $\boxed{Z_4 \text{ is \underline{not} a field}}$

- In Z_6, $\quad 2 \otimes 3 = 6 \pmod 6 = 0$

  but $2 \pmod 6 \neq 0$

  $3 \pmod 6 \neq 0$

So, 2 and 3 are zero divisors in Z_6.

Now since a field can not have zero divisor, we conclude that $\boxed{Z_6 \text{ is \underline{not} a field.}}$

(d) Z_n is not a field if n is not a prime.
(i.e. if n is a composite integer)

If we assume that n is not a prime,

$$\left.\begin{aligned} n = rk \quad & \text{where } 1 < r < n \\ & \text{and } 1 < k < n \end{aligned}\right\}$$

So, $\quad r, k \in Z_n \quad$ and $\quad r \otimes k = n \pmod n = 0$

$r \neq 0, \; k \neq 0$

Thus r and k are zero divisors in Z_n

Hence Z_n is not a field.

10 (a) **TRUE**:

In $\mathbb{R}^{3\times3}$ if A is invertible, then A cannot be a zero divisor.

Assume A is a zero divisor in $\mathbb{R}^{3\times3}$

Then there exists a $B \in \mathbb{R}^{3\times3}$, $B \neq 0$

such that $AB = 0$

Now $A^{-1}(AB) = A^{-1}(0)$ (given that A is invertible)

$\Rightarrow (A^{-1}A)B = 0$

$\Rightarrow I_3 \cdot B = 0 \Rightarrow B = 0$

a contradiction

So, B can't be a nonzero matrix

and so A is not a zero divisor

(b) **TRUE**:

If A is not invertible, by Theorem (1), the homogeneous system $Ax = 0$ has a nontrivial solution

So, there exists $v \in \mathbb{R}^3$ $(v \neq 0)$ such that $Av = 0$ (v is a 3×1 vector)

Let $B \in \mathbb{R}^{3\times3}$ be defined as the matrix whose every column is v (i.e. $B_{3\times3} = [v, v, v]_{3\times3}$)

Then B is a nonzero 3×3 square matrix and

$AB = [Av \; Av \; Av] = 0$ (3×3 zero matrix)

So, A is a zero divisor (according to the definition)

12) (a) $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{bmatrix}$

$\downarrow R_3 \to R_3 - R_2$

$$U = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

So, Row operation are:

$e_1 : R_2 \to R_2 - 2R_1$

$e_2 : R_3 \to R_3 - 3R_1$

$e_3 : R_3 \to R_3 - R_2$

Inverse operations are:

$f_1 : R_2 \to R_2 + 2R_1$

$f_2 : R_3 \to R_3 + 3R_1$

$f_3 : R_3 \to R_3 + R_2$

Then

$L = f_1 f_2 f_3 I$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 + 2R_1}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

and $\boxed{A = LU}$

(b) Now $b_1 = (1, 4, 5)$

$LY = b_1 \Rightarrow$

$y_1 = 1 \Rightarrow y_1 = 1$

$2y_1 + y_2 = 4 \Rightarrow y_2 = 4 - 2 = 2$

$3y_1 + y_2 + y_3 = 5 \Rightarrow y_3 = 5 - 3 - 2 = 0$

So, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Now $Ux = y \Rightarrow$

$$\left.\begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ 2x_2 + 6x_3 &= 2 \\ 0 &= 0 \end{aligned}\right\}$$

This is a consistent system of equation.

Then solution is

$$x_3 = x_3$$

$$2x_2 = 2 - 6x_3 \Rightarrow x_2 = 1 - 3x_3$$

and

$$\begin{aligned} x_1 &= 1 - 5x_3 - 2x_2 \\ &= 1 - 5x_3 - 2(1 - 3x_3) \\ &= 1 - 2 + x_3 = -1 + x_3 \end{aligned}$$

So,

$$\left.\begin{aligned} x_1 &= -1 + x_3 \\ x_2 &= 1 - 3x_3 \\ x_3 &= 0 + x_3 \end{aligned}\right\}$$

Hence

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

The solution set is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

This is the solution set for $Ax = b$.

If $b_2 = (3, 7, 15)$,

We first solve $Ly = b_2$

$$\Rightarrow \quad \begin{aligned} y_1 &= 3 && \Rightarrow y_1 = 3 \\ 2y_1 + y_2 &= 7 && \Rightarrow y_2 = 7 - 6 = 1 \\ 3y_1 + y_2 + y_3 &= 15 && \Rightarrow y_3 = 15 - 3\times 3 - 1 \\ & && \quad = 5 \end{aligned}$$

So, $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$.

Now $Ux = y \Rightarrow \left.\begin{aligned} x_1 + 2x_2 + 5x_3 &= 3 \\ 2x_2 + 6x_3 &= 1 \\ 0 &= 5 \end{aligned}\right\}$

This system is inconsistent and so, $Ax = b$ doesn't have a solution

Clearly A is not invertible.

- $Ax = b_2$ doesn't have a solution

whereas $Ax = b_1$ has infinitely many solutions.

(12) (i)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 6 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix} = U$$

So,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = L$$

<u>Check that</u>

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} = A$$

(b) Now $b = (4, 9, 14)$

So, $Ly = b \Rightarrow$

$$y_1 = 4 \Rightarrow y_1 = 4$$

$$2y_1 + y_2 = 9 \Rightarrow y_2 = 9 - 8 = 1$$

$$y_1 + 3y_2 + y_3 = 14 \Rightarrow y_3 = 14 - 4 - 3\times 1 = 7$$

So, $$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$$

Now $Ux = y \Rightarrow$

$$x_1 + x_2 + x_3 + x_4 = 4$$

$$2x_2 - x_3 - x_4 = 1$$

$$4x_3 + 3x_4 = 7$$

$$4x_3 = 7 - 3x_4 \Rightarrow x_3 = \frac{7}{4} - \frac{3}{4}x_4$$

Now $$2x_2 = 1 + x_3 + x_4 = 1 + \frac{7}{4} - \frac{3}{4}x_4 + x_4$$

$$= \frac{11}{4} + \frac{1}{4}x_4$$

$$\Rightarrow x_2 = \frac{11}{8} + \frac{1}{8}x_4$$

Now $$x_1 = 4 - x_2 - x_3 - x_4 = 4 - \frac{11}{8} - \frac{1}{8}x_4 - \frac{7}{4} + \frac{3}{4}x_4 - x_4$$

$$= \frac{32 - 11 - 14}{8} - \frac{1 - 6 + 8}{8}x_4$$

$$x_1 = \frac{7}{8} - \frac{3}{8}x_4 = \frac{7}{8} - \frac{3}{8}x_4$$

So, the solution is

$$x_1 = \frac{7}{8} - \frac{3}{8} x_4$$

$$x_2 = \frac{11}{8} + \frac{1}{8} x_4$$

$$x_3 = \frac{7}{4} - \frac{3}{4} x_4$$

$$x_4 = x_4$$

$$\Rightarrow x = \begin{bmatrix} \frac{7}{8} \\ \frac{11}{8} \\ \frac{7}{4} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{8} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

So, the set of solution is

$$\left\{ \begin{bmatrix} 7/8 \\ 11/8 \\ 7/4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{8} \\ -3/4 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

There are infinitely many solutions.

(13) No

$\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$

$\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\} \doteq$ set of all two tuples of real numbers.

$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \doteq$ set of all three tuples of real numbers

However the set $\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$ which behaves very much like $\mathbb{R}^2$ but is logically distinct from $\mathbb{R}^2$,

(14) $V = \{ x \in \mathbb{R} \; ; x > 0 \}$

Addition: $x \oplus y = xy \quad \forall x, y \in V$

Scalar multiplication For $\alpha \in \mathbb{R}$, $x \in V$, define $\alpha * x = x^\alpha$

(1) Closure property of addition and scalar multiplication:

Let $x, y \in V$. Then $x, y \in \mathbb{R}$ and $x > 0$, $y > 0$

$\Rightarrow xy \in \mathbb{R}$ and $xy > 0$

$\Rightarrow x \oplus y = xy \in V$

Now Let $\alpha \in \mathbb{R}$, $x \in V$. Then $x \in \mathbb{R}$, $x > 0 \Rightarrow x^\alpha \in \mathbb{R}$ and $x^\alpha > 0$

$\Rightarrow \alpha * x = x^\alpha \in V$

- Commutative property of addition:

For $x, y \in V$, $x \oplus y = xy = yx = y \oplus x$.

- Existence of zero element:

Since, $1 \in \mathbb{R}$ and $1 > 0$, $1 \in V$

and $1 \oplus x = 1.x = x = x \oplus 1$

$\overset{\shortparallel}{(x \oplus 1)}$ $\forall\, x \in V$

Thus 1 is the zero element of V.

- Existence of inverse element:

Let $x \in V$; $\therefore$ Then $x \in \mathbb{R}$ and $x > 0$

$\Rightarrow \frac{1}{x} \in \mathbb{R}$ and $\frac{1}{x} > 0$

$\Rightarrow \frac{1}{x} \in V$ and

$\frac{1}{x} \oplus x = \frac{1}{x} . x = 1 = x \oplus \frac{1}{x}$

$\overset{\shortparallel}{\left(x \oplus \frac{1}{x}\right)}$ $\forall\, x \in V$

- Also $1 * x = x^1 = x$ $\forall\, x \in V$

(2) No.

$\mathbb{R}$ is a vector space over itself (i.e. over the field $\mathbb{R}$) with respect to usual addition and scalar multiplication of real numbers.

(obviously if $\alpha = \frac{1}{2}$, $x = -1$ then $\alpha * x = x^\alpha = (-1)^{1/2} \notin \mathbb{R}$ and so $\mathbb{R}$ cannot be a vector space with respect to the operation $\oplus$ and $*$.)

Now let $x \in V$. Then $x > 0$. Now if we take any $\alpha \in \mathbb{R}$ s.t. $\alpha < 0$, then $\alpha x < 0$ and hence $\alpha x \notin V$

Thus closure property of scalar multiplication is not satisfied.

Also additive identity (zero element) $0 \notin V$.

Hence V is not a subspace of $\mathbb{R}$.