Linear Algebra Lecture 1

First consider an example:

$$x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 = 10$$
$$x_2 - 3x_3 + 3x_4 + x_5 = -5$$
$$x_1 + x_4 - x_5 = 4$$

- The above is a system of 3 linear equations in 5 unknowns
- Such system is important in practical world.

Matrix Formulation:

$$\underset{3\times 5}{\begin{bmatrix} 1 & -7 & 2 & -5 & 8 \\ 0 & 1 & -3 & 3 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}} \underset{5\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 10 \\ -5 \\ 4 \end{bmatrix}}$$

Vector Formulation :

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 4 \end{bmatrix}$$

System of Linear Equations :

- A system of equations of the form :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_i are scalars and the x_j are unknown variables is called a system of m linear equations in n unknowns.

- Any ordered n tuple $(s_1, s_2, \dots, s_n)$ of scalars which satisfies all the m equations is called a solution of the system.
- The set of all solutions is called the "solution set" of the system.
- A system of Linear Equation has either (1) No solution
  OR (2) Exactly one solution
  OR (3) infinitely many solutions
- A system of Linear Equation is said to be consistent if it has either one solution or infinitely many solutions
- A system is called inconsistent if it has no solution

Matrix Formulation :

A system of Linear Equation can be compactly expressed in matrix notation as :

$$\boxed{Ax = b}$$

where $A = [a_{ij}]_{m \times n}$ is called the coefficient matrix and

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

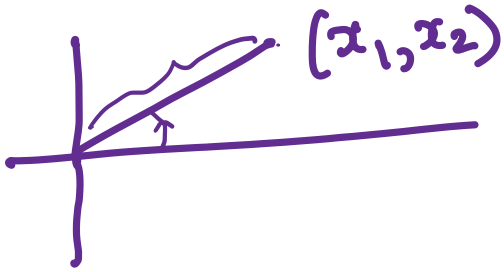
are vectors.

- A vector is an ordered k-tuple of scalars where k is any positive integer

Vectors are denoted as $(x_1, \ldots, x_k)$ or $\underbrace{[x_1, \ldots, x_k]}_{\text{(Row Vector)}}$

OR $\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \text{ or } \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}}_{\text{(Column Vector)}}$

Ex: In two dimensional plane, vectors are ordered pairs.



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Vector Formulation:

A system of Linear Equations can also be written in a vector form:

$$x_1 V_1 + x_2 V_2 + \cdots + x_n V_n = b$$

where x_i's are the scalar unknowns and V_i's are the column vectors formed from the coefficients of the system of Linear equations.

Note (Explanation):

The system of equation $A_{m\times n} X_{n\times 1} = b_{m\times 1}$ can be written as

$$x_1 V_1 + x_2 V_2 + \cdots + x_n V_n = b$$

where $V_1, V_2, \ldots, V_n$ are the n columns of A i.e. $A = [V_1 \; V_2 \cdots V_n]$

because

$$\boxed{AX} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{V_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{V_2} + \cdots\cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{V_n}$$

$$= \boxed{x_1 V_1 + x_2 V_2 + \cdots\cdots + x_n V_n}$$

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Solving System of Linear Equations:

- Small systems of Linear Equations (with two or three variables) can be solved by a method of "elimination" or method of "substitution".

Our Goal: To obtain a more systematic strategy (i.e., an "algorithm") to solve a system.

Note: In this process, the variables play no role.

All the calculations are done with the coefficients and RHS scalars (R.H.S. := Right hand side)

Thus we should directly work with matrices and develope a matrix algorithm.

It has several applications.

Elementary Row Operations:

- Given any $m \times n$ matrix A, we define three elementary row operations:

(1) Multiplication of one row of A by a non-zero scalar c (Scaling)

(2) Replacement of one row of A by the sum of the row and a scalar multiple of a different row (Replacement).

(3) Interchange of two rows of A (Interchange)

Thus by applying an elementary row operation e to A, we get a new matrix which will be denoted by $e(A)$.

Note: To each elementary row operation e, there corresponds an elementary row operation e_1 of the same type such that $e_1(e(A)) = A$

Thus the process is reversible.

Example: Let $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \end{array}$

Scaling:

$$A \xrightarrow{R_1 \to 2R_1} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Big\downarrow R_1 \to \tfrac{1}{2} R_1$$

The reverse operation:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$$

Interchange

$$A \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Big\downarrow R_1 \leftrightarrow R_3$$

The reverse Operation:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$$

Replacement:

$$A \xrightarrow{R_3 \to R_3 + (-5)R_1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & -2 & -6 \end{bmatrix}$$

$$\Big|$$

The reverse operation :

$R_3 \to R_3 + 5R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A$$