$$\boxed{\text{MTH 100 : Lecture 34}}$$

Ex: $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & -1 \\ -3 & -1-\lambda & 1 \\ 6 & 4 & -1-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 - \lambda$$

$$= -\lambda(\lambda^2 - 2\lambda + 1)$$

(as in a previous problem)

$$= (-\lambda)(1-\lambda)^2$$

So, the eigen values are

$\lambda_1 = 1$ with algebraic multiplicity 2

and $\lambda_2 = 0$ with algebraic multiplicity 1

For $\lambda_1 = 1$

$$A - \lambda_1 I = \begin{bmatrix} 4-1 & 2 & -1 \\ -3 & -1-1 & 1 \\ 6 & 4 & -1-1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 6 & 4 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 + R_2} \begin{bmatrix} 0 & 0 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -\frac{1}{3} R_1} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF matrix}$$

So, $(A - \lambda_1 I)x = 0 \Rightarrow x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 0$

$$x_2 = x_2$$

$$x_3 = x_3$$

So, $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Taking, $x_2 = 0$, $x_3 = 3$ we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Taking, $x_2 = 1$, $x_3 = 8$ we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$

<u>Now for $\lambda_2 = 0$</u>

$$A - \lambda_2 I = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{4} \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF matrix}$$

So, $(A - \lambda_2 I)x = 0 \Rightarrow \left.\begin{aligned} x_1 - \tfrac{1}{2}x_3 &= 0 \\ x_2 + \tfrac{1}{2}x_3 &= 0 \\ x_3 &= x_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Taking $x_3 = 2$ we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Now in this example,
geometric multiplicity of λ_1 is equal to its algebraic multiplicity $= 2$

and geometric multiplicity of λ_2 is equal to its algebraic multiplicity $= 1$

Therefore the matrix A is diagonalizable.

Check that $AP = DP$ where $P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix}$

and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$AP = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 8 & 0 \end{bmatrix}$$

and $$PD = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 8 & 0 \end{bmatrix}$$

Therefore $AP = PD$ as desired.

Ex: Look at the matrix of Worksheet 11 (Problem 10)
(H.W.) and find if it is diagonalizable.

Ex: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 3 & 2 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)^2$$

So, the eigen values are

$\lambda_1 = 1$ with algebraic multiplicity 2

and $\lambda_2 = 0$ with algebraic multiplicity 1.

Now for $\lambda_1 = 1$:

$$A - \lambda_1 I = \begin{bmatrix} 1-1 & 0 & 0 \\ 2 & 1-1 & 0 \\ 3 & 2 & 0-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

= RREF matrix

So, $(A-\lambda_1 I)x = 0 \Rightarrow \left.\begin{aligned} x_1 &= 0 \\ x_2 - \tfrac{1}{2}x_3 &= 0 \\ x_3 &= x_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Taking $x_3 = 2$ we get $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = v_1$ (say)

(Check that $Av_1 = 1 \cdot v_1$)

Now for $\lambda_2 = 0$

$$A - \lambda_2 I = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF matrix}$$

So, $(A-\lambda_2 I)x = 0 \Rightarrow \left.\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned}\right\}$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Taking $x_3 = 1$ we get $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as an eigen vector.

Now here

- geometric multiplicity of λ_1 is 1 but its algebraic multiplicity is 2 $\Big\}$ ..... (*)
- geometric multiplicity of λ_2 is 1 and its algebraic multiplicity is also 1

So, the matrix A is not diagonalizable.
(because of (*))

Last case for Diagonalization: Case 3:

An $n \times n$ matrix A has $p < n$ distinct eigenvalues, but even after counting the algebraic multiplicities, there are $< n$ real eigenvalues (p could even be 0).

Then A is not diagonalizable over the real field. If we want to diagonalize, we have to admit complex eigenvalues and eigenvectors.

Remark: Even if we admit complex eigenvalues and eigenvectors, a real matrix does not have to be diagonalizable. The case is quite complicated and we will not go into the details. However we will consider the case of 2×2 real matrix with a complex eigenvalue and describe the nature of such a matrix and its corresponding transformation (i.e. a linear operator on $\mathbb{R}^2$).

Basic Result for Complex Eigenvalues:

Suppose A is a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$, $b \neq 0$ and associated eigenvector v in $\mathbb{C}^2$.

Then $A = PBP^{-1}$ where $P = \begin{bmatrix} \operatorname{Re} v & \operatorname{Im} v \end{bmatrix}$

and $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

- Furthermore, the transformation (left multiplication by B) corresponds to a rotation followed by a scaling.
- The rotation is through the angle ϕ between the positive x-axis and the ray from the origin to (a, b).
  The angle ϕ is called the argument of λ.
- The scaling is by the factor $r = |\lambda| = \sqrt{a^2+b^2}$.
  The quantity $r = |\lambda|$ is known as the modulus of λ.

Ex: Let $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

The characteristic polynomial

$$= \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{vmatrix} = -\lambda(4-\lambda) - (-8)$$

$$= \lambda^2 - 4\lambda + 8$$

Then the eigen values are roots of $\lambda^2 - 4\lambda + 8 = 0$

$$\Rightarrow \quad \lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 8}}{2 \times 1}$$

$$= \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

We take $\lambda = 2 + 2i = 2 - (-2)\,i$ and so $\boxed{a = 2}$ and $\boxed{b = -2}$

Then the matrix $A - \lambda I = \begin{bmatrix} -2-2i & 1 \\ -8 & 4-(2+2i) \end{bmatrix}$

$$= \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}$$

Need to find the Corresponding Eigen Vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$(A - \lambda I)v = 0 \Rightarrow \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (-2-2i)x + y &= 0 \\ -8x + (2-2i)y &= 0 \end{aligned}\right\}$$

- Since the system has a nontrivial solution, its two rows are linearly dependent i.e. the two equations represent the same relationship between x and y.
- Taking the first equation,

$$(-2-2i)x + y = 0$$

$$\Rightarrow \quad y = (2+2i)x$$

Taking $x = 1$, $y = 2+2i$.

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Thus $P = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

$\begin{pmatrix} a = 2 \\ b = -2 \end{pmatrix}$

Now we can verify that $A = PBP^{-1}$

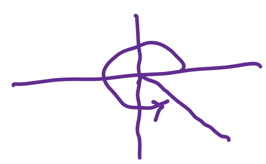
- $PBP^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix} = A$$

- Now $B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \sqrt{8} \begin{bmatrix} \frac{2}{\sqrt{8}} & \frac{2}{\sqrt{8}} \\ -\frac{2}{\sqrt{8}} & \frac{2}{\sqrt{8}} \end{bmatrix}$

$$= \sqrt{8} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The matrix represents a rotation through $\frac{7\pi}{4}$



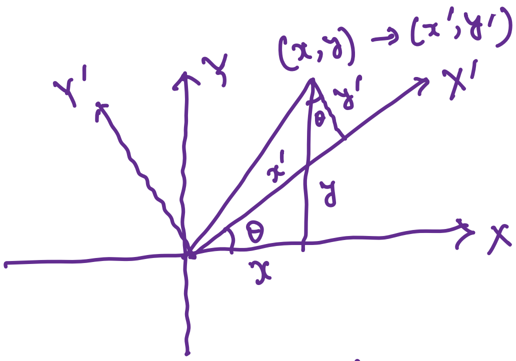
followed by a scaling through $\sqrt{8}$

- If we had taken $\lambda = 2-2i$ we would have obtained a different P and different B.

However $A = PBP^{-1}$ with these new P and B

Ref :

Coordinate geometry:



Then $x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

So, $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

and $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Ref to the tutorial problem :

The Determinant of a Linear Operator

Definition: Let $T: V \longrightarrow V$ be a linear operator where V is a vector space of finite dimension n.
Let α be any basis for V and let A be the matrix of T with respect to the (ordered) basis α. Then $\det T$ is defined as

$$\boxed{\det T = \det A}$$

Note: Suppose that β is any other basis for V and B is the matrix of T with respect to the basis β.

Now $B = PAP^{-1}$ where $P = P_{\alpha \to \beta}$ is the change of basis matrix.

Hence
$$\det B = \det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1})$$
$$= \det A$$

Thus the definition given above is meaningful. (independent of the basis taken)

Eigenvalues of Linear Operators

Definition: An eigenvector of a linear operator $T : V \longrightarrow V$ is a non-zero vector v such that $Tv = \lambda v$ for some scalar λ. Such a scalar is called an eigenvalue of the operator.

Note: If V is finite-dimensional, then the eigenvalues of a linear operator T coincide with the eigenvalues of the matrix of T with respect to any suitable basis of V

- This definition is useful for proving theoretical results and also for infinite-dimensional spaces.