MTH 100 : Lecture 23

Last time: For any $m \times n$ matrix A, we defined $\mathrm{Nul}\, A$, $\mathrm{Col}\, A$ and $\mathrm{Row}\, A$.

- We have seen how to find Bases for all these three spaces.
- Note that there is no containment relationship between $\mathrm{Nul}\, A$, $\mathrm{Col}\, A$ and $\mathrm{Row}\, A$.

In general $\mathrm{Nul}\, A$ and $\mathrm{Col}\, A$ are not even subspaces of the same space because $\mathrm{Col}\, A \subseteq \mathbb{R}^n$ and $\mathrm{Nul}\, A \subseteq \mathbb{R}^n$.

The Rank Theorem:

Definition: If A is an $m \times n$ matrix, the column rank of A is defined to be $\boxed{\dim(\mathrm{Col}\, A)}$.

Similarly, the row rank of A is defined to be $\boxed{\dim(\mathrm{Row}\, A)}$.

- The nullity of A is defined to be $\boxed{\dim(\mathrm{Nul}\, A)}$.

Ex: In the last example of Lecture 24,

A is a 4×4 matrix.

row rank $= 2$, Column rank $= 2$, nullity $= 2$

Theorem: (The Rank Theorem for Matrices):

(a) The row rank and column rank of a matrix A are equal. This number is called the rank of A.

(b) The rank of A is equal to the number of pivot positions in the RREF matrix obtained from A.

(c) $\text{rank}(A) + \text{nullity}(A) = n =$ number of columns of A.

Sketch of a Proof:

- (a) and (b) follow from our discussion of finding the Basis of Col A and Row A
  In each case, the number of basis vectors corresponded to the number of pivot elements in the RREF matrix R of a given matrix A.

- For (c),
  Pivot columns of R will correspond to a basis of Col A (leading variables of the homogeneous system).
  - The remaining columns correspond to a basis of Nul A (free variables of the homogeneous system).

  Since, the total number of columns $= n$
  $=$ number of variables,

we get

$n =$ number of basis vectors in Col A
$\quad +$ number of basis vectors in Nul A

$$\Rightarrow \boxed{n = \text{rank}(A) + \text{nullity}(A)} \qquad \text{(QED)}$$

<u>Note</u>

(1) $\text{Col } A = \mathbb{R}^m$ if and only if the system $Ax=b$ has a solution for each $b \in \mathbb{R}^m$.

(This follows from the description of Col A)

(2) An $m \times m$ matrix A is invertible if and only if its columns form a basis of $\mathbb{R}^m$.

(This follows from Note (1) above and part (d) of the first Theorem of the course.)

<u>Corollary to Rank Theorem:</u>

A square $m \times m$ matrix A is invertible if and only if $\text{rank}(A) = m$ (or equivalently nullity $= 0$)

- In view of today's discussion, an <u>extended version</u> of the <u>first theorem of our course</u> can be given in the following way.

Theorem:

The following are equivalent for an $m \times m$ square matrix A.

(a) A is invertible.

(b) A is row equivalent to the identity matrix.

(c) The homogeneous system $Ax = 0$ has only the trivial solution.

(d) The system of equations $Ax = b$ has at least one solution for every $b \in \mathbb{R}^m$.

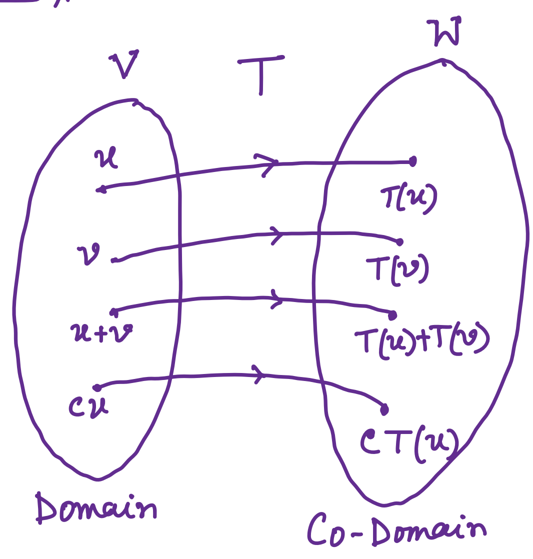
(e) $\text{Nullity}(A) = 0$

(f) $\text{Rank}(A) = m$

(g) The columns of A form a basis for $\mathbb{R}^m$.

(h) $\text{Det } A \neq 0$

Linear Transformations:



Definition: A map or function $T : V \to W$ from a vector space V to a vector space W is called a Linear Transformation (or briefly linear)

if (1) $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in V$

(2) $T(cu) = cT(u) \quad \forall\, u \in V$ and $\forall\, c \in F$

(F is the scalar field)

Note: (1) The space W (the Co-Domain) may be the space V or a subspace of V or may be an entirely different space (but over the same field F).

(2) We may write either $T(v)$ or Tv to indicate the image of the vector v under the transformation T.

(3) Some books use the term homomorphism for a linear transformation (map or function) from a vector space V to a vector space W.

Examples:

(1) The zero transformation $\mathbf{0} : V \longrightarrow W$ defined by $\mathbf{0}(u) = \bar{0}$ (zero vector in W) $\forall\, u \in V$

$\mathbf{0}(u+v) = \bar{0} = \bar{0}+\bar{0} = \mathbf{0}(u) + \mathbf{0}(v)$

$\mathbf{0}(cu) = \bar{0} = c.\bar{0} = c\,\mathbf{0}(u)$ $\quad \forall\, u, v \in V$, $\forall\, c \in \mathbb{R}$

(2) The identity transformation $I : V \longrightarrow V$ defined by $I(u) = u \;\; \forall\, u \in V$

$I(u+v) = u + v = I(u) + I(v)$

$I(cu) = cu = cI(u) \;\; \forall\, u, v \in V \;\; \forall\, c \in \mathbb{R}$

(3) Projection: Define the function $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$P_i(x_1, x_2, \dots, x_i, \dots, x_n) = (0, 0, \dots, 0, x_i, 0, \dots, 0)$$

(all coordinates other than the i-th coordinate are replaced by 0.)

- Then P_i is a linear transformation.
  We can extend this idea by projecting onto any selection of coordinates.

<u>To show that P_i is linear:</u> If $(x_1, x_2, \ldots, x_i, \ldots, x_n), (y_1, y_2, \ldots, y_i, \ldots y_n) \in \mathbb{R}^n$

then $P_i\left[(x_1, x_2, \ldots, x_i, \ldots, x_n) + (y_1, y_2, \ldots, y_i, \ldots, y_n)\right]$

$$= P_i\,(x_1+y_1, x_2+y_2, \ldots, x_i+y_i, \ldots, x_n+y_n)$$

$$= (0, 0, \ldots, 0, x_i+y_i, 0, \ldots, 0)$$

$$= (0, 0, \ldots, 0, x_i, 0, \ldots, 0) + (0, 0, \ldots, 0, y_i, 0, \ldots, 0)$$

$$= P_i\,(x_1, x_2, \ldots, x_i, \ldots, x_n) + P_i\,(y_1, y_2, \ldots, y_i, \ldots, y_n)$$

Now if $c \in \mathbb{R}$ and $(x_1, x_2, \ldots, x_i, \ldots x_n) \in \mathbb{R}^n$

then $P_i\left[c\,(x_1, x_2, \ldots, x_i, \ldots, x_n)\right]$

$$= P_i\,(cx_1, cx_2, \ldots, cx_i, \ldots, cx_n)$$

$$= (0, 0, \ldots, 0, cx_i, 0, \ldots, 0)$$

$$= c\,(0, 0, \ldots, 0, x_i, 0, \ldots, 0)$$

$$= c\,P_i\,(x_1, x_2, \ldots, x_i, \ldots, x_n)$$

Therefore P_i is a linear transformation.