MTH 100 : Lecture 18

Remark 4: A list of non-zero vectors is linearly dependent if and only if atleast one of the vectors is a linear combination of the others.

Proof of Remark 4:

$\Rightarrow$: **given:** A list of vectors $v_1, v_2, \ldots, v_p$ is linearly dependent.

To show that: At least one of the vector is a linear combination of the others.

Since $v_1, v_2, \ldots, v_p$ are linearly dependent, there exist scalars $c_1, c_2, \ldots, c_p \in F$ not all zero such that $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = \bar{0}$

Let us assume $c_k \neq 0$ where $1 \leq k \leq p$

$$c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} + c_k v_k + c_{k+1} v_{k+1} + \cdots + c_p v_p = \bar{0}$$

$$\Rightarrow c_k v_k = -c_1 v_1 - c_2 v_2 - \cdots - c_{k-1} v_{k-1} - c_{k+1} v_{k+1} - \cdots - c_p v_p$$

Now $c_k \neq 0$ and $c_k \in F$; so, $c_k^{-1} \in F$

Then $c_k^{-1}(c_k v_k) = -c_k^{-1}(c_1 v_1) - c_k^{-1}(c_2 v_2) - \cdots - c_k^{-1}(c_{k-1} v_{k-1})$
$- c_k^{-1}(c_{k+1} v_{k+1}) - \cdots - c_k^{-1}(c_p v_p)$

$\Rightarrow (c_k^{-1} c_k) v_k = (-c_k^{-1} c_1) v_1 - (c_k^{-1} c_2) v_2 - \cdots - (c_k^{-1} c_{k-1}) v_{k-1}$
$- (c_k^{-1} c_{k+1}) v_{k+1} - \cdots - (c_k^{-1} c_p) v_p$

$\Rightarrow v_k = (-c_k^{-1} c_1) v_1 + (-c_k^{-1} c_2) v_2 + \cdots + (-c_k^{-1} c_{k-1}) v_{k-1}$
$+ (-c_k^{-1} c_{k+1}) v_{k+1} + \cdots + (-c_k^{-1} c_p) v_p$

where $(-c_k^{-1} c_1), (-c_k^{-1} c_2), \ldots, (-c_k^{-1} c_{k-1}), (-c_k^{-1} c_{k+1}) \cdots (-c_k^{-1} c_p) \in F$

$\Leftarrow$: Given: Atleast one of the vector is a linear combination of the rest of the vectors

To show: The list is linearly dependent.

Let us assume that v_k is a linear combination of the rest of the vectors

Then there exist scalars $c_1, c_2, \ldots, c_{k-1}, c_{k+1}, \ldots c_p \in F$

such that

$$v_k = c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} + c_{k+1} v_{k+1} + \cdots + c_p v_p$$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} - v_k + c_{k+1} v_{k+1} + \cdots + c_p v_p = \bar{0}$$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_{k-1} v_{k-1} + (-1) v_k + c_{k+1} v_{k+1} + \cdots + c_p v_p = \bar{0}$$

where atleast one of the scalar is $(-1) \neq 0$

So, $v_1, v_2, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_p$ are linearly dependent.

Remark 5: Consequently any list of vectors which contains a repeated vector must be linearly dependent. A list which is linearly independent corresponds to a set.

Remark 6: Any list which contains a linearly dependent list is linearly dependent.

Remark 7: Any subset of a linearly independent set is linearly independent.

Proof of Remark 6:

Suppose $v_1, v_2, \ldots, v_p$ is a list of vectors that contains a list

$v_1, v_2, \ldots, v_k$ (where $1 < k \leq p$), which is linearly dependent.

(WLOG, can assume that the linearly dependent vectors are at the beginning of the list.)

Then there exist scalars $c_1, c_2, \ldots, c_k \in F$ (not all of them zeros)

such that $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \overline{0}$

Then $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + 0.v_{k+1} + \cdots + 0.v_p = \overline{0}$

Hence $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_p$ are linearly dependent.

Note: Proofs of Remark 5 and Remark 7 are left as exercises.

Ex: Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Question: Are v_1, v_2, v_3 linearly independent?

Consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = \bar{0}$ where $c_1, c_2, c_3 \in \mathbb{R}$

$$\Rightarrow \quad c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{r} c_1 + 5c_2 + c_3 = 0 \\ 2c_1 + 6c_2 + c_3 = 0 \\ 3c_1 + 7c_2 + c_3 = 0 \\ 4c_1 + 8c_2 + c_3 = 0 \end{array}\right\}$$

The coefficient matrix:

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 3 & 7 & 1 \\ 4 & 8 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1}]{} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -4 & -1 \\ 0 & -8 & -2 \\ 0 & -12 & -3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2}]{} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to \left(-\frac{1}{4} R_2\right)]{} \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 5R_2}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (Here c_3 is a free variable)

So, the solution is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$ where c_3 is any real number.

If $c_3 = 4$, one possible solution is $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

Therefore $\boxed{v_1 - v_2 + 4v_3 = 0}$

Hence v_1, v_2 and v_3 are linearly dependent.

Ex: Let $V = C[0, 2\pi]$

Let $f(x) = 1$, $g(x) = \sin x$, $h(x) = \cos x$

Question: Are f, g, h linearly independent ?

$$c_1 f + c_2 g + c_3 h = \bar{0}(x)$$

$$\Rightarrow \quad c_1 \times 1 + c_2 \sin x + c_3 \cos x = 0 \quad \forall\, x \in [0, 2\pi]$$

If $x = 0 \Rightarrow c_1 + 0 + c_3 \times 1 = 0 \Rightarrow c_1 + c_3 = 0$

If $x = \frac{\pi}{2} \Rightarrow c_1 + c_2 \times 1 + 0 = 0 \Rightarrow c_1 + c_2 = 0$

If $x = \pi \Rightarrow c_1 + 0 - c_3 = 0 \Rightarrow c_1 - c_3 = 0$

The coefficient matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to (-\frac{1}{2} R_3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{RREF matrix}$$

So, the system has only the trivial solution
$c_1 = 0$, $c_2 = 0$ and $c_3 = 0$

Hence f, g and h are linearly independent.

Ex: Let $u_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix}$, $u_3 = \begin{bmatrix} 3 \\ 24 \\ 9 \end{bmatrix}$

Let $A = [u_1 \ u_2 \ u_3]_{3\times 3}$

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 24 \\ 6 & 4 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 6 & 3 \\ 0 & 0 & 18 \\ 0 & -14 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 6 & 3 \\ 0 & -14 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to \left(-\frac{1}{14} R_2\right) \\ R_3 \to \left(\frac{1}{18} R_3\right)}]{R_1 \to \left(\frac{1}{2} R_1\right)} \begin{bmatrix} 1 & 3 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{3}{2} R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - 3R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{RREF matrix}$$

Therefore $c_1 u_1 + c_2 u_2 + c_3 u_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

So, u_1, u_2 and u_3 are linearly independent.

Furthermore for any vector $\bar{b} \in \mathbb{R}^3$, the equation $A\bar{x} = \bar{b}$ has a unique solution and so b can be written as a linear combination of u_1, u_2 and u_3.

Thus $\bar{b} \in \operatorname{Span}\{u_1, u_2, u_3\} \quad \forall\, \bar{b} \in \mathbb{R}^3$

Hence $\boxed{\operatorname{Span}\{u_1, u_2, u_3\} = \mathbb{R}^3}$

Basis and Dimension:

Definition: Let V be a vector space over a field F.

A Basis for V is a linearly independent set S of vectors such that $\boxed{V = \operatorname{Span} S}$

Ex: In the previous example, $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$.

Ex: Consider the vector space $\mathbb{R}^n$ over the field $\mathbb{R}$.

Consider the vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

- $e_1, e_2, \dots, e_n$ are linearly independent.

$$c_1 e_1 + c_2 e_2 + \dots + c_n e_n = \bar{0}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1 = c_2 = \dots = c_n = 0$$

- Any vector $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be written as

$$\bar{x} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

Conclusion: $\{e_1, e_2, \dots, e_n\}$ forms a basis of $\mathbb{R}^n$

Note: In one of the examples considered before we have shown that $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$.

Applying the last example for $n=3$, we can also say that $\{e_1, e_2, e_3\}$ is another basis of $\mathbb{R}^3$.

(Note: Plural of Basis = Bases)

Definition: • A vector space which has a finite basis is called Finite dimensional.

• A vector space which doesn't have a finite basis is called infinite dimensional.

Note: The above definition would also apply to subspaces of V.

Ex: $\mathbb{R}^n$ is a finite dimensional vector space.

Example of an infinite dimensional vector space:

Let $R[t]$ be the vector space of all polynomials (in t) with real coefficients.

$R[t]$ is a vector space over $\mathbb{R}$.

We will show that $R[t]$ is infinite dimensional.

Suppose BWOC that $R[t]$ is finite dimensional.

Then it must have a finite basis,

say $B = \{ p_1(t), p_2(t), \dots, p_n(t) \}$

Let $N = \max \{ \deg p_1(t), \deg p_2(t), \dots, \deg p_n(t) \}$

where $\deg p_k(t)$ = degree of $p_k(t)$ $\left(\text{the } k\text{-th polynomial in } B\right)$

Let $p(t) = t^{N+1}$

Then $p(t) \notin \operatorname{Span} B$, a contradiction.

So, $R[t]$ is infinite dimensional.