MTH100

Mid Semester Exam

Oct 9th, 2024

Time: 1 hour 20 minutes Max Marks: 50

Instructions:

1. Attempt all the five questions. Marks for each question are indicated against it.
2. All your intermediate steps and calculations must be clearly shown.
3. You are not allowed to use determinant in this exam.
4. Marks for proof-type questions will depend on the logical progression of the steps. You may quote without proof any proposition or theorem covered in the lectures and tutorials but it must be clearly identified. Any other results used must be proved.
5. Start each question on a fresh side and clearly indicate if part of the question is done in a different part of the answer book.

Problem 1. (a) Are the vectors

$\overline{\alpha_1} = (1,1,2,4)$, $\overline{\alpha_2} = (2,-1,-5,2)$, $\overline{\alpha_3} = (1,-1,-4,0)$ and $\overline{\alpha_4} = (2,1,1,6)$

linearly independent in $\mathbb{R}^4$? (5 points)

(b) **Show** that the following vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

$\overline{\beta_1} = (1,0,-1)$, $\overline{\beta_2} = (1,2,1)$ and $\overline{\beta_3} = (0,-3,2)$.

Express $\overline{e_1} = (1,0,0)$ and $\overline{e_3} = (0,0,1)$ as linear combinations of $\overline{\beta_1}$, $\overline{\beta_2}$, $\overline{\beta_3}$. (7 points)

Problem 2. (a) Using elementary row operations find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ (if it exists) over $\mathbb{Z}_3$ (4 points)

(Note: You are not allowed to use determinant.)

(b) Extend $\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\}$ to a basis for $\mathbb{R}^{2\times 2}$ where $\mathbb{R}^{2\times 2}$ is the vector space of all (2×2) matrices with real entries. (4 points)

(Note: You need to show proper reasoning.)

Problem 3. (a) Let $C^1[-1,1] = \{f \in C[-1,1] : f'(x) \text{ exists and } f'(x) \text{ is continuous on } [-1,1]\}$

Is $C^1[-1,1]$ a subspace of $C[-1,1]$?

If yes, is $C^1[-1,1]$ a proper subspace of $C[-1,1]$?

(Give reasons) (4 points)

(b) Let $R_3(t)$ be the vector space of all the polynomials (in variable t) of degree ≤ 3 with real coefficients. Let $f(t)$ be a polynomial of degree 3 in $R_3(t)$.

Show that for any $g(t) \in R_3(t)$, there exists scalars c_0, c_1, c_2, $c_3 \in \mathbb{R}$ such that

$g(t) = c_0 f(t) + c_1 f'(t) + c_2 f''(t) + c_3 f'''(t)$ (4 points)

(c) (i) An $n \times n$ matrix A is called idempotent if $A^2 = A$.
Show that the only invertible idempotent $n \times n$ matrix is the identity matrix.

(ii) Prove that if A and B are square matrices and AB is invertible, then both A and B are invertible.

(4 points)

Problem 4. (a) Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots, v_1 + v_2 + \ldots + v_n\}$ is also a basis for V (4 points)

(b) Let M be an $n \times n$ upper triangular matrix with non-zero diagonal entries. Prove that the columns of M are linearly independent. (4 points)

Problem 5. Let A and B be $m \times n$ matrices that are both in RREF form such that $A \neq B$. Suppose that the first $(n-1)$ columns of A and B are identical.

Assume further that neither A nor B have pivot positions in the last column.

Prove or disprove: there exists a vector $\overline{x}$ in $\mathbb{R}^n$ such that $A\overline{x} = \overline{0}$ but $B\overline{x} \neq \overline{0}$. (10 points)

Rubrics for Mid-Sem Exam (Total = 50 points)

1 (a) $\bar{\alpha}_1 = (1,1,2,4)$, $\bar{\alpha}_2 = (2,-1,-5,2)$

$\bar{\alpha}_3 = (1,-1,-4,0)$, $\bar{\alpha}_4 = (2,1,1,6)$

Now $c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3 + c_4\bar{\alpha}_4 = \bar{0}$

$\Rightarrow c_1(1,1,2,4) + c_2(2,-1,-5,2) + c_3(1,-1,-4,0) + c_4(2,1,1,6) = (0,0,0,0)$

$$\Rightarrow \begin{aligned} c_1 + 2c_2 + c_3 + 2c_4 &= 0 \\ c_1 - c_2 - c_3 + c_4 &= 0 \\ 2c_1 - 5c_2 - 4c_3 + c_4 &= 0 \\ 4c_1 + 2c_2 + 0.c_3 + 6c_4 &= 0 \end{aligned}$$

$\Rightarrow$ The coefficient matrix is

+3

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= RREF matrix

+1 From the RREF matrix, we can conclude that the system of the homogeneous equation has nontrivial solutions. (Since there are free variables)

+1 Hence $\overline{\alpha_1}$, $\overline{\alpha_2}$, $\overline{\alpha_3}$ and $\overline{\alpha_4}$ are linearly dependent.

Note:

1. They don't need to give the explicit solution of the system, as long as they can correctly conclude that there is a nontrivial solution (from the RREF matrix)

2. They can start from the coefficient matrix itself.

1 (b) First we will show that $\bar{\beta_1}$, $\bar{\beta_2}$ and $\bar{\beta_3}$ are linearly independent in $\mathbb{R}^3$.

$$c_1\bar{\beta_1} + c_2\bar{\beta_2} + c_3\bar{\beta_3} = \bar{0}$$

$$\Rightarrow \quad c_1(1,0,-1) + c_2(1,2,1) + c_3(0,-3,2) = (0,0,0)$$

$$\Rightarrow \quad \begin{aligned} c_1 + c_2 &= 0 \\ 2c_2 - 3c_3 &= 0 \\ -c_1 + c_2 + 2c_3 &= 0 \end{aligned}$$

The coefficient matrix is

+2

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -3 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{5}R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + \frac{3}{2}R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{RREF matrix.}$$

Therefore $c_1 = c_2 = c_3 = 0$ and so $\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3$ are linearly independent.

+1 Since $\dim(\mathbb{R}^3) = 3$, it follows that $\bar{\beta}_1$, $\bar{\beta}_2$ and $\bar{\beta}_3$ form a basis for $\mathbb{R}^3$.

Now, to solve $c_1 \bar{\beta}_1 + c_2 \bar{\beta}_2 + c_3 \bar{\beta}_3 = \bar{e}_1 = (1,0,0)$

let us apply the same elementary operations on $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{2} R_2]{} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{5} R_3]{} \begin{bmatrix} 1 \\ 0 \\ \frac{1}{5} \end{bmatrix} \xrightarrow[R_2 \to R_2 + \frac{3}{2} R_3]{} \begin{bmatrix} 1 \\ +3/10 \\ \frac{1}{5} \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \begin{bmatrix} 7/10 \\ \frac{3}{10} \\ \frac{1}{5} \end{bmatrix}$$

Thus $c_1 = \frac{7}{10}$, $c_2 = \frac{3}{10}$, $c_3 = \frac{1}{5}$

and $\boxed{\frac{7}{10} \bar{\beta}_1 + \frac{3}{10} \bar{\beta}_2 + \frac{1}{5} \bar{\beta}_3 = (1,0,0)}$

+2

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Now to solve

$$c_1 \bar{\beta}_1 + c_2 \bar{\beta}_2 + c_3 \bar{\beta}_3 = \bar{e}_3 = (0, 0, 1)$$

let us apply the same elementary operations on $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{5} R_3} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \end{bmatrix} \xrightarrow{R_2 \to R_2 + \frac{3}{2} R_3} \begin{bmatrix} 0 \\ \frac{3}{10} \\ \frac{1}{5} \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} -\frac{3}{10} \\ \frac{3}{10} \\ \frac{1}{5} \end{bmatrix}$$

Thus $c_1 = -\frac{3}{10}$, $c_2 = \frac{3}{10}$, $c_3 = \frac{1}{5}$

and $\boxed{-\frac{3}{10} \bar{\beta}_1 + \frac{3}{10} \bar{\beta}_2 + \frac{1}{5} \bar{\beta}_3 = \bar{e}_3 = (0, 0, 1)}$

+2

Note: • Since this problem is rather simple, some students may directly solve the equations instead of using matrix operations.

As long as their solution is correct, they get credit for each part.

(6)

(2) (a) $[A \vdots I] = \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$

$\big\downarrow R_1 \to 2R_1$

$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 4 \pmod 3 & 4 \pmod 3 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$

+4

$\big\downarrow R_2 \to R_2 + R_1$

$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 3 \pmod 3 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$

$\big\downarrow R_1 \to R_1 + 2R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 3 \pmod 3 & 6 \pmod 3 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$

Thus $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ over $\mathbb{Z}_3$.

Note: The students can do it in a different way

such as:

$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to 2R_1} \left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1}$

$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & -2 \pmod 3 = 1 & -4 \pmod 3 = 2 & 1 \end{array}\right]$

$$= \left[\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

(Since $-2 = 3(-1)+1$
$-2 \pmod 3 = 1$
Since $-4 = 3(-2)+2$
$-4 \pmod 3 = 2$
(By Division algorthm))

$$\xrightarrow[R_1 \to R_1 - R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \pmod 3 = 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

(since $-1 = 3(-1)+2$
$-1 \pmod 3 = 2$)

Therefore $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ over $\mathbb{Z}_3$

Note: (1) Please carefully check the modular arithmatic in $\mathbb{Z}_3$

(2) They don't get any credit if they calculate the inverse of the matrix over $\mathbb{R}$

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2(b) First note that

$$c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&1\\1&0\end{bmatrix}+c_3\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

+1

$$\Rightarrow \begin{bmatrix}c_1&c_2-c_3\\c_2+c_3&c_1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow \left.\begin{matrix}c_1=0\\c_2-c_3=0\\c_2+c_3=0\end{matrix}\right\} \Rightarrow c_1=c_2=c_3=0$$

Thus the three matrices are linearly independent on $\mathbb{R}^{2\times 2}$.

+2

It also shows that the span of these three matrices ~~[illegible]~~ is the set

$$\left\{\begin{bmatrix}a&b\\c&a\end{bmatrix} : a,b,c\in\mathbb{R}\right\}$$

i.e. all the matrices whose $(1,1)$th and $(2,2)$th elements are same.

(note that any two elements b and c can be written as

$$b=\frac{c+b}{2}-\frac{c-b}{2} \quad \text{and} \quad c=\frac{c+b}{2}+\frac{c-b}{2}$$
)

Therefore if we take any matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ where $a\neq d$, it will be outside the span of these three matrices and so if we include that, the will be extended to a basis

(It is easy to check that they will be linearly independent and $\dim(\mathbb{R}^{2\times 2})=4$)

For example,

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

+1 is a basis of $\mathbb{R}^{2\times 2}$

Note:

1. They can give different answers (depending on the 4th matrix)

Please check carefully.

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3 (a) (i) Yes, $C^1[-1,1]$ is a subspace of $C[-1,1]$

+1

- The zero function $O(x) = 0 \; \forall x \in [-1,1]$ belongs to $C^1[-1,1]$ because it is differentiable and

$$O'(x) = 0 \quad \forall x \in [-1,1]$$

i.e. $O'(x) = O(x)$ is continuous on $[-1,1]$

- Now suppose $f, g \in C^1[-1,1]$.

+2

Then $f'(x)$ and $g'(x)$ exist and are continuous on $[-1,1]$

Now $(f+g)'(x) = f'(x) + g'(x)$ is continuous on $[-1,1]$

So, $f+g \in C^1[-1,1]$

- Suppose $f \in C^1[-1,1]$ and $c \in \mathbb{R}$ is a scalar.

Then $f'(x)$ exists and is continuous on $[-1,1]$

Now $(cf)'(x) = c f'(x)$ is continuous on $[-1,1]$

So, $cf \in C^1[-1,1]$ for any scalar $c \in \mathbb{R}$

Therefore $C^1[-1,1]$ is a subspace of $C[-1,1]$

- $C^1[-1,1]$ is a proper subspace of $C[-1,1]$

For example Consider $g(x) = |x|$ for $x \in [-1,1]$

+1 Then, $g \in C[-1,1]$, but since g is not differentiable at $x=0$, $g \notin C^1[-1,1]$.

OR:

They can give other examples.

e.g. Consider, $h(x) = 0$ for $-1 \le x \le 0$
$= x$ for $0 < x \le 1$

Then $h \in C[-1,1]$ but h is not differentiable at $x=0$. So, $h \notin C^1[-1,1]$.

3(b) Let $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ where $a_3 \neq 0$

Then
$$f'(t) = a_1 + 2a_2 t + 3a_3 t^2$$
$$f''(t) = 2a_2 + 6a_3 t$$
$$f'''(t) = 6a_3$$

Will show that $f(t)$, $f'(t)$, $f''(t)$ and $f'''(t)$ are linearly independent in $R_3(t)$

$$c_1 f(t) + c_2 f'(t) + c_3 f''(t) + c_4 f'''(t) = 0(t) = 0$$
(Zero polynomial)

$$\Rightarrow c_1 a_3 t^3 + (c_2 3a_3 + c_1 a_2)t^2 + (c_3 6a_3 + c_2 2a_2 + c_1 a_1)t + (c_4 6a_3 + c_3 2a_2 + c_2 a_1 + c_1 a_0) = 0 \ \forall t \in R$$

$\Rightarrow \quad c_1 a_3 = 0 \Rightarrow c_1 = 0$ (since $a_3 \neq 0$)

Then $c_2\, 3a_3 = 0 \Rightarrow c_2 = 0$

Then $c_3\, 6a_3 = 0 \Rightarrow c_3 = 0$

Then $c_4\, 6a_3 = 0 \Rightarrow c_4 = 0$

+2

So, $f(t), f'(t), f''(t), f'''(t)$ are linearly independent in $R_3(t)$

Since $\dim R_3(t) = 4$ ($1, t, t^2, t^3$ is a basis for $R_3(t)$)

it follows that $f(t), f'(t), f''(t)$ and $f'''(t)$ is a basis for $R_3(t)$ (and is therefore is a spanning set for $R_3(t)$)

+1

Hence if $g(t) \in R_3(t)$, it can be written as a linear combination of $f(t), f'(t), f''(t)$ and $f'''(t)$. and so there exists scalars $c_0, c_1, c_2, c_3 \in \mathbb{R}$ such that

+1

$$\boxed{g(t) = c_0 f(t) + c_1 f'(t) + c_2 f''(t) + c_3 f'''(t)}$$

Note:

1. If they conclude that $f(t), f'(t), f''(t)$ and $f'''(t)$ are polynomials of degree 3, degree 2, degree 1 and degree 0 respectively and hence they are linearly independent, they will get credit (+2 points (for this part)) ~~they~~ In that case they don't need to explicitly solve the system of equations.

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3(e): (i) Let A be an $(n\times n)$ idempotent matrix and A is invertible.

Then $A^2 = A$

Since A^{-1} exists, multiplying both sides by A^{-1} we get

$$A^{-1}(A^2) = A^{-1}.A$$

+2

$$\Rightarrow A^{-1}(A.A) = I \Rightarrow (A^{-1}A)A = I$$

$$\Rightarrow I.A = I \Rightarrow \boxed{A = I}$$

So, A is the $(n\times n)$ identity matrix.

(ii) Given AB is invertible (A & B are square matrices)

Let $C = (AB)^{-1}$

Hence $C(AB) = (AB)C = I$

there exists a matrix C s.t.

+1

$$\Rightarrow C(AB) = I$$

Thus CA is the left inverse of B.

Hence by corollary (1.2), $\boxed{B \text{ is invertible}}$

Also, $(AB)C = I \Rightarrow A(BC) = I$

Thus BC is the right inverse of A

+1

Hence by corollary (1.2), $\boxed{A \text{ is invertible}}$

Note: • They can write the proof in different ways.

(1) Consider the homogeneous system $B\bar{x}=\bar{0}$
Multiplying both sides from left by A,

$A(B\bar{x}) = A.\bar{0} \Rightarrow (AB)\bar{x}=\bar{0}$

Since AB is invertible, $\bar{x}=0$

Therefore B is invertible.

So, B^{-1} exists and B^{-1} is also invertible.

Since the product of two invertible matrices is invertible $(AB)B^{-1}$ is invertible

$\Rightarrow A(BB^{-1}) = A$ is invertible.

(2) Some students may try method of contradiction.

So, assume either A or B, or both are not invertible

Case 1: B is not invertible.

Then $B\bar{x}=\bar{0}$ has a nontrivial solution

$\Rightarrow A(B\bar{x})=\bar{0} \Rightarrow (AB)\bar{x}=\bar{0}$ has a nontrivial solution

$\Rightarrow AB$ is not invertible, a contradiction

Case 2: B is invertible but A is not invertible

Since A is not invertible, $A\bar{x}=\bar{0}$ has a nontrivial solution $\bar{y}$ (say)

Since B is invertible,
$B\bar{x}=\bar{y}$ has a solution clearly $\bar{x}\neq\bar{0}$

Now $(AB)\bar{x} = A(B\bar{x}) = A\bar{y}=\bar{0} \Rightarrow (AB)\bar{x}=0$ has a nontrivial solution

$\Rightarrow AB$ is not invertible, contradiction.

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4 (a) We will first show that

$$v_1,\ v_1+v_2,\ v_1+v_2+v_3,\ \dots,\ v_1+v_2+\dots+v_n$$

are linearly independent.

+1

$$c_1v_1 + c_2(v_1+v_2) + c_3(v_1+v_2+v_3) + \dots + c_n(v_1+v_2+\dots+v_n) = \bar{0}$$

$$\Rightarrow (c_1+c_2+c_3+\dots+c_n)v_1 + (c_2+c_3+\dots+c_n)v_2 + (c_3+c_4+\dots+c_n)v_3 + \dots + c_nv_n = \bar{0}$$

+1

$$\Rightarrow \begin{cases} c_1+c_2+c_3+\dots+c_n = 0 \\ c_2+c_3+\dots+c_n = 0 \\ c_3+\dots+c_n = 0 \\ \quad\vdots \\ c_{n-1}+c_n = 0 \\ c_n = 0 \end{cases}$$

(Since $v_1, v_2, \dots, v_n$ are linearly independent.)

+1 Substituting back from the last equation successively we get $c_n = 0,\ c_{n-1} = 0,\ \dots,\ c_3 = 0,\ c_2 = 0,\ c_1 = 0$

Hence $v_1,\ v_1+v_2,\ v_1+v_2+v_3,\ \dots,\ v_1+v_2+\dots+v_n$ are linearly independent.

+1 Since $\dim V = n$ ($\{v_1, v_2, \dots, v_n\}$ is a basis of V) it follows that $\boxed{\{v_1,\ v_1+v_2,\ v_1+v_2+v_3,\ \dots,\ v_1+v_2+\dots+v_n\} \text{ is a basis of } V.}$

(b) Let M be an $n \times n$ upper triangular matrix with non-zero diagonal entries 16

Let $M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ where $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$

Now $c_1 \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

+2

$$\Rightarrow \left.\begin{aligned} c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} &= 0 \\ c_2 a_{22} + \cdots + c_n a_{2n} &= 0 \\ \cdots\cdots & \\ c_n a_{nn} &= 0 \end{aligned}\right\}$$

+2

From the last equation $c_n = 0$ (Since $a_{nn} \neq 0$)

Substituting in the second last equation

$$c_{n-1} a_{n-1\, n-1} + c_n a_{n-1\, n} = 0$$

$$\Rightarrow c_{n-1} a_{n-1\, n-1} + 0 = 0 \Rightarrow c_{n-1} = 0$$

(Since $a_{n-1\, n-1} \neq 0$)

Proceeding in this way $c_1 = 0, c_2 = 0, \ldots, c_{n-1} = 0, c_n = 0$

Therefore the columns of M are linearly independent.

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5

+2 • We will prove the given statement.

• Let A_1 be the $m \times (n-1)$ matrix formed by taking the first $(n-1)$ columns of matrix A, in the same order as in original matrices.

Let $\bar{a}$ be the last column of A.

+3 Consider the non homogeneous system

$$A_1 \bar{y} = \bar{a} \quad \text{where } \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

• The Augmented matrix for the system is:

+3 $$[A_1 \,\vdots\, \bar{a}] = A$$

Since $\bar{a}$ is not a pivot column, the system has a solution $\bar{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix}$ i.e. $A_1 \bar{z} = \bar{a}$

• Let $\bar{x} \in \mathbb{R}^n$ is formed such that

+2 $$\bar{x} = \begin{bmatrix} \bar{z} \\ -1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ -1 \end{bmatrix}$$

Then $A\bar{x} = [A_1 \,\vdots\, \bar{a}] \begin{bmatrix} \bar{z} \\ -1 \end{bmatrix} = A_1 \bar{z} - \bar{a} = \bar{a} - \bar{a} = 0$

but $B\bar{x} = [A_1 \,\vdots\, \bar{b}] \begin{bmatrix} \bar{z} \\ -1 \end{bmatrix} = A_1 \bar{z} - \bar{b} = \bar{a} - \bar{b} \neq 0$

(where $\bar{b}$ is the last column of B and since $A \neq B$, $\bar{a} \neq \bar{b}$)

Another solution:

(+2) • We will prove the given Statement.

• Let the nonzero entries in the last column of matrix A be

$$a_{j_1,n},\ a_{j_2,n},\ \ldots,\ a_{j_k,n}$$

(+2) where $1 \le j_1 < j_2 < \cdots < j_k \le m$

• For $1 \le i \le k$, let l_i denote the column of the matrix A which contains the pivot entry of row j_i

(+2) • Since A is a RREF matrix, the l_i-th column of matrix A is the standard basis vector e_{j_i}

(+2)

• Now let $x_{l_i} = -a_{j_i,n}$ for $i = 1, 2, \ldots, k$

and let $x_n = -1$

(+2) let the remaining entries of the vector $\bar{x}$ to be zero.

Then $A\bar{x} = \bar{0}$

But $B\bar{x} \neq 0$ because if $B\bar{x} = \bar{0}$, then the last column of B will be ~~same~~ the same as last column of A, a contradiction.

(Since $A \neq B$)

Another solution:

(+2)
- We will prove the given statement.
- Let $A = [a_{ij}]$, $B = [b_{ij}]$

and it is given that $a_{ij} = b_{ij}$ for $1 \le j \le n-1$
for $i = 1, 2, \cdots, m$

Since the nth column of A is not a pivot column, the variable x_n is a free variable.
(+2)
So, the system $A\bar{x} = \bar{0}$ has a nontrivial solution.

- When we express the solution of $A\bar{x} = \bar{0}$, the vector corresponding to the free variable x_n has 1 in its nth position corresponding to the dummy equation $x_n = x_n$.

Let this vector be $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$
(+2)

where $u_n = 1$

So, $\bar{u}$ is a nontrivial solution of the equation $A\bar{x} = \bar{0}$

- Since the nth columns of A and B are different, $a_{kn} \neq b_{kn}$ for some $1 \leq k \leq n$

+4

Now the kth entry of $A\bar{u}$

is $a_{k1}u_1 + a_{k2}u_2 + \cdots + a_{kn}u_n = 0$ (Since $A\bar{u} = \bar{0}$)

We claim that $B\bar{u} \neq \bar{0}$.

If $B\bar{u} = \bar{0}$, then

$$b_{k1}u_1 + b_{k2}u_2 + \cdots + b_{kn}u_n = 0$$

$$\Rightarrow a_{k1}u_1 + a_{k2}u_2 + \cdots + a_{kn}u_n = b_{k1}u_1 + b_{k2}u_2 + \cdots + b_{kn}u_n \Rightarrow a_{kn}u_n = b_{kn}u_n$$

(Since $a_{kj} = b_{kj}$ for $1 \leq j \leq n-1$)

$$\Rightarrow a_{kn} = b_{kn}$$

(Since $u_n = 1$)

This contradicts the fact that $a_{kn} \neq b_{kn}$

Therefore $B\bar{u} \neq 0$.