[Solution of Worksheet 8]

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1) Given $\dim V = n$,

$\dim W_1 = n-1$, $\dim W_2 = n-1$.

$\dim(W_1 \cap W_2) = 0$ (Since $W_1 \cap W_2 = \{0\}$)

Now $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$\Rightarrow \dim(W_1+W_2) = n-1+n-1-0$

$\Rightarrow \dim(W_1+W_2) = 2n-2$

Now $W_1 \subseteq W_1+W_2 \subseteq V$. So, either $\dim(W_1+W_2)=n-1$ or $\dim(W_1+W_2)=n$

If $\dim(W_1+W_2)=n-1$

then $n-1 = 2n-2 \Rightarrow 2-1 = 2n-n$

$\Rightarrow \boxed{n=1}$

If $\dim(W_1+W_2) = n$

then $n = 2n-2 \Rightarrow \boxed{n=2}$

So, the value of n is either 1 or 2.

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2. Let $W_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$

$W_2 = \{(0, a, b) : a, b \in \mathbb{R}\}$

(a). • $0 = (0,0,0) \in W_1$

• If $u = (x_1, y_1, 0) \in W_1$ and $v = (x_2, y_2, 0) \in W_1$

then $u + v = (x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0) \in W_1$

• If $c \in \mathbb{R}$ and $v = (x, y, 0) \in W_1$

then $cv = c(x, y, 0) = (cx, cy, 0) \in W_1$

Hence W_1 is a subspace of $\mathbb{R}^3$

• Similarly $0 = (0,0,0) \in W_2$

• If $u = (0, a_1, b_1) \in W_2$ and $v = (0, a_2, b_2) \in W_2$

then $u + v = (0, a_1, b_1) + (0, a_2, b_2) = (0, a_1 + a_2, b_1 + b_2) \in W_2$

• If $c \in \mathbb{R}$ and $v = (0, a, b) \in W_2$

then $cv = c(0, a, b) = (0, ca, cb) \in W_2$

Hence W_2 is a subspace of $\mathbb{R}^3$

(b) First note that $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ are both in W_1 and are linearly independent.

So, $\boxed{\dim W_1 \geq 2 \ldots ①}$. But $e_3 = (0,0,1) \notin W_1$

So, $W_1 \subsetneq \mathbb{R}^3$. So, $\boxed{\dim W_1 < \dim(\mathbb{R}^3) = 3 \ldots ②}$

From 1 and 2, $\boxed{\dim W_1 = 2}$

Similary $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ are both in W_2 and are linearly independent.

So, $\boxed{\dim W_2 \geq 2}$ But $e_1 = (1,0,0) \notin W_2$

So, $W_2 \underset{\neq}{\subseteq} \mathbb{R}^3$ & so $\boxed{\dim W_2 < \dim(\mathbb{R}^3) = 3}$

Combining $\boxed{\dim W_2 = 2}$

Now $e_1, e_2 \in W_1$, $e_3 \in W_2$ and so $e_1, e_2, e_3 \in W_1 + W_2$ and they are linearly independent. $W_1 + W_2 \subseteq \mathbb{R}^3$

So, $\boxed{\dim(W_1 + W_2) = 3}$

Now applying the equality

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad 3 = 2 + 2 - \dim(W_1 \cap W_2)$$

$$\Rightarrow \quad \boxed{\dim(W_1 \cap W_2) = 1}$$

(c) Let $U_1 = \text{Span}\{e_3\}$

Then clearly $W_1 + U_1 = \mathbb{R}^3$ (any vector of $\mathbb{R}^3$ can be written as a sum of a vector from W_1 and a vector for U_1)

Now $\dim \mathbb{R}^3 = \dim(W_1 + U_1) = \dim(W_1) + \dim U_1 - \dim(U_1 \cap W_1)$

$\Rightarrow \quad 3 = 2 + 1 - \dim(U_1 \cap W_1)$

$\Rightarrow \quad \dim(U_1 \cap W_1) = 0 \Rightarrow U_1 \cap W_1 = \{0\}$

So, $\boxed{\mathbb{R}^3 = W_1 \oplus U_1.}$

Now let $v = (0,1,1)$

Then $v \notin \text{Span}\{e_1, e_2\} = W_1$

e_1 & e_2 are linearly independent.

Hence $\{e_1, e_2, v\}$ is linearly independent in $\mathbb{R}^3$

and hence a basis for $\mathbb{R}^3$.

Then $\mathbb{R}^3 = W_1 + U_2$ where $U_2 = \text{Span}\{v\}$

Again $\dim(\mathbb{R}^3) = \dim(W_1 + U_2) = \dim W_1 + \dim U_2 - \dim(W_1 \cap U_2)$

$\Rightarrow 3 = 2 + 1 - \dim(W_1 \cap U_2)$

$\Rightarrow \dim(W_1 \cap U_2) = 0$

$\Rightarrow W_1 \cap U_2 = \{0\}$

So, $\mathbb{R}^3 = W_1 \oplus U_2$

Therefore both U_1 and U_2 are complements of W_1

and $U_1 \neq U_2$ since $\underset{(0,1,1)}{v} \in U_2$ but $v \notin U_1$.

∴ This example illustrates that a subspace can have more than one complement (in a vector space). (5)

(3) Let $A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$

We find the RREF matrix of A

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 6 & 3 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & -1 \\ 2 & 6 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to (-1)R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R_A \text{ matrix.} \rightarrow \text{RREF}$$

(a) Now to find a basis of Nul A we solve the homogeneous system $A\mathbf{x} = 0$

$$\Rightarrow \left.\begin{aligned} x_1 + 3x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= -3x_2 \\ x_2 &= x_2 \\ x_3 &= 0.x_2 \end{aligned} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} x_2$$

So, $\boxed{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for Nul } A}$

Now taking columns of A corresponding to pivot columns of R_A we conclude that

$\boxed{\left\{ \begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 17 \end{bmatrix} \right\} \text{ is a Basis for Col } A.}$

For a basis of Row A, we take the nonzero rows of R_A to conclude that

$\boxed{\left\{ [1 \ 3 \ 0], [0 \ 0 \ 1] \right\} \text{ is a basis for Row } A}$.

(b) To find a Basis of Row A consisting of rows of A, we row-reduce A^T

$$A^T = \begin{bmatrix} 2 & 4 & 13 \\ 6 & 12 & 39 \\ 3 & 5 & 17 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & 4 & 13 \\ 0 & 0 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 2 & 4 & 13 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 4 \\ 2 & 4 & 13 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$= R(A^T)$ (RREF matrix)

So, $\left\{ \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \\ 5 \end{bmatrix} \right\}$ forms a Basis for $Col(A^T)$

Hence $\{[2\ 6\ 3], [4\ 12\ 5]\}$ is a Basis of Row A consisting of rows of A.

Note that $2[1\ 3\ 0] + 3[0\ 0\ 1] = [2\ 6\ 3]$

and $4[1\ 3\ 0] + 5[0\ 0\ 1] = [4\ 12\ 5]$

(The reverse can also be written)

Thus $\text{Span}\{[1\ 3\ 0], [0\ 0\ 1]\}$

$= \text{Span}\{[2\ 6\ 3], [4\ 12\ 5]\}$

(c) A is not invertible

because in (a) we have seen that the homogeneous system $Ax = 0$ has a nontrivial solution.

(Note that By our first theorem (of the course), A is invertible if and only if the homogeneous system $Ax = 0$ has only the trivial solution)

(4) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{3}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_A$ (RREF matrix)

$\boxed{\text{Hence } \left\{ \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix} \right\} \text{ is a basis of Row } A.}$

Now $B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix}$

$\xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix}$

$= R_B$ (RREF matrix)

$$\boxed{\text{Thus } \left\{ \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -\frac{8}{3} \end{bmatrix} \right\} \text{ is a basis of } \operatorname{Row} B}$$

Therefore $\operatorname{Row} A = \operatorname{Row} B$. (The two matrices A & B have the same row space)

(b) Since $W = \operatorname{Span}\{(1, 2, -4, 11), (2, 4, -5, 14)\}$
$= \operatorname{Row} B$

and $\operatorname{Row} A = \operatorname{Row} B$,
we can ~~since~~ see that the two vectors

$(1, 2, -1, 3)$ and $(3, 6, 3, -7) \in W$.

We will check whether $(2, 4, -1, 2)$ belongs to W or not.

Let us consider the matrix

$$C = \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 2 & 4 & -1 & 2 \end{bmatrix}$$

(The matrix containing R_B and $(2, 4, -1, 2)$.)

and row reduce it.

$$C = \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 2 & 4 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & -1 & 4/3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & -4/3 \end{bmatrix} \xrightarrow{R_3 \to \left(-\frac{3}{4} R_3\right)} \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + \frac{8}{3} R_3]{R_1 \to R_1 - \frac{1}{3} R_3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

← (RREF matrix)

Thus, the three rows of the matrix C are linearly independent and hence

$[2\ 4\ -1\ 2] \notin W$

Therefore $U \neq W$ because

Since $U = \text{span}\{[1, 2, -1\ 3]\ [2\ 4\ -1\ 2]\ [3\ 6\ 3\ -7]\}$

and so $[2\ 4\ -1\ 2] \in U$

but $[2\ 4\ -1\ 2] \notin W$.

- Let A be any $m \times n$ matrix.

Let $A = [v_1, \ldots, v_n] = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$ in column and row from respectively.

Now $\text{rank}(A) = \dim(\text{Col } A)$

$= \dim(\text{Span}\{v_1, \ldots, v_n\}) \leq n \quad \ldots (1)$

Similarly $\text{rank}(A) = \dim(\text{Row } A)$

$= \dim(\text{Span}\{r_1, \ldots, r_m\}) \leq m \quad \ldots (2)$

From (1) and (2),

$$\boxed{\text{rank}(A) \leq \min\{m, n\}}$$

- We know that an invertible $(m \times m)$ matrix A satisfies $\text{rank}(A) = m$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $\text{rank } A = 3$

$= \min\{3, 3\}$.

So, equality is achieved ~~is~~ here.

For strict inequality we take $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\left.\begin{array}{l} \text{rank}(A) = 2 \\ \min\{m, n\} = 3 \end{array}\right\}$ So, $\boxed{\text{rank}(A) < \min\{3, 3\} = \min\{m, n\}}$

Note: There are infinitely many possible examples.