MTH 100 : Worksheet 11

1. (a) Let $T : V \to W$ and $U : W \to Z$ be linear transformations, where V, W, Z are finite dimensional vector spaces. Show that $\text{rank}(UT) \leq \min\{\text{ rank}(T), \text{rank}(U)\}$.

   (b) Using 1) or otherwise, show that given any $m \times n$ matrix A and any $n \times k$ matrix B, $\text{rank}(AB) \leq \min\{\text{ rank}(A), \text{rank}(B)\}$.

   Give a non-trivial example (i.e. the matrices A, B should be of non-zero, non-identity and should be of minimum size 2×2) in which equality is achieved, and an example in which strict inequality holds.

2. Let $T : V \to W$ be a bijective linear transformation. Since T is bijective, T is an invertible function i.e. the inverse function of T, $T^{-1} : W \to V$ is well-defined. Prove that $T^{-1} : W \to V$ is also a linear transformation.

   Remark: This holds for finite dimensional as well as infinite dimensional spaces. Thus the proof cannot make use of bases or results for dimension.

3. A linear transformation T from V into W is said to be non-singular if Ker $T = \{0\}$ Prove

   (a) T is non-singular iff T is injective.

   (b) T is non-singular iff T carries every linear independent subset of V into a linear independent subet of W.

   (c) If V and W are finite dimensional with $\dim V = \dim W$, then T is non-singular iff T is invertible.

4. Show that a linear transformation $T : V \to W$ where V and W are finite dimensional with $\dim V = \dim W$, is injective iff it is surjective.

5. Give an example of a vector space V and two operators $T, U : V \to V$ such that T is surjective but not injective and U is injective but not surjective.

6. Show that if A^2 is the zero matrix, then 0 is the only eigen value of A.

7. Show that λ is an eigen value of A iff λ is an eigen value of A^T.

8. Suppose A is an $n \times n$ square matrix such that all the row sums equal the some scalar s. Show that s is an eigen value of A.

9. Suppose that A is an $n \times n$ square matrix and $\text{Rank}(A) = k$.

   Show that A can have atmost $(k+1)$ distinct eigen values.

10. (a) Find the characteristic polynomial $q(\lambda)$ of the matrix A given below, and verify that A satisfies its characteristic polynomial.

    (b) Show that both the polynomials $p(\lambda) = \lambda^2 - 3\lambda + 2$ and $r(\lambda) = \lambda^2 - 4\lambda + 4$ are divisors of $q(\lambda)$.

    Does A satisfy either $p(\lambda)$ or $r(\lambda)$?

    (c) What conclusion can you derive from (2)? Explain briefly.

    (d) Verify that $\lambda = 1$ and $\lambda = 2$ are both eigenvalues of A, and determine atleast three linearly independent eigen vectors of A.

    $$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$