MTH 100 : Worksheet 6

1. Let $F = \mathbf{Z_2}$ and consider the vector space $V = F^4$, the space of all ordered 4-tuples with entries from F.

   (a) Suppose $v \in V$, $v \neq 0$. What can you say about the additive inverse of v?

   (b) Consider the vectors $v_1 = (1, 0, 1, 0), V_2 = (1, 1, 0, 0)$ and $v_3 = (0, 0, 1, 1)$. Determine Span $\{v_1, v_2\}$ and Span$\{v_1, v_2, v_3\}$.

   (c) Construct subspaces U and W of V which have 3 and 5 vectors, respectively.

   (d) Apply what you have learned from (a), (b) and (c) to state and prove a result about the possible orders of subspaces of V. **Note:** For any finite set X, the **order** of X is the number of elements in X, notation $|X|$.

   (e) Generalize your result in (d) to subspaces of F^n for any arbitrary positive integer n.

2. Prove the following : If $S = \{v_1, v_2, ...v_p\}$ is the set of vectors in a vector space V, then Span S=Span$\{v_1, v_2, ...v_p\}$ is the smallest subspace which contains S, i.e.if W is a subspace such that $S \subseteq W$, then $S \subseteq$ Span S$\subseteq W$.

3. Let U and W be the subspaces of the vector space V. Then the sum of U and W is defined as $U + W = \{u + w : u \in U, w \in W\}$. Show that $U + W$ is a subspace of V. Show further that $U + W$ is the smallest subspace of V containing both U and W.

4. Let U and W be subspaces of the vector space V. Show by means of a suitable counterexample that $U \cup W$(set theoretic union) need not be a subspace of V. Then prove that $U \cup W$ is a subspace if and only if either $U \subset W$ or $W \subset U$.**(NB: This results holds whether V is finite dimensional or infinite dimensional. Hence, you can't use any propositions related to basis or dimension in the proof.)**

5. Let W be a real vector space. Let V be a non-empty set and let $f : V \to W$ be a bijection (i.e. an injective and surjective function). For $u, v \in V$, and $c \in \mathbf{R}$, define $u \oplus v = f^{-1}(f(u) + f(v))$ and $c * v = f^{-1}(cf(v)$. Show that V is a real vector space under the operations $\oplus$ and $*$. Why is it necessary for f to be a bijection?

6. Given the following vectors in $\mathbf{R^3}$: $\mathbf{u} = (1,3,5)$, $\mathbf{v} = (1,4,6)$, $\mathbf{w} = (2,-1,3)$ and $\mathbf{b} = (6,5,17)$

   (a) Does $b \in W = \text{Span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$?

   (b) If the answer to first part is yes, express $\mathbf{b}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$

7. Prove Remark related to linear dependence/independence): Any list which contains a linearly dependent list is linearly dependent.

8. Prove Remark related to linear dependence/independence: Any subset of linearly independent set is linearly independent.

9. Determine whether the given matrices in the vector space $\mathbf{R^{2\times 2}}$ are linearly dependent or linearly independent. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

10. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e.functions) are linearly dependent or linearly independent:

    $f_1(x) = 1,\ f_2(x) = \sin(x),\ f_3(x) = \sin(2x)$

    (You must justify your answer)