# MTH 100 : Lecture 16

## Span of a set of Vectors:

Let $V$ be a vector space over a field $F$.

- Then a linear combination of finitely many given vectors is any sum of scalar multiples of the vectors.
- Thus if $\{v_1, v_2, \ldots, v_p\}$ is a finite set of vectors in $V$, then
  $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$ where $c_1, c_2, \ldots, c_p$ are any set of scalars
  is a linear combination of $v_1, v_2, \ldots, v_p$.

## Definition:

Let $S = \{v_1, v_2, \ldots, v_p\}$ be a finite set of vectors in a vector space $V$ over a field $F$.

The span of $S$ is defined as :

$$\text{Span } S = \{ c_1 v_1 + c_2 v_2 + \cdots + c_p v_p : c_1, c_2, \ldots, c_p \in F \}$$

- Clearly $v_i = 1.v_i = 0.v_1 + 0.v_2 + \cdots + 0.v_{i-1} + 1.v_i + 0.v_{i+1} + \cdots + 0.v_p$
  so, $v_i \in \text{Span } S$ for $i = 1, 2, \ldots, n$

Thus Span $S$ is a subset of $V$

- Span $S$ is a subspace of $V$.

Proof: (1) $\bar{0} = 0v_1 + 0.v_2 + \cdots + 0.v_p \in \text{Span } S$

(2) Let $w_1, w_2 \in \text{Span } S$

So, there exist scalars $c_1, c_2, \ldots, c_p \in F$ such that $w_1 = c_1v_1 + c_2v_2 + \cdots + c_pv_p$

and there exist scalars $d_1, d_2, \ldots, d_p \in F$ such that $w_2 = d_1v_1 + d_2v_2 + \cdots + d_pv_p$

Now $w_1 + w_2 = (c_1v_1 + c_2v_2 + \cdots + c_pv_p) + (d_1v_1 + d_2v_2 + \cdots + d_pv_p)$

$$= c_1v_1 + d_1v_1 + c_2v_2 + d_2v_2 + \cdots + c_pv_p + d_pv_p$$

$$= (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_p + d_p)v_p$$

Since $c_1 + d_1, c_2 + d_2, \ldots, c_p + d_p \in F$

So, $w_1 + w_2 \in \text{Span } S$

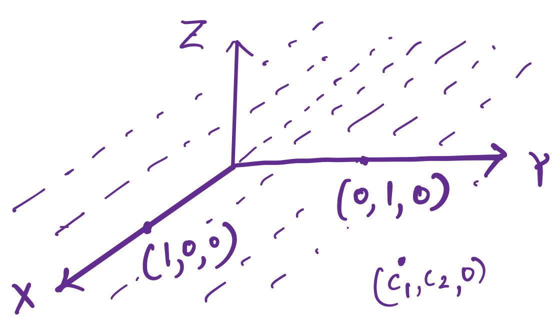
Let $u \in \text{span } S$ and $c \in F$, There exists scalars $c_1, c_2, \ldots, c_p \in F$ such that $u = c_1v_1 + c_2v_2 + \cdots + c_pv_p \Rightarrow cu = c(c_1v_1 + c_2v_2 + \cdots + c_pv_p)$

$$= (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_p)v_p \in \text{span } S$$

Hence span $S$ is a subspace of $V$.

Ex: Let $V = \mathbb{R}^3$

Let $S = \{(1,0,0), (0,1,0)\}$

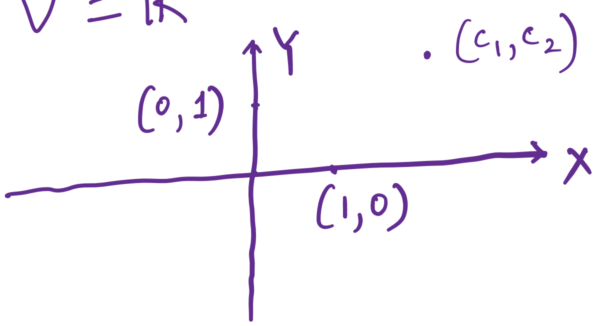


Since for any $c_1, c_2 \in \mathbb{R}$,

$$c_1(1,0,0) + c_2(0,1,0) = (c_1, c_2, 0)$$

$\text{Span } S = \{(x, y, 0) : x, y \in \mathbb{R}\}$ is the XY plane.

Ex: Let $V = \mathbb{R}^2$




Let $S = \{(1,0), (0,1)\}$

Now for $c_1, c_2 \in \mathbb{R}$

$$c_1(1,0) + c_2(0,1) = (c_1, c_2)$$

Hence $\text{Span } S = \mathbb{R}^2$

Ex: Suppose $W_1$ and $W_2$ are two subspaces of a vector space $V$ over a field $F$.

Prove that $W_1 \cap W_2$ is a subspace of $V$

Proof: (1) $\bar{0} \in W_1$, $\bar{0} \in W_2$ (Since $W_1$ & $W_2$ are subspaces of $V$)

$\Rightarrow \bar{0} \in W_1 \cap W_2$

(2) $u, v \in W_1 \cap W_2 \Rightarrow u, v \in W_1$ and $u, v \in W_2$

$\Rightarrow u + v \in W_1$ ($W_1$ is a subspace of $V$)

$\Rightarrow u + v \in W_2$ ($W_2$ is a subspace of $V$)

$\Rightarrow u + v \in W_1 \cap W_2$

(3) Let $c \in F$, $u \in W_1 \cap W_2$

Then $u \in W_1$ and $u \in W_2 \Rightarrow cu \in W_1$ and $cu \in W_2 \Rightarrow cu \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of $V$.

Note: In the same way we can show that intersection of any family of subspaces is a subspace of $V$.

Note: $W_1 \cup W_2$ may not be a subspace of $V$.

Example: Let $V = \mathbb{R}^2$

Let $W_1 = \{(x, 0) : x \in \mathbb{R}\}$

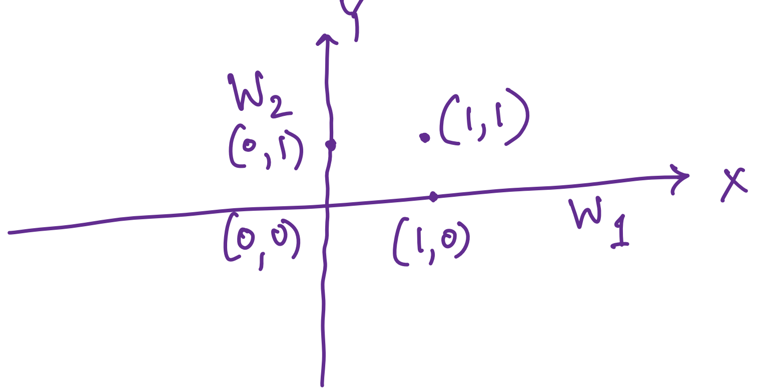
$W_2 = \{(0, y) : y \in \mathbb{R}\}$

Now, $(1, 0) \in W_1$, $(0, 1) \in W_2$

So, $(1, 0) \in W_1 \cup W_2$, $(0, 1) \in W_1 \cup W_2$

but $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$

So, $W_1 \cup W_2$ is not a subspace of $V = \mathbb{R}^2$



## Remarks:

(1) Span $S$ is the smallest subspace of $V$ containing $S$.

$$S \subset \underset{\text{(subspace)}}{\text{Span}(S)} \subset \cdots \subset V$$

$S$ (subset not necessarily a subspace)

Clearly $S \subset \text{Span}(S)$

If $S = \{v_1, v_2, \ldots, v_p\}$ then $v_i = 1.v_i$ for $i = 1, 2, \ldots, p$ and so $v_i \in \text{Span}(S)$

Also if $W$ is a subspace of $V$ such that $S \subset W$ then $\text{span}\, S \subset W$

Proof: Let $u \in \text{Span}\, S$.

Since $S = \{v_1, v_2, \ldots, v_p\}$, there exist scalars $c_1, c_2, \ldots, c_p \in F$ such that $u = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

Now $v_1, v_2, \ldots, v_p \in S$ and $S \subset W \Rightarrow v_1, v_2, \ldots, v_p \in W$

$\Rightarrow c_1 v_1, c_2 v_2, \ldots, c_p v_p \in W$ (Since $W$ is a subspace of $V$)

$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \in W$ (Since $W$ is a subspace of $V$)

$\Rightarrow u \in W$

Therefore $\text{span}\, S \subset W$

(2) Span $S$ is the intersection of all subspaces of $V$ containing $S$. $(S \subset \text{Span}\, S \subset \cdots \subset V)$

Proof: Let $B$ be the intersection of all Subspaces of $V$ containing $S$

Since we have shown that Span $S$ is a subspace of $V$ containing $S$, $\boxed{B \subset \text{Span} S}$ ...... $(\alpha)$

On the other hand, $B$ is also a subspace of $V$ and $S \subset B$,

by Remark(1), $\boxed{\text{Span } S \subset B}$ ...... $(\beta)$

Combining $(\alpha)$ and $(\beta)$, we obtain

Span $S$ = $B$ = intersection of all subspaces of $V$ containing $S$.

Ex: Let $S = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$

where $v_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 3 \\ 25 \\ 9 \end{bmatrix}$

Let $d = \begin{bmatrix} 4 \\ 46 \\ 17 \end{bmatrix}$

Question: Is $d$ in the span $\{v_1, v_2, v_3\}$ ?

Let us solve: $c_1 v_1 + c_2 v_2 + c_3 v_3 = d$ ($c_1, c_2, c_2$ are the unknown scalars)

$$\Rightarrow c_1 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 25 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 46 \\ 17 \end{bmatrix}$$

The Augmented matrix:

$$[A : d] = \left[\begin{array}{ccc|c} 2 & 6 & 3 & 4 \\ 4 & 12 & 25 & 46 \\ 1 & 3 & 9 & 17 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 17 \\ 4 & 12 & 25 & 46 \\ 2 & 6 & 3 & 4 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1}]{} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 17 \\ 0 & 0 & -11 & -22 \\ 0 & 0 & -15 & -30 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to (-\frac{1}{11} R_2) \\ R_3 \to (-\frac{1}{15} R_3)}]{} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 17 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\downarrow R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 17 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 9R_2]{} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{RREF matrix}$$

Since the last column is not a pivot column, the system of equations is consistent.

<u>Solving the system:</u>

$$\left.\begin{array}{r} c_1 + 3c_2 = -1 \\ c_3 = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} c_1 = -1 - 3c_2 \\ c_3 = 2 \end{array}\right\}$$

There are infinitely many solutions.

One solution (By taking $c_2 = 0$) is $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

So, $(-1)\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 12 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 25 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 46 \\ 17 \end{bmatrix} = d$

Hence $d \in \text{span}\{v_1, v_2, v_3\}$

<u>Ex:</u> Let $d_1 = \begin{bmatrix} 4 \\ 46 \\ 18 \end{bmatrix} \in \mathbb{R}^3$

<u>Question:</u> Is $d_1 \in \text{span}\{v_1, v_2, v_3\}$ where $v_1, v_2, v_3$ are given in the previous example?

We perform the same sequence of row operations on $d_1$.

$$d_1 = \begin{bmatrix} 4 \\ 46 \\ 18 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 18 \\ 46 \\ 4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 18 \\ -26 \\ -32 \end{bmatrix} \xrightarrow[\substack{R_2 \to (-\frac{1}{11}R_2) \\ R_3 \to (-\frac{1}{15}R_3)}]{} \begin{bmatrix} 18 \\ \frac{26}{11} \\ \frac{32}{15} \end{bmatrix}$$

$$\downarrow$$

$$\Big\downarrow\, R_3 \to R_3 - R_2$$

$$\begin{bmatrix} -\frac{36}{11} \\ \frac{26}{11} \\ -\frac{38}{165} \end{bmatrix} \xleftarrow{R_1 - 9R_2} \begin{bmatrix} 18 \\ \frac{26}{11} \\ -\frac{38}{165} \end{bmatrix}$$

So, the RREF matrix corresponding to the Augmented matrix $[A : d_1]$ is

$$\left[\begin{array}{ccc:c} 1 & 3 & 0 & -\frac{36}{11} \\ 0 & 0 & 1 & \frac{26}{11} \\ 0 & 0 & 0 & -\frac{38}{165} \end{array}\right]$$

Since the last column is a pivot column (there is a row $\left[0, 0, 0, -\frac{38}{165}\right]$), the system of equations is inconsistent.

Hence $d_1 \notin \operatorname{Span}\{v_1, v_2, v_3\}$