MTH 100 : Worksheet 7

1. Show that $B = \{v_1, v_2, ...v_p\}$ is a basis of the vector space V if and only if every vector $v \in V$ is uniquely expressible as a linear combination of the elements of B.

2. Find a basis for the vector space V of all 2×2 matrices over $\mathbf{R}$. Generalize the idea to find a basis for the vector space of all $m \times n$ matrices over $\mathbf{R}$.

3. Given the standard basis $\mathbf{B} = \{e_1, e_2, e_3\}$ of $\mathbf{R^3}$ and the linearly independent vectors $v_1 = (0, 1, 1)$ and $v_2 = (1, 1, 1)$, apply the method of Steinitz Exchange Lemma to exchange two of the vectors in B and obtain a basis C which includes v_1 and v_2. Show your calculations in detail.

4. Expand the linearly independent set $S = \{u, w\}$ to a basis of $\mathbf{R^3}$, using the propositions proved in the class. Here $u = (3, 3, 7)$ and $w = (10, 9, 21)$. Justify your answer.

5. Show that any $m \times m$ matrix A with real entries is invertible iff its columns form a basis for $\mathbf{R^m}$

6. Given any $m \times m$ (square) matrix A and any polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n (a_n \neq 0)$ of degree n, we say that A **satisfies the polynomial p(t)** if $p(A) = a_0 I_m + a_1 A + a_2 A^2 + \cdots + a_n A^n = \mathbf{0}$,i.e.the zero matrix. Show that any non-zero $m \times m$ matrix A must satisfy atleast one (non-zero) polynomial of degree $\leq m^2$.

7. Consider the space $\mathbf{C}$ of complex numbers as a vector space over the field $\mathbf{R}$ of real numbers.

   (a) Is $\mathbf{C}$ finite dimensional (YES/NO)? If YES, determine the dimension of $\mathbf{C}$

   (b) Prove or disprove: There exists a field F lying strictly between $\mathbf{R}$ and $\mathbf{C}$,i.e there is a field F such that $\mathbf{R} \subseteq F$ but $\mathbf{R} \neq F$, and $F \subseteq \mathbf{C}$ but $F \neq \mathbf{C}$.

8. (a) Show that if the vectors v_1, v_2,..., v_n are linearly independent, then so are the vectors $v_1 - v_2$, $v_2 - v_3$,..., $v_{n-1} - v_n$, v_n, which have been obtained by subtracting from each vector the following vector (except the last one).

   (b) Let V be a vector space over a field F. Suppose $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in V, and $w \in V$. Show that if $v_1 + w, v_2 + w, .., v_n + w$ are linearly dependent in V,then $w \in$ Span(S).

9. Consider the vector space $V = C[a, b]$, the space of all continuous real valued functions with domain closed interval $[a, b]$. Is V finite dimensional (YES/NO)? Justify your answer. If YES, construct a basis for V and determine its dimension.

10. Let $V = \mathbf{R}^\infty$, $W = \{< a_n >$: only finitely many of the terms in $< a_n >$ are non-zero$\}$(NB: *in future, we will use notation* p_∞ for the subspace W defined here)

    (a) Show that W is a subspace of V

    (b) Is W finite dimensional? Justify your answer.

    (c) Is V finite dimensional? Justify your answer.

    (d) Consider c, the vector space of all convergent sequences in $\mathbf{R}^\infty$, is c finite dimensional? Justify your answer.

11. Let $F = Z_2$ and consider the vector space $V = F^n$, the space of all ordered n-tuples with entries from F, where n is an arbitrary but fixed positive integer, Recall that for any finite set X, the order of X is the number of elements in X,notation $|X|$. Let W be any non-zero subspace of V. Determine the possible values of $|W|$. Justify your answer.

12. Expand the linearly independent set $S = \{u, w\}$ to a basis of $\mathbf{R^3}$ using the approach of propositions proved in class. Here $u = (1, 2, 3)$ and $w = (2, 4, 5)$.

    Justify your answer.