Quiz 2

Sep 6th, 2024

Time: 15 minutes

Name: ______________ Roll no.: ______________ Group: ______________

Instructions: Notes, books, computers, cell phones and other electronic devices are not allowed. **Max marks** $= 5$.

Problem 1. Express the following invertible matrix as a product of elementary matrices.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Note: You can't guess. You need to show your work and be careful about the <u>*order of matrices*</u> *in the product.*

Rubrics of Quiz 2

Total points = 5 ①

① Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

- First let us row reduce A

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(+2.5)

So, the elementary row operations are:

e_1 : $R_2 \to R_2 - R_1$
e_2 : $R_3 \to R_3 - R_1$
e_3 : $R_2 \to -\frac{1}{2}R_2$
e_4 : $R_3 \to R_3 + R_2$
e_5 : $R_1 \to R_1 - R_3$
e_6 : $R_1 \to R_1 - 2R_2$

The inverse operations are:

f_1 : $R_2 \to R_2 + R_1$
f_2 : $R_3 \to R_3 + R_1$
f_3 : $R_2 \to -2R_2$
f_4 : $R_3 \to R_3 - R_2$
f_5 : $R_1 \to R_1 + R_3$
f_6 : $R_1 \to R_1 + 2R_2$

Note: They can skip this step as long as they can write the next step correctly.

If $e_i(I) = E_i$ (elementary matrices), then

$(E_6 E_5 E_4 E_3 E_2 E_1) A = I \Rightarrow A = (E_6 E_5 E_4 E_3 E_2 E_1)^{-1}$ +.5

$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$ +.5

$\Rightarrow \boxed{A = F_1 F_2 F_3 F_4 F_5 F_6}$ +.5

where $F_i = f_i(I)$ are the elementary matrices given below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 2R_2]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_6$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 + R_3]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_5$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = F_4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to -2R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = F_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_1$$

+1

Thus $\boxed{A = F_1 F_2 F_3 F_4 F_5 F_6}$ where the elementary matrices F_i's are given above.

Note: There are many paths to row reduce A into the identity matrix.

Accordingly order of these operations may change and so please be careful about the order of the matrices F_i's.

e.g. e_1 and e_2 can be interchanged and so F_1 and F_2 can be interchanged

e_5 and e_6 can be interchanged and so F_5 and F_6 can be interchanged

and so on.

In short:

Their answer for order of elementary matrices in the product should be consistent with the order in which they perform elementary row operations.