MTH 100: Lecture 7

Theorem ①: The following are equivalent for a $n \times n$ square matrix A:

(a) A is invertible

(b) A is row equivalent to the identity matrix.

(c) The homogeneous system $AX = \bar{0}$ has only the trivial solution.

(d) The system of equation $AX = \bar{b}$ has atleast one solution for every $\bar{b} \in \mathbb{R}^n$

Proof: First we will prove (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

$$\begin{array}{ccc} & (a) & \\ \nearrow & & \searrow \\ (b) & \Leftarrow & (c) \end{array}$$

((a) $\Leftrightarrow$ (d) : Later)

- **(a) $\Rightarrow$ (c):**

Given: A is invertible

To show: $AX = \bar{0}$ has only the trivial solution.

- Let u be a solution of $AX = \bar{0}$

$$\Rightarrow Au = \bar{0}$$

$$\Rightarrow A^{-1}(Au) = A^{-1}(\bar{0}) \quad \left(\begin{array}{c}\text{Since } A \text{ invertible} \\ A^{-1} \text{ exists}\end{array}\right)$$

$$\Rightarrow A^{-1}(Au) = \bar{0}$$

$$\Rightarrow (A^{-1}A)u = \bar{0}$$

$$\Rightarrow I.u = \bar{0} \Rightarrow \boxed{u = \bar{0}}$$

So, $AX = \bar{0}$ has only the trivial solution.

(c) $\Rightarrow$ (b): <u>Given:</u> $AX = \bar{0}$ has only the trivial solution.

<u>To show:</u> A is row equivalent to the identity matrix.

- Let R be the RREF matrix of A.

  Then $RX = \bar{0}$ also has only the trivial solution.

  $\Rightarrow$ R has no free variable

  $\Rightarrow$ R has only basic variables

  $\Rightarrow$ R has leading entry as 1 in each row (There are n rows)

  $\Rightarrow$ R has exactly one 1 in each column (There are n columns

  $\Rightarrow$ R is I $\Rightarrow$ $\boxed{A \text{ is row equivalent to } I}$

(b) $\Rightarrow$ (a): <u>Given:</u> A is row equivalent to the identity matrix I

<u>To show:</u> A is invertible

- A is row equivalent to I

  $\Rightarrow$ There are elementary row operations $e_1, e_2, \dots, e_{p-1}, e_p$ such that

  $$e_p(e_{p-1} \cdots (e_2(e_1(A))) \cdots) = I$$

Let E_i be the elementary matrix corresponding to e_i for $i = 1, 2, \dots, p$

(ie. $E_i = e_i(I)$)

Then $E_p(E_{p-1} \cdots (E_2(E_1 A)) \cdots) = I$ (By proposition 5)

$$\Rightarrow (E_p E_{p-1} \cdots E_2 E_1) A = I$$

Let $B = E_p E_{p-1} \cdots E_2 E_1$

Then B is invertible (By observation 4 and Proposition 6)

and we have $BA = I$

Multiplying both sides by B^{-1} from the left, we obtain

$$B^{-1}(BA) = B^{-1}.I$$

$$\Rightarrow (B^{-1}B)A = B^{-1}$$

$$\Rightarrow I.A = B^{-1} \Rightarrow \boxed{A = B^{-1}}$$

So, A is the inverse of an invertible matrix

$\Rightarrow$ $\boxed{A \text{ is invertible}}$ (By Observation 2)

Calculation of Inverse matrix:

Corollary (1.1): An invertible matrix A is a product of elementary matrices.

(**Note:** Any sequence of row operations that reduces A to I also transforms I to A^{-1} (We are using Theorem 1 (b) here))

Proof: If A is invertible, then by Theorem 1(b), A is row equivalent to I.

So, there are some elementary row operations $e_1, e_2, \ldots, e_p$ such that

$$e_p(e_{p-1} \cdots (e_2(e_1(A))) \cdots) = I$$

Let $E_1, E_2, \ldots, E_p$ be the corresponding elementary matrices (i.e. $E_i = e_i(I)$),

then $\quad E_p(E_{p-1} \cdots (E_2(E_1 A))\cdots) = I$

$$\Rightarrow (E_p E_{p-1} \cdots E_2 E_1) A = I$$

$$\Rightarrow A = (E_p E_{p-1} E_2 E_1)^{-1} I$$

$$\Rightarrow A = (E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}) I$$

$$\Rightarrow \boxed{A = E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}}$$

So, A is a product of some elementary matrices

<u>Note:</u> We can say $A^{-1} = (E_1^{-1} E_2^{-1} \cdots E_p^{-1})^{-1}$

$$\Rightarrow A^{-1} = (E_p^{-1})^{-1} \cdots (E_2^{-1})^{-1} (E_1^{-1})^{-1}$$

$$\Rightarrow A^{-1} = E_p \cdots E_2 E_1$$

$$\Rightarrow A^{-1} = (E_p \cdots E_2 E_1) I$$

$$\Rightarrow A^{-1} = E_p(\cdots E_2(E_1 I)\cdots)$$

$$\Rightarrow \boxed{A^{-1} = e_p(\cdots e_2(e_1(I))\cdots)}$$

Thus the same sequence of row operations that reduces A to I also reduces I to A^{-1}.

Method of obtaining A^{-1}:

Form the enlarged matrix $[A : I]$ and carry out elementary row operations till 'A' part becomes I. The final result has the form $[I : A^{-1}]$.

- Corollary (1.2): If A has a left inverse or a right inverse, then it has an inverse.

Note: (1) B is a left inverse of A if $BA = I$

(2) D is a right inverse of A if $AD = I$