Worksheet 10

1. (a) $\beta = \{(1,1,1), (1,2,3), (1,3,6)\}$ is an ordered basis of $\mathbb{R}^3$

For any vector $x \in \mathbb{R}^3$,

$[x]_\beta = P[x]_S$ where $S = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$

where $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1}$

Now

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & +5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

So, $P = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

So if $v_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_5$

then $[v_1]_\beta = P \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 6-9+4 = 1 \\ -6+15-8 = 1 \\ 2-6+4 = 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}_E$

then $[v_2]_\beta = P \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 3+3+2 = 8 \\ -3-5-4 = -12 \\ 1+2+2 = 5 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}$$

<u>Check:</u>

$$1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

and $8 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-12) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

1. $[v]_\beta = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_\beta$

Now $[v]_S = P^{-1} [v]_\beta$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+3+2=7 \\ 2+6+6=14 \\ 2+9+12=23 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 23 \end{bmatrix}$$

Check:

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 23 \end{bmatrix}$$ (as the matrix multiplication)

2. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$

$S = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$

$T(e_1) = T(1,0,0) = (1, 1, -1) = 1e_1 + 1.e_2 + (-1)e_3$

$T(e_2) = T(0,1,0) = (0, 2, 1) = 0e_1 + 2e_2 + 1e_3$

$T(e_3) = T(0,0,1) = (1, 1, 0) = 1.e_1 + 1.e_2 + 0e_3$

Hence $[T]_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

3 (a) Let $S_3 = \{e_1, e_2, e_3\}$ be the standard basis in $\mathbb{R}^3$
and let $S_2 = \{ e_1' = \begin{bmatrix}1\\0\end{bmatrix}, e_2' = \begin{bmatrix}0\\1\end{bmatrix} \}$ be the standard basis in $\mathbb{R}^2$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 \, , \, 2x_3 - x_1)$

Now $Te_1 = T(1,0,0) = (1,-1) = 1 e_1' + (-1) e_2'$
$Te_2 = T(0,1,0) = (1,0) = 1. e_1' + 0. e_2'$
$Te_3 = T(0,0,1) = (0,2) = 0. e_1' + 2 e_2'$

So, $[T]_{S_3 \to S_2} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

(b) To show that $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$
forms a basis of $\mathbb{R}^3$,
it is sufficient to show they are linearly independent (as $\dim \mathbb{R}^3 = 3$)
or equivalently it is sufficient
to show that the homogeneous system $AX = 0$ has only the trivial solution

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

RREF matrix. $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, $Ax = 0$ has only the trivial solution.

$\boxed{\text{So, } \mathcal{B} \text{ is a basis for } \mathbb{R}^3}$

Note: From here we can also conclude that the non homogeneous system $Ax = b$ has a ~~unique~~ solution for every $b \in \mathbb{R}^3$ and hence span$\{(1,0,-1), (1,1,1)$ and $(1,0,0)\}$ $= \mathbb{R}^3$

(c) Let $\mathcal{B} = \{u_1 = (1,0,-1), u_2 = (1,1,1), u_3 = (1,0,0)\}$ and $\mathcal{B}' = \{v_1 = (0,1), v_2 = (1,0)\}$ be ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

Then
$$Tu_1 = T(1,0,-1) = (1,-3) = -3(0,1) + 1(1,0) = -3v_1 + 1.v_2$$
$$Tu_2 = T(1,1,1) = (2,1) = 1(0,1) + 2(1,0) = 1.v_1 + 2.v_2$$
$$Tu_3 = T(1,0,0) = (1,-1) = -1(0,1) + 1(1,0) = -1.v_1 + 1.v_2$$

Hence $[T]_{\mathcal{B} \to \mathcal{B}'} = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

4. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is given by

$$T(x_1, x_2, x_3) = (x_1 + x_3,\ x_1 + 2x_2 + x_3,\ -x_1 + x_2)$$

Let $S = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$.

Then $[T]_S$ has been calculated in problem 2

(a)

$$[T]_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) Now $\beta = \{(1,1,1), (1,2,3), (1,3,6)\}$ is another ordered basis for $\mathbb{R}^3$

In problem 1, the change of basis matrix $P_{S \to \beta}$ has been calculated.

$$P = P_{S \to \beta} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

and

$$P^{-1} = Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Then

$$[T]_\beta = P [T]_S P^{-1}$$

Hence $[T]_\beta = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -5 & 0 \\ 4 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -11 & -16 \\ 14 & 26 & 40 \\ -6 & -11 & -17 \end{bmatrix}$$

(Please check the Calculation !)

<u>Check</u>: $T(1,1,1) = (2,4,0)$

$= (-6)(1,1,1) + 14(1,2,3) + (-6)(1,3,6)$

$T(1,2,3) = (4,8,1) = (-11)(1,1,1) + 26(1,2,3) + (-11)(1,3,6)$

$T(1,3,6) = (7,13,2) = (-16)(1,1,1) + 40(1,2,3) + (-17)(1,3,6)$

To show that:

5 (a) Similarity is an equivalence relation on $\mathbb{R}^{n\times n}$ $(n \geq 2)$

Recall that B is similar to A if there exists an invertible matrix P such that
$B = PAP^{-1}$

Reflexive Property: For any $A \in \mathbb{R}^{n\times n}$,

- $A = I_n \cdot A\, I_n^{-1}$ where I_n is the $n\times n$ identity matrix

$\Rightarrow$ $\boxed{\text{A is similar to A}}$

Symmetric Property:

- Suppose B is similar to A $\Rightarrow$ There exists an invertible matrix P such that $B = PAP^{-1}$

$\Rightarrow P^{-1}BP = A \quad \Rightarrow QBQ^{-1} = A$ where $Q = P^{-1}$ is an invertible matrix

$\Rightarrow$ $\boxed{\text{A is similar to B}}$

Transitive Property

Suppose B is similar to A and C is similar to B

Then there exist invertible matrices P and Q such that $B = PAP^{-1}$ and $C = QBQ^{-1}$

Now, $C = QBQ^{-1} = QPAP^{-1}Q^{-1} = (QP)A(QP)^{-1}$
$= RAP^{-1}$ where $R = QP$ is an invertible matrix

Hence $\boxed{\text{C is similar to A}}$

Prove

Note that

(b) The only matrix similar to I_n is I_n itself

Therefore

If $A = I_n$ and B is any invertible matrix other than I_n, then B is row equivalent to $I_n = A$ but B is not similar to I_n because for any invertible $(n\times n)$ matrix P,

$$P I_n P^{-1} = I_n \neq B$$

An example for $n=2$: $B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

B is row equivalent to $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

but B is not similar to I_2.

(c) Prove

If A and B are the matrices of a linear operator T with respect to the bases α and β respectively, then B is similar to A.

In fact $B = PAP^{-1}$ where $P = P_{\alpha\to\beta}$

Now let $T = P_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection on the first coordinate

ie $P_1(x,y) = (x,0)$

Let $\alpha = \{e_1, e_2\}$ = standard basis. Then $A = [T]_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

If we take the ordered basis $\beta = \{e_2, e_1\}$

then $B = [T]_\beta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Now clearly A and B are not row equivalent to each other

but A and B are similar (because $B = PAP^{-1}$)

(a) $U : V = \mathbb{R}^{2\times 2} \longrightarrow V = \mathbb{R}^{2\times 2}$

$U(A) = A + A^T \ \forall A \in V$

For $A, B \in V$, $U(A+B) = (A+B) + (A+B)^T$

$= (A+B) + (A^T + B^T) = (A + A^T) + (B + B^T)$

$= U(A) + U(B)$

For $c \in \mathbb{R}$ and $A \in V$,

$U(cA) = (cA) + (cA)^T = cA + cA^T$

$\Rightarrow \quad U(cA) = c(A + A^T) = c\,U(A)$

So, U is a linear operator

(b) $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is an ordered basis of $\mathbb{R}^{2\times 2}$ (standard basis)

$U(E_{11}) = E_{11} + (E_{11})^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 2E_{11}$

$U(E_{12}) = E_{12} + (E_{12})^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_{12} + E_{21}$

$U(E_{21}) = E_{21} + (E_{21})^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{12} + E_{21}$

$U(E_{22}) = E_{22} + (E_{22})^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 2E_{22}$

Thus $[U]_\beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Since $\dim V = 4$, $[U]_\beta$ will be a 4×4 matrix

(c) To find a basis for $\ker U$,

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker U$

Then $U(A) = A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$= \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}$

Now $U(A) = 0 \Rightarrow \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $2a = 0$, $2d = 0$ $\Rightarrow a = 0, d = 0$

$b + c = 0 \Rightarrow c = -b$

So, $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

So, $A \in \ker U \Leftrightarrow A$ is a scalar multiple of

$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\ker U = \left\{ cQ = c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : c \in \mathbb{R} \right\}$ = space of all 2×2 skew-symmetric matrices

$\ker U = 1$ and a basis of $\ker U$ is $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

Note that Rank U + Nullity U = $\dim V = 4$

$\Rightarrow$ Rank $U = 4 - 1 = 3 \Rightarrow \dim(\text{Range } U) = 3$

Furthermore if $X \in$ Range U, then there exists some $A \in V$ such that $X = A + A^T$

Now $X^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$
$= X$

$\Rightarrow X$ is a symmetric matrix

$\Rightarrow X \in Sym_2(\mathbb{R})$

Thus $\text{Range } U \subseteq Sym_2(\mathbb{R}) \subseteq V$

$\Rightarrow \quad \text{Rank}(U) \leq \dim\left[Sym_2(\mathbb{R})\right] \leq \dim V$

$\Rightarrow \quad 3 \leq \dim\left[Sym_2(\mathbb{R})\right] \leq 4$

Since not all matrices in $\mathbb{R}^{2\times 2}$ are symmetric,

$$Sym_2(\mathbb{R}) \neq V$$

and so $\dim Sym_2(\mathbb{R}) < 4$

Thus $\dim\left[Sym_2(\mathbb{R})\right] = 3$

Since $\text{Rank}[U] = 3$

it follows that $\boxed{\text{Range } U = Sym_2(\mathbb{R})}$

Thus to find a basis of Range U, we will find a basis of $Sym_2(\mathbb{R})$

Let us consider $\{E_{11}, E_{22}, D\}$

where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

They are all symmetric and they are linearly independent, because

$$c_1 E_{11} + c_2 E_{22} + c_3 E_{33} = 0 \Rightarrow \begin{bmatrix} c_1 & c_3 \\ c_3 & c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow c_1 = 0, \; c_2 = 0, \; c_3 = 0$$

$$\boxed{\text{Therefore } \{E_{11}, E_{22}, D\} \text{ forms a basis of Range } U = Sym_2(\mathbb{R})}$$

Note that $V = \text{Ker } U \oplus \text{Range } U$

Every matrix in $\mathbb{R}^{2\times 2}$ can be uniquely expressed as a sum of symmetric and skew symmetric matrix

(d) To find the $\dim(Sym_n(\mathbb{R}))$, note that if $A = [a_{ij}]$ is symmetric, then $a_{ij} = a_{ji}$

i.e. entries symmetric with respect to the diagonal are equal.

Such entries can be obtained from

$c[E_{ij} + E_{ji}]$ for $i < j$

where c is a constant.

Hence we get matrices of the form $E_{ij} + E_{ji}$ & there are $\binom{n}{2} = \frac{n(n-1)}{2}$ matrices of this type

Furthermore since the diagonal elements can take any value, we get n addition basis matrices say D_i where D_i is a diagonal matrices with 1 in the ith position on the diagonal and 0's elsewhere.

Hence we get $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ basis matrices

So, $\boxed{\dim(Sym_2(\mathbb{R})) = \frac{n(n+1)}{2}}$ ---- (1)

Additional Note

- A square matrix A is called skew symmetric if $A = -A^T$
- A square matrix B is called symmetric if $B = B^T$
- Any matrix A can be written as a sum of symmetric and skew symmetric matrix in the

following way (14)

$$A = \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{Symmetric}} + \underbrace{\left(\frac{A-A^T}{2}\right)}_{\text{Skew-symmetric}}$$

- For a skew symmetric matrix, the diagonal entries are all zero.

So, the space of skew symmetric matrices has basis matrices of the form $E_{ij} - E_{ji}$ for $i<j$.
There are $\binom{n}{2} = \frac{n(n-1)}{2}$ such matrices.

$$\boxed{\dim(\text{Skew Sym}_n(\mathbb{R})) = \frac{n(n-1)}{2}} \quad \text{---} \quad (2)$$

Adding (1) & (2) we get

$$\frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2 = \dim(\mathbb{R}^{n\times n})$$

Thus $\boxed{\mathbb{R}^{n\times n} = \text{Sym}_n(\mathbb{R}) \oplus \text{Skew-Sym}_n(\mathbb{R})}$

(7) Let $T: V \longrightarrow W$ be a linear transformation and $\dim V = \dim W = n < \infty$

$\Rightarrow$: Assume that T is 1-1
Want to show T is onto

T is 1-1 $\Rightarrow$ $\ker T = \{0\}$ $\Rightarrow$ nullity $T = 0$

By Rank Theorem $\quad$ Rank T + nullity $T = \dim V = n$

$\Rightarrow$ Rank $T = n$ $\Rightarrow$ $\dim(\text{Range } T) = n$

$\Rightarrow$ Range $T = W$

$\Rightarrow$ T is onto

$\Leftarrow$: Assume that T is onto

Want to show that T is 1-1

T is onto $\Rightarrow$ Range $T = W \Rightarrow \dim(\text{Range } T) = \dim W = n$

$\Rightarrow$ Rank $T = n$

By Rank theorem,

Rank T + nullity T = $\dim V = n$

$\Rightarrow n +$ nullity $T = n \Rightarrow$ nullity $T = 0$

$\Rightarrow \dim(\text{Ker } T) = 0 \Rightarrow \text{Ker } T = \{0\}$

$\Rightarrow T$ is 1-1

(8) $S_P : V \longrightarrow V$ $\qquad V = F^{n \times n}$

$S_P(A) = PAP^{-1}$ $\qquad$ (P is a fixed invertible matrix)

- For any $A, B \in V$,

$S_P(A+B) = P(A+B)P^{-1} = PAP^{-1} + PBP^{-1} = S_P(A) + S_P(B)$

For any $c \in F$ and any $A \in V$,

$S_P(cA) = P(cA)P^{-1} = c(PAP^{-1}) = cS_P(A)$

So, S_P is a linear transformation.

- $S_P(A) = [0]$ (zero matrix) $\Rightarrow PAP^{-1} = [0]$

$\Rightarrow A = P^{-1}[0]P = [0]$

$\Rightarrow A = [0]$

So, $\ker(S_P) = \{[0]\}$

Therefore S_P is 1-1.

Now for any $B \in F^{n \times n}$,

$$C = P^{-1} B P \in F^{n \times n},$$

and $$S_P(C) = P C P^{-1} = P(P^{-1} B P) P^{-1}$$
$$= (P P^{-1}) B (P P^{-1}) = B$$

$\Rightarrow B \in \text{Range}(S_P)$

Hence S_P is surjective.

- Now for any $A, B \in F^{n \times n}$,

$$S_P(A) S_P(B) = (P A P^{-1})(P B P^{-1}) = (P A)(P^{-1} P) B P^{-1}$$
$$= (PA)(BP^{-1}) = P(AB)P^{-1} = S_P(AB)$$

Thus $S_P(AB) = S_P(A) S_P(B) \;\; \forall A, B \in F^{n \times n}$

Hence S_P is an isomorphism and also a multiplicative transformation.

(9) $V = \mathbb{R}^2$, $\alpha = \{u_1, u_2\} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 11 \\ 4 \end{pmatrix} \right\}$

and $\beta = \{v_1, v_2\} = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix} \right\}$

(a) To find $P_{\alpha \to \beta}$, we note that columns of P are β-coordinate vectors of the basis α (i.e. old basis α in terms of new basis β)

Let us assume $u_1 = c_1 v_1 + c_2 v_2$ and $u_2 = d_1 v_1 + d_2 v_2$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$\Rightarrow$ Augmented matrix $= \left[\begin{array}{cc|c} 3 & 7 & 3 \\ 2 & 5 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3} R_1} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & 1 \\ 2 & 5 & 1 \end{array}\right]$

$\downarrow R_2 \to R_2 - 2R_1$

$\left[\begin{array}{cc|c} 1 & \frac{7}{3} & 8 \\ 0 & 1 & -3 \end{array}\right] \xleftarrow{R_1 \to R_1 - \frac{7}{3} R_2} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & 1 \\ 0 & 1 & -3 \end{array}\right] \xleftarrow{R_2 \to 3R_2} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & 1 \\ 0 & \frac{1}{3} & -1 \end{array}\right]$

RREF matrix

$\Rightarrow \boxed{c_1 = 8 , \ c_2 = -3}$

Similarly $\begin{bmatrix} 11 \\ 4 \end{bmatrix} = d_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + d_2 \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$\Rightarrow$ Augmented matrix $= \left[\begin{array}{cc|c} 3 & 7 & 11 \\ 2 & 5 & 4 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{3} R_1} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & \frac{11}{3} \\ 2 & 5 & 4 \end{array}\right]$

$\downarrow R_2 \to R_2 - 2R_1$

$\left[\begin{array}{cc|c} 1 & 0 & 27 \\ 0 & 1 & -10 \end{array}\right] \xleftarrow{R_1 \to R_1 - \frac{7}{3} R_2} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & \frac{11}{3} \\ 0 & 1 & -10 \end{array}\right] \xleftarrow{R_2 \to 3R_2} \left[\begin{array}{cc|c} 1 & \frac{7}{3} & \frac{11}{3} \\ 0 & \frac{1}{3} & -\frac{10}{3} \end{array}\right]$

$\Rightarrow \boxed{d_1 = 27 , \ d_2 = -10}$

So, $\boxed{P_{\alpha \to \beta} = \begin{bmatrix} 8 & 27 \\ -3 & -10 \end{bmatrix}}$

Another method :

For any $x \in V$, $[x]_\beta = P_{\alpha \to \beta} [x]_\alpha$.

[redacted]

Taking $S = \{e_1, e_2\}$ as the standard basis of $\mathbb{R}^2$,

$$[x]_\beta = P_{\alpha \to \beta} [x]_\alpha = P_{\alpha \to \beta} P_{S \to \alpha} [x]_S \quad \text{---} (1)$$

Also $[x]_\beta = P_{S \to \beta} [x]_S \quad \text{----} (2)$

From (1) and (2) we get $P_{\alpha \to \beta} P_{S \to \alpha} = P_{S \to \beta}$

$$\Rightarrow P_{\alpha \to \beta} = P_{S \to \beta} \left(P_{S \to \alpha}\right)^{-1}$$

$$= P_{S \to \beta} P_{\alpha \to S}$$

$$= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 27 \\ -3 & -10 \end{bmatrix}$$

$$\Rightarrow \boxed{P_{\alpha \to \beta} = \begin{bmatrix} 8 & 27 \\ -3 & -10 \end{bmatrix}}$$

(b) Given $[v]_\alpha = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

$$[v]_\beta = P_{\alpha \to \beta} [v]_\alpha$$

$$= \begin{bmatrix} 8 & 27 \\ -3 & -10 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 620 \\ -230 \end{bmatrix}_\beta$$

To check the answer for (b),

(c) We need to show that $[v]_\alpha$ and $[v]_\beta$ refer to the same vector $v \in V$.

~~To~~ Let us find $[v]_S$ from both the expressions.

$[v]_\alpha = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ So, $[v]_S = 10u_1 + 20u_2$

$$= 10\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 20\begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

It is same as $P_{\alpha \to S}[v]_\alpha$
$= \begin{bmatrix} 3 & 11 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix}$

$$= \begin{bmatrix} 250 \\ 90 \end{bmatrix}_S$$

$[v]_\beta = \begin{bmatrix} 620 \\ -230 \end{bmatrix}$ So, $[v]_S = 620\begin{bmatrix} 3 \\ 2 \end{bmatrix} + (-230)\begin{bmatrix} 7 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} 250 \\ 90 \end{bmatrix}_S$$

Same as $P_{\beta \to S}[v]_\beta$
$= \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 620 \\ -230 \end{bmatrix}$