MTH 100 : Worksheet 12

1. Find the eigen values and corresponding eigenvectors for the matrix A given below. Is A diagonalizable? Justify your answer.

   $$A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

2. For each matrix find all eigenvalues and a basis of each eigenspace. Which matrix can be diagonalized and why? If yes, indicate the diagonal matrix D and the invertible matrix P such that $A = PDP^{-1}$. [*Hint:* $\lambda = 4$ *is an eigenvalue.*]

   (a) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

3. A 7×7 matrix A has three eigenvalues. One eigenspace is $2-$ dimensional and one of the others is $3-$ dimensional. Is it possible for A to be not diagonalizable? Justify your answer.

4. (a) If A is row-equivalent to the identity matrix, then A must be diagonalizable. Is this statement TRUE or FALSE?

   (b) Justify your answer to 1). Give a proof if TRUE or a concrete counter-example if FALSE. In the second case, you should verify that your counter-example is row equivalent to identity matrix but not diagonalizable.

5. For the given matrix A, find the invertible matrix P and the matrix B which has the form given below such that $A = PBP^{-1}$. In other words, find the values a and b. Finally, express B as a rotation followed by a scaling. $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

6. For the matrix A given in previous question diagonalize it over a complex field.

7. Let $V = C^{\infty}[\mathbf{R}]$, the vector space of real functions having continuous derivatives of all orders. Let D be the differentiation operator on V. Determine the eigenvalues and corresponding eigenvectors of D.

8. Recall the interpolation inner product on $R_n[t]$:

   Let $t_0, t_1, t_2, ..., t_n$ be distinct real numbers.(Note that there are $(n+1)$ numbers.)

   For any two polynomials p and q in $R_n[t]$, we define:

   $< p, q >= p(t_0)q(t_0) + p(t_1)q(t_1) + ... + p(t_n)q(t_n)$

   Verify the inner product axioms for this example and explain why we need $(n+1)$ distinct numbers.

9. Let $V = C[a, b]$. Verify the inner product properties for the inner product given by:

   $< f, g >= \int_a^b f(t)g(t)dt$

10. Use the Gram-Schmidt process to find an orthonormal basis given the basis $\{x_1 = (2, 1, 2), x_2 = (4, 1, 0), x_3 = (3, 1, -1)\}$ for $\mathbf{R^3}$.

11. Let V be the vector space $R_2[t]$ of polynomials of degree ≤ 2 with real coefficients with the inner product $< p, q >= p(0)q(0) + p(-2)q(-2) + p(2)q(2)$, i.e. the interpolation inner product.

    (a) Find an orthogonal basis for V starting from the standard basis $\{1, t, t^2\}$ using the Gram-Schimdt process.

    (b) Find the coordinates of $p(t) = 1 + 2t + 3t^2$ with respect to the orthogonal basis found in previous part.

12. Let W be the subspace of $\mathbf{R^3}$ spanned by the vector $v = (1, 2, 3)$. Find the orthogonal bases for W and $W^{\perp}$ respectively. Is the union of these two bases a basis for $\mathbf{R^3}$?

13. Let S be a finite subset of an inner product vector space V, and define $S^{\perp} = \{v \in V :< v, u >= 0$ for every $u \in S\}$,i.e. $S^{\perp}$ is the set of vectors orthogonal to S. Show that in fact $S^{\perp}$ is a subspace of V. If $W =$ Span S, what is the relationship between $S^{\perp}$ and $W^{\perp}$? Justify your answer.

14. Let $A \in R^{m \times n}$,i.e. A is an $m \times n$ matrix with real entries. Show that Nul A is the orthogonal complement of Row A.

15. (a) Let V be a complex inner product space,i.e. the usual symmetry property is replaced by the property:$< u, v >= \overline{< v, u >}$. Show that $< u, cv >= \bar{c} < u, v >$. Here the bar indicates the complex conjugate.

    (b) Suggest a suitable inner product for $\mathbf{C^n}$ regarded as a vector space over C and verify the inner product properties.