MTH 100 : Worksheet 5

1. Let U and W be two subspaces of the vector space V. Show that $U \cap V$ is also a subspace of V.

2. In the following is W a subspace of V? (Base field is $\mathbf{R}$ in all.) Justify your answer.

   (a) $V = \mathbf{R_n}[t]$ = vector space of all polynomials of degree $\leq n$,$W = \{p(t) \in V$:deg p(t)$= n\} \cup \{\mathbf{0(t)}\}$.Here $\mathbf{0(t)}$ indicates the zero polynomial.

   (b) $V = \mathbf{R^3}$,$W = \{(x, y, z) : x, y, z \in \mathbf{Q}\}$

   (c) $V = \mathbf{R^3}$,$W = \{(x, y, z) : xy = 0\}$

   (d) $V = \mathbf{R^3}$,$W = \{(x, y, z) : x^2 + y^4 + z^6 = 0\}$

3. Consider the space V of all 2×2 matrices over $\mathbf{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.

   (a) All symmetric matrices (Definition: For any $m \times n$ matrix $A = [a_{ij}]$, its transpose is the $n \times m$ matrix $B = [b_{ij}]$,given by $b_{ij} = a_{ji}$. The standard notion for the transpose of A is A^T. A matrix is symmetric if $A = A^T$)
   (b) All A such that $AB = BA$ where B is some fixed matrix in V.
   (c) All A such that $BA = 0$ where B is some fixed matrix in V.
   (d) Would the above results hold for all $n \times n$ matrices where n is a general positive integer?

4. Consider the space V of all $n \times n$ matrices over $\mathbf{R}$ and let W be the subset consisting of all upper triangular matrices.

   (a) Show that W is a subspace of V.

   (b) Show further that W satisfies closure with regard to products and multiplicative inverses,i.e. if $A, B \in W$,then $AB \in W$, and if $A \in W$ happens to be invertible, then $A^{-1} \in W$.

5. Let V be a vector space. Prove the following:

   (a) The additive inverse vector of any vector $\mathbf{u}$ is unique; we use the notation **-u** for the inverse vector.

   (b) $0\mathbf{u} = \mathbf{0}$ for every vector u.

   (c) $c\mathbf{0} = \mathbf{0}$ for every scalar c.

(d) Cancellation Law,i.e. show that if **u**+**v** = **u**+**w**, for **u**,**v**,**w** $\in V$, then **v**=**w**.

6. Give an example of a set X and an operation involving elements of X, which does not satisfy the cancellation law. Briefly justify your answer.

7. Show that the set $\mathbf{Q}[\sqrt{\mathbf{2}}] = \{a + b\sqrt{2}, a, b \in \mathbf{Q}\}$ is a field.

   Remark: Note that $\mathbf{Q}[\sqrt{\mathbf{2}}]$ is a subset of $\mathbf{R}$; the wording for this situation is :$\mathbf{Q}[\sqrt{\mathbf{2}}]$ is a subfield of $\mathbf{R}$. (Hint: The key step is to show that nonzero elements of $\mathbf{Q}[\sqrt{\mathbf{2}}]$ have multiplicative inverses in $\mathbf{Q}[\sqrt{\mathbf{2}}]$.)

8. (a) Is $\mathbf{R}$ a vector space over $\mathbf{Q}$? Justify your answer in brief.

   (b) Is $\mathbf{C}$ a vector space over $\mathbf{R}$? Justify your answer in brief.

   (c) Can you generalize the answers to 1) and 2) above to a statement about fields and vector spaces? Explain briefly.

9. Modular arithmetic and fields: Let n be a fixed but arbitrary positive integer, $n \geq 2$. Put $Z_n = \{0, 1, 2, ..., n-1\}$. Define the operations of modular addition and modular multiplication on Z_n by $x \bigoplus y = (x + y)$ (mod n) and $x \bigotimes y = xy$ (mod n)

   NB: Recall that z (mod n)= remainder after the division of z by n for all z $\in \mathbf{Z}$. Note that we have $0 \leq$ remainder $< n$,i.e., z(mod n)$\in Z_n$ for all z $\in \mathbf{Z}$.

   (a) Show that if $x \in Z_n$, then x has an inverse in Z_n with regard to the operation $\bigoplus$(i.e. additive inverse.)

   (b) ***We have already shown in class that $Z_2$is a field.*** Now show that Z_3 and Z_5 are fields.***(Hint: you may assume that $\bigoplus$ and $\bigotimes$ satisfy closure, associativity, commutativity and distributivity on Z_n. This is straightforward but a little lengthy.Also see the hint of question 8)***

   (c) Are Z_4 and Z_6 fields? Justify your answer briefly.

   (d) Can you generalize the above to state a condition for Z_n not to be a field? Briefly justify your statement.

10. Consider the system $R^{3\times3}$ of 3×3 (square) matrices with real entries. A non-zero matrix A is said to be a zero-divisor if there exists some non-zero matrix B such that $AB = 0$, the zero matrix.

    (a) If A is invertible, then it cannot be a zero divisor.TRUE OR FALSE? Justify your answer.

    (b) If A is not invertible, then it must be a zero divisor.TRUE OR FALSE? Justify your answer.

11. (a) Obtain an LU decomposition of the matrix A given below.

(b) Solve the non-homogeneous system $Ax = b$, for b_1 and b_2 given below, using the LU decomposition obtained in first part. Take b_1 and b_2 as column vectors. Explain the difference in the answers for these two vectors b_1 and b_2.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \; b_1 = (1, 4, 5) \; b_2 = (3, 7, 15)$$

12. (a) Obtain an LU decomposition of the matrix A given below.

(b) Solve the non-homogeneous system $Ax = b$, for b given below, using the LU decomposition obtained in first part. Take b as column vector.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \; b = (4, 9, 14)$$

13. Is $\mathbf{R^2}$ a subspace of $\mathbf{R^3}$?(YES/NO) Justify your answer briefly.

14. Let $V = \{x \in \mathbf{R} : x > 0\}$. Define the addition for V by $x \bigoplus y = xy$ and scalar multiplication by any $\alpha \in \mathbf{R}$ by $\alpha * x = x^\alpha$

(a) Verify the closure axioms, the commutative, zero and inverse properties for addition and the property $1 * x = x$ for all $x \in V$

(*Remark: V is in fact a vector space over the field* **R**.However, you need not verify the other properties of a vector space.)

(b) Is V a subspace of **R** regarded as a vector space over itself?(YES/NO) Justify your answer clearly.

(*This question was given as an exam problem for a previous batch.*)