Worksheet 3

1. Determine the inverse of the given matrix A using row reduction.
   $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

2. TRUE OR FALSE?

   (a) The sum of two invertible matrices(square matrices of same order) is always invertible.

   (b) If matrices A and B commute ,i.e. $AB = BA$, then invertibility of A implies invertibility of B.

   Justify your answer. Prove if TRUE or give counter-example if FALSE.

3. Suppose $AB = AC$,where B and C are $n \times k$ matrices and A is invertible. Show that $B = C$. Is this true in general when A is not invertible? Justify your answer (prove if true, give counter-example if false).

4. (a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

   (b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

   (c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

5. For each of the following clearly state True or False (prove if true, counter example if false)

   - For any square matrix A, if A^k is invertible for some positive integer $k > 1$, then A itself is invertible.
   - If a 3×3 square matrix A satisfies $A^3 = 0$, then $A = 0$.(Here 0 indicates the zero matrix.)

6. Check whether A is invertible and find A^{-1} if it exists. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

7. Suppose A is 2×1 matrix and B is 1×2 matrix.Prove that $C = AB$ is not invertible.

8. Prove the following generalization of previous problem.

   If A is $m \times n$ matrix and B is $n \times m$ matrix and $n < m$ then prove that AB is not invertible.

9. Let A be an $n \times n$ (square) matrix.Prove the following statements:

   - If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.
   - If A is not invertible then there exists an $n \times n$ matrix B such that $AB = 0$, but $B \neq 0$

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Prove using elementary row operations that A is invertible iff $ad - bc \neq 0$

11. Prove that an upper triangular(square) matrix is invertible iff every entry on the main diagonal is different from zero.(An $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ is called upper triangular if $a_{ij} = 0$ for $i > j$,i.e. every entry below the main diagonal is zero.)

12. Given an $m \times n$ matrix A and $n \times k$ matrix B, the product $AB = \begin{bmatrix} Av_1 & Av_2 & .... & Av_k \end{bmatrix}$ in column form where $B = \begin{bmatrix} v_1 & v_2 & .... & v_k \end{bmatrix}$ is in column form. Construct an example to illustrate this rule. The matrix A in your example should be atleast 3×3 and B should be atleast 3×2.

13. Prove the following proposition in general case,i.e. for any row operation e and any matrix A.

    Proposition If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A, $e(A) = EA$.

    (NB: the three cases of scaling, replacement and interchange require separate proofs.)