MTH 100 : Lecture 22

From Last time :

Sum of two Subspaces :

If U and W are two subspaces of a vector space V, then their sum $U+W$ is defined by :

$$U+W = \{u+w : u \in U, w \in W\}$$

$U+W$ is a subspace of V.

- If U and W are finite dimensional subspaces of a vector space V, then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Direct Sums

Definition: V is said to be the direct sum of the subspaces U and W if every vector $v \in V$ is uniquely expressible in the form $v = u + w$ where $u \in U$, $w \in W$.

Notation: We will use the notation

$V = U \oplus W$ to indicate that V is the direct sum of U and W.

Proposition: If U and W are subspaces of a vector space V,

then $V = U \oplus W$ if and only if

$V = U + W$ and $U \cap W = \{0\}$

Proof: Exercise (Try it !!)

Remark: The subspace W in the above is often called a Complement or Complementary subspace of U.

Corollary: If V is the direct sum of the finite dimensional subspaces U and W, then $\dim V = \dim(U \oplus W) = \dim U + \dim W$

Fundamental Subspaces

Definition: The null space of an $m \times n$ matrix A is the set of all solutions to the homogeneous system $Ax = 0$.

It is denoted by $\mathrm{Nul}\, A$.

Thus $\mathrm{Nul}\, A = \{ x \in \mathbb{R}^n : Ax = 0 \}$

So, $\mathrm{Nul}\, A$ is a subset of $\mathbb{R}^n$.

Proposition: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Proof: Let $\bar{0} \in \mathbb{R}^n$ be the zero vector of $\mathbb{R}^n$.

- Now $A \underset{\text{(zero vector in } \mathbb{R}^n)}{\bar{0}} = \underset{\text{(zero vector in } \mathbb{R}^m)}{\bar{0}}$ and so $\bar{0} \in \mathrm{Nul}\, A$

- If $\bar{u}, \bar{v} \in \mathrm{Nul}\, A$, then $A\bar{u} = \bar{0}$, $A\bar{v} = \bar{0}$

  Then $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v} = \bar{0} + \bar{0} = \bar{0}$

  $\Rightarrow \bar{u} + \bar{v} \in \mathrm{Nul}\, A$

- Finally if $\bar{u} \in \mathrm{Nul}\, A$ and $c \in \mathbb{R}$ (any scalar)

  then $A(c\bar{u}) = c(A\bar{u}) = c(\bar{0}) = \bar{0} \Rightarrow c\bar{u} \in \mathrm{Nul}\, A$

Therefore $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

Remark:

- We need to take a homogeneous system of equations to get a subspace.
- The solution set of a non-homogeneous system is not a subspace.
- Nul A is defined implicitly.
  To describe Nul A explicitly, we need to solve the system of linear equation
  $$A\overline{x} = \overline{0}.$$

How to find a Basis for Nul A:

- Reduce A to an RREF matrix.
- Express the solution vector of the simplified system as a linear combination where the coefficients are the free variables.
- The spanning set produced by this method is a basis for Nul A.

Remark: **Either** Nul A is the zero subspace

or dim(Nul A) = Number of free variables in the solution.

Column Space:

Let $A_{m\times n} = [a_1, a_2, \dots, a_n]$, $a_i \in \mathbb{R}^m$ for $i = 1, 2, \dots, n$

Then column space of A (denoted by $\text{Col } A$) is the set of all linear combinations of the columns of A.

i.e. $\text{Col } A = \text{Span}\{a_1, a_2, \dots, a_n\}$

Proposition: $\text{Col } A$ is a subspace of $\mathbb{R}^m$.

Proof: Since A is an $m\times n$ matrix, its columns are vectors in $\mathbb{R}^m$.
Since $\text{Col } A$ is the span of the columns of A, it is a subspace of $\mathbb{R}^m$.

Remark: $\boxed{\text{Col } A = \{b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n\}}$

If $b \in \text{Col } A$, b can be written as

$$b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

where x_i's are scalars for $i = 1, 2, \dots, n$

Hence
$$\begin{aligned} b &= x_1 a_1 + x_2 a_2 + \dots + x_n a_n \\ &= [a_1, a_2, \dots, a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Ax \end{aligned}$$

where, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Basis for col A:

Proposition: The pivot columns of a matrix A form a basis for col A.

Proof: Any linear dependence relationship between the columns of A can be written in the form $Ax = 0$

(Note that $x_1 a_1 + \cdots + x_n a_n = 0 \Rightarrow [a_1, \cdots, a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$

$$\Rightarrow \quad Ax = 0 \text{)}$$

When the matrix A is row reduced to R, the columns of A change but the equation $Rx = 0$ has the same set of solutions as $Ax = 0$.

Thus row reduction does not change the dependence relations between the columns.

The pivot columns of A must be linearly independent since the pivot columns R are linearly independent.

Also, non-pivot columns are linear combinations of the preceding (i.e. left) pivot columns.

Hence Pivot columns of A form a basis for Col A.

(QED)

Note: We must take the columns of A for Basis (Not of its RREF matrix R)

Comparison between Nul A and Col A:

Nul A	Col A
• Nul A is a subspace of $\mathbb{R}^n$	• Col A is a subspace of $\mathbb{R}^m$
• Nul A is defined implicitly	• Col A is defined explicitly.
• To find vectors in Nul A, we have to solve an equation	• Vectors in Col A can be found directly.
• There is no obvious relation between Nul A and entries of A.	• There is a definite relation between Col A and entries of A.
• If $v \in$ Nul A, $Av = 0$	• If $v \in$ Col A, the system $Ax = v$ is consistent.
• Given a specific vector v, we can easily test whether it is in Nul A.	• Given a specific vector v, to test whether it is in Col A, we have to solve an equation.
• Nul $A = \{0\}$ if and only if $Ax = 0$ has only the trivial solution.	• Col $A = \mathbb{R}^m$ if and only if $Ax = b$ has a solution for every $b \in \mathbb{R}^m$.

Row space: Let A be an $m \times n$ matrix. The Row space of A (denoted by Row A) is the set of all linear combinations of the rows of A.

If $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$, then Row $A = \text{Span}\{r_1, r_2, \ldots, r_m\}$

Proposition: Row A is a subspace of $\mathbb{R}^n$.

Proposition: Row equivalent matrices have the same row space.

Proof: Elementary row operations replace rows of the original matrix by rows which are the same or linearly dependent on them.

Hence row space does not get enlarged by row operations.

Thus if B is obtained from A by an elementary row operation, then Row $B \subseteq$ Row A

But since elementary row operations are reversible, we also have Row $A \subseteq$ Row B

Therefore $\boxed{\text{Row } B = \text{Row } A}$

(QED)

How to find a Basis for Row A:

- Given a matrix A, reduce it to an RREF matrix R.
- The non-zero rows of R are linearly independent and they form a Basis for the row space of R and also for the row space of A.

Alternate Method:

- Since the rows of A are the columns of A^T (Transpose of A), we can find a Basis for Row A by using the method to find a Basis for Col A^T.
- This method can be used to find a Basis for Row A consisting of actual rows of A.

Ex: Let $A = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 6 & 4 & -6 \\ 3 & 9 & 7 & -11 \\ 8 & 24 & 9 & -10 \end{bmatrix} = [v_1, v_2, v_3, v_4]$ (say)

Find a Basis for Nul A, a Basis for Col A and a Basis for Row A.

$$A=\begin{bmatrix}1&3&2&-3\\2&6&4&-6\\3&9&7&-11\\8&24&9&-10\end{bmatrix}\xrightarrow[\substack{R_3\to R_3-3R_1\\R_4\to R_4-8R_1}]{R_2\to R_2-2R_1}\begin{bmatrix}1&3&2&-3\\0&0&0&0\\0&0&1&-2\\0&0&7&-14\end{bmatrix}$$

$$\xrightarrow{R_4\to R_4+7R_3}\begin{bmatrix}1&3&2&-3\\0&0&0&0\\0&0&1&-2\\0&0&0&0\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1&3&2&-3\\0&0&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-2R_2}\begin{bmatrix}1&3&0&1\\0&0&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}=R=\begin{bmatrix}r_1\\r_2\\0\\0\end{bmatrix}\text{ (say)}$$

$$\begin{array}{cccc}\downarrow&&\downarrow&\\\text{Basic}&&\text{Basic}&\end{array}$$

The corresponding system $R\bar{x}=\bar{0}$ is

$$\left.\begin{aligned}x_1+3x_2+x_4&=0\\x_2&=x_2\\x_3-2x_4&=0\\x_4&=x_4\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}x_1&=-3x_2-x_4\\x_2&=x_2\\x_3&=2x_4\\x_4&=x_4\end{aligned}\right\}$$

Thus $\bar{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = x_2 u_1 + x_4 u_2$ (say)

- Then a Basis for Nul $A = \{u_1, u_2\}$

$$= \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

- A Basis for Col $A = \{v_1, v_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ 9 \end{bmatrix} \right\}$$

- A Basis for Row $A = \{r_1, r_2\}$

$$= \left\{ \begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -2 \end{bmatrix} \right\}$$