MTH 100 : Lecture 27

Coordinate Systems:

Let V be a finite dimensional vector space and $B = \{v_1, \dots, v_n\}$ be an ordered basis of V.

Then any vector $u \in V$ can be uniquely written as $u = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$

The vector $[u]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called the coordinate vector of u (relative to B) and is denoted by $[u]_B$

- Now the mapping or correspondence between V and F^n given by: $u \longmapsto [u]_B$ is called the coordinate mapping determined by B

- It is an one-to-one correspondence.
  i.e. each vector has a unique corresponding n-tuple and each n-tuple has a unique corresponding vector.

- The sum of two vectors corresponds to the sum of the two n-tuples.

- The scalar multiple of a vector corresponds to a scalar multiple of the n-tuple.

Therefore the coordinate mapping is actually an isomorphism from an n-dimensional vector space V over the field F to F^n.

- Note that we get a <u>different isomorphism for each choice of an ordered basis for V.</u>

(Recall proposition of last class).

<u>Ex:</u> Let $V = \mathbb{R}^3$

Now, $E = \{e_1, e_2, e_3\}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

is an ordered Basis of $\mathbb{R}^3$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let $v = \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix} \in V \in \mathbb{R}^3$

$$[v]_E = \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix}_E \quad \left(\text{because } \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix} = \underbrace{14}_{\downarrow} e_1 + \underbrace{11}_{\downarrow} e_2 + \underbrace{7}_{\downarrow} e_3 \right)$$

coordinates

Now $B = \{v_1, v_2, v_3\}$ be another ordered Basis of V

where $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now

$$v = \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, $[v]_B = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}_B$

Similarly

since $w = \begin{bmatrix} 12 \\ 15 \\ 9 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

we have $[w]_B = \begin{bmatrix} -3 \\ 6 \\ 9 \end{bmatrix}_B$

—— x —— x —— x —— x —— x —— x —— x

In general if $z \in V = \mathbb{R}^3$, then to find $[z]_B$, we need to find coefficients x_1, x_2, x_3 such that

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = z$$

$$\Rightarrow \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = z$$

$$\Rightarrow \quad Ax = z \quad \text{where } A = [v_1 \; v_2 \; v_3]$$
$$\text{and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \quad x = A^{-1}z$$

(Note that A is invertible as its columns are linearly independent)

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (Please calculate and check !!)

Now for $z = v = \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix}$

$$[z]_B = A^{-1}z = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 11 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}_B \quad \text{(as expected)}$$

Ex:

If $z_1 = \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$, check that $[z_1]_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ 5 \\ 9 \end{bmatrix}_B$$

Matrix of a Linear Transformation

Suppose V and W are finite dimensional vector spaces over the field F and $T: V \to W$ is a linear transformation.

Suppose $\dim V = n$ and $\dim W = m$

Let $B = \{v_1, \ldots, v_n\}$ be an ordered basis of V and $C = \{w_1, \ldots, w_m\}$ be an ordered basis of W.

Since $Tv_1, Tv_2, \ldots, Tv_n \in W$, we can express them uniquely as linear Combinations of $w_1, \ldots, w_m$.

Thus we can write
$$\begin{aligned} Tv_1 &= A_{11} w_1 + A_{21} w_2 + \cdots + A_{m1} w_m \\ Tv_2 &= A_{12} w_1 + A_{22} w_2 + \cdots + A_{m2} w_m \\ &\cdots\cdots\cdots\cdots \\ Tv_n &= A_{1n} w_1 + A_{2n} w_2 + \cdots + A_{mn} w_m \end{aligned}$$

We now form the $m \times n$ matrix A with these coefficients as columns.

i.e.
$$A_{m \times n} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

- The matrix A is called the matrix of T with respect to the bases B and C and is denoted by $[T]_{B \to C}$
- For any vector $v \in V$, we can find the coordinates of Tv in W by left multiplying the coordinate vector of v by the matrix $A = [T]_{B \to C}$

- In terms of coordinate vectors, we can write : $\boxed{[T(v)]_C = [T]_{B\to C}[v]_B}$

- In the special case of a linear operator, i.e. a linear transformation from V into itself, the bases B and C are usually taken as the same, and the matrix A is called the B-matrix for T, written $[T]_B$.

Then the above equation becomes:

$$\boxed{[T(v)]_B = [T]_B [v]_B}$$

Ex: Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$T(x,y,z) = (x+y+z, \; x+2y+3z)$$

Let $B = \left\{ e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

and $C = \left\{ e_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Then B is an ordered basis of $\mathbb{R}^3$
and C is an ordered basis of $\mathbb{R}^2$

$$Te_1 = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\,e_1' + 1\,e_2'$$

$$Te_2 = T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\,e_1' + 2\,e_2'$$

$$Te_3 = T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\,e_1' + 3\,e_2'$$

So, the matrix of T:

$$[T]_{B\to C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$$

If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$,

then $$[T(v)]_C = [T]_{B\to C} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x+y+z \\ x+2y+3z \end{bmatrix} = [Tv]_C$$

Ex: Let $D : \mathbb{R}_3[t] \longrightarrow \mathbb{R}_2[t]$
be defined by $D[p(t)] = p'(t)$ for any $p(t) \in \mathbb{R}_3(t)$

Note that $\dim(\mathbb{R}_3[t]) = 4$ and $\dim(\mathbb{R}_2[t]) = 3$

and

$B = \{1, t, t^2, t^3\}$ is an ordered basis of $\mathbb{R}_3[t]$

and $C = \{1, t, t^2\}$ is an ordered basis of $\mathbb{R}_2[t]$

Now

$$D(1) = 0 = 0.1 + 0.t + 0.t^2$$

$$D(t) = 1 = 1.1 + 0.t + 0.t^2$$

$$D(t^2) = 2t = 0.1 + 2.t + 0.t^2$$

$$D(t^3) = 3t^2 = 0.1 + 0.t + 3.t^2$$

Therefore $[D]_{B \to C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Now let us take a particular polynomial

$$p_1(t) = 4 + 5t + 2t^2 + 3t^3$$

Then $[p_1(t)]_B = \begin{bmatrix} 4 \\ 5 \\ 2 \\ 3 \end{bmatrix}$

Now $[T]_{B \to C} [p_1(t)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix}_C$$

Note that $D[p_1(t)] = 5 + 4t + 9t^2$

whose coordinate vector with respect to C is

$$\begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix}_C$$