$\boxed{\text{MTH 100: Lecture 6}}$

Invertible Matrices:

An $m \times m$ (square) matrix A is called invertible if there exists another square matrix B such that $\boxed{BA = AB = I_m}$ ($m \times m$ identity matrix)

B is called an inverse of A.

- Another terminology: Invertible matrices are also called Non singular.

Matrices which are not invertible are called Singular.

Observation 1: The inverse of A if it exists is unique. (Notation: A^{-1})

Let $AB = BA = I$, $AC = CA = I$ i.e. B & C are two inverses of A. Then

$$\left.\begin{aligned} BAC &= (BA)C = I.C = C \\ BAC &= B(AC) = B.I = B \end{aligned}\right\} \Rightarrow B = C$$

So, the inverse is unique.

Observation 2: If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$ (Since $A^{-1}A = AA^{-1} = I$)

Observation 3: If A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

because

$$\left.\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I \\ \text{and } (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}(I)B = (B^{-1}I)B = B^{-1}B = I \end{aligned}\right\}$$

Observation 4 (Generalization of 3):

The product of invertible matrices is invertible and the inverse of the product is the product of the inverses taken in reverse order.

In other words, if $A_1, A_2, \dots A_n$ $(n \geq 2)$ are invertible matrices, then $C = A_1 A_2 \cdots A_n$ is an invertible matrix and $C^{-1} = A_n^{-1} \dots A_2^{-1} A_1^{-1}$

Elementary Matrices:

- An $m \times m$ (square) matrix is said to be an elementary matrix if it is obtained from the $m \times m$ identity matrix I_m by an elementary row operation.

Proposition ⑤: If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A

$$\boxed{e(A) = EA}$$

Thus applying an elementary row operation is the same as left multiplication by the corresponding elementary matrix.

Proof: Exercise

Note: The three types of elementary row operation have to be treated separately.

Ex:

(This is not a proof) Let $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Let e be the replacement operation
$$e : R_3 \longrightarrow R_3 + 2R_1$$

Then $e(A) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{bmatrix}$

$$E = e(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{bmatrix} = e(A)$$

- This is an <u>illustration</u> of proposition 5.
  (However, <u>this is not a proof</u>)
- You can also try with other types of operation.

Proposition ⑥: Every elementary matrix is invertible.

Proof: Let E be an elementary matrix.
Let e be the corresponding row operation.
So, $e(I) = E$

- We know that there is another row operation of the same type (we call it f) that reverses the action of e.
- Let F be the elementary matrix corresponding to f i.e. $f(I) = F$

Now, $FE = (FE)I = F(EI) = F(e(I))$ (By proposition ⑤)

$= f(e(I))$ (By proposition ⑤)

$= I$ (since f is the reverse operation of e)

$EF = (EF)I = E(FI) = E(f(I))$ (By proposition ⑤)

$= e(f(I))$ (By proposition ⑤)

$= I$ (since e is the reverse operation of f)

- So, $EF = FE = I$
  Thus E is invertible and $E^{-1} = F$

Note: The inverse of an elementary matrix is also an elementary matrix of the same type.

Ex: An example of finding the inverse of a matrix by row reduction:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

: We will take the enlarged matrix $[A:I]$

$$[A:I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to (-1)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

I

This will be A^{-1}

Check :

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: This method is preferable to the adjoint/ Determinant formula which requires approximately $(n!)$ calculations.
Gauss-Jordan elimination requires approximately $\left(\frac{3}{2}n^3\right)$ operations.

Theorem (1): The following are equivalent for an $m \times m$ square matrix A.

(a) A is invertible

(b) A is row equivalent to the identity matrix.

(c) The homogeneous system $AX = 0$ has only the trivial solution.

(d) The system of equation $AX = b$ has atleast one solution for every $b \in \mathbb{R}^m$

Proof: