MTH 100 : Lecture 33

Theorem (Diagonalization Theorem):

(a) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

(b) In this case, $A = PDP^{-1}$ where the columns of P are n linearly independent eigen vectors of A, and the diagonal entries of D are eigenvalues corresponding to these eigenvectors.

Another way to express the above Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has enough (linearly independent) eigenvectors to form a basis of $\mathbb{R}^n$.

Such a basis is called an eigenvector basis.

Note: Check if the matrix considered in the last lecture is diagonalizable.

Proof of the Diagonalization Theorem:

(a) $\Rightarrow$: Suppose A is diagonalizable
want to show that A has n linearly independent eigen vectors.

Now $A = PDP^{-1}$ for some diagonal matrix D and some invertible matrix P.

Therefore $AP = PD$ ...... (1)

Let $P = [v_1 \; v_2 \; \dots \; v_n]$ in column form

and Let $D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$ where the λ's need not be distinct.

Then (1) becomes

$$A[v_1 \; v_2 \; \dots \; v_n] = [v_1 \; v_2 \; \dots \; v_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\Rightarrow [Av_1 \; Av_2 \; \dots \; Av_n] = [\lambda_1 v_1 \; \lambda v_2 \; \dots \; \lambda v_n]$$

$$\Rightarrow \quad Av_i = \lambda_i v_i \quad \text{for } i = 1, 2, \dots, n$$

Therefore $v_1, v_2, \dots, v_n$ are eigen vectors corresponding to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

Also the vectors $v_1, v_2, \dots, v_n$ being columns of an invertible matrix P are linearly independent.

$\underline{\Leftarrow}$: Suppose A has n linearly independent eigen vectors.

Want to show that A is diagonalizable.

Let $v_1, v_2, \dots, v_n$ be the n linearly independent eigen vectors of A corresponding to the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$ (not necessarily distinct).

Then $Av_i = \lambda_i v_i$ for $i = 1, 2, \dots, n$

We form the matrix P with the v_i's as columns.

i.e. $P = [v_1 \ v_2 \ \dots\dots \ v_n]$

Then $$AP = A[v_1 \ v_2 \ \dots\dots \ v_n]$$
$$= [Av_1 \ \ Av_2 \ \dots\dots \ Av_n]$$
$$= [\lambda_1 v_1 \ \ \lambda_2 v_2 \dots\dots, \lambda_n v_n]$$
$$= [v_1 \ \ v_2 \ \dots\dots \ v_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$
$$= PD \quad \text{where } D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Now since $v_1, v_2, \dots, v_n$ are linearly independent, the matrix $P = [v_1 \ v_2 \dots\dots v_n]$ is invertible and $\boxed{A = PDP^{-1}}$. Hence A is diagonalizable.

(b) Part (b) of the theorem has been proved enroute to proving part (a) of the theorem. (QED)

In practice, we can distinguish three cases:

- Case 1: An $n \times n$ matrix A has n distinct (real) eigen values.

Then we have the following result.

Proposition: An $n \times n$ matrix A with n distinct eigen values is diagonalizable.

Proof: By an earlier proposition, eigen vectors corresponding to distinct eigen-values are linearly independent. Therefore in this case A has n linearly independent eigen vectors. Hence by Diagonalization Theorem, A is diagonalizable.

Two Preliminary Definitions:

Given an eigenvalue λ_1 for a matrix A we define

- The algebraic multiplicity of λ_1 is the power of the factor $(\lambda - \lambda_1)$ in the characteristic polynomial of A.

- The <u>geometric multiplicity of λ_1</u> is the dimension of the eigen space corresponding to λ_1.

<u>Note:</u> Algebraic multiplicity applies to polynomials in general (not only characteristic polynomial). The geometric multiplicity applies specifically to the characteristic polynomial (since its roots are eigenvalues which have corresponding eigenspaces).

<u>Case 2:</u> An $n \times n$ matrix A has $p < n$ distinct eigenvalues, but counting the (algebraic) multiplicities, there are <u>n real</u> eigenvalues (<u>not distinct</u>).

We then have a weaker result for this case.

<u>Proposition:</u> Let A be an $n \times n$ matrix with n (real) eigenvalues (counting algebraic multiplicities) of which only $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct $(p < n)$.

Then the following hold:

(a) For $1 \leq k \leq p$, the <u>geometric multiplicity</u>

of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) A is diagonalizable if and only if the sum of the dimensions of the distinct eigen spaces is n and this happens if and only if the geometric multiplicity for each λ_k equals its algebraic multiplicity.

(c) If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in $B_1, B_2, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Note: This result is weaker because unlike case 1, A is not automatically diagonalizable. A has to satisfy the additional condition (b) and this may not happen for all matrices.

Ex (1) Let $A = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix}$

Then $\det(A-\lambda I) = \begin{vmatrix} 42-\lambda & -33 \\ 22 & -13-\lambda \end{vmatrix}$

$= (42-\lambda)(-13-\lambda) + 22(33)$

$= -546 - 42\lambda + 13\lambda + \lambda^2 + 726$

$= 180 - 29\lambda + \lambda^2 = (\lambda-20)(\lambda-9)$

Thus here there are 2 distinct eigen values

$\lambda_1 = 20$, $\lambda_2 = 9$

For $\lambda_1 = 20$ $\quad A - \lambda_1 I = \begin{bmatrix} 42-20 & -33 \\ 22 & -13-20 \end{bmatrix}$

$= \begin{bmatrix} 22 & -33 \\ 22 & -33 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 22 & -33 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{22} R_1} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$ = RREF matrix

So, $(A - \lambda_1 I)x = 0 \Rightarrow \left. \begin{array}{l} x_1 = \frac{3}{2} x_2 \\ x_2 = x_2 \end{array} \right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Let us take $v_1 = 2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as an eigen vector corresponding to $\lambda_1 = 20$

(by taking $x_2 = 2$)

(Check $A v_1 = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix} = 20 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 20 v_1$)

For $\lambda_2 = 9$ $\quad A - \lambda_2 I = \begin{bmatrix} 42-9 & -33 \\ 22-9 & -13-9 \end{bmatrix} = \begin{bmatrix} 33 & -33 \\ 22 & -22 \end{bmatrix}$

$$\begin{bmatrix} 33 & -33 \\ 22 & -22 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{33} R_1} \begin{bmatrix} 1 & -1 \\ 22 & -22 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 22R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{RREF matrix}$$

So, $(A - \lambda_2 I)x = 0 \Rightarrow \left.\begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let us take $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue $\lambda_2 = 9$
(By taking $x_2 = 1$)

(Check $A v_2 = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix} = 9 v_2$)

Note that we should get $A = PDP^{-1}$
where $D = \begin{bmatrix} 20 & 0 \\ 0 & 9 \end{bmatrix}$ and $P = [v_1, v_2]$
$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

Easier to check $AP = PD$

Now $AP = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 60 & 9 \\ 40 & 9 \end{bmatrix}$

and $PD = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 60 & 9 \\ 40 & 9 \end{bmatrix}$

So, $AP = PD$ as desired.