Tutorial 2 [redacted]

Ex 1:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$A:I = \left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right]$$

$R_2 \to \frac{1}{2} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + \frac{1}{2} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{5}{2} & \frac{1}{4} & 1 \end{array}\right]$$

$R_3^* \to (-4) R_3$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 + \left(-\frac{1}{2}\right) R_3 \\ R_1 \to R_1 + \frac{1}{2} R_3}]{} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{11}{2} & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -5 & 1 & +2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

So, $\boxed{A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}}$

2 (a) False

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

A & B are invertible but

$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and is not invertible.

(b) False: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then A is invertible but B is not invertible.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

So, $AB = BA$

(3) Given $AB = AC$ and A is invertible, B and C are $n \times k$ matrices.

$$AB = AC$$
$$\Rightarrow A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I.B = I.C$$
$$\Rightarrow B = C$$

This result does not hold in general if A is not invertible.

(Continued)

Example Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Now $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC$

But $B \neq C$

Note that A is not invertible.

4 (a) Recall that a matrix $A = (a_{ij})_{n \times n}$ is unit lower triangular

$$\left.\begin{array}{l} \text{if } \quad a_{ii} = 1 \\ \text{and } \quad a_{il} = 0 \quad \text{for } l > i \end{array}\right\} \cdots (*)$$

Now E is obtained from I after the operation "Replacement of Row i (R_i) by Row i + k Row j ($R_i + kR_j$) where $j < i$

Now if we consider E, only ith row has been affected by this replacement. Other rows are rows of I which satisfy condition (*)

Now in the ith row, $e_{ii} = 1 + k(0)$
$= 1$

and $e_{il} = 0 + k(0) = 0$ for $l > i$

So, $\boxed{E \text{ is unit Lower triangular}}$

(b) Let A and B be unit Lower triangular matrices and let $C = AB$

Assume that all the matrices are $m \times m$.

Now for any row i of C,

$$c_{ii} = \sum_{i=1}^{m} a_{ik} b_{ki} = a_{i1} b_{1i} + \cdots + a_{ii} b_{ii} + a_{i(i+1)} b_{i(i+1)} + \cdots + a_{im} b_{mi}$$

Now A & B are lower triangular

$\Rightarrow$ $b_{1i}, b_{2i}, \ldots, b_{(i-1)i} = 0$

and $a_{i(i+1)}, \ldots a_{im} = 0$

So, $c_{ii} = a_{ii} b_{ii} = 1 \times 1 = 1$ (Since $a_{ii} = 1$ & $b_{ii} = 1$)

Now for $j > i$,

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

For $k > i$ $a_{ik} = 0$
and for $k < j$, $b_{kj} = 0$

Now Since $i < j$, for any k, either $a_{ik} = 0$ or $b_{kj} = 0$

So, $c_{ij} = 0$

Thus $c_{ii} = 1$
and $c_{ij} = 0$ for $j > i$ $\Big\}$ So, C is unit Lower triangular

(c) Let A be unit Lower triangular $m \times m$ matrix.

Let
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & & & 1 \end{bmatrix}$$

Consider the homogeneous system
$$Ax = 0$$
Then the system of equation becomes
$$\begin{aligned} x_1 &= 0 \\ a_{21}x_1 + x_2 &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + x_m &= 0 \end{aligned}$$
Solving by forward substitution we get $x_1 = x_2 = \cdots = x_m = 0$

Thus $Ax = 0$ has only trivial solution.

So, (by Theorem 1) A is invertible.

Note: The above can also be shown by using the fact that $\det(A) = 1$ and so A is invertible.

Now to show that A^{-1} is also unit lower triangular, we use the result which states that the same sequence of elementary row operation which row reduces A to I also row reduces I to A^{-1}.

Now since $a_{ii} = 1$ for all i,
we don't require any row interchange while row reducing A to echelon form.
Any entry above $a_{ii} = 0$ since A is unit Lower triangular.

Thus, $(E_p \cdots E_1) A = I$ where each elementary matrix E_i is ~~[illegible]~~ obtained from I by a row replacement operations that add a multiple of a row to a row below it.
So, each E_i is unit lower triangular and by part (b), $(E_p \cdots E_1)$ is unit lower triangular.

But $$A^{-1} = (E_p \cdots E_1) I$$

$$\Rightarrow A^{-1} = E_p \cdots E_1$$

and E^{-1} is unit Lower triangular.

True

(5) (a) Let A be a square matrix, $K > 1$ and A^k is invertible.

Then we know that product of some square matrices is invertible if and only if each of its factor matrices is invertible.

Thus $A^k = A.A \ldots k$ times is invertible implies that A must be invertible.

Another way: A^k is invertible

$\Rightarrow$ $\exists$ a unique matrix B s.t

$$A^k B = B A^k = I$$

$$\Rightarrow \quad A(A^{k-1}B) = (BA^{k-1})A = I$$

So, A has a right inverse (or left inverse)

and So A is inverse.

Thus A is invertible.

(b) False:

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

but $A \neq 0$

7. Given that A is a 2×1 matrix and B is a 1×2 matrix.

To Prove that $C = AB$ is not invertible.

Let $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$

Then $C = AB = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$

- If any of a_1, a_2, b_1, b_2 is zero, then C has a zero row or zero column and so C is not invertible.
- Now assume $a_1, a_2, b_1, b_2 \neq 0$.

Now $C = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{a_2} R_2]{R_1 \to \frac{1}{a_1} R_1} \begin{bmatrix} b_1 & b_2 \\ b_1 & b_2 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}$ (zero row)

So, C is not invertible.

(8) Let B be a $n \times m$ matrix, $n < m$.

So, if X is a $m \times 1$ vector,

then $BX = 0$ is a system of n equations with m unknowns and $n < m$

So, it has a nontrivial solution X_0 (say)

Now $CX_0 = (AB)X_0 = A(BX_0) = A \cdot 0 = 0$

So, the system $CX = 0$ has a nontrivial solution X_0.

So, C is not invertible.

(9) (a) Let A be invertible and

$AB = 0$

Then $A^{-1}(AB) = A^{-1} 0 \Rightarrow (A^{-1}A)B = 0$

$\Rightarrow I \cdot B = 0 \Rightarrow \boxed{B = 0}$

(b) Let X be a $n \times 1$ vector and A is not invertible.

Then $AX = 0$ has a nontrivial solution

say $B_1 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \neq 0$

Let us form a $n \times n$ matrix

$$B = [\underbrace{B_1 \; B_1 \; \cdots \; B_1}_{n \text{ columns}}] = \begin{bmatrix} b_1 & b_1 & \cdots & b_1 \\ b_2 & b_2 & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ b_n & b_n & \cdots & b_n \end{bmatrix}$$

Then $AB = A[B_1 \ B_2 \cdots B_n]$

$= [AB_1 \ AB_2 \cdots AB_n] = [0 \ 0 \cdots 0]$

$\Rightarrow \boxed{AB = 0 \quad \text{But } B \neq 0}$

10 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

To show that A is invertible $\Leftrightarrow ad - bc \neq 0$

Case 1 Let $a = 0$.

Now if $c = 0$, then A has a zero column & so A is not invertible.

Thus $c \neq 0$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$

$\xrightarrow{R_1 \to \frac{1}{c} R_1} \begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix} = B$

Now if $b = 0$, B has a zero row and hence not invertible.

and so A is not invertible (since A is row equivalent to A)

Thus $b \neq 0$

So, $A \longrightarrow B = \begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{b} R_2} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - \frac{d}{c} R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus if $a = 0$,

then A is invertible $\Leftrightarrow b \neq 0$ and $c \neq 0$

$\Leftrightarrow ad - bc = -bc \neq 0$

<u>Case 2:</u> $a \neq 0$.

If $c = 0$ let $A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

But Case (1), A' is invertible

$\Leftrightarrow cb - ad \neq 0$

i.e. $ad - bc \neq 0$

But A is row equivalent to A'

So, A is invertible

$\Leftrightarrow ad - bc \neq 0$.

Thus we can assume $c \neq 0$ in case 2.

Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{c}R_2]{} \begin{bmatrix} a & b \\ 1 & \frac{d}{c} \end{bmatrix} \xrightarrow[R_1 \to \frac{1}{a}R_1]{} \begin{bmatrix} 1 & b/a \\ 1 & d/c \end{bmatrix}$

$\xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{d}{c} - \frac{b}{a} \end{bmatrix}$

$= D$ (say)

So, for A (or D) to be invertible,
we must have $\frac{d}{c} - \frac{b}{a} \neq 0$

$\Leftrightarrow ad - bc \neq 0$

In that case

$A \longrightarrow D = \begin{bmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{ac} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus in all Cases A is invertible
$\Leftrightarrow ad - bc \neq 0$.

(11) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be

an upper triangular matrix.

Consider the system $AX = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\cdots\cdots$$
$$a_{nn}x_n = 0$$

Now if $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$
then using back substitution, we get ~~a_{nn}~~ $x_n = 0$,
$x_{n-1} = 0, \ldots, x_1 = 0$

ie $AX = 0$ has only the trivial solution.

$\Rightarrow$ $\boxed{\text{A is invertible.}}$

Conversely if one of the diagonal element $a_{KK} = 0$, then the corresponding column of A will not be a pivot column

Thus if we now reduce A, the corresponding RREF will not be an identity matrix

So, A is not row equivalent to identity matrix and so A is not invertible.

Thus $\boxed{\text{A is invertible} \Rightarrow \text{all the diagonal elements are non zero.}}$

12 A is an $m \times n$ matrix &
B is an $n \times k$ matrix.

Then $AB = [Av_1 \; Av_2 \cdots Av_k]$
where $B = [v_1 \; v_2 \cdots v_k]$

Let us take $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -6 & 1 \\ 5 & 4 & -3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ -2 & -6 & 1 \\ 5 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 4 & -4 \\ 16 & 48 & -10 \\ -6 & -40 & 3 \end{bmatrix}$$

Now $B = [v_1 \; v_2 \; v_3]$ where $v_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$

and $v_3 = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$

Now,

$$Av_1 = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \\ -6 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 48 \\ -40 \end{bmatrix}$$

and $$Av_3 = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \\ 3 \end{bmatrix}$$

So, we get
$AB = [Av_1 \; Av_2 \; Av_3]$

Note: One can have many examples.

13) Let A be an $m \times n$ matrix

We want to show that $EA = e(A)$, I is the $m \times m$ identity matrix

Consider three cases.

(1) Let e = scaling the kth row by a number $c \in \mathbb{R}$, $c \neq 0$

Then in $e(A)$, all the entries are unchanged except the rth row where the entries are $c\,a_{kj}$ $j = 1, 2, \ldots n$.

Now in $E = e(I)$, all entries are unchanged except (k,k)th entry becomes c. (instead of 1)

So,
$$EA = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \\ 0 & 0 & \cdots & k & \cdots & 0 \\ \vdots & & & \vdots & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In the kth row of EA, the elements are $c\,a_{k1}$, $c\,a_{k2}, \ldots c\,a_{kn}$

Therefore comparing with $e(A)$, we conclude $e(A) = EA$ in this case.

(2) Replacement : Suppose the kth row is replaced by kth row plus c times the pth row. $c \neq 0$

Then $e(A)$, all entries are unchanged except in the kth row, where the entries are $a_{kj} + c\, a_{pj}$ $j = 1, 2, \ldots n$

Now in $E = e(I)$ all entries are unchanged except that ~~in the kth row, the~~ (k, p)th entry is now c (instead of 0)

$$\text{So, } EA = \begin{array}{c} \\ \text{kth row} \leftarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & c \cdots & 0 \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

In EA all row are unchanged except the kth row where the elements are

$$a_{kj} + c\, a_{pj} \quad j = 1, 2, \ldots, n$$

Comparing with the above, we get

$$e(A) = EA$$

(3): Interchange:

Suppose kth row and pth row are interchanged for $k < p$.

Then in $e(A)$, the entries in kth row is a_{pj}, $j = 1, \dots n$
and in pth row is a_{kj}, $j = 1, 2, \dots, n$

All other rows are unchanged.

Now in $E = e(I)$,
the kth row is all zeros except $a_{kp} = 1$
and the pth row is all zeros except $a_{pk} = 1$

Now

$$EA = \begin{array}{c} \\ k\text{th row} \leftarrow \\ p\text{th row} \leftarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots 1 & \cdots & 0 \\ 0 & \cdots 0 & 1 \cdots & 0 & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The only rows ~~the~~ of A that are affected are the kth and pth rows.

The entries of the kth row of EA are
$a_{p1}, a_{p2} \dots a_{pn}$ (i.e. a_{pj} $j = 1, 2, \dots n$)

The entries of the pth row of EA are
$a_{k1}, a_{k2}, \dots a_{kn}$ (i.e. a_{kj}, $j = 1, \dots n$)

Comparing with above, we conclude that in this case also $\boxed{e(A) = EA}$