MTH 100 : Lecture 17

Ex: Let $u_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix}$, $u_3 = \begin{bmatrix} 3 \\ 24 \\ 9 \end{bmatrix}$

Show that $\operatorname{Span}\{u_1, u_2, u_3\} = \mathbb{R}^3$.

Let $A = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 24 \\ 1 & 4 & 9 \end{bmatrix}$

Let us row reduce A

$$\begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 24 \\ 1 & 4 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 9 \\ 4 & 12 & 24 \\ 2 & 6 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 4 & 9 \\ 0 & -4 & -12 \\ 0 & -2 & -15 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \left(-\frac{1}{4}R_2\right)} \begin{bmatrix} 1 & 4 & 9 \\ 0 & 1 & 3 \\ 0 & -2 & -15 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 4 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix} \xrightarrow{R_3 \to \left(-\frac{1}{9}R_3\right)} \begin{bmatrix} 1 & 4 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 - 9R_3} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, A is row equivalent to the identity matrix I_3.

Hence $A\bar{x} = \bar{b}$ has a solution for every $b \in \mathbb{R}^3$.
Therefore any vector $\bar{b} \in \mathbb{R}^3$ can be written as a linear combination of the columns of A.
(Viz. u_1, u_2, u_3)

$\Rightarrow \quad b \in \operatorname{span}\{u_1, u_2, u_3\}$

Hence $\boxed{\operatorname{span}\{u_1, u_2, u_3\} = \mathbb{R}^3}$

<u>Note:</u>

$$A\bar{x} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 u_1 + x_2 u_2 + x_3 u_3$$

is a linear combination of u_1, u_2 and u_3 where the scalars $x_1, x_2, x_3 \in \mathbb{R}$

- Linear independence and dependence:

Definition: Let $v_1, v_2, \ldots, v_p$ be a finite list of vectors in a vector space V over a field F. Then the vectors are said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_p \in F$ not all zero such that $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = \bar{0}$

Definition: If a list of vectors is not linearly dependent, they are called linearly independent.

Thus if $\{v_1, v_2, \ldots, v_p\}$ is linearly independent and $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = \bar{0}$ then $c_1 = c_2 = \cdots = c_p = 0$

Ex: Consider the following elements of $\mathbb{R}^{2\times 2}$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Question: Are A, B, C linearly independent ?

Let $c_1 A + c_2 B + c_3 C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ where $c_1, c_2, c_3 \in \mathbb{R}$

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + c_3 & c_1 + c_3 \\ c_1 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} & c_1 + c_2 + c_3 = 0 \\ & c_1 = 0 \quad \Rightarrow \quad c_1 = 0 \\ & c_1 + c_3 = 0 \quad \Rightarrow \quad 0 + c_3 = 0 \Rightarrow c_3 = 0 \\ & c_1 + c_2 = 0 \quad \Rightarrow \quad 0 + c_2 = 0 \Rightarrow c_2 = 0 \end{aligned}\right\}$$

So, $c_1 A + c_2 B + c_3 C = \bar{0}$ (zero matrix)

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

Hence A, B are C are linearly independent.

Ex: Let $V = C[0, 2\pi]$ (This is a vector space over $\mathbb{R}$)

Let $f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \sin(2x)$

Question: Are f_1, f_2 and f_3 linearly independent?

Let $c_1 f_1 + c_2 f_2 + c_3 f_3 = \bar{0}(x)$ (The zero function of $C[0, 2\pi]$)

($\bar{0}(x) = 0 \ \forall\, x \in [0, 2\pi]$)

$$\Rightarrow \quad c_1(1) + c_2 \sin x + c_3 \sin 2x = \bar{0}(x) = 0 \quad \forall\, x \in [0, 2\pi]$$

Let $x = 0$: Then $c_1 + c_2 \sin(0) + c_3 \sin(2 \times 0) = 0$

$$\Rightarrow c_1 + c_2 \times 0 + c_3 \times 0 = 0 \Rightarrow \boxed{c_1 = 0}$$

Let $x = \frac{\pi}{2}$: Then $c_2 \sin\left(\frac{\pi}{2}\right) + c_3 \sin\left(2 \times \frac{\pi}{2}\right) = 0$

$$\Rightarrow c_2 \times 1 + c_3 \times 0 = 0$$

$$\Rightarrow \boxed{c_2 = 0}$$

Let $x = \frac{\pi}{4}$: $c_3 \sin\left(2 \times \frac{\pi}{4}\right) = 0 \Rightarrow c_3 \times 1 = 0$

$$\Rightarrow \boxed{c_3 = 0}$$

Thus $c_1 f_1 + c_2 f_2 + c_3 f_3 = \bar{0}(x) \ \forall\, x \in [0, 2\pi] \Rightarrow c_1 = c_2 = c_3 = 0$

Hence f_1, f_2 and f_3 are linearly independent.

Note: We can use other points in $[0, 2\pi]$ to solve for the scalars c_1, c_2 and c_3.

Remark 1: Any list which contains the zero vector has to be linearly dependent.

Suppose $v_1, v_2, \ldots, v_p$ is a list of vectors such that $v_i = \overline{0}$ for some $1 \le i \le p$

Now $0.v_1 + 0.v_2 + \cdots + 0.v_{i-1} + 1.v_i + 0.v_{i+1} + \cdots + 0.v_p = v_i = \overline{0}$

So, we have the above linear combination of $v_1, v_2, \ldots, v_p$ to be a zero vector where one of the scalar is $1 \neq 0$

Hence $v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_p$ is linearly dependent.

Remark 2: A single non zero vector is linearly independent.

Suppose $v \neq \overline{0}$ and $c.v = \overline{0}$ where the scalar $c \in F$

Then $c = 0$

(Suppose $c \neq 0$. Then $c^{-1} \in F$ and $c^{-1}(cv) = c^{-1}(\overline{0})$

$\Rightarrow (c^{-1}c)v = \overline{0}$

$\Rightarrow 1.v = \overline{0} \Rightarrow v = \overline{0}$, a contradiction)

Remark 3: A list of two non zero vector is linearly dependent only if one of the vectors is a scalar multiple of the other.

Suppose two non zero vectors v_1 and v_2 are linearly dependent. Then there exists scalars $c_1, c_2 \in F$ (atleast one of them is non zero) such that

$$c_1 v_1 + c_2 v_2 = \overline{0}$$

Without any loss of generality (WLOG), we assume that $c_1 \neq 0$

Then $c_1 v_1 + c_2 v_2 = \overline{0} \Rightarrow c_1 v_1 = -c_2 v_2 \Rightarrow c_1^{-1}(c_1 v_1) = c_1^{-1}(-c_2 v_2)$ (Since $c_1 \neq 0$, $c_1^{-1} \in F$)

$\Rightarrow (c_1^{-1} c_1) v_1 = -(c_1^{-1} c_2) v_2 \Rightarrow 1.v_1 = (-c_1^{-1} c_2) v_2$

$\Rightarrow v_1 = (-c_1^{-1} c_2) v_2$. Thus v_1 is a scalar multiple of v_2.