Quiz 3

Sep 13th, 2024

Time: 15 minutes

Name: ______________ Roll no.: ______________ Group: ______________

Instructions: Notes, books, computers, cell phones and other electronic devices are not allowed. **Max marks** $= 5$**.**

Problem 1. 1. Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} 7 & -1 & 0 \\ 14 & 0 & 1 \\ 7 & -3 & 3 \end{bmatrix}$$

2. Using the above factorization, solve the linear system $A\overline{x} = \overline{b}$ where $\overline{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Note: You are not allowed to use any other method to solve the system.

Rubrics for Quiz 3

Total points = 5

1 (a)

$$A = \begin{bmatrix} 7 & -1 & 0 \\ 14 & 0 & 1 \\ 7 & -3 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

+1.5

The row operations are

$e_1 : R_2 \to R_2 - 2R_1$

$e_2 : R_3 \to R_3 - R_1$

$e_3 : R_3 \to R_3 + R_2$

The inverse operations are

$f_1 : R_2 \to R_2 + 2R_1$

$f_2 : R_3 \to R_3 + R_1$

$f_3 : R_3 \to R_3 - R_2$

+.5

So, $\boxed{L = (f_1 f_2 f_3) I}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xleftarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

+1

So, $A = LU$ or

$$\boxed{\begin{bmatrix} 7 & -1 & 0 \\ 14 & 0 & 1 \\ 7 & -3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U}}$$

(b) $b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

First we solve $L\bar{y} = \bar{b}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow$

$y_1 = 0 \Rightarrow \boxed{y_1 = 0}$

$2y_1 + y_2 = 1 \Rightarrow \boxed{y_2 = 1}$

$y_1 - y_2 + y_3 = -1 \Rightarrow y_3 = -1 - y_1 + y_2$

$= -1 + 0 + 1$

$= \boxed{0}$

+1

So, $\bar{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now $U\bar{x} = \bar{y} \Rightarrow \begin{bmatrix} 7 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \left. \begin{aligned} 7x_1 - x_2 &= 0 \\ 2x_2 + x_3 &= 1 \\ 4x_3 &= 0 \end{aligned} \right\} \Rightarrow \boxed{x_3 = 0}$, then $2x_2 = 1 \Rightarrow \boxed{x_2 = \frac{1}{2}}$

and then $7x_1 - \frac{1}{2} = 0 \Rightarrow 7x_1 = \frac{1}{2}$

$\Rightarrow x_1 = \frac{1}{7} \times \frac{1}{2} = \boxed{\frac{1}{14}}$

+1

So, $\boxed{\bar{x} = \begin{bmatrix} \frac{1}{14} \\ \frac{1}{2} \\ 0 \end{bmatrix}}$

Note: There is an alternative way to find L (while looking at the reduction from A to U)

- The first column of L will be the first column of A divided by the pivot

$$\begin{bmatrix} 7/7 \\ 14/7 \\ 7/7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \boxed{\begin{matrix} 7 \\ 14 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -2 \end{matrix}} & \boxed{3} \end{bmatrix}$$

(divide by 7)

The second column will be

$$\begin{bmatrix} 0 \\ 2/2 \\ -2/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

(divide by 2)

The third column is

$$\begin{bmatrix} 0 \\ 0 \\ 4/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(divide by 4)

So,
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

+1.5

So, if they get L in this way they will get full credit for L (i.e. +1.5)