Worksheet 9

1. Given that A and B are two $m \times n$ matri[ces]

Let $A = [a_1, \dots, a_n]$ and $B = [b_1, \dots, b_n]$ in the column form.

Then $A+B = [a_1+b_1, \dots a_n+b_n]$

Now if $v \in \text{Col}(A+B)$ (column space of $A+B$)

then $v \in \text{Span}\{a_1+b_1, \dots, a_n+b_n\}$

$\Rightarrow v = c_1(a_1+b_1) + \dots + c_n(a_n+b_n)$ where $c_i \in F$ for $i = 1, 2, \dots, n$

$\Rightarrow v = (c_1 a_1 + \dots + c_n a_n) + (c_1 b_1 + \dots + c_n b_n)$

$\Rightarrow v = a + b$ where $a = c_1 a_1 + \dots + c_n a_n \in \text{Col}\,A$ and $b = c_1 b_1 + \dots + c_n b_n \in \text{Col}\,B$

$\Rightarrow v \in \text{Col}\,A + \text{Col}\,B$

Thus $\text{Col}(A+B) \subseteq \text{Col}\,A + \text{Col}\,B$

Infact $\text{Col}(A+B)$ is a subspace of $\text{Col}\,A + \text{Col}\,B$ (as both are subspaces)

Hence
$$\begin{aligned}
\text{rank}(A+B) &= \dim(\text{Col}(A+B)) \\
&\leq \dim(\text{Col}\,A + \text{Col}\,B) \\
&= \dim(\text{Col}\,A) + \dim(\text{Col}\,B) - \dim((\text{Col}\,A) \cap (\text{Col}\,B)) \\
&\leq \dim(\text{Col}\,A) + \dim(\text{Col}\,B) \\
&= \text{rank}\,A + \text{rank}\,B
\end{aligned}$$

Therefore ~~Hence~~, $\boxed{\text{rank}(A+B) \leq \text{rank}\, A + \text{rank}\, B}$

Example for equality

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{rank}\, A = 1$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{rank}\, B = 1$$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{rank}(A+B) = 2$$

So, $\boxed{\text{rank}(A+B) = \text{rank}\, A + \text{rank}\, B}$

Example for strict inequality :

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{rank}\, A = 1$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{rank}\, B = 1$$

$$A+B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{rank}(A+B) = 1$$

(since first row and second row of A+B are linearly dependent)

So, $\boxed{\text{rank}(A+B) < \text{rank}\, A + \text{rank}\, B}$

(2) (a) Yes:

If (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in \mathbb{R}^3$,

then $T[(x_1, y_1, z_1) + (x_2, y_2, z_2)]$

$= T[(x_1 + x_2, y_1 + y_2, z_1 + z_2)]$

$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - z_1 - z_2)$ .... (1)

Now, $T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= (x_1 + y_1, x_1 - z_1) + (x_2 + y_2, x_2 - z_2)$

$= (x_1 + y_1 + x_2 + y_2, x_1 - z_1 + x_2 - z_2)$ ..... (2)

From (1) & (2) we get

$T[(x_1, y_1, z_1) + (x_2, y_2, z_2)]$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

Also if $c \in \mathbb{R}$, then

$T[c(x_1, y_1, z_1)] = T[(cx_1, cy_1, cz_1)]$

$= (cx_1 + cy_1, cx_1 - cz_1)$

$= (c(x_1 + y_1), c(x_1 - z_1)) = c(x_1 + y_1, x_1 - z_1)$

$= c\,T[(x_1, y_1, z_1)]$

Therefore T is a linear transformation.

(b) No.

Let us take $(x, y, z) = (1, 1, 1)$
and $c = 2$

Then $T[c(x,y,z)] = T[2(1,1,1)]$
$= T[(2,2,2)] = (2+2, 2^2)$
$= (4, 4)$

But $cT[(x,y,z)] = 2\,T[(1,1,1)]$
$= 2(1+1, 1^2) = 2(2,1)$
$= (4,2)$

Therefore $T[2(1,1,1)] \neq 2\,T[(1,1,1)]$

So, T is not a linear transformation.

(c) Yes:

If $A, B \in \mathbb{R}^{n\times n}$ then
$U(A+B) = (A+B)^T = A^T + B^T = U(A) + U(B)$

Also for any $c \in \mathbb{R}$,
$U(cA) = (cA)^T = cA^T = c\,U(A)$

Therefore U is a linear transformation.

(d) Yes:

If $p(t), q(t) \in R[t]$ then,

$$M[p(t)+q(t)] = t[p(t)+q(t)]$$
$$= tp(t) + tq(t) = M[p(t)] + M[q(t)]$$

Also if $c \in \mathbb{R}$,

$$M[cp(t)] = t[cp(t)] = c[tp(t)]$$
$$= cM[p(t)].$$

Therefore T is a linear transformation.

(3) Let $T : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ be any linear transformation.

Let $T(1) = c$ (the image of 1 under T)

Then $c \in \mathbb{R}^1$

and for any $x \in \mathbb{R}^1$,

$$T(x) = T(1.x) = xT(1) = x.c = cx.$$

Thus each real number $c \in \mathbb{R}^1$, determines a linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^1$ and these are the only linear transformations from $\mathbb{R}^1$ to itself.

4. $D_\epsilon : C[R] \longrightarrow C[R]$ is defined as:

$D_\epsilon(f) = f_\epsilon$ where $f_\epsilon(x) = f(x+\epsilon)$ for all $x \in \mathbb{R}$

For $f, g \in C[R]$,

$$D_\epsilon(f+g) = (f+g)_\epsilon = f_\epsilon + g_\epsilon = D_\epsilon(f) + D_\epsilon(g) \quad \ldots (1)$$

(Since $(f+g)_\epsilon(x) = (f+g)(x+\epsilon) = f(x+\epsilon) + g(x+\epsilon)$
$= f_\epsilon(x) + g_\epsilon(x)$

we have $(f+g)_\epsilon = f_\epsilon + g_\epsilon$ for all $x \in \mathbb{R}$)

For $c \in \mathbb{R}$ and $f \in C[R]$,

$$D_\epsilon(cf) = (cf)_\epsilon = c f_\epsilon = c D_\epsilon(f) \quad \ldots (2)$$

(Since $(cf)_\epsilon(x) = (cf)(x+\epsilon) = c f(x+\epsilon) = c f_\epsilon(x)$ for all $x \in \mathbb{R}$

we have $(cf)_\epsilon = c . f_\epsilon$)

From (1) and (2) we can conclude that D_ϵ is a linear transformation.

5

Suppose there exist a linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ such that

$$\operatorname{Ker} T = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : \begin{array}{l} x_1 = 3x_2 \\ x_3 = x_4 = x_5 \end{array} \right\}$$

then elements of $\operatorname{Ker} T$ satisfies the system of homogeneous equation:

$$\left.\begin{array}{r} x_1 - 3x_2 = 0 \\ x_3 - x_4 = 0 \\ x_3 - x_5 = 0 \\ x_4 - x_5 = 0 \end{array}\right\}$$

The Coefficient matrix of the system

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 + R_3 \\ R_4 \to R_4 - R_3 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R_A$$

(RREF matrix)

Note that R_A has three basic variables columns

So, $\dim(\text{Col } A) = \text{rank } A = 3$

and $\text{nullity}(A) = 5 - 3 = 2$

Hence $\text{Nullity}(T) = 2$, So, $\text{Rank}(T) = 5 - 2 = \boxed{3}$

However $\text{rank}(T) = \dim(\text{Range } T)$

$\leq \boxed{2}$ (since $\text{Range } T \subseteq \mathbb{R}^2$)

a contradiction.

Hence such a linear transformation is not possible.

(6) If $z = a + ib$ and $\omega = c + id \in \mathbb{C}$,

then $\phi(z+\omega) = \overline{(z+\omega)} = \overline{\{(a+ib) + (c+id)\}}$

$= \overline{\{(a+c) + i(b+d)\}} = (a+c) - i(b+d)$

$= (a-ib) + (c-id) = \bar{z} + \bar{\omega} = \phi(z) + \phi(\omega)$

Now if $p \in \mathbb{R}$

$\phi(pz) = \phi(p(a+ib)) = \overline{p(a+ib)} = \overline{(pa+ipb)}$

$= pa - ipb = p(a-ib) = p\bar{z}$

$= p\,\phi(z)$

Hence ϕ is a linear transformation from $\mathbb{C}$ to $\mathbb{C}$.

If $z = a+ib$, $\omega = c+id$, then

$$\phi(z\omega) = \phi\big((a+ib)(c+id)\big)$$

$$= \overline{z\omega} = \overline{(a+ib)(c+id)}$$

$$= \overline{(ac-bd)+i(ad+bc)}$$

$$= ac-bd - i(ad+bc)$$

$$= (a-ib)c - id(a-ib)$$

$$= (a-ib)(c-id) = \bar{z}\,\bar{\omega} = \phi(z)\phi(\omega)$$

So, ϕ is a multiplicative function.

Now ψ is another multiplicative linear transformation on $\mathbb{C}$ to $\mathbb{C}$ which is not a zero transformation.

Then there exist some complex number $z \neq 0$ such that $\psi(z) \neq 0$.

But then $\psi(z) = \psi(1.z) = \psi(1)\psi(z)$

$\Rightarrow$ dividing both sides by $\psi(z)$ $(\neq 0)$, we get $\boxed{\psi(1) = 1}$

Then $\psi(-1) = \psi(-1.1) = (-1)\psi(1) = (-1).1 = -1$

Now $-1 = \psi(-1) = \psi(i^2) = \psi(i.i) = \psi(i)\psi(i)$

$\Rightarrow \{\psi(i)\}^2 = -1 \Rightarrow$ either $\psi(i) = i$

or $\psi(i) = -i$

If $\psi(i)=i$, then for any $z=a+ib$,

$$\psi(z)=\psi(a+ib) = \psi(a)+\psi(ib)$$
$$= a\psi(1)+\psi(i)\psi(b) = a\psi(1)+\psi(i)b\psi(1)$$
$$= a\times 1 + i\times b\times 1$$
$$= a+ib = z$$

So, ψ is the identity transformation

If $\psi(i)=-i$, then for any $z=a+ib$,

$$\psi(z)=\psi(a+ib)= \psi(a)+\psi(ib) = \psi(a\cdot 1)+\psi(i)\psi(1b)$$
$$= a\psi(1)+\psi(i)\,b\psi(1)$$
$$= a\times 1 - i\,b\times 1 = a-ib = \bar{z}$$
$$= \phi(z)$$

So, if ψ is not zero or identity transformation, then $\psi=\phi$

7 By a proposition in the class we know that given a basis $\{v_1,\dots,v_n\}$ for V and a list $w_1,\dots,w_n$ (not necessarily distinct) of vectors in W there exists a unique linear transformation $T: V \to W$ such that $T(v_i)=w_i$ for $i=1,2,\dots,n$.

Here let us take $V = \mathbb{R}^2$, $W = \mathbb{R}^3$

We take $\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ as our basis for $\mathbb{R}^2$

Consider:

$i = 1$ Define $T_1 e_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $T_1 e_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Thus for any arbitrary vector $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $v = x e_1 + y e_2$

$$T_1 v = T_1 (x e_1 + y e_2) = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} x + 2y \\ x + 2y \\ x + 2y \end{bmatrix}$$

Rank $T_1 = 1$ (Range T_1 is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

Consider $i = 2$

Define $T_2 e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $T_2 e_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Then $T_2 v = T_2 (x e_1 + y e_2) = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} x + y \\ y \\ 0 \end{bmatrix}$$

Rank $T_2 = 2$

Consider $i = 3$ Such a transformation T_3 is not possible. By the rank Theorem of linear transformation, any $T : V \to W$ satisfies $\text{rank}(T) + \text{nullity}(T) = \dim V$

$\Rightarrow \text{rank } T \leq \dim V$

But here $\dim V = \dim \mathbb{R}^2 = 2$. So, rank $T_i = 3$ is not possible.

8) Given $T: V \longrightarrow V$ is a linear transformation (linear operator)

$\dim V = n.$

Let $\operatorname{rank}(T) = \boxed{m}$

Then ~~[illegible]~~ $\dim(\text{Range}(T)) = m$

Since it is given that $\text{Range}(T) = \ker T$,

$\dim(\ker T) = m$

$\Rightarrow \boxed{\text{nullity}(T) = m}$

Now by the Rank Theorem

$\text{Rank}(T) + \text{nullity}(T) = \dim V$

$\Rightarrow m + m = n$

$\Rightarrow n = 2m$

$\Rightarrow \boxed{n \text{ is even.}}$

Example: Let $V = \mathbb{R}^2$

~~[illegible]~~ T should be such that Range $T = \ker T$

~~[illegible]~~

$\{e_1, e_2\}$ forms a basis of $V = \mathbb{R}^2$

Define $\boxed{Te_1 = 0 \text{ and } Te_2 = e_1}$,

Now any $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ can be written as $v = xe_1 + ye_2$

$Tv = T(xe_1 + ye_2) = xT(e_1) + yT(e_2) = 0 + ye_1$

$= \boxed{ye_1}$

So, $\boxed{\begin{array}{l} \text{Range } T = \text{span}\{e_1\} = \{(x, 0) : x \in \mathbb{R}\} \\ \ker T = \{(x, 0) : x \in \mathbb{R}\} \quad \text{i.e. Range } T = \ker T \end{array}}$

$\left(Tv = 0 \Leftrightarrow y = 0 \atop v = \begin{bmatrix} x \\ y \end{bmatrix}\right)$