MTH100: Lecture 15

Subspace: Let V be a vector space over a field F. A (vector) subspace of V is a nonempty subset of V which is also a vector space over F with respect to the same operations of vector addition and scalar multiplication taken from V.

Test for Subspaces

Proposition: Let V be a vector space over a field F.

A subset W of V is a subspace if and only if it satisfies the following three properties:

(1) The zero vector $\bar{0}$ is in W

(2) W is closed under addition

i.e. $u+v \in W \quad \forall\, u, v \in W$

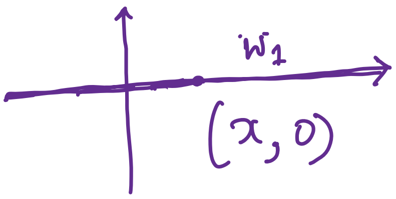
(3) W is closed under scalar multiplication

i.e. $cu \in W \quad \forall\, c \in F$ and $\forall\, u \in W$

Note: (1) can be replaced by (1')

(1'): W is nonempty.

Ex(1) $\mathbb{R}^2$ is a vector space over $\mathbb{R}$



$W_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$ is a subspace.

Check:

(1) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W_1$

(2) If $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \in W_1$ then

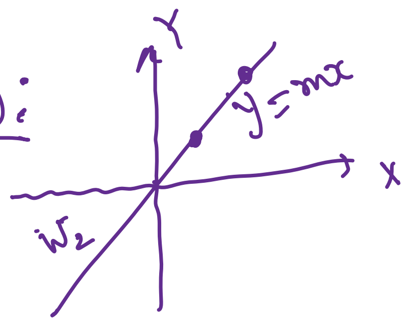
$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \in W_1$$

(3) If $\begin{pmatrix} x \\ 0 \end{pmatrix} \in W_1$ and $c \in \mathbb{R}$,

then $c \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ 0 \end{pmatrix} \in W_1$

Hence, W_1 is a subspace of $\mathbb{R}^2$

Ex(2):



$W_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = mx, \; x, y \in \mathbb{R} \right\}$

Check

(1) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W_2$ (since $0 = m.0$)

(2) If $\begin{pmatrix} x_1 \\ mx_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ mx_2 \end{pmatrix} \in W_2$

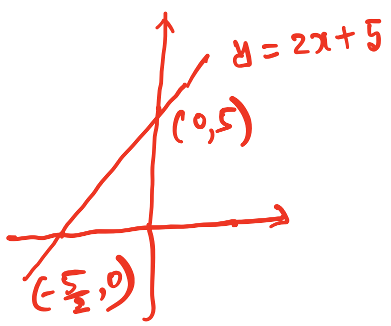
then $\begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ mx_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ mx_1 + mx_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ m(x_1 + x_2) \end{pmatrix} \in W_2$

(3) If $\begin{pmatrix} x \\ mx \end{pmatrix} \in W_2$ and $c \in \mathbb{R}$

then $c\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} cx \\ c(mx) \end{pmatrix} = \begin{pmatrix} cx \\ m(cx) \end{pmatrix} \in W_2$

Hence W_2 is a subspace of $\mathbb{R}^2$

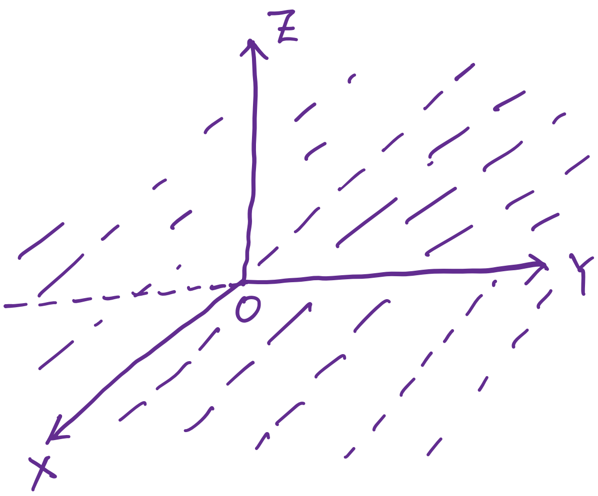
Ex ③: Is $W_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 2x + 5 ,\ x, y \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$



$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin W_3$ So, W_3 is not a subspace of $\mathbb{R}^2$.

Ex ④: Let $W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathbb{R}^3$

Show that W is a subspace of $\mathbb{R}^3$



(1) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$

(2) If $\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \in W$

then $\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{pmatrix} \in W$

(3) If $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in W$ and $c \in \mathbb{R}$

then $c \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ 0 \end{pmatrix} \in W$

So, W is a subspace of $\mathbb{R}^3$

Ex 5: Consider the set W of all solutions of the system of equations $A\bar{x}=\bar{0}$ where A is a $m \times n$ matrix.

- W is a subset of $\mathbb{R}^n$
- W is also a subspace of $\mathbb{R}^n$

(1) $\bar{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in W$ because $A\bar{0}=\bar{0}$

(2) If $\bar{x}$ and $\bar{y} \in W$ then $A\bar{x}=\bar{0}$, $A\bar{y}=\bar{0}$

So, $A(\bar{x}+\bar{y}) = A\bar{x} + A\bar{y} = \bar{0} + \bar{0} = \bar{0}$

So, $\bar{x}+\bar{y} \in W$

(3) If $\bar{x} \in W$ and $c \in \mathbb{R}$

then $A\bar{x} = \bar{0}$

Now $A(c\bar{x}) = c(A\bar{x}) = c.\bar{0} = \bar{0}$

So, $c\bar{x} \in W$

So, W is a subspace of $\mathbb{R}^n$.

Ex ⑥: Let W be the set of all solutions of the system $A\bar{x} = \bar{b}$ where A is a $m \times n$ matrix and $\bar{b}_{m\times 1} \neq \bar{0}$

Is this a subspace of $\mathbb{R}^n$?

$A\bar{0} = \bar{0} \neq \bar{b}$ (given)

So, $\bar{0} \notin W$

Hence W is not a subspace of $\mathbb{R}^n$.

Ex ⑦: Let $\mathbb{R}^{n\times n}$ be the set of all $n\times n$ matrices over $\mathbb{R}$.

Then $\mathbb{R}^{n\times n}$ is a vector space over $\mathbb{R}$ with respect to matrix addition and scalar multiplication.

Let W be the set of all $n\times n$ symmetric matrices over $\mathbb{R}$

i.e. $W = \{A \in \mathbb{R}^{n\times n} : A^t = A\}$ (A^t = Transpose of A)

- W is a subspace of $\mathbb{R}^{n\times n}$

<u>Check:</u> ① $\mathbf{0}_{n\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}_{n\times n} \in W$ (since $\mathbf{0}^t = \mathbf{0}$)

② Let $A, B \in W$; Then $A^t = A$ and $B^t = B$

$\Rightarrow (A+B)^t = A^t + B^t = A + B$

$\Rightarrow A + B \in W$

③ Let $A \in W$ and $c \in \mathbb{R}$; Then $A^t = A$

Now $(cA)^t = cA^t = cA$

So, $cA \in W$

Thus W is a subspace of $\mathbb{R}^{n\times n}$.

<u>Ex:</u> $\mathbb{R}^{\infty}$ (The set of all sequences with real entries) is a vector space over $\mathbb{R}$.

Let C be the set of all <u>convergent</u> sequences with real entries.

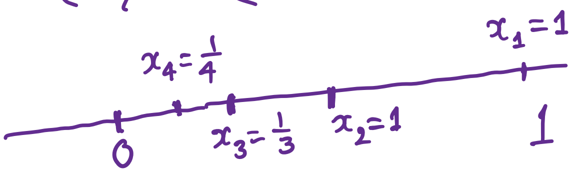
<u>Note:</u> Let (x_n) be a sequence of real numbers $(x_1, x_2, \ldots)$.

If there exists $l \in \mathbb{R}$ such that

$\lim_{n\to\infty} x_n = l$, then (x_n) is called convergent and we say x_n converges to l.

Ex:

Let $(x_n) = \left(\frac{1}{n}\right)$



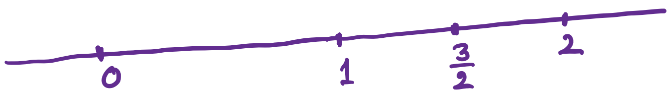
$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n}$

$= 0$

x_n converges to 0

Ex: Let $(x_n) = \left(1+\frac{1}{n}\right)$

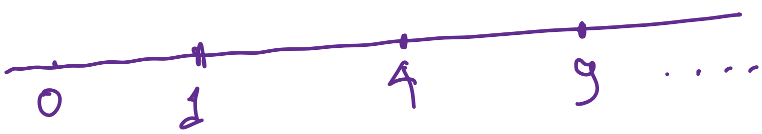
$\lim_{n\to\infty} x_n = 1.$



x_n converges to 1.

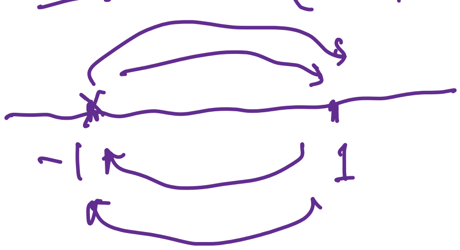
Ex: $(x_n) = (n^2)$

Not Convergent



Ex: $(x_n) = (-1)^n$

Not Convergent



$x_1 = -1$

$x_2 = 1$

$x_3 = -1$

$x_4 = 1$

(In fact it is oscillating)

Ex: We know $C \subset \mathbb{R}^\infty$. C is also a subspace of $\mathbb{R}^\infty$.

check:

(1) $(0, 0, 0, \ldots) \in C$

(2) $(x_n), (y_n) \in C$

Thus (x_n) is convergent, (y_n) is convergent.

Then $(x_n + y_n)$ is also convergent.

(If $x_n \to l$ and $y_n \to m$, then $x_n + y_n \to l + m$)

(3) Let $d \in \mathbb{R}$, $(x_n) \in C$

So, (x_n) is convergent. Then (dx_n) is also convergent.

(If $x_n \to l$, then $dx_n \to dl$)

Hence C is a subspace of $\mathbb{R}^\infty$.

Ex: $R_0(t) \subset R_1(t) \subset R_2(t) \subset \quad \subset R_n(t) \subset \quad \subset R(t)$

Show that $R_n(t)$ is a subspace of $R(t)$.

Ex: Find some subspaces of $C[a,b]$

$C^1[a,b]$?? $C^1[a,b] = \{f:[a,b] \to \mathbb{R} : f$ is differentiable and f' is continuous$\}$

$C^\infty[a,b]$?? Similarly $C^2[a,b], \ldots, C^\infty[a,b]$ can be defined.

Another Test for Subspaces

Proposition 9: A non-empty subset W of V (V is a vector space over a field F) is a subspace if and only if for each u and v in W and each scalar c in F,

$$cu + v \in W$$

(i.e. $cu + v \in W \ \forall\, u, v \in W$ and $\forall c \in F$)

Exercise: Show that the two tests are equivalent.