MTH 100 : Lecture 28

Change of Basis

- We would like to know what happens to the matrix of a linear transformation if the basis gets change.
- We will only consider the case when T is a linear operator from a finite dimensional vector space V to V

Preliminary Result:

Proposition: Let $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be two ordered bases of a vector space V. Then there is an invertible $n \times n$ matrix P such that

$$[x]_C = P\,[x]_B \quad \text{for any } x \in V.$$

Proof: Will be given as a note. (rather technical)

Note: The columns of P are the C-coordinate vectors of the basis B.

The matrix P is called the change of coordinate matrix from B to C and is denoted by $P_{B \to C}$

Remark: To change coordinates between two bases, we need the coordinate vectors of the old basis B relative to the new basis C. These become the columns of the change of matrix P.

- In practice $P = Q^{-1}$ where Q has its columns the coordinate vectors of the new basis C relative to the old basis B.

In most of the applications, the old basis is the standard basis for $\mathbb{R}^n$ and so Q can be found directly.

- Recall the first example of last lecture.

Ex (First part):

$$\text{Let } V = \mathbb{R}^2$$

Let old (ordered) basis $\alpha = \{e_1, e_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

(standard basis in $\mathbb{R}^2$)

and Let

new (ordered) basis $\beta = \{u_1, u_2\} = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}\right\}$

(Should be clear that this is a basis)

Step 1: Construct the matrix Q

$$= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

New basis in terms of old basis

(Since $u_1 = 2e_1 + 1e_2$ and $u_2 = 5e_1 + 3e_2$ we have $[u_1]_\alpha = \begin{bmatrix}2\\1\end{bmatrix}_\alpha$ and $[u_2]_\alpha = \begin{bmatrix}5\\3\end{bmatrix}_\alpha$)

Step 2 Change of Basis Matrix $= P = \boxed{P_{\alpha\to\beta} = Q^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}}$

check: Let us determine $[v]_\beta$ for a specific vector v, say $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha$

Then $[v]_\beta = P[v]_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta$

Note that $\begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta = -7u_1 + 3u_2 = -7\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = v$

Verification that columns of P are the coordinate vectors of the old basis in terms of the new basis.

First column of $P = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta = 3u_1 + (-1)u_2 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$

i.e. $[e_1]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta$

Similarly,

Second column of $P = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_\beta = 5u_1 + (-2)u_2 = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$

i.e. $[e_2]_\beta = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_\beta$

Similarity of Matrices

- A $n \times n$ matrix B is called similar to an $n \times n$ matrix A if there exists an invertible matrix P such that $\boxed{B = PAP^{-1}}$.

Proposition: Similarity of matrices is an equivalence relation on $F^{n\times n}$
($F^{n\times n}$ is the set of $n\times n$ matrices with entries taken from a field F)

Remarks (1): If A is similar to B, then B is similar to A.

(So, we will say A and B are similar matrices)

(2) If A and B are similar matrices then $\boxed{\det(A) = \det(B)}$

Effect of change of Basis

Proposition: Suppose A and B are the matrices of the linear operator T relative to the ordered basis α and β respectively.

Then A and B are similar matrices.

Infact, $\boxed{B = PAP^{-1}}$, where $P = P_{\alpha\to\beta}$ is the change of Basis matrix.

$$\boxed{\text{i.e. } \underbrace{[T]_\beta}_{B} = P \underbrace{[T]_\alpha}_{A} P^{-1}}$$

<u>Proof:</u> If P is the change of basis matrix from α to β, then P^{-1} is the change of basis matrix from β to α.

Let $[T]_\alpha = A$ and $[T]_\beta = B$

Then for any $v \in V$, $(PAP^{-1})[v]_\beta = (PA)(P^{-1}[v]_\beta)$

$$= (PA)[v]_\alpha = P(A[v]_\alpha)$$

$$= P([T]_\alpha [v]_\alpha) = P[Tv]_\alpha$$

$$= [Tv]_\beta = [T]_\beta [v]_\beta = B[v]_\beta$$

Since the above holds for all vectors $v \in V$, it follows that

$$\boxed{PAP^{-1} = B = [T]_\beta}$$

Ex: (2nd part): Let $V = \mathbb{R}^2$

Let $\alpha = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be the standard basis of $\mathbb{R}^2$

Let $\beta = \{x_1, x_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$ be another basis of $\mathbb{R}^2$

We have seen that $Q = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and the change of basis matrix

$$P = P_{\alpha \to \beta} = Q^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Define a linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix};$$

Since $Te_1 = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1e_1 + 3e_2$

and $Te_2 = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2e_1 + 4e_2$,

We have $[T]_\alpha = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$ (say)

By the proposition,

$$B = [T]_\beta = PAP^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}$$

Verification with a specific vector:

$$\text{Let } v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta$$

Now, $$[Tv]_\alpha = [T]_\alpha [v]_\alpha = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha$$

and $$[Tv]_\beta = [T]_\beta [v]_\beta = \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta$$

Note that

$$\begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta = -40 u_1 + 17 u_2$$
$$= -40 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 17 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha$$

So, we get the same vector but expressed in two different coordinate system.