# MTH 100 : Lecture 12

Last time we have defined field and Vector space over a field

## Examples of Vector spaces

**Ex ①:** The space

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for } i = 1, 2, \dots n \right\}$$

(For any $n \geq 1$

The base field is $\mathbb{R}$

**Addition:**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

**Scalar Multiplication:**

For any $c \in \mathbb{R}$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx \\ \vdots \\ cx_n \end{bmatrix}$$

Want to show that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

## Closure Property:

Let $u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $v = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$

and Let $c \in \mathbb{R}$

$$u + v = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

$$c\,u = \begin{bmatrix} c x_1 \\ \vdots \\ c x_n \end{bmatrix} \in \mathbb{R}^n$$

- Let $w = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^n$

$$(u + v) + w = \left( \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} (x_1 + y_1) + z_1 \\ \vdots \\ (x_n + y_n) + z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + (y_1 + z_1) \\ \vdots \\ x_n + (y_n + z_n) \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ \vdots \\ y_n + z_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \left( \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \right)$$

$$= u + (v + w)$$

- $\bar{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$ and

$$\bar{0} + u = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 + x_1 \\ \vdots \\ 0 + x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = u$$

for every $u \in \mathbb{R}^n$

Similarly $u + \bar{0} = u$ for every $u \in \mathbb{R}^n$

- For every $u = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$\begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix} \in \mathbb{R}^n$$

$$= (-u \text{ (say)})$$

such that $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} x_1 + (-x_1) \\ \vdots \\ x_n + (-x_n) \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $u + v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$= \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ \vdots \\ y_n + x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= v + u$$

**To show:** $c(u+v) = cu + cv$

$$c\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = c\begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$= \begin{bmatrix} c(x_1+y_1) \\ \vdots \\ c(x_n+y_n) \end{bmatrix} = \begin{bmatrix} cx_1+cy_1 \\ \vdots \\ cx_n+cy_n \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} cy_1 \\ \vdots \\ cy_n \end{bmatrix}$$

$$= c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + c\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= cu + cv$$

- **To show that $(c+d)u = cu + du$:**

$$(c+d)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (c+d)x_1 \\ \vdots \\ (c+d)x_n \end{bmatrix}$$

$$= \begin{bmatrix} cx_1+dx_1 \\ \vdots \\ cx_n+dx_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix} + \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + d\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = cu + du$$

- To show that $c(du) = (cd)u$

$$c(du) = c\left(d\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = c\begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= \begin{bmatrix} c(dx_1) \\ \vdots \\ c(dx_n) \end{bmatrix} = \begin{bmatrix} (cd)x_1 \\ \vdots \\ (cd)x_n \end{bmatrix} = (cd)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (cd)u$$

To show:

- $1.u = u$ where $1 \in \mathbb{R}$

$$1.\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1.x_1 \\ \vdots \\ 1.x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = u$$

We can now conclude that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

Ex: Is $\mathbb{R}^n$ a vector space over the base field $\mathbb{Q}$ ?

Ex: Is $\mathbb{R}^n$ a vector space over the base field $\mathbb{C}$ ?

- $\mathbb{R}^n$ is frequently referred to as Euclidean space.

Ex ②: The space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices with real entries.

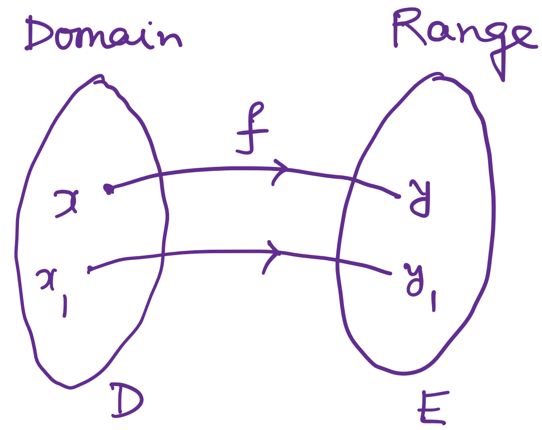
This is a vector space over $\mathbb{R}$. (Show that!)

Note: This vector space is useful in image processing:

$$\downarrow$$

# Function :

A function is a correspondence between two sets called Domain and Range such that for every element in the domain, there is a corresponding element in the range.




$y = f(x)$
($y$ is called the image of $x$ under $f$)

- More than one element can correspond to the same element in the range but one element can not correspond to more than one element in the range.

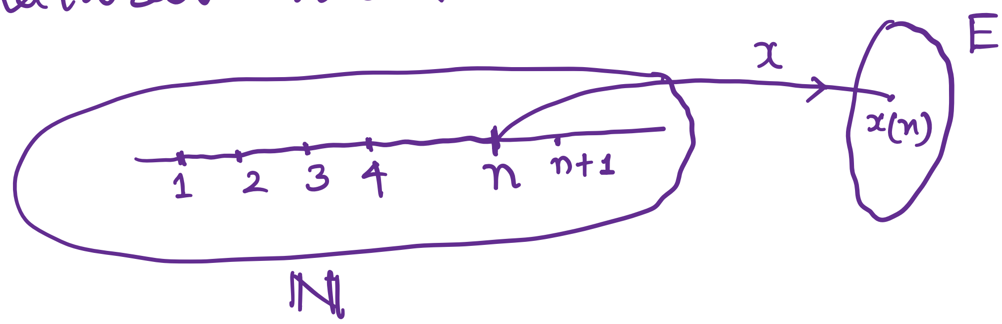
- If the range is a subset of $\mathbb{R}$ (or $\mathbb{C}$) we call it real valued (complex valued) function.

  In that case if the domain D is also a subset of $\mathbb{R}$ (or $\mathbb{C}$), we can talk about continuity and differentiability of the functions.

• <u>Note:</u> Two functions $f$ and $g$ are equal if the values (images) of $f$ and $g$ are equal in every point of the domain.

<u>Definition:</u> If the domain of a function is $\mathbb{N}$ (the set of natural numbers) we call it a sequence

Thus a sequence is a function of natural numbers.




If $x$ is a sequence, the image of $n$ under $x$ is often denoted by

$$x(n) = x_n$$

Then we can denote a sequence by $\{x_1, x_2, x_3, \ldots\}$ or by $\{x_n\}$

Thus we can count the terms of a sequence one by one (it is countably infinite)

- If we have a sequence $\{x_n\}$ we can see if $\lim_{n\to\infty} x_n$ exists.

If $\lim_{n\to\infty} x_n$ exists, it is called a convergent sequence.

<u>Ex:</u> ① Let $\{x_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

So, the sequence $\{x_n\}$ <u>converges to 0</u>

② Let $\{x_n\} = \{n+1\} = \{2, 3, 4 \ldots\}$

$\{x_n\}$ doesn't converge

(3) Let $\{x_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$

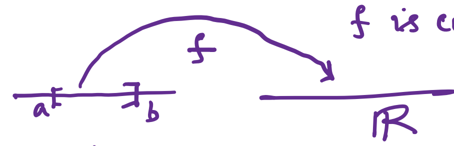
$\{x_n\}$ doesn't converge.

<u>Ex ③:</u> Let $[a, b]$ be a fixed closed interval on $\mathbb{R}$.


Let $C[a, b]$ be the set of all continuous functions from $[a, b]$ to $\mathbb{R}$.

This is a vector space over $\mathbb{R}$.

## Vector addition :




$f$ is continuous

If $f$ and $g \in C[a,b]$

We define

$f+g$ by $(f+g)(x) = f(x)+g(x)$

for every $x \in [a,b]$

## Scalar multiplication

If $c \in \mathbb{R}$, $f \in C[a,b]$

We define

$cf$ by $(cf)(x) = c.f(x)$

for every $x \in [a,b]$

**Show that $C[a,b]$ is a vector space over $\mathbb{R}$.**

**Note:** Often we take $[a,b]$ as $[0,1]$ or $[0,2\pi]$

This vector space is important in Signal and System.

Ex ④: The space $\mathbb{R}^\infty$ of real sequences is a vector space over $\mathbb{R}$.

$$\mathbb{R}^\infty = \{ \{a_n\} : \{a_n\} \text{ is a sequence of real numbers} \}$$

Addition: $\{a_n\} + \{b_n\} = \{a_n + b_n\}$

scalar multiplication: $c\{a_n\} = \{c\,a_n\}$

- Note: This is useful in discrete or digital signals.

- Of more interest is

$$\underbrace{C}_{\text{Set of convergent sequences}} \subset \mathbb{R}^\infty$$

(It is a subset of $\mathbb{R}^\infty$)

- $C$ is also a vector space over $\mathbb{R}$