MTH100 : Lecture 4

Example on Proposition 3:

Consider the system of equations $AX = 0$

where $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 5 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(RREF matrix)

So, the corresponding equivalent system of equation is:

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\}$$

So, the system has a unique solution

$X = \bar{0}$ i.e. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Non-homogeneous System:

For a Non-homogeneous system $Ax=b$, we work with the augmented matrix $[A:b]_{m\times(n+1)}$ and reduce it to an RREF matrix, say R.

Proposition 4 (Existence and Nature of solutions):

The system $Ax=\bar{b}$ is consistent if and only if the rightmost column of R is not a pivot column.

i.e. there is no row of the form $[0,0,\dots,0,p]$ with p nonzero.

If the system is consistent, then it has either (1) a Unique solution if there are no free variables

or (2) infinitely many solutions when there is atleast one free variable.

Proof: Exercise.

Examples of Non-homogeneous Systems

① Consider $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \quad \text{and} \quad \bar{b} = \begin{bmatrix} 7 \\ 9 \\ 30 \end{bmatrix}$$

The Augmented matrix

$$[A : b] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 2 & -1 & 3 & 9 \\ 4 & 1 & 8 & 30 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow[R_3 \to (-1)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\Rightarrow$ The corresponding equivalent system of equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

So, the solution of the system is :

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}\right\} \quad \text{i.e.} \quad X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- So, we have a unique solution (There are no free variables)

<u>Check</u>: $AX = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 30 \end{bmatrix} = \bar{b}$ (as expected)

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(2) $A\mathbf{x} = \bar{b}$ where $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix}$

and $\bar{b} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix}$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 3 & 8 & 16 & 11 \\ 8 & 20 & 40 & 28 \end{array}\right] \xrightarrow[R_3 \to R_3 - 8R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 4 & 8 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = R \text{ (RREF matrix)}$$

(Pivot columns)

So, x_3 is a free variable.

The corresponding system is: $\left. \begin{array}{l} x_1 = 1 \\ x_2 + 2x_3 = 1 \end{array} \right\}$

$$\Rightarrow \quad \left. \begin{array}{l} x_1 = 1 + 0 \cdot x_3 \\ x_2 = 1 - 2x_3 \\ x_3 = 0 + x_3 \end{array} \right\}$$

$\to$ (dummy equation)

So, $X = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}_{\bar{w}}$ is a solution of $AX = \bar{b}$

<u>Check</u>: $A\bar{u} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix} = \bar{b}$

$$A\bar{w} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \bar{0}$$

Thus the solution set is $S = \{ \bar{u} + t\bar{w} : t \in \mathbb{R} \}$ where $\bar{u}$ is a solution of the non-homogeneous system and $\bar{w}$ is a solution of the associated homogeneous system $AX = \bar{0}$

- Since x_3 (or t) acts as a parameter, we get <u>infinitely many solutions.</u>

③ Consider $A x = \bar{b}$ when

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} 4 \\ 11 \\ 28 \end{bmatrix}$$

The Augmented matrix

$$[A : b] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 3 & 8 & 16 & 11 \\ 8 & 20 & 40 & 28 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 8R_1}]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 2 & 4 & -1 \\ 0 & 4 & 8 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow[R_3 \to -\frac{1}{2} R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_2 + \frac{1}{2} R_3 \\ R_1 \to R_1 - 5R_3}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So, the last row is of the form
$[0, 0, \dots, 0, p]$ when $p \neq 0$.

Thus the system is inconsistent and there is no solution.
(By proposition 4).

- The corresponding system becomes

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 1 \longrightarrow \text{not true} \end{aligned}\right\}$$

So, the system is inconsistent.

Vector Interpretation of Solutions:

Let $Ax = b$ be a non-homogeneous system and let $Ax = 0$ be its associated homogeneous system.
Assume that the non-homogeneous system is consistent so that it has atleast one solution u. By necessity $u \neq 0$.
Now the relationship between solutions

of the two systems is given in the following observation (RS):

- If a vector u is a given solution of $Ax=b$, then another vector is a solution of $Ax=b$ if and only if it is of the form $u+v$ where v is a solution of the associated homogeneous system.

- In case $Ax=0$ has only trivial solution (i.e. $v=0$), then there is a unique solution u.

(otherwise we have infinitely many solutions.)

Proof: (H.W.)