MTH 100 : Lecture 3

Application to Determinants: (Will be taken up later)

We note the following :

- If A is an $n \times n$ matrix and B is an echelon form $n \times n$ matrix obtained from A by Gaussian reduction without applying any scaling operation,

then $\det(A) = (-1)^k \det B = (-1)^k b_{11} b_{22} \cdots b_{nn}$

where k is the number of interchange operation.

- This is the preferred algorithm to calculate the determinant. That is why in software for matrix calculations, the two phases of the Row Reduction algorithm are carried out separately and we can obtain the determinant on the way.

Back to System of Linear Equations:

- Consider a system of Linear Equation in matrix form : $Ax = b$
- If $b = \bar{0}$, the system is called Homogeneous.
  A homogeneous system always has the trivial solution consisting of all zeros.
- If $b \neq \bar{0}$, the system is called non-homogeneous. A nonhomogeneous system may or may not have any solution.
- A system which has atleast one solution is called consistent.
  Otherwise it is said to be inconsistent.

- Now for System of Linear Equations, we will directly work with matrices.
- For Homogeneous System, we will work with coefficient matrix A.
- For Non-homogeneous System we will work with the Augmented matrix $[A : b]$.
  It is obtained by putting a column

corresponding to b as an additional column (the $(n+1)$st column).

Observation :

If we obtain a row equivalent matrix to either the coefficient matrix (in the case of homogeneous system) or the augmented matrix (in the non-homogeneous case), then the solution sets of the two corresponding systems are same. In this case we say the systems are equivalent.

Homogeneous System :

Suppose that we have row-reduced the coefficient matrix A to an RREF matrix R.

- The leading variables in each nonzero row of R correspond to pivot columns. These are called Basic Variables. Remaining variables (if any) are called Free Variables.

- If we write the matrix equation $Rx = \bar{0}$ as a system of Linear equations, we can obtain the general solution of the system.
(The system $Rx = \bar{0}$ is equivalent to original system $Ax = \bar{0}$)
<u>The general solution is best expressed in Vector terms.</u>

(To the next page)

Example: Homogeneous System:

①
$$\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 6x_2 - 4x_3 &= 0 \end{aligned}\right\}$$

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -2 \\ 3 & 6 & -4 \end{bmatrix}$

Let us row reduce A

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -2 \\ 3 & 6 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} = R \text{ (say)}$$

(RREF matrix)

Pivot columns

The basic variables are x_1 and x_3

The free variable is x_2

So, the system becomes $R\,x=\bar{0}$ i.e. $R\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow \quad \left.\begin{aligned} x_1+2x_2&=0\\ x_3&=0\end{aligned}\right\}$$

Let us express basic variables in terms of free variables and introduce a dummy equation :

$$\begin{aligned} x_1&=-2x_2\\ x_2&=x_2 \rightarrow \text{(dummy equation)}\\ x_3&=0=0.x_2\end{aligned}$$

We can write the solution in vector form :

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-2x_2\\x_2\\0\end{pmatrix}=x_2\begin{pmatrix}-2\\1\\0\end{pmatrix}$$ where x_2 acts as a parameter.

- There are <u>infinitely many solutions</u>.
- The solution set can be concisely described as:

$$S=\left\{t\begin{bmatrix}-2\\1\\0\end{bmatrix} : t\in\mathbb{R}\right\}=\left\{t\bar{u} : \bar{u}=\begin{bmatrix}-2\\1\\0\end{bmatrix},\ t\in\mathbb{R}\right\}$$

<u>Check</u> : $$A\bar{u}=\begin{bmatrix}1&2&-3\\2&4&-2\\3&6&-4\end{bmatrix}\begin{bmatrix}-2\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Ex ②: $AX = \bar{0}$ where $A = \begin{bmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$

We note that A is a RREF matrix

The Basic Variables are x_1 and x_3
The free variables are x_2 and x_4

The system reduces to:

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}\right\}$$

Expressing Basic Variables in terms of free variables:

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_2 &= x_2 + 0 \quad \text{(dummy equation)} \\ x_3 &= 0 - x_4 \\ x_4 &= 0 + x_4 \quad \text{(dummy equation)} \end{aligned}$$

So, in the vector form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The solution set is:

$$S = \left\{ t \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + s \underbrace{\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\bar{w}} : t, s \in \mathbb{R} \right\}$$

$= \{ t\bar{u} + s\bar{w} : t, s \in \mathbb{R} \}$

- There are infinitely many solutions.

- There are infinitely many solutions.

Remark:

It is possible to obtain the solution set in a different form:

e.g.: $S_1 = \{ r\bar{u}_1 + s\bar{u}_2 + t\bar{u}_3 : r, s, t \in \mathbb{R} \}$

where $\bar{u}_1 = \begin{bmatrix} -8 \\ 1 \\ -2 \\ 2 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -11 \\ 1 \\ -3 \\ 3 \end{bmatrix}$, $\bar{u}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ -3 \end{bmatrix}$

We can verify (later) that $S = S_1$, regarded as a set of vectors.

So, to avoid such difficulties (which arises because the solution set is infinite), we will always obtain the solution via the RREF matrix.

The reasons are:

(1) The RREF matrix is unique.

(2) The number of vectors obtained

on RHS via the RREF is least possible. (i.e. any other method can not provide fewer vectors.)

— x — x — x — x

Recall Homogeneous System $Ax = \bar{0}$,
A is the coefficient matrix and assume that A is row reduced to an RREF matrix R by elementary row operations.

Then the system $Rx = \bar{0}$ is equivalent to the original system $Ax = \bar{0}$.

We can solve the system $Rx = \bar{0}$ by using Basic Variables and Free Variables. and the solution is best expressed in vector form:

Observations:

(1) If the number of nonzero rows r of R is less than the number of variables n, the system has a non-trivial solution

- We express the Basic Variables in terms of free variables

- Free variables behave like parameters; i.e. we can choose any values for them and each such choice gives a solution. This way we get infinitely many solutions.

(2) <u>Special case of (1)</u>:

If A is an $m \times n$ matrix with $m < n$, then the homogeneous system $Ax = 0$ must have a nontrivial solution. (actually infinitely many solutions because in this case, there have to be free variables)

(3) If the number of non-zero rows of R is equal to the number of variables (i.e. the number of columns), then there are no free variables and the system has a unique solution (Only the trivial solution of all zeros).

<u>Proposition (3)</u>: If A is a square matrix, then A is row equivalent to the identity matrix if and only if

the homogeneous system $Ax = 0$ has only the trivial solution.

Proof: Exercise.

Summary of Homogeneous Systems:

(1) System is always consistent.

(2) If the system has a unique solution then it is the trivial solution of all zeros: In this case the RREF is either the $n \times n$ identity matrix I_n or has I_n as its upper portion with only zero rows below.

(3) Else, the system contains free variables and has infinitely many solutions (one of which is the trivial solution): This happens when number of non-zero rows in the RREF is less than the number of variables.

(4) If number of equations is less than the number of variables,

then the system has infinitely many solutions. This is a special case of 3.