MTH : Lecture 10

LU algorithm:

- Input a $m \times n$ matrix A
- Step 1: Row reduce A, if possible to an echelon form matrix U, using only row replacement operations that add a multiple of a row to a row below it.
- Step 2: Place entries in L such that the same sequence of row operations reduces L to I.

Remark: (a) The computational efficiency of the LU approach depends on the fact that L is obtained without doing any significant extra work.

(b) Step 1 is not always possible but if it is, then the theoretical justification indicates why an LU factorization is then obtained.

Theoretical Justification :

- Suppose it is possible to row reduce A to an echelon form matrix U using only row replacement operations that add a multiple of a row to a row below it.
- Then there are unit lower triangular elementary matrices $E_1, E_2, \dots, E_p$ such that $E_p \cdots E_2 E_1 A = U$

$E_i = e_i(I)$
Note that E_i is unit lower triangular because e_i is a replacement operation.

- Hence $A = (E_p \cdots E_1)^{-1} U$

$$\Rightarrow A = LU$$

where $L = (E_p \cdots E_1)^{-1}$ is clearly invertible.

Also note that inverses and products of unit lower triangular matrices are also unit lower triangular matrices.

(Verify!!) (Exercise)

- Finally $L = (E_p \cdots E_1)^{-1}$

$$\Rightarrow L = (E_p \cdots E_1)^{-1} I$$

$$\Rightarrow L = (E_1^{-1} E_2^{-1} \cdots E_p^{-1}) I = (F_1 F_2 \cdots F_p) I$$
$$= f_1(f_2 \cdots f_p(I) \ldots)$$

Also $\boxed{I = (E_p \ldots E_1) L}$

i.e. the same sequence of row operations that reduces A to U reduces L to I.

LU factorization (General Case)

- In practical work, row interchanges are almost always used for computational stability.

So, this situation can be handled in nearly the same way except that the resultant L is not necessarily unit Lower triangular but is permuted unit Lower triangular.

(i.e. we can make L into a unit Lower triangular matrix by a permutation of the row.)

Ex: $A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + \frac{1}{2}R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U \text{ (say)}$$

Row operations

$e_1 : \quad R_2 \to R_2 - 3R_1$

$e_2 : \quad R_3 \to R_3 + \frac{1}{2}R_1$

$e_3 : \quad R_3 \to R_3 + 2R_2$

Inverse operations

$f_1 : \quad R_2 \to R_2 + 3R_1$

$f_2 : \quad R_3 \to R_3 - \frac{1}{2}R_1$

$f_3 : \quad R_3 \to R_3 - 2R_2$

Now L will be a 3x3 unit lower triangular matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\downarrow R_3 \to R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \xleftarrow{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}$$

$= L$ (say)

So, $A = LU$ (check !!)

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} = A$$

Ex: Solve $A\bar{x} = \bar{b}$ where $\bar{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

and A is given in the previous example.

We will use the result of the previous example : $A = LU$

First we solve $L\bar{y} = \bar{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad y_1 = \boxed{2}$

and $3y_1 + y_2 = 1 \Rightarrow y_2 = 1 - 3y_1 = 1 - 3(2)$
$= \boxed{-5}$

and $-\frac{1}{2}y_1 - 2y_2 + y_3 = 4 \Rightarrow y_3 = 4 + \frac{1}{2}y_1 + 2y_2$

$= 4 + \frac{1}{2}(2) + 2(-5)$

$= 5 - 10 = \boxed{-5}$

So, $\bar{y} = \begin{bmatrix} 2 \\ -5 \\ -5 \end{bmatrix}$

Now we solve $U\bar{x} = \bar{y}$

$$\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 - 4x_2 + 4x_3 - 2x_4 = 2 \\ 3x_2 - 5x_3 + 3x_4 = -5 \\ x_3 = x_3 \rightarrow \text{(dummy equation)} \\ 5x_4 = -5 \end{cases}$$

Note that x_3 is a free variable here.

$$\Rightarrow \quad x_4 = -1 = \boxed{-1 + 0 . x_3}$$

and $x_3 = \boxed{0 + 1 . x_3}$

and $3x_2 = -5 + 5x_3 - 3x_4$

$\Rightarrow 3x_2 = -5 + 5x_3 - 3(-1)$

$\Rightarrow 3x_2 = -2 + 5x_3$

$\Rightarrow \boxed{x_2 = -\frac{2}{3} + \frac{5}{3}x_3}$

and $2x_1 = 2 + 4x_2 - 4x_3 + 2x_4$

$\Rightarrow 2x_1 = 2 + 4\left(-\frac{2}{3} + \frac{5}{3}x_3\right) - 4x_3 + 2(-1)$

$$\Rightarrow 2x_1 = 2 - \frac{8}{3} + \frac{20}{3}x_3 - 4x_3 - 2$$

$$\Rightarrow 2x_1 = -\frac{8}{3} + \frac{8}{3}x_3$$

$$\Rightarrow \boxed{x_1 = -\frac{4}{3} + \frac{4}{3}x_3}$$

So, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ 0 \\ -1 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}}$

So, the set of solutions can be written as:

$$S = \{ \bar{u} + t\bar{v} \ : \ t \in \mathbb{R} \}$$

<u>Check:</u>

$$A\bar{u} = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{3} + \frac{8}{3} + 0 + 2 \\ -8 + 6 + 3 \\ \frac{4}{3} + \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \bar{b}$$

and

$$A\bar{v} = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{3} - \frac{20}{3} + 4 + 0 \\ 8 - 15 + 7 + 0 \\ -\frac{4}{3} - \frac{20}{3} + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \bar{0}$$

Ex: Let $A = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix}$

$$\begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_1$
$R_3 \longrightarrow R_3 - \frac{3}{2} R_1$
$R_4 \longrightarrow R_4 + 3R_1$
$R_5 \longrightarrow R_5 - 4R_1$

$R_3 \to R_3 + 2R_2$
$R_4 \to R_4 - 2R_2$
$R_5 \to R_5 + 3R_2$

$$\downarrow$$

$$\begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U \text{ (say)}$$

Now L will be a 5×5 Lower triangular matrix.

Row operations	Inverse Operations
$e_1 : R_2 \longrightarrow R_2 + 2R_1$	$f_1 : R_2 \longrightarrow R_2 - 2R_1$
$e_2 : R_3 \longrightarrow R_3 - \frac{3}{2}R_1$	$f_2 : R_3 \longrightarrow R_3 + \frac{3}{2}R_1$
$e_3 : R_4 \longrightarrow R_4 + 3R_1$	$f_3 : R_4 \longrightarrow R_4 - 3R_1$
$e_4 : R_5 \longrightarrow R_5 - 4R_1$	$f_4 : R_5 \longrightarrow R_5 + 4R_1$
$e_5 : R_3 \longrightarrow R_3 + 2R_2$	$f_5 : R_3 \longrightarrow R_3 - 2R_2$
$e_6 : R_4 \longrightarrow R_4 - 2R_2$	$f_6 : R_4 \longrightarrow R_4 + 2R_2$
$e_7 : R_5 \longrightarrow R_5 + 3R_2$	$f_7 : R_5 \longrightarrow R_5 - 3R_2$

$$I_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_5 \to R_5 - 3R_2 \\ R_4 \to R_4 + 2R_2 \\ R_3 \to R_3 - 2R_2}]{} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_5 \to R_5 + 4R_1 \\ R_4 \to R_4 - 3R_1 \\ R_3 \to R_3 + \frac{3}{2}R_1 \\ R_2 \to R_2 - 2R_1}]{} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \frac{3}{2} & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix} = L$$

So, $A = LU$ where L and U are given above.

Let $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$

and we solve for $\bar{y}$ in $L\bar{y} = \bar{b}$

$$L\bar{y} = \bar{b} \Rightarrow \boxed{y_1 = b_1}$$

and $$-2y_1 + y_2 = b_2 \Rightarrow y_2 = b_2 + 2y_1$$

$$\Rightarrow \boxed{y_2 = b_2 + 2b_1}$$

and $$\frac{3}{2}y_1 - 2y_2 + y_3 = b_3 \Rightarrow y_3 = b_3 - \frac{3}{2}y_1 + 2y_2$$

$$\Rightarrow y_3 = b_3 - \frac{3}{2}b_1 + 2(b_2 + 2b_1)$$

$$\Rightarrow \boxed{y_3 = b_3 + 2b_2 + \frac{5}{2}b_1}$$

and $$-3y_1 + 2y_2 + y_4 = b_4 \Rightarrow y_4 = b_4 + 3y_1 - 2y_2$$

$$\Rightarrow y_4 = b_4 + 3b_1 - 2(b_2 + 2b_1)$$

$$\Rightarrow \boxed{y_4 = b_4 - 2b_2 - b_1}$$

and $$4y_1 - 3y_2 + y_5 = b_5$$

$$\Rightarrow y_5 = b_5 - 4y_1 + 3y_2$$

$$\Rightarrow y_5 = b_5 - 4b_1 + 3(b_2 + 2b_1)$$

$$\Rightarrow \boxed{y_5 = b_5 + 3b_2 + 2b_1}$$

So, $$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 + 2b_1 \\ b_3 + 2b_2 + \frac{5}{2}b_1 \\ b_4 - 2b_2 - b_1 \\ b_5 + 3b_2 + 2b_1 \end{bmatrix}$$

• First we choose $b_1 = 2$, $b_2 = 3$, $b_3 = 5$, $b_4 = 8$, $b_5 = -13$

Then $y_1 = \boxed{2}$, $y_2 = 3 + 2(2) = \boxed{7}$, $y_3 = 5 + 2(3) + \frac{5}{2}(2) = \boxed{16}$

$y_4 = 8 - 2(3) - 2 = \boxed{0}$, $y_5 = -13 + 3(3) + 2(2) = \boxed{0}$

Now $U\bar{x} = \bar{y} \Rightarrow \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 16 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{aligned} 2x_1 - 6x_2 + 6x_3 &= 2 \\ -7x_2 + 5x_3 &= 7 \\ 0 &= 16 \leftarrow \text{a Contradiction} \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

So, for this choice of $\bar{b}$, the system doesn't have any solution.

• Now we choose $b_1 = 2$, $b_2 = 3$, $b_3 = -11$, $b_4 = 8$, $b_5 = -13$

Then $y_1 = \boxed{2}$, $y_2 = 3 + 2(2) = \boxed{7}$

$y_3 = -11 + 2(3) + \frac{5}{2}(2) = -11 + 11 = \boxed{0}$

$y_4 = 8 - 2(3) - 2 = 8 - 8 = \boxed{0}$

$y_5 = -13 + 3(3) + 2(2) = -13 + 13 = \boxed{0}$

Now $U\bar{x} = \bar{y} \Rightarrow \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} 2x_1 - 6x_2 + 6x_3 &= 2 \\ -7x_2 + 5x_3 &= 7 \\ 0 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}\right\}$$

$\Rightarrow \quad -7x_2 = 7 - 5x_3$ (Here x_3 will be a free variable)

$\Rightarrow \quad x_2 = -1 + \frac{5}{7}x_3$

and $\quad 2x_1 = 2 + 6x_2 - 6x_3$

$\Rightarrow \quad 2x_1 = 2 + 6\left(-1 + \frac{5}{7}x_3\right) - 6x_3$

$\Rightarrow \quad 2x_1 = -4 - \frac{12}{7}x_3$

$\Rightarrow \quad x_1 = -2 - \frac{6}{7}x_3$

$$\Rightarrow \quad \begin{aligned} x_1 &= -2 - \frac{6}{7}x_3 \\ x_2 &= -1 + \frac{5}{7}x_3 \\ x_3 &= \qquad\quad x_3 \end{aligned}$$

$\longrightarrow$ (dummy equation)

$$\Rightarrow \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} -\frac{6}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix}}_{\bar{v}}$$

Then the set of solutions can be written as:

$$S = \{ \bar{u} + t\bar{v} \text{ where } t \in \mathbb{R} \}$$

Check: $A\bar{u} = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+6=2 \\ 8-5=3 \\ -6-5=-11 \\ 12-4=8 \\ -16+3=-13 \end{bmatrix}$

$$\Rightarrow A\bar{u} = \begin{bmatrix} 2 \\ 3 \\ -11 \\ 8 \\ -13 \end{bmatrix} = \boxed{\bar{b}}$$

and $A\bar{v} = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \begin{bmatrix} -\frac{6}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{12}{7} - \frac{30}{7} + 6 = 0 \\ \frac{24}{7} + \frac{25}{7} - 7 = 0 \\ -\frac{18}{7} + \frac{25}{7} - 1 = 0 \\ \frac{36}{7} + \frac{20}{7} - 8 = 0 \\ -\frac{48}{7} - \frac{15}{7} + 9 = 0 \end{bmatrix}$

$$\Rightarrow A\bar{v} = \boxed{\bar{0}}$$