MTH 100 : Worksheet 9

1. Given any two $m \times n$ matrices A and B, prove that $\text{rank}(A+B) \leq \text{rank}(A)+\text{rank}(B)$. Give a non-trivial example in which equality is achieved and a non-trivial example in which strict inequality holds.

2. Determine whether the following are linear transformations(Yes or No) Justify your answers.

   (a) $T : \mathbf{R^3} \to \mathbf{R^2}$ given by $T(x,y,z) = (x+y, x-z)$

   (b) $T : \mathbf{R^3} \to \mathbf{R^2}$ given by $T(x,y,z) = (x+y, z^2)$

   (c) $U : \mathbf{R^{n\times n}} \to \mathbf{R^{n\times n}}$ given by $U(A) = A^T$ where A^T indicates the transpose of the matrix A.

   (d) $M : \mathbf{R}[t] \to \mathbf{R}[t]$ given by $M(p(t)) = tp(t)$ for all polynomials $p(t) \in \mathbf{R}[t]$.

3. Determine all linear transformations $T : \mathbf{R^1} \to \mathbf{R^1}$

   (N.B.:$\mathbf{R^1}$ is the vector space consisting of all 1-tuples with real entries; it is essentially the same as $\mathbf{R}$, however regarded as only a vector space rather than a field.)

4. Consider the space $V = C[\mathbf{R}]$ and consider the mapping $D_\epsilon : V \to V$ given by $D_\epsilon(f) = f_\epsilon$ where $f_\epsilon(x) = f(x+\epsilon)$ for all x.

   Here ϵ is an arbitrary but fixed real number. Is D_ϵ a linear transformation? Justify your answer.

5. Prove that there does not exist a linear transformation $T : \mathbf{R^5} \to \mathbf{R^2}$ such that Ker T= $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R^5}, x_1 = 3x_2, x_3 = x_4 = x_5\}$

6. Consider the field $\mathbf{C}$ of complex numbers as a vector space over the field $\mathbf{R}$. Show that the function

   $\phi : \mathbf{C} \to \mathbf{C}$ given by $\phi(z) = \bar{z}$ is a linear transformation. Here $\bar{z}$ indicates the complex conjugate of z i.e. if $z = a+ib$, then $\bar{z} = a-ib$. Show that complex conjugation is actually a multiplicative function i.e. if $w, z \in \mathbf{C}$, then $\phi(wz) = \phi(w)\phi(z)$. Finally show that ϕ is the only multiplicative linear transformation from $\mathbf{C}$ to $\mathbf{C}$ other than the zero and identity transformations.

7. Applying a proposition proved in class, construct three linear transformations, T_1, T_2, T_3 with domain $\mathbf{R^2}$ and codomain $\mathbf{R^3}$ such that rank $(T_i) = i$ for $i = 1, 2, 3$.

8. Let V be an n-dimensional space and let T be a linear operator V such that Range(T) = Kernel (T)

   Show that n must be even.

   Give an example of such an operator. (Note: A linear operator T on V is a linear transformation $T : V \to V$ i.e. the codomain is the same as domain.)