Worksheet 6

(1) Let $V = F^4$ where $F = \mathbb{Z}_2$.

Then V is a vector space over the field F with respect to the following operation:

If $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4) \in V$,

then $u + v = \big((u_1+v_1) \bmod 2, (u_2+v_2) \bmod 2, (u_3+v_3) \bmod 2, (u_4+v_4) \bmod 2 \big)$

$\big(u_i, v_i \in \mathbb{Z}_2 \ \forall\, i = 1,2,3,4 \big)$

If $c \in F$ (i.e. $c = 0$ or 1)

then $cu = (cu_1, cu_2, cu_3, cu_4)$

clearly $u + v \in F^4 = V$ and $cu \in F^4 = V$

(a) If $v = (v_1, v_2, v_3, v_4) \in V$

then $v_i \in \mathbb{Z}_2 \ \forall\, i = 1,2,3,4$.

Now in $\mathbb{Z}_2$, $v_i + v_i = 0$ for $i = 1,2,3,4$.

So, if u is the additive inverse of v in V,

and $u = (u_1, u_2, u_3, u_4)$,

and $u + v = v + u = (0,0,0,0)$

then $u_i + v_i = 0$ for $i = 1,2,3,4$

$\Rightarrow u_i = v_i$ in $\mathbb{Z}_2$

$\boxed{\text{So, additive inverse of } v \text{ is } v \text{ itself.}}$

(b) Let $v_1 = (1,0,1,0)$, $v_2 = (1,1,0,0)$
$v_3 = (0,0,1,1)$

Now $\text{span}\{v_1, v_2\} = \{av_1 + bv_2 : a, b \in \mathbb{Z}_2\}$
$= \{(a+b, b, a, 0) : a, b \in \mathbb{Z}_2\}$

Now there are four possibilities:

$$\left.\begin{array}{l} a=0, b=0 \longrightarrow (0,0,0,0) \\ a=0, b=1 \longrightarrow (1,1,0,0) \\ a=1, b=0 \longrightarrow (1,0,1,0) \\ a=1, b=1 \longrightarrow (0,1,1,0) \end{array}\right\}$$

So, $\text{span}\{v_1, v_2\} = \{(0,0,0,0), (1,1,0,0), (1,0,1,0), (0,1,1,0)\}$.

Now
$\text{span}\{v_1, v_2, v_3\} = \{av_1 + bv_2 + cv_3 : a, b, c \in \mathbb{Z}_2\}$
$= \{(a+b, b, a+c, c) : a, b, c \in \mathbb{Z}_2\}$

Now, there are eight possibilities:

$$\left.\begin{array}{l} a=0, b=0, c=0 \longrightarrow (0,0,0,0) \\ a=0, b=0, c=1 \longrightarrow (0,0,1,1) \\ a=0, b=1, c=0 \longrightarrow (1,1,0,0) \\ a=0, b=1, c=1 \longrightarrow (1,1,1,1) \\ a=1, b=0, c=0 \longrightarrow (1,0,1,0) \\ a=1, b=0, c=1 \longrightarrow (1,0,0,1) \\ a=1, b=1, c=0 \longrightarrow (0,1,1,0) \\ a=1, b=1, c=1 \longrightarrow (0,1,0,1) \end{array}\right\}$$

Thus $\text{span}\{v_1, v_2, v_3\}$
$= \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)\}$

(e) Consider a subspace W of V.

(i) If W is the zero subspace i.e. $W = \{(0,0,0,0)\}$, then W has 1 element.

(ii) If W contains a nonzero element say u then $\{0, u\}$ is a subspace of V (u is its own additive inverse)

So, it has 2 elements.

(iii) If W contains a nonzero element other than u (say v) such that u and v are linearly independent, then $\{0, u, v, u+v\}$ is a subspace which has 4 elements.

(iv) If W contains another nonzero element z (say) such that $z \neq u$, $z \neq v$, $z \neq u+v$ then $\{0, u, v, z, u+v, u+z, v+z, u+v+z\}$ is a subspace and it has 8 elements.

(v) Thus, whenever we add a nonzero element to the subspace (not included in the above cases) the number of elements is multiplied by 2.

Thus the other subspace will have $2^4 = 16$ elements which is $F^4 = V$ itself. ($V = F^4$ has 16 elements)

Thus any subspace of F^4 can have possibly 1, 2, 4, 8 or 16 elements.

$\boxed{\text{Therefore there does not exist subspaces with 3 or 5 vectors.}}$

(d) The possible orders of subspaces of V are 1, 2, 4, 8, 16.

(e) Let F^n be the space of n-tuples, with entries from Z_2.

Then any subspace W of F^n has 2^k elements where $k = 0, 1, 2, \ldots, n$

- If $W = \{0\}$, W has $2^0 = 1$ element.
- If we add one non zero element to this zero subspace, we get a subspace W_1 with $2^1 = 2$ elements.
- If we add another non zero element which is not in W_1, we get a subspace W_2 with $2^2 = 4$ elements.

Proceeding in this way, we will get a subspace with 2^n element which is F^n itself.

Thus F^n can have subspaces of order 2^k for $k = 0, 1, 2, \ldots, n$.

2) Let V be a vector space over a field F.
Let $S = \{v_1, \dots, v_p\}$ be a finite set of vectors in V.
Assume that W is a subspace of V such that $S \subset W$.
Let $x = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$ be any arbitrary element in span(S) where $c_i \in F$ $(i = 1, 2, \dots, p)$
Now $v_1, v_2, \dots, v_p \in W \Rightarrow c_1 v_1 + c_2 v_2 + \dots + c_p v_p \in W$.

$\Rightarrow x \in W$

So, $\boxed{\text{span}(S) \subseteq W}$

Thus span(S) is the smallest subspace of V containing S.

Note: We can also prove that span(S) is the intersection of all subspaces of V containing S.
Note that intersection of any arbitrary collection of subspaces of V is a subspace of V (can be proved using Test for Subspaces)
Let $\mathcal{S}$ be the collection of all subspaces of V which contains S.
i.e. $\mathcal{S} = \{ W : W \text{ is a subspace of } V \text{ and } S \subseteq W \}$

Let $X = \bigcap_{W \in \mathcal{S}} W$

Now span$(S) \in \mathcal{S} \Rightarrow X \subseteq$ span(S). ---- ①

On the other hand if $W \in \mathcal{S}$, span$(S) \subseteq W$ (By exercise ②)

$$\Rightarrow \quad Span(S) \subseteq \bigcap_{W \in \mathcal{S}} W \quad \Rightarrow Span(S) \subseteq X \quad \text{--- (2)}$$

Combining (1) and (2), $\quad X = Span(S)$

i.e. $\boxed{\text{Span}(S) \text{ is the intersection of all subspaces of } V \text{ that contain } S}$

(I have proved the above in my notes (in class) where I used words instead of notation for intersection)

Note: Thus in different situation, we can use different characterizations of Span(S).

Viz. Span(S) = set of all linear combinations of vectors in S

= Smallest subspace of V containing S

= intersection of all subspaces of V containing S.

3 4 7

To show that $U+W$ is a subspace, note that

(i) $\underset{\text{(vector)}}{0} = 0+0$ where $0 \in U$ and $0 \in W$

So, $0 \in U+W$.

(ii) If $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ are elements of $U+W$, where $u_1, u_2 \in U$ and $w_1, w_2 \in W$,

then $v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2)$
$= (u_1 + u_2) + (w_1 + w_2)$

where $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$
(since U and W are subspaces of V)

$\Rightarrow v_1 + v_2 \in W$

(iii) If $c \in F$ and $v = u + w \in U+W$ (where $u \in U$, $w \in W$),

then $cv = cu + cw$ where $cu \in U$ and $cw \in W$

and so $cv \in U+W$

So, $\boxed{U+W \text{ is a subspace of } V}$

Now, if X is any other subspace of V such that $U, W \subseteq X$,

then any $v \in U+W$, ~~~~ can be written as $v = u + w$ where $u \in U$, $w \in W$.

This implies $u \in U \subseteq X$, $w \in W \subseteq X$

$\Rightarrow u + w \in X$ (X is a subspace) $\Rightarrow v \in X$

So, $\boxed{U+W \subseteq X}$ $\boxed{\text{So, } U+W \text{ is the smallest subspace of } V \text{ containing } U \text{ and } W.}$

(4) Let $V = \mathbb{R}^2$,

$U = \{(x, 0) : x \in \mathbb{R}\}$ (The x-axis)

$W = \{(0, y) : y \in \mathbb{R}\}$ (The y-axis)

Then U and W are subspaces of V (Check!)

But $U \cup W$ is not a subspace of $\mathbb{R}^2$

Because $(1,0) \in U$, $(0,1) \in W$

$\Rightarrow (1,0), (0,1) \in U \cup W$

But $(1,0) + (0,1) = (1,1) \notin U \cup W$.

So, $U \cup W$ is not closed under addition

and so $U \cup W$ is not a subspace of $\mathbb{R}^2$.

We will prove that

$U \cup W$ is a subspace of V

if and only if either $U \subseteq W$ or $W \subseteq U$.

Proof: '$\Leftarrow$'.

First assume that either $U \subseteq W$ or $W \subseteq U$

Want to show $U \cup W$ is a subspace

If $U \subseteq W$, then $U \cup W = W$ which is a subspace.

If $W \subseteq U$, then $U \cup W = U$ which is a subspace

Thus in either case $U \cup W$ is a subspace.

$\Rightarrow$

Conversely assume that $U \cup W$ is a subspace.
Want to show that $U \subseteq W$ or $W \subseteq U$.

If $U \subseteq W$, then we are done.

So, we assume that $U \subseteq W$ does not hold.
We will show that $W \subseteq U$ holds.

Since $U \nsubseteq W$, there exists a vector $u \in U$ such that $u \notin W$.

Now let $w \in W$ be any element of W.

Since $u \in U \Rightarrow u \in U \cup W$
$w \in W \Rightarrow w \in U \cup W$

Since $U \cup W$ is a subspace, $u + w \in U \cup W$

So, either $u + w \in U$ or $u + w \in W$

If $u + w \in W$, $(u + w) + (-w)$ $\left(\begin{matrix}\text{Since}\\ -w \in W\end{matrix}\right)$
$= u + (w + (-w)) = u \in W$

But $u \notin W$ a contradiction.

Thus $u + w \in U$.

Then $(-u) + (u + w) = w \in U$ $\left(\begin{matrix}\text{Since}\\ -u \in U\end{matrix}\right)$

Since w is any arbitrary element of W, it follows that $\boxed{W \subseteq U}$

(QED)

(5) Let $f: V \longrightarrow W$ be a bijection. $(V \neq \phi)$

W is a real vector space.

For $u, v \in V$ define $u \oplus v = f^{-1}(f(u)+f(v))$

For $c \in \mathbb{R}$, $u \in V$, define $c * u = f^{-1}(cf(u))$

<u>Closure property :</u>

For $u, v \in V$, $f(u) \in W$, $f(v) \in W$

$\Rightarrow f(u)+f(v) \in W$ (Since W is a vector space over $\mathbb{R}$)

Since f is a bijection, there exists a unique $p \in V$

s.t. $f(p) = f(u)+f(v)$

$\Rightarrow p = f^{-1}(f(u)+f(v))$

So, $u \oplus v$ is this unique $p \in V$

Similarly if $v \in V$, $c \in \mathbb{R}$, $f(v) \in W$

$\Rightarrow cf(v) \in W$ (Since W is a vector space over $\mathbb{R}$)

Since f is a bijection, there exists

a unique $q \in V$ s.t. $f(q) = cf(v)$

$\Rightarrow q = f^{-1}(cf(v))$

So, $c * v$ is this unique $q \in V$.

Thus closure property of addition and scalar multiplication is satisfied.

Commutative property:

Let $u, v \in V$

Then $u \oplus v = f^{-1}(f(u) + f(v))$

$= f^{-1}(f(v) + f(u))$ (By Commutative property of addition in W)

$= v \oplus u$

(we used the property that f is a bijection)

Zero property:

Let $0 \in W$ be the zero vector in W.

~~Let~~ Then there exists a unique element $r \in V$ such that $f(r) = 0$

Then for every $v \in V$,

$r + v = f^{-1}(f(r) + f(v))$

$= f^{-1}(0 + f(v))$ (since 0 is the zero element in W)

$= f^{-1}(f(v))$

$= v$ (since f is bijection)

So, r is the zero vector in V.

Inverse property

For any $v \in V$, $f(v) \in W$ and there exists an unique inverse $-f(v) \in W$ such that $-f(v) + f(v) = 0$ vector

Let $s = f^{-1}(-f(v))$ be the unique element in V.

then $v \oplus s = f^{-1}(f(v) + f(s)) = f^{-1}(f(v) - f(v))$

$= f^{-1}(0) = r$ (f is bijective)

Thus s is the additive inverse of v
we will denote $s = f^{-1}(-f(v))$ as $-v$.

<u>Associative property:</u>

For $u, v, w \in V$,

$$(u \oplus v) \oplus w$$

$$= f^{-1}\left(f(u \oplus v) + f(w) \right)$$

$$= f^{-1}\left(\left(f(u) + f(v)\right) + f(w) \right) \quad \left(\begin{array}{l} \text{Since} \\ u \oplus v = f^{-1}\left(f(u) + f(v)\right) \\ f(u \oplus v) = f(u) + f(v) \end{array} \right)$$

$$= f^{-1}\left(f(u) + \left(f(v) + f(w)\right) \right) \quad \left(\text{By associative property in } W\right)$$

$$= f^{-1}\left(f(u) + f(v \oplus w) \right)$$

$$= u \oplus (v \oplus w) \quad \left(\text{we have used bijective property of } f\right)$$

Similarly we can verify the other properties of scalar multiplication.
(Left as exercises for students).

Note:

To define $u \oplus v$ and $c * v$ we require the <u>inverse function</u> f^{-1} and for that we require f to <u>be bijective</u>.

(However, from the verification, we can see that we can ~~have~~ require f to be injective and $f(V)$ to be a subspace of $\bar{W}$ (needed for closure & zero property))

(6) $u = (1,3,5)$, $v = (1,4,6)$, $w = (2,-1,3)$, $b = (6,5,17)$

Let us Consider

$$c_1 u + c_2 v + c_3 w = b$$

$$\Rightarrow c_1\begin{bmatrix}1\\3\\5\end{bmatrix} + c_2\begin{bmatrix}1\\4\\6\end{bmatrix} + c_3\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}6\\5\\17\end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + c_2 + 2c_3 &= 6 \\ 3c_1 + 4c_2 - c_3 &= 5 \\ 5c_1 + 6c_2 + 3c_3 &= 17 \end{aligned}$$

The Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ 5 & 6 & 3 & 17 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 1 & -7 & -13 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent

$$c_1 + 9c_3 = 19 \Rightarrow c_1 = -9c_3 + 19$$

$$c_2 + 7c_3 = -13 \qquad c_2 = 7c_3 - 13$$

$$c_3 = c_3 \qquad c_3 = c_3$$

So, $\begin{bmatrix} 19 \\ -13 \\ 0 \end{bmatrix}$ is a Solution

Check;

$$19\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - 13\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}$$

(a) So, $b \in \text{Span}\{u, v, w\}$

(b) and $\boxed{b = 19u - 13v + 0.w}$

7) Suppose $v_1, v_2, \ldots, v_p$ is a list of vectors which contains a list $v_1, v_2, \ldots, v_k$ (without any loss of generality we can assume this notation) which is linearly dependent. (ofcourse $k \leq p$)

Then there exist scalars $c_1, c_2, \ldots, c_k$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

Now $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$

$$+ 0.v_{k+1} + \cdots + 0.v_p = 0$$

where not all the scalars are zero.

Hence $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_p$ is linearly dependent.

8 Suppose $\{v_1, v_2, \ldots, v_p\}$ is a linearly independent set in a vector space V

Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ be a subset of the above set of vectors $(k \le p)$

Assume that this subset is linearly dependent.

Then there exists scalars $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ not all zeros such that

$$c_{i_1} v_{i_1} + c_{i_2} v_{i_2} + \cdots + c_{i_k} v_{i_k} = 0$$

For all $i \notin \{i_1, i_2, \ldots, i_k\}$, we take $c_i = 0$

Then $\quad c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$,

(where for some index i_j, $c_{i_j} \neq 0$)

where atleast one scalar is nonzero

Thus we arrive at a contradiction since $\{v_1, v_2, \ldots, v_p\}$ are linearly independent.

Note: In the above, if you are uncomfortable with notation, you can assume that without any loss of generality the subset $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent and then proceed the same way.

10 9 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Now let $c_1 A + c_2 B + c_3 C = 0 \rightarrow$ (zero 2×2 matrix)

$$\Rightarrow c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + c_3 & c_1 + c_3 \\ c_1 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow c_1 + c_2 + c_3 = 0, \quad c_1 + c_3 = 0,$$
$$c_1 = 0 \text{ and } c_1 + c_2 = 0$$

$\Rightarrow c_1 = 0$ & $c_1 + c_2 = 0 \Rightarrow c_2 = 0$

$c_1 + c_3 = 0$ & $c_1 = 0 \Rightarrow c_3 = 0$

Thus $c_1 = c_2 = c_3 = 0$

So, A, B, C are linearly independent.

(10) $V = C[0, 2\pi]$

$f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \sin(2x)$

Then $f_1, f_2, f_3 \in C[0, 2\pi]$

Suppose $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0(x) \rightarrow$ (Zero function)

So, $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \forall x \in [0, 2\pi]$

$\Rightarrow c_1 1 + c_2 \sin x + c_3 \sin 2x = 0 \quad \forall x \in [0, 2\pi]$

In particular if we take $x = 0$,

then $c_1 + c_2 \times 0 + c_3 \times 0 = 0 \Rightarrow c_1 = 0$

if we take $x = \frac{\pi}{2}$, then $c_1 + c_2 \times 1 + c_3 \times 0 = 0$

$\Rightarrow c_1 + c_2 = 0 \Rightarrow 0 + c_2 = 0$

$\Rightarrow c_2 = 0$

if we take $x = \frac{\pi}{4}$ then

$c_1 + c_2 \frac{1}{\sqrt{2}} + c_3 \times 1 = 0$

$\Rightarrow 0 + 0 + c_3 = 0 \Rightarrow c_3 = 0$

So, $c_1 = c_2 = c_3 = 0$

and hence f_1, f_2 and f_3 are linearly independent.