Quiz 7

Nov 5th, 2024

Time: 15 minutes Max marks = 5

Name: ______________ Roll no.: ______________ Group: ______________

Instructions: Notes, books, computers, cell phones and other electronic devices are not allowed.

Problem 1. Let A be an $n \times n$ matrix such that $A^2 = 0$ (zero matrix). Prove that rank $(A) \leq \frac{n}{2}$.
(Hint: First show that Col$(A) \subseteq$ Nul (A).)

MTH 100 : Rubrics of Quiz 7 | Total points = 5

(1) Let A be an $n \times n$ matrix such that $A^2 = 0$ (zero matrix)

Prove that $\text{rank}(A) \leq \frac{n}{2}$

First we will show that
$$Col(A) \subset null(A)$$

Let $\bar{y} \in Col(A)$. Then $\bar{y}$ is a linear combination of columns of A.

Thus if $A = [a_1 \, a_2 \cdots a_n]$ (where $a_1, a_2, \ldots, a_n$ are columns of A)

then $\bar{y} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$
for some scalars $x_1, x_2, \ldots, x_n \in \mathbb{R}$

(*) ~~Now~~ Thus $\bar{y} = [a_1 \, a_2 \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(Please see the note at the end.)

$= A\bar{x}$ where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

+2

Now $A\bar{y} = A(A\bar{x}) = A^2\bar{x} = 0 \cdot \bar{x} = 0$

$\Rightarrow \bar{y} \in null(A)$

Therefore $Col(A) \subset null(A)$

$\Rightarrow \dim(Col(A)) \leq \dim(null(A))$

$\Rightarrow \text{rank } A \leq \text{nullity } A$ ...... (1)

+1

Now by Rank-nullity Theorem

$$\text{Rank}(A) + \text{nullity}(A) = n$$

$$\Rightarrow \text{rank } A + \text{rank } A \leq \text{rank}(A) + \text{nullity } A = n \quad (\text{By } 1)$$

$$\Rightarrow \quad 2\,\text{rank}(A) \leq n \Rightarrow \boxed{\text{rank}(A) \leq \frac{n}{2}}$$

+2

Note: To show $\text{col}(A) \subset \text{null}(A)$,
if a students takes any element $\bar{y} \in \text{Col}(A)$
and say that it is of the
form $\bar{y} = A\bar{x}$ for some $\bar{x} \in \mathbb{R}^n$,
without explicitly taking or mentioning
columns of A (i.e. $a_1, a_2, \ldots, a_n$),
it will still be $\boxed{\text{O.K.}}$