MTH 100 : Worksheet 10

1. (a) Find the coordinates of the vectors $v_1 = (2, 3, 4)$ and $v_2 = (1, -1, 2)$ with respect to the ordered basis $\beta = \{(1, 1, 1), (1, 2, 3), (1, 3, 6)\}$
   (NB: the vectors have been written as 3-tuples, but should be regarded as column vectors.)

   (b) If $[v]_\beta = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_\beta$, find $[v]_S$ where S is the standard basis for $\mathbf{R^3}$.

2. Find the matrix relative to the standard basis of the linear operator T on $\mathbf{R^3}$ given by:

   $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$

3. Let $T : \mathbf{R^3} \to \mathbf{R^2}$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$

   (a) Find the matrix of T with respect to standard basis for $\mathbf{R^3}$ and $\mathbf{R^2}$

   (b) Verify that $\beta = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ is a basis for $\mathbf{R^3}$

   (c) Now, determine the matrix of T with respect to the ordered bases β and $\beta' = \{(0, 1), (1, 0)\}$ for $\mathbf{R^3}$ and $\mathbf{R^2}$ respectively.

4. (a) Find the matrix relative to the standard basis of the linear operator T on $\mathbf{R^3}$ given by:
   $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$

   (b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1, 1, 1), (1, 2, 3), (1, 3, 6)\}$
   [**NB: the change of basis matrix $P_{S \to \beta}$ for this basis has been calculated in Question 1.**]

5. (a) Prove that similarity is an equivalence relation on the set $R^{n \times n}$ of square $n \times n$ matrices (n≥ 2).

   (b) Prove or disprove: There exist square matrices (atleast 2×2) A and B such that B is row-equivalent to A, but B is not similar to A.

   (c) Prove or disprove: There exist square matrices (atleast 2×2) A and B such that B is similar to A, but B is not row-equivalent to A.

6. Let $V = R^{2\times 2}$ = vector space of 2×2 matrices with real entries and consider the function $U : V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T is the transpose of A.

   (a) Show that U is a linear operator.

   (b) Determine the matrix of U with regard to any suitable ordered basis β of V.

   (c) Determine a basis for Ker U and determine a basis for Range U.

   (d) Determine the dimension of $\mathrm{Sym}_n(R)$, the space of symmetric $n \times n$ matrices with real entries. Briefly explain your answer.

7. Show that a linear transformation $T : V \to W$, where V and W are finite dimensional with dim V = dim W, is injective iff it is surjective.

8. Let $V = F^{n\times n}$ for a fixed $n \geq 2$, and let $P \in V$ be a fixed but arbitrary invertible matrix. Then the mapping $S_P : V \to V$ given by $S_P(A) = PAP^{-1}$ is known as similarity transformation induced by P. Show that S_P is an isomorphism. Further, show that S_P is a multiplicative transformation, i.e. $S_P(AB) = S_P(A)S_P(B)$ for all $A, B \in V$

9. Let $V = \mathbf{R^2}$ and consider the ordered bases $\alpha = \{u_1, u_2\}$ and $\beta = \{v_1, v_2\}$, where the vectors are as given below.(NB: regard all vectors as column vectors in V.)

   (a) Find the change of basis matrix $P_{\alpha\to\beta}$

   (b) Hence find $[\mathbf{v}]_\beta$ given that $[\mathbf{v}]_\alpha = (10, 20)$.

   (c) Is there some way to check your answer to 2)? Explain your method and use it to check your answer.

   $u_1 = (3, 1), u_2 = (11, 4), v_1 = (3, 2), v_2 = (7, 5)$