Solution of Tutorial 11 | Worksheet 11

1 (a) Let $T: V \to W$, $U: W \to Z$ be linear transformations (V, W and Z are finite dimensional vector spaces)

Let $U.T : V \to Z$ be denoted by S.

Now if $z \in$ range S, there exists $v \in V$ s.t. $z = S(v) \Rightarrow z = UT(v) \Rightarrow z = U(Tv)$

$\Rightarrow z = U(w)$ where $w = Tv \in W$

$\Rightarrow z \in$ range U

Hence range $S \subseteq$ range U

$\Rightarrow$ rank $S \leq$ rank U ------ (1)

Now if $v \in \ker T$, then $Tv = 0 \Rightarrow U(Tv) = U(0) = 0$

$\Rightarrow UT(v) = 0 \Rightarrow S(v) = 0$

$\Rightarrow v \in \ker(S)$

Hence $\ker T \subseteq \ker S$

$\Rightarrow$ nullity $T \leq$ nullity S ---- (2)

Now by Rank Theorem

$$\dim V = \text{rank}(S) + \text{nullity}(S)$$
$$= \text{rank}(T) + \text{nullity}(T)$$

$\Rightarrow$ rank(S) + nullity(S) = rank(T) + nullity(T) ---- (3)

From (2) and (3), rank$(S) \leq$ rank(T) ---- (4)

Now from (1) and (4) we can conclude

rank$(S) \leq \min\{$rank(U), rank$(T)\}$

i.e. $\boxed{\text{rank}(UT) \leq \min\{\text{rank}(U), \text{rank}(T)\}}$

(b) If A is an $m\times n$ matrix and B is an $n\times k$ matrix, we can define linear transformations

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \text{ by } T_A(v) = Av \ \forall\, v \in \mathbb{R}^n$$

$$T_B : \mathbb{R}^k \to \mathbb{R}^n \text{ by } T_B(w) = Bw \ \forall\, w \in \mathbb{R}^k$$

Now $T_A \cdot T_B : \mathbb{R}^k \to \mathbb{R}^m$ defined by

$$T_A \cdot T_B(w) = A(Bw) = (AB)w$$

So, $T_A \cdot T_B = T_{AB}$

Now $\operatorname{rank}(T_A) = \operatorname{rank} A$, $\operatorname{rank}(T_B) = \operatorname{rank} B$

$\operatorname{rank}(T_{AB}) = \operatorname{rank}(AB)$

Now by (a) $\operatorname{rank}(AB) = \operatorname{rank}(T_{AB}) = \operatorname{rank}(T_A T_B)$

$$\leq \min\{\operatorname{rank}(T_A), \operatorname{rank}(T_B)\}$$

$$\Rightarrow \operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$$

(c) For equality we take A and B to be two invertible $n\times n$ matrices.

e.g. $A = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 10 & 4 \\ 2 & 5 \end{bmatrix}$

Then AB is also invertible.

$\operatorname{rank} A = 2 = \operatorname{rank} B$, $\quad \operatorname{rank} AB = 2$

So, $\operatorname{rank}(AB) = \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

Note: Can take any two $n\times n$ invertible matrices.

Now, for strict inequality we take $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Then $\operatorname{rank} A = 1$ $\operatorname{rank} B = 1$, So, $\min\{\operatorname{rank}(A), \operatorname{rank}(B)\} = 1$

but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and so $\operatorname{rank}(AB) = 0$

Thus in this case $\boxed{\operatorname{rank}(AB) < \min\{\operatorname{rank} A, \operatorname{rank} B\}}$

2 Given: $T : V \longrightarrow W$ is a bijective (and hence invertible) linear transformation.

Let $T^{-1} : W \to V$ be the inverse function.

Since T is bijective, T^{-1} is well-defined.

Let $w_1, w_2 \in W$

Since T is bijective, there exist unique $v_1, v_2 \in V$ such that $Tv_1 = w_1$, $Tv_2 = w_2$ (i.e. $v_1 = T^{-1}w_1$ and $v_2 = T^{-1}w_2$)

Now $v_1 + v_2 \in V$ and

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$

Hence $T^{-1}(w_1 + w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$

i.e. $\boxed{T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)}$

Now if $c \in F$ and $w \in W$,
there exists a vector $v \in V$
such that $Tv = w$ (i.e. $v = T^{-1}(w)$)

Now $cv \in V$ and $T(cv) = cT(v)$
$\Rightarrow T(cv) = cw$
$\Rightarrow T^{-1}(cw) = cv = cT^{-1}(w)$
$\Rightarrow \boxed{T^{-1}(cw) = cT^{-1}(w)}$

Therefore T^{-1} is a linear transformation from W ~~onto~~ to V.

Note: The above result shows that if $T: V \to W$ is an isomorphism, so is T^{-1}

Therefore we can show that isomorphism is an equivalence relation on the family of all vector spaces over a particular field F.
(Show it!!)

3 (a): Given $T: V \to W$ linear transformation.
$\Rightarrow$ Assume T is nonsingular
i.e. $\ker T = \{0\}$

Now for $v_1, v_2 \in V$, if $Tv_1 = Tv_2$
then $T(v_1 - v_2) = Tv_1 - Tv_2 = 0$
$\Rightarrow v_1 - v_2 \in \ker T$
$\Rightarrow v_1 - v_2 = 0$
$\Rightarrow v_1 = v_2$

So, T is injective.

~~Now~~ $\Leftarrow$: Now assume that T is injective

Now $T0 = 0$ i.e. $0 \in \ker T$

If $v \in V$ and $v \neq 0$, then $Tv \neq 0$ (since T is injective)

hence $v \notin \ker T$

Therefore $\ker T = \{0\}$ and so T is nonsingular.

(b) $\Rightarrow$: Assume that T is non singular and let $v_1, v_2, \ldots, v_n$ be any linearly independent set in V

Consider $Tv_1, Tv_2, \ldots, Tv_n \in W$

Now $c_1 Tv_1 + c_2 Tv_2 + \cdots + c_n Tv_n = 0$

$\Rightarrow T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n \in \ker T$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n = 0$ (as $\ker T = \{0\}$ since T is nonsingular)

$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ (since $v_1, v_2, \ldots, v_n$ are linearly independent)

Therefore $Tv_1, Tv_2, \ldots, Tv_n$ are linearly independent.

$\Leftarrow$: Conversely suppose T carries every linearly independent subset of V into a linearly independent subset of W.

Let $v \in V$ and $v \neq 0$

Then $\{v\}$ (set consisting of only v) is a linearly independent subset of V

By the given condition,
$\{Tv\}$ is a linearly independent subset of W

$\Rightarrow Tv \neq 0$

$\Rightarrow v \notin \ker T$

Hence $\ker T = \{0\}$ (Recall that $0 \in \ker T$ for any linear transformation T)

So, T is nonsingular.

(c) T is nonsingular

$\Leftrightarrow \ker T = \{0\}$

$\Leftrightarrow$ Nullity $T = 0$

$\Leftrightarrow$ Rank $T = \dim V$ (since By Rank Theorem, Rank T + Nullity $T = \dim V$)

$\Leftrightarrow$ Rank $T = \dim W$ (since $\dim V = \dim W$)

$\Leftrightarrow$ Range $T = W$

$\Leftrightarrow T$ is surjective

By (a), T is nonsingular $\Leftrightarrow T$ is injective

Thus combining
T is nonsingular $\Leftrightarrow T$ is bijective
& so T is nonsingular $\Leftrightarrow T$ is invertible.

4 $T: V \longrightarrow W$, V and W are finite-dimensional with $\dim V = \dim W = n$ (say)

T is injective $\Rightarrow$ $\ker T = \{0\}$

$\Rightarrow$ nullity $T = 0$ $\Rightarrow$ By Rank Theorem

rank $T = n$ $\left(\begin{array}{l} \text{rank } T + \text{nullity } T = \dim V \\ \Rightarrow \text{rank } T + 0 = n \end{array}\right)$

$\Rightarrow$ Range $T = W$

$\Rightarrow$ T is surjective

Furthermore,

T is surjective $\Rightarrow$ Range $T = W$

$\Rightarrow$ rank $T = n$

$\Rightarrow$ By Rank Theorem

nullity $T = 0$ $\left(\begin{array}{l} \text{rank } T + \text{nullity } T = \dim V = n \\ \Rightarrow n + \text{nullity } T = n \end{array}\right)$

$\Rightarrow$ $\ker T = \{0\}$

$\Rightarrow$ T is injective.

5 Let $V = \mathbb{R}[t]$ space of polynomial with real coefficients

(In view of 4, the V we are looking for must be infinite dimensional)

Let $T: \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$ be the differentiation operator

given by $T(p(t)) = p'(t)$

Then T is surjective because for any

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in \mathbb{R}[t],$$

$$q(t) = a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \cdots + \frac{a_n}{n+1} t^{n+1} \in \mathbb{R}[t]$$

and $T(q(t)) = p(t)$.

But T is not injective because

$1+t \in \mathbb{R}[t]$, $2+t \in \mathbb{R}[t]$,

$1+t \neq 2+t$ but $T(1+t) = T(2+t) = 1$.

Now let $U : \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$ be the "multiplication by t" operator

given by $U(p(t)) = t\,p(t)$.

Then $U(p(t)) = 0$ (zero polynomial)

$\Rightarrow t\,p(t) = 0 \quad \forall\, t \in \mathbb{R}$

$\Rightarrow p(t) = 0 \quad \forall\, t \in \mathbb{R} \Rightarrow \ker U = \{0\}$

So, U is injective.

However U is not surjective because $q(t) = 1$ is a constant polynomial but there is no polynomial $p(t) \in \mathbb{R}[t]$ such that $U(p(t)) = q(t)$

Infact Range U is the set of all polynomials with constant coefficient zero.

6. Let A be an $n\times n$ matrix such that

$$A^2 = O_{n\times n} \quad (\text{Zero matrix})$$

If λ is any eigenvalue of A, there exists a non zero eigen vector $v\ (\neq 0)$ such that

$$Av = \lambda v$$

Now $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda . \lambda v$

$\Rightarrow \quad A^2 v = \lambda^2 v \Rightarrow \lambda^2 v = \mathbf{0}$ (zero vector) (Since $A^2 = O_{n\times n}$, $A^2 v = \mathbf{0}$ zero vector)

Since $v \neq 0$ (a non zero vector),

$$\lambda^2 = 0 \Rightarrow \lambda = 0$$

Thus 0 is the only eigen value of A.

7. First note that

$$(A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I^T = (A^T - \lambda I)$$

and $\det(A-\lambda I) = \det\left[(A-\lambda I)^T\right] = \det(A^T - \lambda I)$

Now λ is an eigen value of A

$\Leftrightarrow \det(A-\lambda I) = 0 \Leftrightarrow \det\left[(A-\lambda I)^T\right] = 0$

$\Leftrightarrow \det(A^T - \lambda I) = 0$

$\Leftrightarrow \lambda$ is an eigen value of A^T.

(8) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be an $n \times n$ matrix

and $\sum_{j=1}^{n} a_{ij} = s$ for all $1 \leq i < n$

ie.
$$\left.\begin{aligned} a_{11} + a_{12} + \cdots + a_{1n} &= s \\ a_{21} + a_{22} + \cdots + a_{2n} &= s \\ \cdots\cdots\cdots\cdots &\cdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} &= s \end{aligned}\right\}$$

Now characteristic polynomial of A

$$= \det(A - \lambda I)$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$C_1 \to C_1 + C_2 + \cdots + C_n$

$$= \begin{vmatrix} a_{11}+a_{12}+\cdots+a_{1n}-\lambda & a_{12} \cdots a_{1n} \\ a_{21}+a_{22}+\cdots+a_{2n}-\lambda & a_{22}-\lambda \cdots a_{2n} \\ \cdots & \cdots \\ a_{n1}+a_{n2}+\cdots+a_{nn}-\lambda & a_{n2} \cdots a_{nn}-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} s-\lambda & a_{12} & \cdots & a_{1n} \\ s-\lambda & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ s-\lambda & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (s-\lambda) \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Thus one of the solution of the equation $\det(A - \lambda I) = 0$

is $\lambda = s$

Hence s is an eigen value of the matrix A.

9 Given that A is an $n \times n$ square matrix and Rank $A = k$

Therefore $\dim(\text{Row } A) = \dim(\text{Col } A) = k$

Assume that A has $(k+2)$ or more distinct eigen values. Then A must have atleast $(k+1)$ nonzero distinct eigen values.

Let $\{\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}\}$ be $(k+1)$ non zero distinct eigen values of A.

Thus for each $1 \leq i \leq k+1$, there exists a non-zero vector v_i such that

$$A v_i = \lambda_i v_i$$

$$\Rightarrow A\left(\frac{v_i}{\lambda_i}\right) = v_i \qquad \left(\text{note that } \lambda_i \neq 0\right)$$

$$\Rightarrow v_i \in \text{Col}(A)$$

Now $v_1, \dots v_{k+1}$ are eigen vectors corresponding to distinct eigen values $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$ and therefore $\{v_1, v_2, \dots, v_{k+1}\}$ is linearly independent.

Hence $\dim \text{Col}(A) \geq k+1$

a contradiction to the fact that $\dim(\text{Col } A) = k$

Therefore A can have atmost $(k+1)$ distinct eigen values.

10 (a) $A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

characteristic polynomial

$$= q(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\{(1-\lambda)(1-\lambda) - 1\} + (-1)\{(-1)(1) - 1(1-\lambda)\}$$

$$+ (-1)\{1(-1) - 1(1-\lambda)\}$$

$$= (3-\lambda)(\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}) - 1(\lambda - 2) - 1(\lambda - 2)$$

$$= (3-\lambda)(\lambda^2 - 2\lambda) - 2(\lambda - 2)$$

$$= 3\lambda^2 - 6\lambda - \lambda^3 + 2\lambda^2 - 2\lambda + 4$$

$$= \boxed{-\lambda^3 + 5\lambda^2 - 8\lambda + 4}$$

Verification:

$$A^2 = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -7 & -7 \\ 7 & 1 & -7 \\ 7 & -7 & 1 \end{bmatrix}$$

Now $q(A)$

$= -A^3 + 5A^2 - 8A + 4I_3$

$$= -\begin{bmatrix} 15 & -7 & -7 \\ 7 & 1 & -7 \\ 7 & -7 & 1 \end{bmatrix} + 5\begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix} - 8\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0_{3\times 3}$$

So, A satisfies its characteristic polynomial

(b) Note that

$q(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$

$= (3-\lambda)(\lambda^2 - 2\lambda) - 2(\lambda - 2)$ (from the earlier step)

$= (3-\lambda)\lambda(\lambda-2) - 2(\lambda-2)$

$= (\lambda-2)(3\lambda - \lambda^2 - 2)$

$= -(\lambda-2)(\lambda^2 - 3\lambda + 2) = -(\lambda-2)(\lambda-2)(\lambda-1)$

$= -(\lambda-2)^2(\lambda-1)$

Now $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$

and $r(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

So, $p(\lambda)$ and $r(\lambda)$ both are divisors of $q(\lambda)$

Now

$$p(A) = A^2 - 3A + 2I$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0_{3\times3}$$

$$r(A) = A^2 - 4A + 4I = \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix} - 4\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \neq 0_{3\times3}$$

Thus A satisfies $p(\lambda)$ but does not satisfy $r(\lambda)$.

(c) We observe that a matrix satisfies its characteristic polynomial but not necessarily satisfies the divisors of the characteristic polynomials. However A satisfies a divisor of the characteristic polynomial consisting of all eigen values of A as its roots

(d) Eigen values of A are roots of $\det(A-\lambda I)=0$

$$\Rightarrow -(\lambda-2)^2(\lambda-1)=0$$

$$\Rightarrow \lambda = 1, 2, 2$$

For eigen vectors of A

For $\lambda=1$, $(A-\lambda I)X=0$

$$\Rightarrow (A-I)x=0$$

$$\Rightarrow \begin{bmatrix} 3-1 & -1 & -1 \\ 1 & 1-1 & -1 \\ 1 & -1 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[R_3\to R_3-\frac{1}{2}R_1]{R_2\to R_2-\frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow[R_3\to R_3+R_2]{} \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2\to 2R_2]{R_1\to \frac{1}{2}R_1} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1\to R_1+\frac{1}{2}R_2]{} \text{RREF matrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the solution is
$$\left.\begin{aligned} x_1 - x_3 = 0 &\Rightarrow x_1 = x_3 \\ x_2 - x_3 = 0 &\Rightarrow x_2 = x_3 \\ & \quad\; x_3 = x_3 \end{aligned}\right\}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda=1$

<u>Now for $\lambda = 2$</u>

$$(A - \lambda I)\, x = 0 \Rightarrow (A - 2I)\, x = 0$$

$$\Rightarrow \begin{bmatrix} 3-2 & -1 & -1 \\ 1 & 1-2 & -1 \\ 1 & -1 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF matrix}$$

So, the solution is $x_1 - x_2 - x_3 = 0$

$$\Rightarrow \quad x_1 = x_2 + x_3$$
$$x_2 = x_2 + 0.x_3$$
$$x_3 = 0.x_2 + x_3$$

So, $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} \text{choosing } x_2 = 1, x_3 = 0 \\ \text{and } x_2 = 0, \ x_3 = 1 \end{array}\right\}$ we get $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors corresponding to the eigen value 2.

$\left.\begin{array}{l} c_1 v_2 + c_2 v_3 = 0 \\ \Rightarrow \ c_1 + c_2 = 0 \\ \quad c_1 = 0 \\ \quad c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0 \Rightarrow v_2$ & v_3 are linearly independent.

Since v_1 is an eigen vector corresponding to the eigen value 1,

$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ are linearly independent.