UNIT-4

CHAPTER 7

THREE DIMENSIONAL CURVES

CONTENTS

7.0 OBJECTIVES:

After reading this chapter you will be able to:

- Learn about interpolating curves
- Learn Approximation curves
- Acquire knowledge about Bezier curves
- Define B-Spline curves
- Know about fractals

7.1 INTRODUCTION:

Three dimensional graphics is much more complex then two dimensional graphics. When we see a picture in a computer screen, it is a two dimensional picture. So, three dimensional to two dimension transformation is necessary. This is called projection. The 3D objects are transferred to 2D reference coordinates using projection. There are different types of projections like parallel projection, depth projection etc. Also we have to project only the visible surfaces. The hidden surface should be removed from the picture. To add realism we also have to consider the effect of light on the object and the properties of the object surface (glossy, transparent or solid).

The fundamentals of different three dimensional objects are the different curves and simple surfaces. Polygons and quadratic surfaces are used to design different three dimensional objects. In this chapter we shall discuss different curves and their generation techniques.

7.2 SPLINES:

A spline is flexible smooth curve passing through n+1 data points. Mathematically a spline is a piecewise cubic polynomial curve whose first and second order derivatives are continuous across the various curve sections. In computer graphics a spline curve means any curve formed with polynomial section satisfying the specified continuity conditions.

7.2.1 SOME IMPORTANT DEFINITION:

Control points: The points given by the user that describes the general shape of the curve are called control points.

Interpolating Curve: When the polynomial sectors are fitted in such a way that the curve passes through the control points then curve is called interpolating curve.

Approximation Curve: When the curve is fitted in such a way that the curve not necessarily passes through the control pints, then the curve is called approximation curve.

Control Graph: The set of control points are often connected with poly-lines to give the designer an idea of the ordering of the points. The set of connected line segments are referred to as control graph. It is also called control polygon or characteristics polygon.

Blending Functions: Blending functions are functions which are used to represent the given curve as a sum of these functions. If f(x) is a curve to be designed using the blending functions $\Psi_i(x)$, i = 0, 1, 2… n then

$$f(x) = \sum_{i=0}^{n} a_i \psi_i(x) \text{ where}$$

$a_0, a_2, \ldots\ldots, a_n$ are constants called weights.

There are several blending functions. One of the important blending functions is polynomial. We generally use polynomial of degree three as blending function which is also called cubic polynomial.

7.3 SPLINE INTERPOLATION:

This class of splines is used for setting the paths for a motion object or drawing of an object etc. As it is interpolation method, the curve will pass through every control points. We shall generally use cubic polynomial as the blending function because; n-degree polynomials are more complex to calculate and more storage is required to store the data. n-degree polynomials require more time to compute. Cubic curves on the other hand require less time to compute. Storage required is also less. The cubic polynomials are very efficient and also manage the smoothness of the curve. We shall now discuss cubic splines.

Given a set of n+1 control points, cubic interpolation spline is obtained by generating the piecewise cubic polynomials that passes through every control point. Let us suppose we have n+1 control points (x_0, y_0, z_0), (x_1, y_1, z_1) ………… (x_n, y_n, z_n). Clearly a cubic polynomial that will fit in the above n+1 points will have three components, the x component, y component and the z component, each of which should be determined separately and each of which will be a polynomial of degree three. It is to be fitted between each pair of control points. The equations of the components can be given by

$$x(\lambda) = a_{0x} + a_{1x}\lambda + a_{2x}\lambda^2 + a_{3x}\lambda^3$$

$$y(\lambda) = a_{0y} + a_{1y}\lambda + a_{2y}\lambda^2 + a_{3y}\lambda^3 \qquad \text{where } 0 \le \lambda \le 1 \qquad \ldots\ldots\ldots\ldots 7.1$$

$$z(\lambda) = a_{0z} + a_{1z}\lambda + a_{2z}\lambda^2 + a_{3z}\lambda^3$$

i.e.,

$$x(\lambda) = \sum_{i=0}^{3} a_{ix}\lambda^i$$

$$y(\lambda) = \sum_{i=0}^{3} a_{iy}\lambda^i \qquad 0 \le \lambda \le 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 7.2$$

$$z(\lambda) = \sum_{i=0}^{3} a_{iz}\lambda^i$$

To solve each equation, we need to find out the values of four unknowns a_0, a_1, a_2, and a_3. We have to solve the equations between each pair of control points. For the two end points of the segment, we shall get tow equations and for the remaining two equations, we need another two equations. These two equations can be obtained from the boundary conditions. These different boundary conditions will give different splines.

7.3.1 NATURAL SPLINE:

When two adjacent curve sections have same first and second order derivatives at their common boundary, the spline is called natural cubic spline or natural spline.

We have n+1 control points. So, the total number of segments is n. so the total number of coefficients for all segments is 4n, (4 unknowns for each segment). At each interior control point, we shall get four equations.

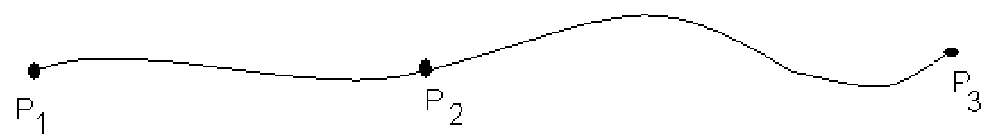

Fig-7.1 Natural Spline

From the figure-7.1, the scenario will be clear. There are two segments P_1P_2 and P_2 P_3. So, for each direction of the vector λ i.e., in x(λ) or y(λ) or z(λ), there will be four unknown coefficients, a_0, a_1, a_2 and a_3. So, total there are total eight unknowns in each directions. For n–1 interior points (in this case n-1 =1 i.e. P_2 point only), there will be 4n-4 equations. (in this case n = 2, so 4n-4 = 4 unknown).

(i) The point P_2 will pass through segment P_1P_2

(ii) The point P_2 will pass through segment P_2P_3

(iii) At point P_2, the first order derivative of P_1P_2 is equal to first order derivative of P_2P_3

(iv) At point P_2, the second order derivative of P_1P_2 is equal to second order derivative of P_2P_3

The remaining four equation will formulated from first point P_0 and the last point P_n.

(i) The segment P_0 P_1will pass through segment P_0

(ii) The segment $P_{n-1}P_n$ will pass through segment P_n

(iii) The second order derivative at P_0 =0.

(iv) The second order derivative at P_n =0.

A curve with these conditions is called natural spline. Through it good to describe mathematically, it has a major disadvantage. If one control point is shifted the entire curve is affected. There is no local control.

7.3.2 HERMITE SPLINE:

Hermite spline is a piecewise cubic spline where each control point has a specified tangent. Let us discuss it.

To derive the Hermite Spline we have to rewrite the equations 7.2 in the following form–

$$P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i \quad \text{where } 0 \le \lambda \le 1 \qquad \ldots\ldots 7.3$$

where x-component of $P(\lambda)$ can be written as

$$x(\lambda) = \sum_{i=0}^{n} a_{ix} \lambda^i \quad \text{where } 0 \le \lambda \le 1$$

Similarly other components can be written.
Expanding equation 7.3 we get

$$P(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \qquad 0 \le \lambda \le 1 \qquad \ldots\ldots 7.4$$

$\therefore P(\lambda)$ can be written in matrix form in the following way

$$P(\lambda) = [\lambda^3 \ \lambda^2 \ \lambda \ 1] \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} \qquad \ldots\ldots 7.5$$

Now, derivative at point P(u) with respect to u can be given by

$$P'(\lambda) = [\ 3\lambda^2 \ \ 2\lambda \ \ 1 \ \ 0\ ] \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} \qquad \ldots\ldots 7.6$$

Let P_k and P_{k+1} be two adjacent control points on the spline curve. Then the boundary conditions for Hermite spline can be given by

$$P(0) = P_k$$
$$P(1) = P_{k+1}$$
$$P'(0) = DP_k \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 7.7$$
$$P'(1) = D\,P_{k+1}$$

Substituting the value of $\lambda = 0$ and $\lambda = 1$ in Equations 7.5 and 7.6 we get the Hermite boundary conditions as follows.

$$\begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} . \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 7.8$$

Solving the Equation 7.8 we get

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} . \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{k+1} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 7.9$$

$$= M_H . \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{k+1} \end{bmatrix}$$

where M_H is the inverse matrix and it is called Hermite matrix.

Therefore $P(\lambda)$ can be written as

$$P(\lambda) = \begin{bmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \end{bmatrix} M_H . \begin{bmatrix} P_k \\ P_{k+1} \\ DP_k \\ DP_{k+1} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 7.10$$

This is Hermite spline where the coefficient of P_k, P_{k+1}, DP_k and DP_{k+1} are called blending functions. One of the major advantages of Hermiote spline is Hermite spline can be adjusted locally because each segment depends on only its endpoints. It has some disadvantages also. One disadvantage is that the derivative values are needed at each end point which sometimes becomes difficult to supply.

7.4 BEZIER CURVES:

This is a kind of spline approximation curve i.e., it will not pass through all of its control points as in case of interpolation. The curve was developed by French engineer Pierre Bezier. This curve is very popular in engineering design. We shall now discuss Bezier curve mathematically. A Bezier curve can be specified by with boundary conditions, with a characterizing matrix or with blending function.

7.4.1 DEFINITION OF BEZIER CURVE:

Suppose there are n+1 control points $P_i = (x_i, y_i, z_i)$ with i = 0, 1, 2 …., n. Then the Bezier curve between P_0 and P_n can be given by

$$P(\lambda) = \sum P_i Bz_{i,n}(\lambda) \text{ where } 0 \le \lambda \le 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..7.11$$

Where Bz(λ) is the Blending function and it can be described by the Bernstein Polynomials as

$$Bz_{i,n}(\lambda) = {}^nC_i \lambda^i (1-\lambda)^{n-i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad 7.12$$

where ${}^nC_i = \dfrac{n!}{i!(n-i)!}$ ……………………………………………………7.13

The Bezier Blending function can recursively be written as follows

$$Bz_{i,n}(\lambda) = (1-\lambda) = Bz_{i,n-1}(\lambda) + \lambda\, Bz_{i-1,\, n-1}(\lambda); \qquad n > i \ge 1 \qquad \ldots\ldots..7.14$$

With $Bz_{i,i} = \lambda^i$ and $Bz_{0,i} = (1-\lambda)^i$

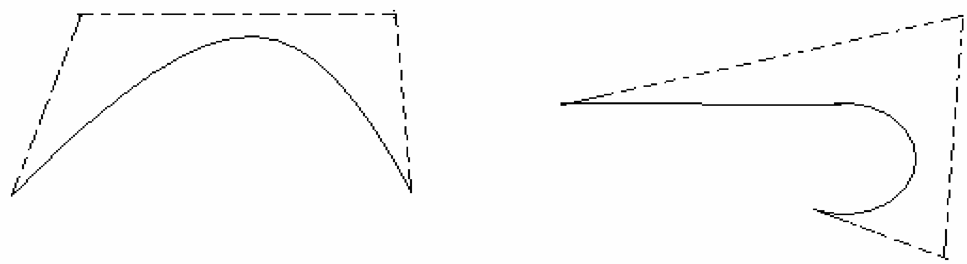

Fig-7.2: Example of Bezier Curves

7.4.2 PROPERTIES OF BEZIER CURVES:

These are several important properties of Bezier curves. These can be listed below

a) Bezier Curve is a polynomial of degree, one less than member of control points used.

b) A very important property of Bezier curve is that it always passes through the first and last control points.

c) The slope at the beginning of the curve is along the line joining the first two points and slope at the end is along the line joining the last two points.

d) Bezier curves lie on the convex hull formed by the control points.

e) The basis functions of Bezier curve are real.

7.4.3 BEZIER SURFACES:

Bezier surface can be generated from Bezier curves. Two sets of orthogonal Bezier curves can be used to design a Bezier surface. The parametric vector function can be given by

$$P(\lambda,\gamma)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{i,j}Bz_{i,m}(\lambda).Bz_{j,n}(\gamma) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 7.15$$

with $P_{j,\,k}$ specifying the location of m+1 by n+1 control points. Bezier surface has the same property as Bezier curves.

7.5 B-SPLINE CURVES:

The Bezier curve with Bernstein basis has two disadvantages. First, the degree of the curve is determined by the number of control points. A curve with four control points will always be a curve of degree three. Secondly a Bezier curve with Bernstein basis function is global in nature. i.e., the change in one vertex will change the shape of the entire curve.

B-spline curves, on the other hand, overcome the above mentioned disadvantages. It is an approximation curve where (i) degree of the polynomial can be set independently of the number of control points and (ii) B-spline curves allow local control over the shape of the curve.

7.5.1 DEFINITION OF B-SPLINE CURVES:

Let us consider that there are n+1 control points $P_i = (x_i, y_i, z_i)$; i = 0, 1, 2..., n. Let $P(\lambda)$ be the position vector along the B-spline curve as a parameter of λ. Then $P(\lambda)$ can be given by

$$P(\lambda) = \sum_{i=0}^{n} P_i B_{i,d}(\lambda) \quad \text{where } \lambda_{\min} \le \lambda \le \lambda_{\max} \text{ and } 2 \le d \le n+1 \qquad \text{……………………..7.16}$$

The blending function for B-spline curves are defined by Cox-deBoor recursion formula as follows.

$$B_{i,1}(\lambda) = \begin{cases} 1 & \text{If } \lambda_i \le \lambda \le \lambda_{i+1} \\ 0 & \text{Otherwise} \end{cases}$$

and

$$B_{i,d}(\lambda) = \frac{\lambda - \lambda_i}{\lambda_{i+d-1} - \lambda_i} B_{i,d-1}(\lambda) + \frac{\lambda_{i+d} - \lambda}{\lambda_{i+d} - \lambda_{i+1}} B_{i+1,d-1}(\lambda) \qquad \text{……………..7.17}$$

where each blending function is defined over d subintervals of the total range of λ. Selected set of subinterval endpoints λ_i are referred to as knot vectors. We can select any values for the subinterval endpoints satisfying the relation $\lambda_i \leq \lambda_{i+1.}$ Values for λ_{min} and λ_{max} then depend on the number of control points we select, the value for d and how we set up the knot vectors.

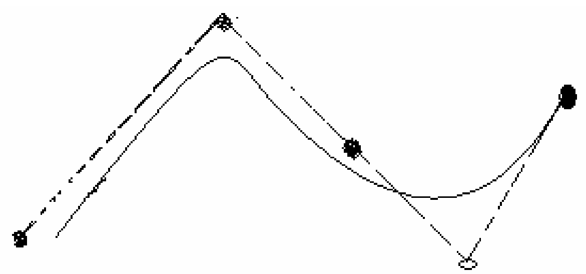

Fig-7.3: B-Spline Curve

7.5.2 PROPERTIES OF B-SPLINE CURVES:

The B-sline curves have the following properties.

(a) Polynomial has the degree d-1 and continuity C^{d-2} over λ.

(b) For n+1 control points, the curve has n+1 blending functions.

(c) Each section of the curve is influenced by d control points.

(d) Any one control point can affect the shape of at most d curve sections.

7. 6 FRACTALS:

Mathematical equations are good for describing many objects in computer graphics, but they don't show good results for natural objects. Natural objects such as clouds, rivers, roads have irregular and fragmented features and mathematical equations fail to describe them realistically. Fortunately there is a method which can model this type of realistic things into computer graphics. This is called fractals. And the method is known as fractal-geometry method. Fractals use procedure rather than equations to generate natural objects.

These are tow special properties of fractals one is self-similarity and the other is infinite detail. The self-similarity means a part of the fractal is features wise similar with the whole fractal object. The infinite detail means if we zoom the fractal, it will give us more details no matter what is the zooming factor.

7.6.1 TYPES OF FRACTALS:

There are various kinds of fractals depending upon their generation methods. To enlarge a part of the fractal, we need to use the scaling. The scaling with scaling factor greater than one will enlarge the fractal and we will have greater details. If we apply statistical methods for this scaling, the fractals are called statistically self-similar. For designing trees these fractals are used. If we apply different scaling parameters for different parts then the fractals are called self affine fractals. If in self affine fractals we use statistical properties, then they are called statistically –self affine fractals. Using non-linear transformations we can generate Invariant fractals. Self squaring and self inverse fractals are the examples of Invariant fractals.

7.7: KEY WORDS:

- Approximating Curve
- Bezier Curve
- Bezier Surface
- Control Graph
- Cubic Spline
- Fractals
- Interpolating Curve

- Natural Spline
- Spline

7.8 SAMPLE QUESTIONS:

7.8.1 Define a spline with a suitable example.

7.8.2 Define the following terms (i) Control Point (ii) Interpolating Curve (iii) Approximation curve (iv) Control graph.

7.8.3 What do you mean by Blending functions?

7.8.4 What do you mean by spline interpolation? Explain the mathematical definition of natural spline.

7.8.5 What is Hermite spline? Explain.

7.8.6 What is Bezier curve? Explain the properties of Bezier curve.

7.8.7 Define Bezier surface.

7.8.8 What do you mean by B-Spline curves? Explain its properties.

7.8.9 What are fractals? Why fractals are needed?

7.8.10 What are the properties of fractals?

7.8.11 Discuss different types of fractals.