QUIZ - I

MA423: Matrix Computations | **Time : 40 minutes** | **20 marks**

September 8, 2023

Answer ALL questions

1. Let $x \in F(\beta, t, L, U)$ be such that $\beta^{e-1} < x < \beta^e$. Let $y \in \mathbb{R}$. If $|y| < \mathrm{ulp}(x)/2$ then show that $\mathrm{fl}(x \pm y) = x$. For $(\beta, t, L, U) = (10, 4, -30, 30)$ and $x = 10^{-9}$, what is the smallest $y > 0$ such that $\mathrm{fl}(x + y) > x$? Next when $x = 10^{15}$, what is the smallest $y > 0$ such that $\mathrm{fl}(x + y) > x$. Here $\mathrm{fl}(x)$ uses round to nearest rounding mode. **5 marks**

2. Let $A \in \mathbb{R}^{n \times n}$ is nonsingular. Describe an efficient algorithm (outline only the steps) based on Gaussian elimination for solving the problem $A^{10}x = b$ and determine the flop count.

   5 marks

3. Consider the system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is nonsingular and $b \in \mathbb{R}^n$. Suppose that $A + \Delta A$ is nonsingular. Now consider the system $(A + \Delta A)\hat{x} = b + \Delta b$. Show that

$$\frac{\|x - \hat{x}\|}{\|\hat{x}\|} \leq \mathrm{cond}(A)\left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|A\|\,\|\hat{x}\|}\right),$$

   where $\mathrm{cond}(A) := \|A\|\,\|A^{-1}\|$. **5 marks**

4. Consider $A := \begin{bmatrix} 10^{-4} & 1 \\ 1 & a \end{bmatrix}$, where $0 < a \leq 4$. Prove that in the floating point system $F(10, 3, -10, 10)$, the computed LU factorization of A using GENP is independent of a. Is GENP backward stable for computing LU factorization of A? Justify your answer. **5 marks**

*** End ***

QUIZ - II

MA423: Matrix Computations **Time : 40 minutes** **20 marks**

November 7, 2023

Answer ALL questions

1. Let $A \in \mathbb{C}^{m\times n}$ be such that $\operatorname{rank}(A) = n$. Let $\sigma_{\min}(A)$ denote the smallest singular value of A. Show that $\|(A^*A)^{-1}A^*\|_2 = 1/\sigma_{\min}(A)$ and $\|A(A^*A)^{-1}A^*\|_2 = 1$. **6 marks**

2. Let $u \in \mathbb{R}^n$ be a unit vector. Consider the Householder reflector $H := I - 2uu^\top$. Determine all the eigenvalues of H and their algebraic multiplicities. Also determine all the eigenvectors. **5 marks**

3. Let $A \in \mathbb{C}^{n\times n}$ and $\sigma > 0$. Show that σ is a singular value of A $\iff$ $\begin{bmatrix} A & -\sigma I_n \\ -\sigma I_n & A^* \end{bmatrix}$ is singular. **4 marks**

4. Let $A \in \mathbb{R}^{n\times n}$ be proper upper Hessenberg and A_3 be the result of 3 basic QR steps on A, that is, $A_{k-1} = Q_k R_k$, $A_k := R_k Q_k$ for $k = 1, 2, 3$, with $A_0 := A$. Set $R := R_3 R_2 R_1$ and $Q := Q_1 Q_2 Q_3$. Show that $A^3 = QR$ and $A_3 = Q^* A Q$. **5 marks**

*** End ***

$Q_3^{-1} Q_2^{-1} Q_1^{-1} A^3 = R_3 R_2$

End Semester Examination

MA 423 : Matrix Computations **Time : 3 hours** **40 marks**

November 22, 2023

Answer ALL questions

1. Let $A \in \mathbb{C}^{m\times n}$ be such that $\operatorname{rank}(A) = r < \min(m, n)$ and $b \in \mathbb{C}^m$. Define the Moore-Penrose pseudo-inverse A^+ of A and show that A^+b is a unique solution of the LSP $Ax \approx b$ having the smallest norm. Use SVD of A to show that $\lim_{\mu\to 0+}(A^*A + \mu I_n)^{-1}A^* = A^+$. **6 marks**

2. Let $A := \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$. Use QR factorization of A to prove the Hadamard's determinant inequality
$$|\det(A)| \leq \prod_{j=1}^{n} \|\mathbf{a}_j\|_2,$$
where $\det(A)$ is the determinant of A. **4 marks**

3. Let $A \in \mathbb{C}^{m\times n}, b \in \mathbb{C}^m$ and $\Delta b \in \mathbb{C}^m$. Let x and $\hat{x}$ be unique solutions (with smallest norms) of the LSP $Ax \approx b$ and LSP $A\hat{x} \approx b + \Delta b$, respectively. Show that
$$\frac{\|x - \hat{x}\|_2}{\|x\|_2} \leq \frac{\operatorname{cond}(A)}{\cos\theta}\frac{\|\Delta b\|_2}{\|b\|_2},$$
where θ is the acute angle between b and Ax, and $\operatorname{cond}(A) := \|A^+\|_2\,\|A\|_2$. Here A^+ is the Moore-Penrose pseudo-inverse of A. **4 marks**

4. Compute QR factorization of $A := \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ in two different ways using Householder reflectors and classical Gram-Schmidt method. **6 marks**

5. Let $A \in \mathbb{R}^{n\times n}$ be upper Hessenberg. Consider the QR step $A - \mu I = QR$ and $A_1 = RQ + \mu I$. Show that A_1 is upper Hessenberg when μ is NOT an eigenvalue of A. Show that if the QR decomposition of $A - \mu I$ is computed using rotations then A_1 is upper Hessenberg even if μ is an eigenvalue of A. **5 marks**

$(x+\bar{\sigma})^2$

6. Let $A \in \mathbb{R}^{n\times n}$ and $s \in \mathbb{C}$. Suppose that s is not an eigenvalue of A. Perform two steps of explicit QR algorithm on A with shifts s and its complex conjugate $\bar{s}$:
$$\begin{aligned} A - sI &= Q_1R_1,\ A_1 := R_1Q_1 + sI \\ A_1 - \bar{s}I &= Q_2R_2,\ A_2 := R_2Q_2 + \bar{s}I. \end{aligned}$$
Show that Q_1 and Q_2 can be chosen such that Q_1Q_2 is real. Deduce that in such a case A_1 is complex and A_2 is real. Describe an algorithm (outline the steps) for computing A_2 directly from A without computing A_1 thereby avoiding complex arithmetic. **5 marks**

7. Describe an algorithm (outline the steps) for reducing a matrix $A \in \mathbb{R}^{n\times n}$ to an upper bidiagonal form B. Describe a single step of Golub-Kahan implicit QR algorithm for computing SVD of B when B is proper upper bidiagonal. **5 marks**

8. Write an algorithm (outline the steps) that implements inverse power method for computing an eigenvalue and a corresponding eigenvector of nonsingular matrix $A \in \mathbb{C}^{n\times n}$. Determine the flop count for performing ℓ steps of inverse power method. Suppose that A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $|\lambda_1| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$. Let $Av_n = \lambda_n v_n$ and $v_n \neq 0$. Let x_0 be a nonzero vector such that $\langle x_0, v_n\rangle \neq 0$. Show that the iterate x_j obtained from inverse power method with starting vector x_0 converges to a scalar multiple of v_n at the rate $\dfrac{|\lambda_{n-1}|}{|\lambda_n|}$. **5 marks**

*** End ***