CHAPTER

14 Fuzzy Arithmetic and Fuzzy Measures

LEARNING OBJECTIVES

- Basic concepts of fuzzy arithmetic.
- How interval analysis is performed for uncertain values.
- A note on fuzzy numbers, fuzzy ordering and fuzzy vectors.
- Discusses on extension principle for generalizing crisp sets into fuzzy sets.
- A description on belief, plausibility, probability, possibility and necessity measures.
- Gives a view on fuzzy integrals.

14.1 INTRODUCTION

In this chapter, we will discuss the basic concepts involved in fuzzy arithmetic and fuzzy measures. Fuzzy arithmetic is based on the operations and computations of fuzzy numbers. Fuzzy numbers help in expressing fuzzy cardinalities and fuzzy quantifiers. Fuzzy arithmetic is applied in various engineering applications when only imprecise or uncertain sensory data are available for computation. In this chapter we will discuss various forms of fuzzy measures such as belief, plausibility, probability and possibility. A representation of uncertainty can be done using fuzzy measure. All the measures to be discussed are functions applied to crisp subsets, instead of elements, of a universal set.

14.2 FUZZY ARITHMETIC

In the present scenario, we experience many applications which perform computation using ambiguous (imprecise) data. In all such cases, the imprecise data from the measuring instruments are generally expressed in the form of intervals, and suitable mathematical operations are performed over these intervals to obtain a reliable data of the measurements (which are also in the form of intervals). This type of computation is called *interval arithmetic* or *interval analysis*. Fuzzy arithmetic is a major concept in possibility theory. Fuzzy arithmetic is also a tool for dealing with fuzzy quantifiers in approximate reasoning (Chapter 15). Fuzzy numbers are an extension of the concept of intervals. Intervals are considered at only one unique level. Fuzzy numbers consider them at several levels varying from 0 to 1.

14.2.1 Interval Analysis of Uncertain Values

Consider a data set to be uncertain. We can locate this uncertain value to be lying on a real line, *R*, inside a closed interval, i.e., $x \in [a_1, a_2]$ where $a_1 \le a_2$. The value of x is greater than or equal to a_1 and smaller than or equal to a_2. In interval analysis, the uncertainty of the data is limited between the intervals specified by the lower bound and upper bound. This can be represented as

$$\underset{\sim}{A} = [a_1, a_2] = \{x \mid a_1 \le x \le a_2\}$$

where $\underset{\sim}{A}$ represents an interval $[a_1, a_2]$. Generally, the values a_1 and a_2 are finite. In few cases, $a_1 = -\infty$ and/or $a_2 = +\infty$. If value of x is singleton in R then the interval form is $x = [x, x]$. In general, there are four types of intervals which are as follows:

1. $[a_1, a_2] = \{x \mid a_1 \le x \le a_2\}$ is a closed interval.
2. $[a_1, a_2) = \{x \mid a_1 \le x < a_2\}$ is an interval closed at the left end and open at right end.
3. $(a_1, a_2] = \{x \mid a_1 < x \le a_2\}$ is an interval open at left end and closed at right end.
4. $(a_1, a_2) = \{x \mid a_1 < x < a_2\}$ is an open interval, open at both left end and right end.

The set operations performed on the intervals are shown in Table 14-1. Here $[a_1, a_2]$ and $[b_1, b_2]$ are the upper bounds and lower bounds defined on the two intervals $\underset{\sim}{A}$ and $\underset{\sim}{B}$, respectively, i.e.,

$$\underset{\sim}{A} = [a_1, a_2], \quad \text{where } a_1 \le a_2$$
$$\underset{\sim}{B} = [b_1, b_2], \quad \text{where } b_1 \le b_2$$

TABLE 14-1 SET OPERATIONS ON INTERVALS

Conditions	Union, $\cup$	Intersection, $\cap$
$a_1 > b_2$	$[b_1, b_2] \cup [a_1, a_2]$	ϕ
$b_1 > a_2$	$[a_1, a_2] \cup [b_1, b_2]$	ϕ
$a_1 > b_1, a_2 < b_2$	$[b_1, b_2]$	$[a_1, a_2]$
$b_1 > a_1, b_2 < a_2$	$[a_1, a_2]$	$[b_1, b_2]$
$a_1 < b_1 < a_2 < b_2$	$[a_1, b_2]$	$[b_1, a_2]$
$b_1 < a_1 < b_2 < a_2$	$[b_1, a_2]$	$[a_1, b_2]$

The mathematical operations performed on intervals are as follows:

1. **Addition (+):** Let $\underset{\sim}{A} = [a_1, a_2]$ and $\underset{\sim}{B} = [b_1, b_2]$ be the two intervals defined. If $x \in [a_1, a_2]$ and $y \in [b_1, b_2]$, then

$$(x + y) \in [a_1 + b_1, a_2 + b_2]$$

This can be written as

$$\underset{\sim}{A} + \underset{\sim}{B} = [a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$$

2. **Subtraction (–):** The subtraction for the two intervals of confidence is given by

$$\underset{\sim}{A} - \underset{\sim}{B} = [a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$$

That is, we subtract the larger value out of b_1 and b_2 from a_1 and the smaller value out of b_1 and b_2 from a_1.

3. **Multiplication (·):** Let the two intervals of confidence be $\underset{\sim}{A} = [a_1, a_2]$ and $\underset{\sim}{B} = [b_1, b_2]$ defined on non-negative real line. The multiplication of these two intervals is given by

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = [a_1, a_2] \cdot [b_1, b_2] = [a_1 \cdot b_1, a_2 \cdot b_2]$$

If we multiply an interval with a non-negative real number α, then we get

$$\alpha \cdot \underset{\sim}{A} = [\alpha, \alpha] \cdot [a_1, a_2] = [\alpha \cdot a_1, \alpha \cdot a_2]$$
$$\alpha \cdot \underset{\sim}{B} = [\alpha, \alpha] \cdot [b_1, b_2] = [\alpha \cdot b_1, \alpha \cdot b_2]$$

4. **Division (÷):** The division of two intervals of confidence defined on a non-negative real line is given by

$$\underset{\sim}{A} \div \underset{\sim}{B} = [a_1, a_2] \div [b_1, b_2] = \left[\frac{a_1}{b_2}, \frac{a_2}{b_1}\right]$$

If $b_1 = 0$ then the upper bound increases to $+\infty$. If $b_1 = b_2 = 0$, then interval of confidence is extended to $+\infty$.

5. **Image** $(\bar{A})$: If $x \in [a_1, a_2]$ then its image $-x \in [-a_2, -a_1]$. Also if $\underset{\sim}{A} = [a_1, a_2]$ then its image $\bar{A} = [-a_2, -a_1]$. Note that

$$\underset{\sim}{A} + \bar{\underset{\sim}{A}} = [a_1, a_2] + [-a_2, -a_1] = [a_1 - a_2, a_2 - a_1] \neq 0$$

That is, with image concept, the subtraction becomes addition of an image.

6. **Inverse** (A^{-1}): If $x \in [a_1, a_2]$ is a subset of a positive real line, then its inverse is given by

$$\left(\frac{1}{x}\right) \in \left[\frac{1}{a_2}, \frac{1}{a_1}\right]$$

Similarly, the inverse of $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A}^{-1} = [a_1, a_2]^{-1} = \left[\frac{1}{a_2}, \frac{1}{a_1}\right]$$

That is, with inverse concept, division becomes multiplication of an inverse. For division by a non-negative number $\alpha > 0$, i.e., $(1/\alpha) \cdot \underset{\sim}{A}$, we obtain

$$\underset{\sim}{A} \div \alpha = \underset{\sim}{A} \cdot \left[\frac{1}{\alpha}, \frac{1}{\alpha}\right] = \left[\frac{a_1}{\alpha}, \frac{a_2}{\alpha}\right]$$

7. **Max and min operations:** Let two intervals of confidence be $\underset{\sim}{A} = [a_1, a_2]$ and $\underset{\sim}{B} = [b_1, b_2]$. Their max and min operations are defined by

$$\text{Max: } \underset{\sim}{A} \vee \underset{\sim}{B} = [a_1, a_2] \vee [b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2]$$
$$\text{Min: } \underset{\sim}{A} \wedge \underset{\sim}{B} = [a_1, a_2] \wedge [b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]$$

The algebraic properties of the intervals are shown in Table 14-2.

TABLE 14-2 ALGEBRAIC PROPERTIES OF INTERVALS

Property	Addition (+)	Multiplication (·)
Commutativity	$\underset{\sim}{A} + \underset{\sim}{B} = \underset{\sim}{B} + \underset{\sim}{A}$	$\underset{\sim}{A} \cdot \underset{\sim}{B} = \underset{\sim}{B} \cdot \underset{\sim}{A}$
Associativity	$(\underset{\sim}{A} + \underset{\sim}{B}) + \underset{\sim}{C} = \underset{\sim}{A} + (\underset{\sim}{B} + \underset{\sim}{C})$	$(\underset{\sim}{A} \cdot \underset{\sim}{B}) \cdot \underset{\sim}{C} = \underset{\sim}{A} \cdot (\underset{\sim}{B} \cdot \underset{\sim}{C})$
Neutral number	$\underset{\sim}{A} + 0 = 0 + \underset{\sim}{A} = \underset{\sim}{A}$	$\underset{\sim}{A} \cdot 1 = 1 \cdot \underset{\sim}{A} = \underset{\sim}{A}$
Image and inverse	$\underset{\sim}{A} + \bar{\underset{\sim}{A}} = \bar{\underset{\sim}{A}} + \underset{\sim}{A} \neq 0$	$\underset{\sim}{A} \cdot \underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1} \cdot \underset{\sim}{A} \neq 1$

14.2.2 Fuzzy Numbers

A fuzzy number is a normal, convex membership function on the real line R. Its membership function is piecewise continuous. That is, every λ-cut set A_λ, $\lambda \in [0, 1]$, of a fuzzy number $\underset{\sim}{A}$ is a closed interval of R and the highest value of membership of $\underset{\sim}{A}$ is unity. For two given fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in R, for a specific $\lambda_1 \in [0, 1]$, we obtain two closed intervals:

$$\underset{\sim}{A}_{\lambda_1} = \left[a_1^{(\lambda_1)}, a_2^{(\lambda_2)}\right] \text{ from fuzzy number } \underset{\sim}{A}$$
$$\underset{\sim}{B}_{\lambda_1} = \left[b_1^{(\lambda_1)}, b_2^{(\lambda_2)}\right] \text{ from fuzzy number } \underset{\sim}{B}$$

The interval arithmetic discussed can be applied to both these closed intervals. Fuzzy number is an extension of the concept of intervals. Instead of accounting intervals at only one unique level, fuzzy numbers consider them at several levels with each of these levels corresponding to each λ-cut of the fuzzy numbers. The notation $\underset{\sim}{A}_\lambda = [a_1^{(\lambda)}, a_2^{(\lambda)}]$ can be used to represent a closed interval of a fuzzy number $\underset{\sim}{A}$ at a λ-level.

Let us discuss the interval arithmetic for closed intervals of fuzzy numbers. Let (*) denote an arithmetic operation, such as addition, subtraction, multiplication or division, on fuzzy numbers. The result $\underset{\sim}{A} * \underset{\sim}{B}$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are two fuzzy numbers is given by

$$\mu_{\underset{\sim}{A}*\underset{\sim}{B}}(z) = \bigvee_{z=x*y} [\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)]$$

Using extension principle (see Section 14.3), where $x, y \in R$, for min ($\wedge$) and max ($\vee$) operation, we have

$$\mu_{\underset{\sim}{A}*\underset{\sim}{B}}(z) = \sup_{z=x*y} [\mu_{\underset{\sim}{A}}(x) * \mu_{\underset{\sim}{B}}(y)]$$

Using λ-cut, the above two equations become

$$(\underset{\sim}{A} * \underset{\sim}{B})_\lambda = \underset{\sim}{A}_\lambda * \underset{\sim}{B}_\lambda \text{ for all } \lambda \in [0,1]$$

where $\underset{\sim}{A}_\lambda = [a_1^{(\lambda)}, a_2^{(\lambda)}]$ and $\underset{\sim}{B}_\lambda = [b_1^{(\lambda)}, b_2^{(\lambda)}]$. Note that for $a_1, a_2 \in [0,1]$, if $a_1 > a_2$, then $\underset{\sim}{A}_{a_1} \subset \underset{\sim}{A}_{a_2}$.

On extending the *addition and subtraction* operations on intervals to two fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in R, we get

$$\underset{\sim}{A}_\lambda + \underset{\sim}{B}_\lambda = [a_1^\lambda + b_1^\lambda, a_2^\lambda + b_2^\lambda]$$
$$\underset{\sim}{A}_\lambda - \underset{\sim}{B}_\lambda = [a_1^\lambda - b_2^\lambda, a_2^\lambda - b_1^\lambda]$$

Similarly, on extending the *multiplication and division* operations on two fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in R^+ (non-negative real line) $= [0, \infty]$, we get

$$\underset{\sim}{A}_\lambda \cdot \underset{\sim}{B}_\lambda = [a_1^\lambda \cdot b_1^\lambda, a_2^\lambda \cdot b_2^\lambda]$$
$$\underset{\sim}{A}_\lambda \div \underset{\sim}{B}_\lambda = \left[\frac{a_1^\lambda}{b_2^\lambda}, \frac{a_2^\lambda}{b_1^\lambda}\right], \quad b_2^\lambda > 0$$

The *multiplication* of a fuzzy number $\underset{\sim}{A} \subset R$ by an ordinary number $\beta \in R^+$ can be defined as

$$(\beta \cdot \underset{\sim}{A})_\lambda = [\beta a_1^\lambda, \beta a_2^\lambda]$$

The *support* for a fuzzy number, say $\underset{\sim}{A}$, is given by

$$\text{supp } \underset{\sim}{A} = \{x \mid \mu_{\underset{\sim}{A}}(x) > 0\}$$

which is an interval on the real line, denoted symbolically as A. The support of the fuzzy number resulting from the arithmetic operation $\underset{\sim}{A} * \underset{\sim}{B}$, i.e.,

$$\underset{\underset{\sim}{A}*\underset{\sim}{B}}{\text{supp}}(z) = A * B$$

is the arithmetic operation on the two individual supports, A and B, for fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$, respectively.

In general, arithmetic operations on fuzzy numbers based on λ-cut are given by (as mentioned earlier)

$$(\underset{\sim}{A} * \underset{\sim}{B})_\lambda = A_\lambda * B_\lambda$$

The algebraic properties of fuzzy numbers are listed in Table 14-3. The operations on fuzzy numbers possess the following properties as well.

TABLE 14-3 ALGEBRAIC PROPERTIES OF ADDITION AND MULTIPLICATION ON FUZZY NUMBERS

Property	Addition	Multiplication
Fuzzy numbers	$A, B, C \subset R$	$A, B, C \subset R^{+}$
Commutativity	$A + B = B + A$	$A \cdot B = B \cdot A$
Associativity	$(A + B) + C = A + (B + C)$	$(A \cdot B) \cdot C = A \cdot (B \cdot C)$
Neutral number	$A + 0 = 0 + A = A$	$A \cdot 1 = 1 \cdot A = A$
Image and inverse	$A + \bar{A} = \bar{A} + A \neq 0$	$A \cdot A^{-1} = A^{-1} \cdot A \neq 1$

1. If A and B are fuzzy numbers in R, then $(A+B)$ and $(A-B)$ are also fuzzy numbers. Similarly if A and B are fuzzy numbers in R^{+}, then $(A \cdot B)$ and $(A \div B)$ are also fuzzy numbers.
2. There exist no image and inverse fuzzy numbers, $\overline{A}$ and A^{-1}, respectively.
3. The inequalities given below stand true:

$$(A-B)+B \neq A \quad \text{and} \quad (A+B) \cdot B \neq A$$

14.2.3 Fuzzy Ordering

There exist several methods to compare two fuzzy numbers. The technique for fuzzy ordering is based on the concept of possibility measure.

For a fuzzy number $\underset{\sim}{A}$, two fuzzy sets $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are defined. For this number, the set of numbers that are possibly greater than or equal to $\underset{\sim}{A}$ is denoted as $\underset{\sim}{A}_1$ and is defined as

$$\mu_{\underset{\sim}{A}_1}(w) = \prod_{\underset{\sim}{A}}(-\infty, w) = \sup_{u \leq w} \mu_{\underset{\sim}{A}}(u)$$

In a similar manner, the set of numbers that are necessarily greater than $\underset{\sim}{A}$ is denoted as $\underset{\sim}{A}_2$ and is defined as

$$\mu_{\underset{\sim}{A}_2}(w) = N_{\underset{\sim}{A}}(-\infty, w) = \inf_{u \geq w}[1 - \mu_{\underset{\sim}{A}}(u)]$$

where $\prod_A$ and N_A are possibility and necessity measures (see Section 14.4.3). Figure 14-1 shows the fuzzy number and its associated fuzzy sets $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$.

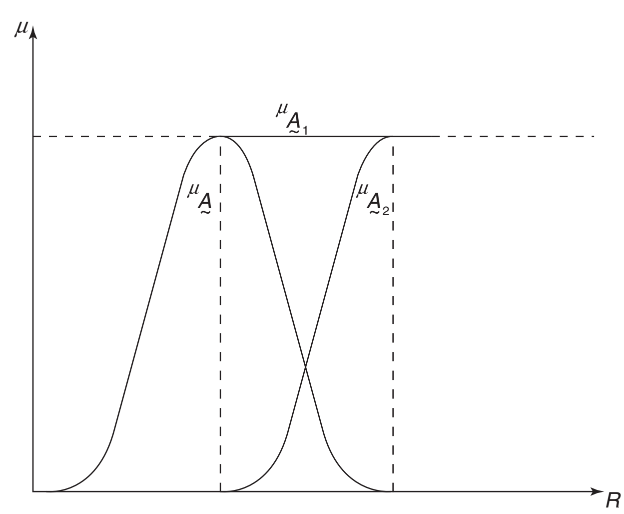

Figure 14-1 Fuzzy number $\underset{\sim}{A}$ and its associated fuzzy sets.

When we try to compare two fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$ to check whether $\underset{\sim}{A}$ is greater than $\underset{\sim}{B}$, we split both the numbers into their associated fuzzy sets. We can compare $\underset{\sim}{A}$ with $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ by index of comparison such as the possibility or necessity measure of a fuzzy set. That is, we can calculate the possibility and necessity measures, in the set $\mu_{\underset{\sim}{A}}$, of fuzzy sets $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$. On the basis of this, we obtain four fundamental indices of comparison which are given below.

1. $\prod_{\underset{\sim}{A}}(\underset{\sim}{B}_1) = \sup_u \min\,(\mu_{\underset{\sim}{A}}(u), \sup_{v\le u} \mu_{\underset{\sim}{B}}(v)) = \sup_{u\ge v} \min(\mu_{\underset{\sim}{A}}(u),\ \mu_{\underset{\sim}{B}}(v))$

   This shows the possibility that the largest value X can take is at least equal to smallest value that Y can take.
2. $\prod_{\underset{\sim}{A}}(\underset{\sim}{B}_2) = \sup_u \min\,(\mu_{\underset{\sim}{A}}(u), \inf_{v\ge u}[1-\mu_{\underset{\sim}{B}}(v)]) = \sup_u \inf_{v\ge u} \min\,(\mu_{\underset{\sim}{A}}(u),[1-\mu_{\underset{\sim}{B}}(v)])$

   This shows the possibility that the largest value X can take is greater than the largest value that Y can take.
3. $N_{\underset{\sim}{A}}(\underset{\sim}{B}_1) = \inf_u \max\,(1-\mu_{\underset{\sim}{A}}(v), \sup_{v\le u} \mu_{\underset{\sim}{B}}(v)) = \inf_u \sup_{v\le u} \max\,(1-\mu_{\underset{\sim}{A}}(u),\ \mu_{\underset{\sim}{B}}(v))$

   This shows the possibility that the smallest value X can take is at least equal to smallest value that Y can take.
4. $N_{\underset{\sim}{A}}(\underset{\sim}{B}_2) = \inf_u \max\,(1-\mu_{\underset{\sim}{A}}(u), \inf_{v\ge u}\,[1-\mu_{\underset{\sim}{B}}(v)]) = 1-\sup_{u\le v} \min\,[\mu_{\underset{\sim}{A}}(u),\ \mu_{\underset{\sim}{B}}(v)]$

   This shows the possibility that the smallest value X can take is greater than the largest value that Y can take.

14.2.4 Fuzzy Vectors

A vector $\underset{\sim}{P} = (P_1, P_2, \ldots, P_n)$ is called a fuzzy vector if for any element we have $0 \le P_i \le 1$ for $i = 1$ to n. Similarly, the transpose of the fuzzy vector $\underset{\sim}{P}$, denoted by $\underset{\sim}{P}^{\mathrm{T}}$, is a column vector if $\underset{\sim}{P}$ is a row vector, i.e.,

$$\underset{\sim}{P}^{\mathrm{T}} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

Let us define $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ as fuzzy vectors of length n and $\underset{\sim}{P}\cdot\underset{\sim}{Q}^{\mathrm{T}} = \bigvee_{i=1}^{n}(P_i \wedge Q_i)$ as the fuzzy inner product of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$. Then the fuzzy outer product of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ is defined by

$$\underset{\sim}{p} \oplus \underset{\sim}{Q}^{\mathrm{T}} = \bigwedge_{i=1}^{n}(P_i \vee Q_i)$$

The component of the fuzzy vector is defined as

$$\overline{\underset{\sim}{P}} = (1-P_1, 1-P_2, \ldots, 1-P_n) = (\overline{P_1}, \overline{P_2}, \overline{P_3}, \ldots, \overline{P_n})$$

The fuzzy complement vector $\overline{\underset{\sim}{P}}$ has the constraint $0 \le \overline{P_i} \le 1$, for $i = 1$ to n, and it is also a fuzzy vector.

The largest component $\hat{P}$ in the fuzzy vector $\underset{\sim}{P}$ is defined as its upper bound, i.e.,

$$\hat{P} = \max_i(P_i)$$

The smallest component $\underset{\wedge}{P}$ of the fuzzy vector $\underset{\sim}{P}$ is defined by its lower bound, i.e.,

$$\underset{\wedge}{P} = \min_i(P_i)$$

The properties that the two fuzzy vectors $\underset{\sim}{P}$ and $\underset{\sim}{Q}$, both of length n, are given as follows:

1. $\overline{\underset{\sim}{P} \cdot \underset{\sim}{Q}^{\mathrm{T}}} = \overline{\underset{\sim}{P}} \oplus \overline{\underset{\sim}{Q}}^{\mathrm{T}}$
2. $\overline{\underset{\sim}{P} \oplus \underset{\sim}{Q}^{\mathrm{T}}} = \overline{\underset{\sim}{P}} \cdot \overline{\underset{\sim}{Q}}^{\mathrm{T}}$
3. $\underset{\sim}{P} \cdot \underset{\sim}{Q}^{\mathrm{T}} \leq (\hat{P} \wedge \hat{Q})$
4. $\underset{\sim}{P} \oplus \underset{\sim}{Q}^{\mathrm{T}} = (\underset{\wedge}{P} \vee \underset{\wedge}{Q})$
5. $\underset{\sim}{P} \cdot \underset{\sim}{P}^{\mathrm{T}} = \hat{P}$
6. $\underset{\sim}{P} \oplus \underset{\sim}{P}^{\mathrm{T}} \geq \underset{\wedge}{P}$
7. If $\underset{\sim}{P} \subseteq \underset{\sim}{Q}$ then $\underset{\sim}{P} \cdot \underset{\sim}{Q}^{\mathrm{T}} = \hat{P}$ and if $\underset{\sim}{Q} \subseteq \underset{\sim}{P}$ then $\underset{\sim}{P} \oplus \underset{\sim}{Q}^{\mathrm{T}} = \underset{\wedge}{P}$
8. $\underset{\sim}{P} \cdot \overline{\underset{\sim}{P}} \leq \frac{1}{2}$
9. $\underset{\sim}{P} \oplus \overline{\underset{\sim}{P}} \leq \frac{1}{2}$

It should be noted that when two separate fuzzy vectors are identical, i.e., $\underset{\sim}{P} = \underset{\sim}{Q}$, the inner product $\underset{\sim}{P}\underset{\sim}{Q}^{\mathrm{T}}$ reaches a maximum value while the outer product $\underset{\sim}{P} \oplus \underset{\sim}{Q}^{\mathrm{T}}$ reaches a minimum value.

14.3 EXTENSION PRINCIPLE

Extension principle was introduced by Zadeh in 1978 and is a very important tool of fuzzy set theory. This extension principle allows the generalization of crisp sets into the fuzzy set framework and extends point-to-point mappings to mappings for fuzzy sets. This principle allows any function f – that maps an n-tuple $(x_1, x_2, \ldots, x_n)$ in the crisp set U to a point in the crisp set V – to be generalized as a set that maps n fuzzy subsets in U to a fuzzy set in V. Thus, any mathematical relationship between nonfuzzy crisp elements can be extended to deal with fuzzy entities. The extension principle is also useful to deal with set-theoretic operations for higher order fuzzy sets.

Given a function $f: M \rightarrow N$ and a fuzzy set in M, where

$$\underset{\sim}{A} = \frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \cdots + \frac{\mu_n}{x_n}$$

the extension principle states that

$$f(\underset{\sim}{A}) = f\left(\frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \cdots + \frac{\mu_n}{x_n}\right) = \frac{\mu_1}{f(x_1)} + \frac{\mu_2}{f(x_2)} + \cdots + \frac{\mu_n}{f(x_n)}$$

If f maps several elements of M to the same element y in N (i.e., many-to-one mapping), then the maximum among their membership grades is taken. That is,

$$\mu_{f(\underset{\sim}{A})}(y) = \max_{\substack{x_i \in M \\ f(x_i) = y}} [\mu_{\underset{\sim}{A}}(x_i)]$$

where x_i's are the elements mapped to same element y. The function f maps n -tuples in M to a point in N.

Let M be the Cartesian product of universes $M = M_1 \times M_2 \times \cdots \times M_n$ and $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ be n fuzzy sets in $M_1, M_2, \ldots, M_n$, respectively. The function f maps an n-tuple $(x_1, x_2, \ldots, x_n)$ in the crisp set M to a point y in the crisp set V, i.e., $y = f(x_1, x_2, \ldots, x_n)$. The function $f(x_1, x_2, \ldots, x_n)$ to be extended to act on the n fuzzy subsets of M, $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ is permitted by the extension principle such that

$$\underset{\sim}{I} = f(\underset{\sim}{A})$$

where $\underset{\sim}{I}$ is the fuzzy image of $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ through $f(\cdot)$. The fuzzy set $\underset{\sim}{B}$ is defined by

$$\underset{\sim}{B} = \{(y, \mu_{\underset{\sim}{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_n) \in M\}$$

where

$$\mu_{\underset{\sim}{B}}(y) = \sup_{\substack{(x_1, x_2, \ldots, x_n) \in M \\ y = f(x_1, x_2, \ldots, x_n)}} \min[\mu_{\underset{\sim}{A}_1}(x_1), \mu_{\underset{\sim}{A}_2}(x_2), \ldots, \mu_{\underset{\sim}{A}_n}(x_n)]$$

with a condition that $\mu_{\underset{\sim}{B}}(y) = 0$ if there exists no $(x_1, x_2, \ldots, x_n) \in M$ such that $y = f(x_1, x_2, \ldots, x_n)$.

The extension principle helps in propagating fuzziness through generalized relations that are discrete mappings of ordered pairs of elements from input universes to ordered pairs of elements from other universe. The extension principle is also useful for mapping fuzzy inputs through continuous-valued functions. The process employed is same as for a discrete-valued function, but it involves more computation.

14.4 FUZZY MEASURES

A fuzzy measure explains the imprecision or ambiguity in the assignment of an element a to two or more crisp sets. For representing uncertainty condition, known as ambiguity, we assign a value in the unit interval [0, 1] to each possible crisp set to which the element in the problem might belong. The value assigned represents the degree of evidence or certainty or belief of the element's membership in the set. The representation of uncertainty of this manner is called fuzzy measure. In sum, a fuzzy measure assigns a value in the unit interval [0, 1] to each classical set of the universal set signifying the degree of belief that a particular element x belongs to the crisp set. In this section several different fuzzy measures such as belief measures, plausibility measure, probability measure, necessity measure and possibility measure are covered. All these measures are functions applied to crisp subsets, instead of elements of a universal set.

The difference between a fuzzy measure and a fuzzy set on a universe of elements is that, in fuzzy measure, the imprecision is in the assignment of an element to one of two or more crisp sets, and in fuzzy sets, the imprecision is in the prescription of the boundaries of a set.

A fuzzy measure is defined by a function

$$g : P(X) \to [0,1]$$

which assigns to each crisp subset of a universe of discourse X a number in the unit interval [0, 1], where $P(X)$ is power set of X. A fuzzy measure is obviously a set function. To qualify a fuzzy measure, the function g should possess certain properties. A fuzzy measure is also described as follows:

$$g : B \to [0,1]$$

where $B \subset P(X)$ is a family of crisp subsets of X. Here B is a Borel field or a σ field. Also, g satisfies the following three axioms of fuzzy measures:

Axiom 1: Boundary Conditions (g1)

$$g(\phi) = 0;\ g(X) = 1$$

Axiom 2: Monotonicity (g2) – For every classical set $A, B \in P(X)$, if $A \subseteq B$, then $g(A) \le g(B)$.

Axiom 3: Continuity (g3) – For each sequence $(A_i \in P(X) \mid i \in N)$ of subsets of X, if either $A_1 \subseteq A_2 \subseteq \ldots$ or $A_1 \supseteq A_2 \supseteq \ldots$, then

$$\lim_{i \to \infty} g(A_i) = g(\lim_{i \to \infty} A_i)$$

where N is the set of all positive integers.

A σ field or Borel field satisfies the following properties:

1. $X \in B$ and $\phi \in B$.
2. If $A \in B$, then $\overline{A} \in B$.
3. B is closed under set union operation, i.e., if $A \in B$ and $B \in B$ (σ field), then $A \cup B \in B$ (σ field)

The fuzzy measure excludes the additive property of standard measures, h. The additive property states that when two sets A and B are disjoint, then

$$h(A \cup B) = h(A) + h(B)$$

The probability measure possesses this additive property. Fuzzy measures are also defined by another weaker axiom: subadditivity. The other basic properties of fuzzy measures are the following:

1. Since $\underset{\sim}{A} \subseteq \underset{\sim}{A} \cup \underset{\sim}{B}$ and $\underset{\sim}{B} \subseteq \underset{\sim}{A} \cup \underset{\sim}{B}$, and because fuzzy measure g possesses monotonic property, we have

$$g(\underset{\sim}{A} \cup \underset{\sim}{B}) \geq \max[g(\underset{\sim}{A}), g(\underset{\sim}{B})]$$

2. Since $\underset{\sim}{A} \cap \underset{\sim}{B} \subseteq \underset{\sim}{A}$ and $\underset{\sim}{A} \cap \underset{\sim}{B} \subseteq \underset{\sim}{B}$, and because fuzzy measure g possesses monotonic property, we have

$$g(\underset{\sim}{A} \cap \underset{\sim}{B}) \leq \min[g(\underset{\sim}{A}), g(\underset{\sim}{B})]$$

14.4.1 Belief and Plausibility Measures

The belief measure is a fuzzy measure that satisfies three axioms g1, g2 and g3 and an additional axiom of subadditivity. A belief measure is a function

$$\text{bel}: B \to [0,1]$$

satisfying axioms g1, g2 and g3 of fuzzy measures and subadditivity axiom. It is defined as follows:

$$\begin{aligned}\text{bel}\ (A_1 \cup A_2 \cup \cdots \cup A_n) \geq & \sum_i \text{bel}\ (A_i) - \sum_{i<j} \text{bel}\ (A_i \cap A_j) \\ & + \cdots + (-1)^{n-1}\ \text{bel}\ (A_1 \cap A_2 \cap \cdots \cap A_n)\end{aligned}$$

for every $n \in N$ and every collection of subsets of X. N is set of all positive integer. This is called axiom 4 (g4).

For $n = 2$, g4 is of the form

$$\text{bel}\ (A_1 \cup A_2) \geq \text{bel}\ (A_1) + \text{bel}\ (A_2) - \text{bel}\ (A_1 \cap A_2)$$

For $n = 2$, if $A_1 = A$ and $A_2 = \overline{A}$, axiom g4 indicates

$$\text{bel}\ (A_1 \cup A_2) = \text{bel}\ (A \cup \overline{A})$$

$$\text{bel}\ (A \cup \overline{A}) \geq \text{bel}\ (A) + \text{bel}\ (\overline{A}) - \text{bel}\ (A \cap \overline{A})$$

Since $A \cup \overline{A} = X$ and $A \cap \overline{A} = \phi$, we have

$$\text{bel}\ (X) \geq \text{bel}\ (A) + \text{bel}\ (\overline{A})$$

$$\text{bel}\ (A) + \text{bel}\ (\overline{A}) \leq 1$$

On the basis of the belief measure, one can define a plausibility measure Pl as

$$\text{Pl}\ (A) = 1 - \text{bel}\ (\overline{A})$$

for all $A \in B(CP(X))$. On the other hand, based on plausibility measure, belief measure can be defined as

$$\text{bel } (A) = 1 - \text{Pl } (\overline{A})$$

Plausibility measure can also be defined independent of belief measure. A plausibility measure is a function

$$\text{Pl}: B \rightarrow [0,1]$$

satisfying axioms g1, g2, g3 of fuzzy measures and the following additional subadditivity axiom (axiom g5):

$$\text{Pl } (A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_i \text{Pl } (A_i) - \sum_{i<j} \text{Pl } (A_i \cup A_j) + \cdots + (-1)^{n-1} \text{ Pl } (A_1 \cup A_2 \cup \cdots \cup A_n)$$

for every $n \in N$ and all collection of subsets of X. For $n = 2$, consider $A_1 = A$ and $A_2 = \overline{A}$, then we have

$$\text{Pl } (A \cap \overline{A}) \leq \text{Pl } (A) + \text{Pl } (\overline{A}) - \text{Pl}(A \cup \overline{A})$$

$$\Rightarrow \quad \text{Pl } (A) + \text{Pl } (\overline{A}) \geq 1$$

The belief measure and the plausibility measure are mutually dual, so it will be beneficial to express both of them in terms of a set function m, called a basic probability assignment. The basic probability assignment m is a set function,

$$m: B \rightarrow [0,1]$$

such that $m(\phi = 0)$ and $\sum_{A \in B} m(A) = 1$. The basic probability assignments are not fuzzy measures. The quantity $m(A) \in [0,1], A \in B(CP(X))$, is called A's basic probability number. Given a basic assignment m, a belief measure and a plausibility measure can be uniquely determined by

$$\text{bel } (A) = \sum_{B \subseteq A} m(B)$$

$$\text{Pl } (A) = \sum_{B \cap A \neq 0} m(B)$$

for all $A \in B(\text{CP}(X))$.

The relations among $m(A)$, bel (A) and Pl (A) are as follows:

1. $m(A)$ measures the belief that the element $(x \in X)$ belongs to set A alone, not the total belief that the element commits in A.
2. bel (A) indicates total evidence that the element $(x \in X)$ belongs to set A and to any other special subsets of A.
3. Pl (A) includes the total evidence that the element $(x \in X)$ belongs to set A or to other special subsets of A plus the additional evidence or belief associated with sets that overlap with A.

Based on these relations, we have

$$\text{Pl } (A) \geq \text{bel } (A) \geq m(A) \quad \forall \ A \in B \, (\sigma \text{ field})$$

Belief and plausibility measure are dual to each other. The corresponding basic assignment m can be obtained from a given plausibility measure Pl:

$$m(A) = \sum_{B \subseteq A} (-1)^{(A-B)} [1 - \text{Pl } (\overline{B})] \quad \forall A \in B \, (\sigma \text{ field})$$

Every set $A \in B(CP(X))$ for which $m(A) > 0$ is called a focal element of m. Focal elements are subsets of X on which the available evidence focuses.

14.4.2 Probability Measures

On replacing the axiom of subadditivity (axiom g4) with a stronger axiom of additivity (axiom g6),

$$\text{bel}\,(A \cup B) = \text{bel}\,(A) + \text{bel}\,(B) \text{ whenever } A \cap B = \phi; A, B \in B(\sigma \text{ field})$$

we get the crisp probability measures (or Bayesian belief measures). In other words, the belief measure becomes the crisp probability measure under the additive axiom.

A probability measure is a function

$$P : B \rightarrow [0, 1]$$

satisfying the three axioms g1, g2 and g3 of fuzzy measures and the additivity axiom (axiom g6) as follows:

$$P(A \cup B) = P(A) + P(B) \text{ whenever } A \cap B = \phi, A, B \in B$$

With axiom g6, the theorem given below relates the belief measure and the basic assignment to the probability measure.

> *"A belief measure bel on a finite σ- field B, which is a subset of $P(X)$, is a probability measure if and only if its basic probability assignment m is given by $m(\{x\}) = \text{bel}\,(\{x\})$ and $m(A) = 0$ for all subsets of X that are not singletons".*

The theorem mentioned is very significant. The theorem indicates that a probability measure on finite sets can be represented uniquely by a function defined on the elements of the universal set *X* rather than its subsets. The probability measures on finite sets can be fully represented by a function,

$$P : X \rightarrow [0,1] \quad \text{such that} \quad P(x) = m(\{x\})$$

This function $P(X)$ is called probability distribution function. Within probability measure, the total ignorance is expressed by the uniform probability distribution function

$$P(x) = m(\{x\}) = \frac{1}{|X|} \text{ for all } x \in X$$

The plausibility and belief measures can be viewed as upper and lower probabilities that characterize a set of probability measures.

14.4.3 Possibility and Necessity Measures

In this section, let us discuss two subclasses of belief and plausibility measures, which focus on nested focal elements. A group of subsets of a universal set is nested if these subsets can be ordered in a way that each is contained in the next; i.e., $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n, A_i \in P(X)$ are nested sets. When the focal elements of a body of evidence $(E,\ m)$ are nested, the linked belief and plausibility measures are called consonants, because here the degrees of evidence allocated to them do not conflict with each other. The belief and plausibility measures are characterized by the following theorem:

Theorem: Consider a consonant body of evidence (E, m), the associated consonant belief and plausibility measures posses the following properties:

$$\text{bel}\,(A \cap B) = \min\,[\text{bel}\,(A), \text{bel}\,(B)]$$

$$\text{Pl}\,(A \cup B) = \max\,[\text{Pl}\,(A), \text{Pl}(B)]$$

for all $A_1 B \in B(\text{CP}(X))$.

Consonant belief and plausibility measures are referred to as necessity and possibility measures and are denoted by *N* and $\prod$, respectively. The possibility and necessity measures are defined independently as follows:

The possibility measure $\prod$ and necessity measure N are functions

$$\prod : B \to [0,1]$$
$$N : B \to [0,1]$$

such that both $\prod$ and N satisfy the axioms g1, g2 and g3 of fuzzy measures and the following additional axiom (g7):

$$\prod(A \cup B) = \max\,(\prod(A), \prod(B)) \qquad \forall A, B \in B$$
$$N(A \cap B) = \min(N(A), N(B)) \qquad \forall A, B \in B$$

As necessity and possibility measures are special subclasses of belief and plausibility measures, respectively, they are related to each other by

$$\prod(A) = 1 - N(\overline{A})$$
$$N(A) = 1 - \prod(\overline{A}) \quad \forall A \in \sigma \text{ field}$$

The properties given below are based on the axiom $g7$ and above set of equations.

1. $\min[N(A), N(\overline{A})] = N(A \cap \overline{A}) = 0$. This implies that $A \; or \; \overline{A}$ is not necessary at all.
2. $\max[\prod(A), \prod(\overline{A})] = \prod(A \cup \overline{A}) = \prod(X) = 1$. This implies that either A or $\overline{A}$ is completely possible.
3. $\prod(A) \geq N(A) \; \forall A \subseteq \sigma$ field.
4. If $N(A) > 0$ then $\prod(A) = 1$ and if $\prod(A) < 1$ then $N(A) = 0$.

The two equations indicate that if an event is necessary then it is completely possible. If it is not completely possible then it is not necessary. Every possibility measure $\prod$ on $B \subset P(x)$ can be uniquely determined by a possibility distribution function

$$\prod : x \to [0,1]$$

using the formula

$$\prod(A) = \max_{x \in A} \prod(x) \qquad \forall x \in \sigma \text{ field}$$

The necessity and possibility measure are mutually dual with each other. As a result we can obtain the necessity measure from the possibility distribution function. This is given as

$$N(A) = 1 - \prod(\overline{A}) = 1 - \max_{x \notin A} \prod(x)$$

The total ignorance can be expressed in terms of the possibility distribution by $\prod(x_n) = 1$ and $\prod(x_i) = 0$ for $i = 1$ to $n - 1$, corresponding to $\prod(A_n) = \prod(X) = 1$ and $\prod(A) = 0$.

14.5 MEASURES OF FUZZINESS

The concept of fuzzy sets is a base frame for dealing with vagueness. In particular, the fuzzy measures concept provides a general mathematical framework to deal with ambiguous variables. Thus, fuzzy sets and fuzzy measures are tools for representing these ambiguous situations. Measures of uncertainty related to vagueness are referred to as measures of fuzziness.

Generally, a measure of fuzziness is a function

$$f : \underset{\sim}{P}(X) \to R$$

where R is the real line and $P(X)$ is the set of all fuzzy subsets of X. The function f satisfies the following axioms:

1. *Axiom 1* (f1): $f(A)=0$ if and only if A is a crisp set.
2. *Axiom 2* (f2): If A (shp) B, then $f(A) \leq f(B)$, where A (shp) B denotes that A is sharper than B.
3. *Axiom 3* (f3): $f(A)$ takes the maximum value if and only if A is maximally fuzzy.

Axiom f1 shows that a crisp set has zero degree of fuzziness in it. Axioms f2 and f3 are based on concept of "sharper" and "maximal fuzzy", respectively.

1. The first fuzzy measure can be defined by the function:

$$f(A)=-\sum_{x\in A}\left\{\mu_A(x)\log_2\left[\mu_A(x)\right]\left[1-\mu_A(x)\right]\log_2[1-\mu_A(x)]\right\}$$

   It can be normalized as

$$f'(A)=\frac{f(A)}{|x|}$$

   where $|x|$ is cardinality of universal set X. This measure of fuzziness can be considered as the entropy of a fuzzy set

2. A (shp) B, A is sharper than B, is defined as

$$\begin{aligned}&\mu_A(x)\leq\mu_B(x) \quad \text{for } \mu_B \leq 0.5\\ &\mu_A(x)\geq\mu_B(x) \quad \text{for } \mu_B(x)\geq 0.5 \quad \forall x\in X\end{aligned}$$

3. A is maximally fuzzy if

$$\mu_A(x)=0.5 \qquad \text{for all } x\in X$$

14.6 FUZZY INTEGRALS

Sugeno in the year 1977 defined fuzzy integral using fuzzy measures based on a Lebesque integral, which is defined using "measures".

Let K be a mapping from X to [0,1]. The fuzzy integral, in the sense of fuzzy measure g, of K over a subset A of X is defined as

$$\int_A K(x)\cdot g=\sup_{\alpha\in[0,1]}\min[\beta,g(A\cap H_\beta)]$$

where $H_\beta=\{x\in x\,|\,K(x)\geq\beta\}$. Here, A is called the domain of integration. If $k=a\in[0,1]$ is a constant, then its fuzzy integral over X is "a" itself, because $g(X\cap H_\beta)=1$ for $\beta\leq a$ and $g(X\cap H_\beta)=0$ for $\beta>a$, i.e.,

$$\int_x a\cdot g=a, \qquad a\in[0,1]$$

Consider X to be a finite set such that $X=\{x_1,x_2,\ldots,x_n\}$. Without loss of generality, assuming the function to be integrated, k can be obtained such that $k(x_1)\geq k(x_2)\geq\cdots\geq k(x_n)$. This is obtained after proper ordering. The basic fuzzy integral then becomes

$$\int_X k(x)\cdot g=\max_{i=1\text{ to }n}\min[k(x_i),g(H_i)]$$

where $H_i=\{x_1,x_2,\ldots,x_i\}$. The calculation of the fuzzy measure "g" is a fundamental point in performing a fuzzy integration.

14.7 SUMMARY

In this chapter we discussed *fuzzy arithmetic*, which is considered as an extension of interval arithmetic. The chapter provides a general methodology for extending crisp concepts to address fuzzy quantities, such as real algebraic operations on fuzzy numbers. One of the important tools of fuzzy set theory introduced by Zadeh is the extension principle, which allows any mathematical relationship between nonfuzzy elements to be extended to fuzzy entities. This principle can be applied to algebraic operations to define set-theoretic operations for higher order fuzzy sets. The operations and properties of fuzzy vectors were discussed in this chapter for their use in similarity metrics. Also, we have discussed the concept of fuzzy measures and the axioms that must be satisfied by a set function in order for it to be a fuzzy measure. We also discuss *belief and plausibility measures* which are based on the dual axioms of subadditivity. The belief and plausibility measures can be expressed by the basic probability assignment m, which assigns degree of evidence or belief indicating that a particular element of X belongs only to set A and not to any subset of A. Focal elements are the subsets that are assigned with nonzero degrees of evidence. The main characteristic of *probability measures* is that each of them can be distinctly represented by a probability distribution function defined on the elements of a universal set apart from its subsets. Also the *necessity and possibility measures*, which are consonant belief measures and consonant plausibility measures, respectively, are characterized distinctly by functions defined on the elements of the universal set rather than on its subsets. The fuzzy integrals defined by Sugeno (1977) are also discussed. Fuzzy integrals are used to perform integration of fuzzy functions. The measures of fuzziness were also discussed. The definitions of measures of fuzziness dealt in this chapter can be extended to noninfinite supports by replacing the summation by integration appropriately.

14.8 SOLVED PROBLEMS

1. Perform the following operations on intervals:

(a) [3, 2] + [4, 3] **(b)** [2, 1] × [1, 3]
(c) [4, 6] ÷ [1, 2] **(d)** [3, 5] – [4, 5]

Solution: The operations were performed on the basis of the interval analysis.

(a) $$\begin{aligned}[3,2]+[4,3] &= [a_1,a_2]+[b_1,b_2]\\ &= [a_1+b_1, a_2+b_2]\\ &= [3+4, 2+3] = [7,5]\end{aligned}$$

(b) $$\begin{aligned}[2,1]\times[1,3] &= [a_1,a_2]\cdot[b_1,b_2]\\ &= [a_1\cdot b_1, a_2\cdot b_2]\end{aligned}$$

(c) $$\begin{aligned}[4,6]\div[1,2] &= [a_1,a_2]\div[b_1,b_2]\\ &= [2\cdot 1, 1\cdot 3] = [2,3]\\ &= \left[\frac{a_1}{b_2},\frac{a_2}{b_1}\right]\\ &= \left[\frac{4}{2},\frac{6}{1}\right] = [2,6]\end{aligned}$$

(d) $$\begin{aligned}[3,5]-[4,5] &= [a_1,a_2]-[b_1,b_2]\\ &= [a_1-b_2, a_2-b_1]\\ &= [3-5, 5-4] = [-2,1]\end{aligned}$$

2. For the interval $\underset{\sim}{A} = [5,3]$, find its image and inverse.

Solution: The given interval is

$$\underset{\sim}{A} = [5,3] = [a_1, a_2]$$

(a) Image $\overline{A} = [-a_2, -a_1] = [-3, -5]$

(b) Inverse $$A^{-1} = \left[\frac{1}{a_2},\frac{1}{a_1}\right] = \left[\frac{1}{3},\frac{1}{5}\right] = [0.333,\ 0.2]$$

3. Given the two intervals $\underset{\sim}{E} = [2,4]$, $\underset{\sim}{F} = [-4,5]$, perform the max and min operations over these intervals.

Solution: The given intervals are $\underset{\sim}{E} = [a_1,a_2] = [2,4]$ and $\underset{\sim}{F} = [b_1,b_2] = [-4,5]$.

(a) Max operation

$$\begin{aligned}\underset{\sim}{E}\vee\underset{\sim}{F} &= [a_1,a_2]\vee[b_1,b_2] = [a_1\vee b_1, a_2\vee b_2]\\ &= [2\vee -4, 4\vee 5] = [2,5]\end{aligned}$$

(b) Min operation

$$\begin{aligned}\underset{\sim}{E}\wedge\underset{\sim}{F} &= [a_1,a_2]\wedge[b_1,b_2] = [2,4]\wedge[-4,5]\\ &= [2\wedge -4, 4\wedge 5] = [-4,4]\end{aligned}$$

4. Consider a fuzzy number $\underset{\sim}{1}$, the normal convex membership function defined on integers

$$\underset{\sim}{1} = \left\{\frac{0.5}{0} + \frac{1}{1} + \frac{0.5}{2}\right\}$$

Perform addition of two fuzzy numbers, i.e., add $\underset{\sim}{1}$ to $\underset{\sim}{1}$ using extension principle.

Solution:

$$\underset{\sim}{1}+\underset{\sim}{1}=\left(\frac{0.5}{0}+\frac{1}{1}+\frac{0.5}{2}\right)+\left(\frac{0.5}{0}+\frac{1}{1}+\frac{0.5}{2}\right)$$

$$\underset{\sim}{2}=\left\{\frac{\min(0.5,0.5)}{0}+\frac{\max[\min(0.5,1),\min(1,0.5)]}{1}+\frac{\max\left[\min(0.5,0.5),\min(1,1),\min(0.5,0.5)\right]}{2}+\frac{\max[\min(1,0.5),\min(0.5,1)]}{3}+\frac{\min(0.5,0.5)}{4}\right\}$$

$$=\left\{\frac{0.5}{0}+\frac{\max[0.5,0.5]}{1}+\frac{\max[0.5,1,0.5]}{2}+\frac{\max[0.5,0.5]}{3}+\frac{\min(0.5,0.5)}{4}\right\}$$

$$\underset{\sim}{2}=\left\{\frac{0.5}{0}+\frac{0.5}{0}+\frac{1}{2}+\frac{0.5}{3}+\frac{0.5}{4}\right\}$$

5. The two fuzzy vectors of length 4 are defined as

$$\underset{\sim}{a}=(0.5,0.2,1.0,0.8)$$
$$\text{and } \underset{\sim}{b}=(0.8,0.1,0.9,0.3)$$

Find the inner product and outer product for these two fuzzy vectors.

Solution:

(a) *Inner product*:

$$\underset{\sim}{a}\cdot\underset{\sim}{b}^{\mathrm{T}}=(0.5,0.2,1.0,0.8)\begin{pmatrix}0.8\\0.1\\0.9\\0.3\end{pmatrix}$$
$$=(0.5\wedge 0.8)\vee(0.2\wedge 0.1)\vee(1.0\wedge 0.9)\vee(0.8\wedge 0.3)$$
$$=0.5\vee 0.1\vee 0.9\vee 0.3=0.9$$

(b) *Outer product*:

$$\underset{\sim}{a}\oplus\underset{\sim}{b}^{\mathrm{T}}=(0.5,0.2,1.0,0.8)\begin{pmatrix}0.8\\0.1\\0.9\\0.3\end{pmatrix}$$
$$=(0.5\vee 0.8)\wedge(0.2\vee 0.1)\wedge(1.0\vee 0.9)\wedge(0.8\vee 0.3)$$
$$=(0.8)\wedge(0.2)\wedge(1.0)\wedge(0.8)=0.2$$

6. Let X be the universal set and let A, B, and C be the subsets of X. The basic assignments for the corresponding focal elements are mentioned in the following table. Determine the corresponding belief measure.

Focal elements	***m*(·)**
P	0.04
B	0.04
E	0.04
$P\cup B$	0.12
$P\cup E$	0.08
$B\cup E$	0.04
$P\cup B\cup E$	0.64

Solution: The belief measures are obtained as follows:

$$\text{bel}(P)=m(P)=0.04$$
$$\text{bel}(B)=m(B)=0.04$$
$$\text{bel}(E)=m(E)=0.04$$
$$\text{bel}(P\cup B)=m(P\cup B)+m(P)+m(B)=0.12+0.04+0.04=0.2$$
$$\text{bel}(P\cup E)=m(P\cup E)+m(P)+m(E)=0.08+0.04+0.04=0.16$$
$$\text{bel}(B\cup E)=m(B\cup E)+m(B)+m(E)=0.04+0.04+0.04=0.12$$
$$\text{bel}(P\cup B\cup E)=m(P\cup B\cup E)+m(P\cup B)+m(P\cup E)+m(B\cup E)+m(P)+m(B)+m(E)$$
$$=0.64+0.12+0.08+0.04+0.04+0.04+0.04$$
$$=1.0$$

14.9 REVIEW QUESTIONS

1. State the importance of fuzzy arithmetic.
2. How is an interval analysis obtained in fuzzy arithmetic?
3. List the set operations performed on intervals.
4. Discuss the mathematical operations performed on intervals.
5. What are the properties of performing addition and multiplication on intervals?
6. Define fuzzy numbers.
7. Mention the properties of addition and multiplication on fuzzy numbers.
8. Write short note on fuzzy ordering.
9. Explain in detail the concept of fuzzy vectors.
10. State the extension principle in fuzzy set theory.
11. What are fuzzy measures?
12. Explain in detail the belief and plausibility measures.
13. How are necessity and possibility measures obtained from belief and plausibility measures?
14. Discuss in detail:
    - Probability measure;
    - Fuzzy integrals.
15. Mention the measures of fuzziness in detail.

14.10 EXERCISE PROBLEMS

1. Perform the following operations on intervals
   (a) $[5,3]+[4,2]$ **(b)** $[6,9]-[2,4]$
   (c) $[1,2]\times[5,3]$ **(d)** $[7,3]\div[3,6]$
   (e) $[5,3]$ **(f)** $[6,5]^{-1}$

2. Perform the max and min operations over the intervals $\underset{\sim}{F}=[5,6]$ and $\underset{\sim}{G}=[9,2]$.

3. Given the following fuzzy numbers and using Zadeh's extension principle, calculate $\underset{\sim}{K}=\underset{\sim}{A}\cdot\underset{\sim}{B}$ and show why $\underset{\sim}{1}0$ is nonconvex.

$$\underset{\sim}{A}=\underset{\sim}{5}=\frac{0.2}{2}+\frac{1}{3}+\frac{0.1}{4}$$
$$\underset{\sim}{A}=\underset{\sim}{2}=\frac{0.1}{1}+\frac{1}{2}+\frac{0.2}{3}$$

4. Given

$$\underset{\sim}{A}=\frac{0.4}{0.2}+\frac{1}{0.4}+\frac{0.4}{0.6}$$
$$\underset{\sim}{B}=\frac{1}{0.2}+\frac{0.4}{0.4}+\frac{0.5}{0.6}$$

   calculate the following: $\underset{\sim}{A}+\underset{\sim}{B},\underset{\sim}{A}-\underset{\sim}{B},\underset{\sim}{A}*\underset{\sim}{B},\underset{\sim}{A}\div\underset{\sim}{B}$.

5. For the two triangular fuzzy numbers $\underset{\sim}{A}$ and $\underset{\sim}{B}$, whose membership functions are respectively

$$\mu_{\underset{\sim}{A}}(x)=\begin{cases}2-x & \text{if } -1\le x\le 0\\ \dfrac{x-2}{5} & \text{if } 0\ge x\ge 0\\ 0 & \text{otherwise}\end{cases}$$

$$\mu_{\underset{\sim}{B}}(x)=\begin{cases}x+1 & \text{if } -1\le x\le 0\\ \dfrac{3-x}{5} & \text{if } 0\ge x\ge 0\\ 0 & \text{otherwise}\end{cases}$$

   compute the following:

   (a) $\underset{\sim}{A}+\underset{\sim}{B},\ \underset{\sim}{A}-\underset{\sim}{B}$

   (b) $\underset{\sim}{A}\wedge\underset{\sim}{B},\ \underset{\sim}{A}\vee\underset{\sim}{B}$

   (c) $\underset{\sim}{A}\div\underset{\sim}{B},\ \underset{\sim}{A}\div\underset{\sim}{B}$

6. Consider the three fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ and their membership functions:

$$\mu_{\underset{\sim}{A}}(x)=\frac{1}{1+10x},\quad \mu_{\underset{\sim}{B}}(x)=\left(\frac{1}{1+x}\right),$$
$$\mu_{\underset{\sim}{C}}(x)=\left(\frac{1}{1+2x}\right)^{0.5}$$

   Order the fuzzy sets. Take $x\ge 0$.

7. The two fuzzy vectors of length 6 are defined as

$$\underset{\sim}{a}=(0.5,\ 0.7,\ 0.2,\ 0.3,\ 1,\ 0.8)$$
$$\underset{\sim}{b}=(0,\ 0.2,\ 0.1,\ 0.4,\ 0.6,\ 1.0)$$

   Find the inner product and outer product of two vectors.

8. Determine the corresponding belief and plausibility measures from the table below:

Focal elements	**m**
P	0.05
B	0.05
E	0.05
$P \cup B$	0.50
$P \cup E$	0.15
$B \cup E$	0.05
$P \cup B \cup E$	0.15

9. Consider the possibility distribution induced by the proposition "*x* is an even integer" is

$$\prod_x = \{(1,1),(2,3),(3,0.5),(4,0.4),(5,0.6),(6,0.3)\}$$

If $A = \{1, 2, 3\}$ is a crisp set, then find the possibility and necessity measures of A

10. With suitable example, show that the maximum measure of fuzziness is $|X|$.