Discrete Mathematics BCSC1010

Module 1

Recurrence Relations (Lecture10)

Lecture Notes by Dr. Praveen Mittal

Methods of solving RR

- Iteration
- Characteristic Roots
- Generating Functions

Iteration Method

- Solve the recurrence relation $\mathbf{a_n = a_{n-1} + 2,}$ $n \geq 2$ subject to initial condition $a_1 = 3$.
- Given $a_n = a_{n-1} + 2$ ....(1)

  then $a_{n-1} = a_{n-2} + 2$

  from(1) $a_n = (a_{n-2} + 2) + 2 = a_{n-2} + 2*2$ .....(2)

  also $a_{n-2} = a_{n-3} + 2$

  from(2) $a_n = (a_{n-3} + 2) + 2*2 = a_{n-3} + 3*2$

  In general, $a_n = a_{n-k} + k*2$

  put $k = n-1$; $a_n = a_1 + (n - 1) * 2 = 3 + 2n - 2$

therefore, $\boxed{\mathbf{a_n = 2n + 1}}$

Characteristic Roots Method

- In this method, we assume that solution to homogeneous recurrence relation $\mathbf{C_0a_n + C_1a_{n-1} + C_2a_{n-2} + \ldots\ldots\ldots + C_ka_{n-k} = 0}$ is of the form $\mathbf{a_n = Ar^n}$.

  Then, putting this in given recurrence relation

  $C_0 Ar^n + C_1 Ar^{n-1} + C_2 Ar^{n-2} + \ldots\ldots\ldots + C_k Ar^{n-k} = 0$

  or, $Ar^{n-k} [C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \ldots\ldots\ldots + C_k] = 0$

  or, $\mathbf{C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \ldots\ldots\ldots + C_k = 0}$

  This is called as characteristic equation of the recurrence relation and solutions to this are called as characteristic roots.

Characteristic Roots Method

- A characteristic equation of degree k has k characteristic roots.

- If these roots (e.g. r_1, r_2, r_3, ……, r_k) are all distinct and real, then general form of the solutions for homogeneous recurrence relation is:

  $$\mathbf{a_n = A_1\, r_1^{\,n} + A_2\, r_2^{\,n} + A_3\, r_3^{\,n} + \ldots\ldots\ldots + A_k\, r_k^{\,n}}$$

  where A_1, A_2, A_3, ……., A_k are constants which may be chosen to satisfy any initial conditions.

Solve the recurrence relation $a_n = a_{n-1} + 2\, a_{n-2}$, $n \geq 2$ with the initial conditions $a_0 = 0$, $a_1 = 1$.

Given $a_n = a_{n-1} + 2\, a_{n-2}$

i.e. $a_n - a_{n-1} - 2\, a_{n-2} = 0$

Then the characteristic eqn. is **$r^2 - r - 2 = 0$**

or, $(r - 2)(r + 1) = 0$

or, $r = 2, -1$

Therefore, the general solution is

$$\mathbf{a_n = A_1 (2)^n + A_2 (-1)^n}$$

Given $a_0 = 0$ then $A_1 + A_2 = 0$ .....(1)

and $a_1 = 1$ then $2A_1 - A_2 = 1$ .....(2)

Solving (1) and (2) we get $A_1 = ⅓$ and $A_2 = -⅓$

Hence the solution to the given homogeneous RR is:

$$\mathbf{a_n = \tfrac{1}{3} (2)^n - \tfrac{1}{3} (-1)^n}$$

Next Topic

- How to solve Recurrence Relations(Contd.)