Discrete Mathematics BCSC1010

Module 1
Dr. Praveen Mittal
Relations (Lecture 8)

Properties of Relations

- Reflexive Relation
- Irreflexive Relation
- Symmetric Relation
- Transitive Relation

Transitive Relation

- A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

- The relation R on a set A is transitive if we have

$$\forall a \forall b \forall c((a, b) \in R \text{ and } (b, c) \in R) \rightarrow (a, c) \in R).$$

Example

- R = {(2, 1), (3,1), (3, 2), (4,1), (4, 2), (4, 3)}
- R is transitive

Summary

Let R is a relation on set X. It is

- *Reflexive*
  - if $(a,a) \in R$, for each $a \in X$.
- *Symmetric*
  - if $(a,b) \in R$ implies $(b,a) \in R$.
- *Transitive*
  - if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.
- *Irreflexive*
  - if $(a,a) \notin R$, for each $a \in X$.

Representing Relations Using Matrices

- A relation between finite sets can be represented using a **zero-one** matrix
- Suppose that R is a relation from A = $\{a_1, a_2, \ldots, a_m\}$ to

B = $\{b_1, b_2, \ldots, b_n\}$.

- The rel nted by the matrix $M_R =$ $[m_{ij}]$, w

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to R. The 0s show that no other pairs belong to R. ◀

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Equivalence Relation

- A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example

- The relation "is equal to", denoted "=", is an equivalence relation on the set of real numbers since for any x, y, z ∈ R:
- 1. (Reflexivity) x = x,
- 2. (Symmetry) if x = y then y = x,
- 3. (Transitivity) if x = y and y = z then x = z.

Example

R be a relation on set of integers Z defined by

$R = \{(x, y) \mid x\text{-}y \text{ is divisible by } 3\}$. Show that R is an equivalence relation.

Solution

Consider any $a,b,c \in Z$

1. Since $a-a=0=3.0 \Rightarrow (a-a)$ is divisible by 3.

   $\Rightarrow (a,a) \in R$

   $\Rightarrow$ is reflexive.

Contd…

2. Let $(a,b) \in R \Rightarrow (a-b)$ is divisible by 3.

$\Rightarrow a-b=3q$

$\Rightarrow a-b=3q$ for some $q \in Z$

$\Rightarrow b-a=3(-q)$

$\Rightarrow (b-a)$ is divisible by 3 $(\because q \in Z \Rightarrow -q \in Z \Rightarrow -q \in Z)$

Thus, $(a,b) \in R \Rightarrow (b,a) \in R \Rightarrow R$ is symmetric.

3. Let $(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow (a-b)$ is divisible by 3 and $(b-c)$ is divisible by 3

$\Rightarrow a-b=3q$ and $b-c=3q'$ for some $q,q' \in Z$

$\Rightarrow (a-b)+(b-c)=3(q+q')$

$\Rightarrow a-c=3(q+q')$

$\Rightarrow (a-c)$ is divisible by 3

$(\because q.q' \in Z \Rightarrow q+q' \in Z)$

$\Rightarrow (a,c) \in R$

Thus, $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \Rightarrow R$ is transitive.

Therefore, the relation R is reflexive, symmetric and transitive, and hence it is an equivalence relation

Next Topic..

- Recurrence Relations