Statistics and complex Analysis

BMAS 0106

Chapter-1 (Analytic function)

Tutorial Sheet-1.

Problem: Type (1) Check the analyticity of $f(z)$

1. Discuss the analyticity of the following function and find the corresponding derivatives if possible

(a) $f(z) = z^2$ (b) $f(z) = z^4$ (c) $f(z) = e^{\bar{z}}$

(d) $f(z) = \cos z$ (e) $f(z) = \sinh z$ (d) $f(z) = \cosh z$

2. Determine a, b, c, d so that the function

$f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

3. Find the constants a, b, c such that the function

$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic.

4. Show that $f(z) = xy + iy$ is everywhere continuous but nowhere analytic

5. Show that the functions $f(z) = z + 2\bar{z}$ and $g(z) = z|z|$ nowhere analytic.

6. Prove that if $f(z)$ is analytic and $Re[f(z)] = $ constant then $f(z)$ is a constant function.

(ii) If $f(z)$ is an analytic function, show that $|f(z)|$ is not a harmonic ...

9. (i) Show that the function $u(x, y) = 2x + y^3 - 3x^2y$ is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$.

(ii) Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$. (U.P.T.U. 2009)

(iii) Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate of the function $u(x, y) = 2x(1 - y)$. (U.K.T.U. 2011)

(iv) Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic.

10. (i) Show that the function $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

(ii) Determine constant '$b$' such that $u = e^{bx} \cos 5y$ is harmonic.

11. Determine the analytic function $f(z)$ in terms of $z$ whose real part is

(i) $\frac{1}{2} \log (x^2 + y^2)$ (U.K.T.U. 2011)    (ii) $\cos x \cosh y$

(iii) $e^{-x} (x \cos y + y \sin y)$ ; $f(0) = 1$

(iv) $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$    (v) $\dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

12. Find the regular function $f(z)$ in terms of $z$ whose imaginary part is

(i) $\dfrac{x - y}{x^2 + y^2}$    (ii) $\cos x \cosh y$    (iii) $\sinh x \cos y$

(iv) $6xy - 5x + 3$    (v) $\dfrac{x}{x^2 + y^2} + \cosh x \cos y$.

13. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function $v$ such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of $z$.

14. (i) An electrostatic field in the $xy$-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

(ii) If the potential function is $\log (x^2 + y^2)$, find the flux function and the complex potential function.

15. (i) In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\frac{y}{x}\right)$, find the velocity potential $\phi$.

(ii) If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function $\phi$.

(iii) If $u = (x - 1)^3 - 3xy^2 + 3y^2$, determine $v$ so that $u + iv$ is a regular function of $x + iy$.

[U.K.T.U. 2010]

16. If $f(z)$ is an analytic function of $z$, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |R\, f(z)|^2 = 2\, |f'(z)|^2$.

17. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.

18. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of $z$ if

(i) $u - v = e^x (\cos y - \sin y)$    (ii) $u + v = \dfrac{x}{x^2 + y^2}$, when $f(1) = 1$

[U.P.T.U. (C.O.) 2008]

(iii) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ when $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.

Tutorial sheet - 2

Chapter - 1 (Analytic function)

Problem: Type II (Harmonic function)

1. check whether the following functions are harmonic or not

(I) $u(x,y) = x^2 - y^2$ (II) $u(x,y) = x^3 + 3xy^2$

(III) $v(x,y) = e^x \sin y$ (IV) $v(x,y) = \cosh x \sin y$

2. Determine constant 'b' such that $u(x,y) = e^{bx} \cos 5y$ is harmonic.

3. Find 'm' such that the function $u(x,y) = 2x - x^2 + my^2$ is harmonic.

Ans: 1 (I) Y. (II) N. (III) Y. (IV) Y.

Ans: 2. $b = \pm 5$ 3. $m = 1$

Problem: Type III (Milne-Thomson Method)

1. Determine an analytic function $f(z)$ in terms of $z$ whose real part is $u(x,y) = e^{-x}(x\sin y - y\cos y)$

Ans: $f(z) = ize^{-z} + c$

2. Find an analytic function $f(z)$ whose imaginary part is $\cos x \cosh y$. Ans: $f(z) = \cos z + c$

3. Determine analytic function $f(z)$ in terms of $z$ whose real part is

(I) $e^{-x}(x\cos y + y\sin y)$ and $f(0)=1$ Ans: $f(z) = 1 + ze^{-z}$

(II) $\dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ Ans: $f(z) = \tan z + c$

(III) $e^{2x}(x\cos 2y - y\sin 2y)$ Ans: $f(z) = ze^{2z} + c$

4. Determine analytic function $f(z)$ in terms of $z$ whose imaginary part is

(I) $6xy - 5x + 3$ Ans: $f(z) = 3z^2 - 5iz + c$

(II) $\sinh x \cos y$ Ans: $f(z) = i\sinh z + c$

(III) $\dfrac{x}{x^2+y^2} + \cosh x \cos y$ Ans: $f(z) = \dfrac{i}{z} + i\cosh z + c$

5. If $f(z) = u(x,y) + iv(x,y)$ is analytic, find $f(z)$ in $z$ if

(I) $u - v = e^{x}(\cos y - \sin y)$ Ans: $f(z) = e^{z} + c$

(II) $u - v = \dfrac{e^{y} - \cos x + \sin x}{\cosh y - \cos x}$ when $f\left(\frac{\pi}{2}\right) = \left(\frac{3-i}{2}\right)$

$f(z) = \cot\left(\frac{z}{2}\right) + \frac{(1-i)}{2}$

(III) $u + v = (x+y)(2 - 4xy + x^2 + y^2)$ Ans: $f(z) = 2z + iz^3 + c$

(IV) $u + v = \dfrac{x}{x^2+y^2}$ and $f(1) = 1$ $f(z) = \dfrac{1}{i+1}\left(\dfrac{i}{z} + 1\right)$

6. Show that the function $u(x,y) = 2x + y^3 - 3x^2y$ is harmonic. find its harmonic Conjugate function $v(x,y)$ and also find $f(z)$ in terms of $z$.

Ans: $f(z) = 2z + iz^3 + c$, $v(x,y) = 2y - 3x^2y + x^3 + c$

7. Show that the function $v(x,y) = e^x \sin y$ is harmonic. Further find $u(x,y)$ such that $f(z) = u + iv$ is analytic

Ans: $f(z) = e^z + c$, $u(x,y) = e^x \cos y + c$

8. If $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$, determine $v(x,y)$ Such that $f(z) = u + iv$ is analytic.

Ans: $v = 3y(1+x^2) - y^3$

9. Find an analytic function $f(z) = u + iv$ given that

$$v = \left(r - \frac{1}{r}\right) \sin\theta \quad r \neq 0$$

Ans: $f(z) = z + \frac{1}{z} + c$

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Tutorial Sheet - 3

chapter - 1

Analytic Functions

Problems - Type III (M.T.M.)

Ques (1) If $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of $z$ if

(I) $3u + v = 3\sin x \cosh y + \cos x \sinh y$

Ans: $f(z) = \sin z + c$

(II) $u - 2v = \cos x \cosh y + 2\sin x \sinh y$

Ans: $f(z) = \cos z + c$

Ques (2) If $\omega = \phi + i\psi$ represents the complex potential for an electrical field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine $\omega$ and $\phi$.

Ans: $\omega = i\left(z^2 + \frac{1}{z}\right) + c$

$$\phi(x,y) = -2xy + \frac{y}{x^2+y^2} + c$$

Note: For Real and Im. part of $\frac{1}{z}$ $\Rightarrow$ Real Part Im. part

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x-iy}{|z|^2} = \frac{x-iy}{x^2+y^2} = \left(\frac{x}{x^2+y^2}\right) - i\,\frac{y}{x^2+y^2}$$

Ques(3) Show that the function $u(r,\theta) = r^2 \cos 2\theta$ is harmonic. Find its harmonic conjugate and $f(z)$ in terms of $z$.

Ans: $v(r,\theta) = r^2 \sin 2\theta + c$ $\quad f(z) = z^2 + c$

Problems - Type (IV) $f(z)$ satisfy C-R Eq$^n$s at $(0,0)$ but $f'(0)$ does not exist

Ques(4) Show that the function
$$f(z) = \begin{cases} \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

satisfy C-R Eq$^n$s at $(0,0)$ but $f'(0)$ does not exist.

Ques (5) Show that the function
$$f(z) = \begin{cases} \dfrac{x^3 y (y - ix)}{x^6 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Satisfy C-R Eq$^n$s at $(0,0)$ but not diff. at $z=0$.

# Tutorial sheet: 4

## Chapter-2

## Complex Integration

Ques (1): Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+3y^2)\,dx + 2(x^2+3xy+4y^2)\,dy$

(I) along $y=x^2$ (II) along $y=x$.

Does the value of integral depend upon the path?

Ans: (I) $\frac{26}{3}$ (II) $\frac{26}{3}$ , No.

Ques (2): Evaluate $\int_0^{1+i} (x^2-iy)\,dz$ along the paths

(I) $y=x$ (II) $y=x^2$

Ans: (I) $\frac{5}{6}-\frac{i}{6}$ (II) $\frac{5}{6}+\frac{i}{6}$

Ques (3): Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$, along

(I) the real axis from $z=0$ to $z=2$ and then along a line parallel to $y$ axis from $z=2$ to $z=2+i$

(II) along the line $2y=x$

Ans: (I) $\frac{14}{3}+\frac{11}{3}i$ (II) $\frac{10}{3}-\frac{5}{3}i$

Ques (4): Evalute $\int_0^{3+i} z^2 dz$, along

(I) the line $y = \frac{x}{3}$ (II) the real axis from $z=0$ to $z=3$ and then vertically from $z=3$ to $z=3+i$ (III) parabola $x=3y^2$

Ans: (I) $6 + \frac{26}{3}i$ (II) $6 + \frac{26}{3}i$ (III) $6 + \frac{26}{3}i$

Note: $f(z) = z^2$ being an analytic $f^n$ $\int_c f(z)dz$ will be path independent.

Ques (5): Evaluate the integral $\int_c Re(z^2)\, dz$ from 0 to $2+4i$ along the line segment joining the points $(0,0)$ to $(2,4)$.

Ans: $-8(1+2i)$

Ques (6): Evaluate $\int_c (z-z^2)dz$ where C is upper half of circle $|z|=1$. What will be the value of this integral if C is the lower half of the given circle.

Ans: $\frac{2}{3}$ , $-\frac{2}{3}$

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# Tutorial sheet : 5

## Power series

## (Taylor's and Laurent's)

Ques (1) Find R.O.C. of the following series

(I) $\sum_{n=1}^{\infty} \frac{z^n}{n^p}$ (II) $\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}$ (III) $\sum_{n=0}^{\infty} (5+12i)^n z^n$

(IV) $\sum_{n=0}^{\infty} \frac{2n+3}{(2n+5)(n+5)} z^n$

Ans: (I) $R=1$ (II) $R=3$ (III) $R=\frac{1}{13}$

(IV) $R=1$

Ques (2) Expand the following functions as a Taylor's series

(I) $f(z) = \ln(z+1)$ about $z=0$

(II) $f(z) = \tan^{-1} z$ about $z=0$

(III) $f(z) = \sin z$ about $z=\frac{\pi}{4}$.

Ans: (I) $f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \text{------}$

(II) $f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \text{------}$

(III) $f(z) = \frac{1}{\sqrt{2}}\left[1 + \left(z-\frac{\pi}{4}\right) - \frac{1}{2}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6}\left(z-\frac{\pi}{4}\right)^3 + \cdots\right]$

Ques (3) Expand the function $f(z) = \frac{\sin z}{z-\pi}$ about $z=\pi$.

Ques (4) Expand the following function in a Laurent's Series

(I) $f(z) = \frac{e^z}{(z-1)^2}$ about $z=1$ (II) $f(z) = \frac{1}{z(z-1)(z-2)}$ for $|z-1|<1$

Ques (5) Find the Taylor's or Laurent's Series which represents the function $f(z) = \frac{1}{(1+z^2)(z+2)}$ when

(I) $|z|<1$ (II) $1<|z|<2$ (III) $|z|>2$.

Ques (6) Expand the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ when

(I) $|z|<2$ (II) $2<|z|<3$ (III) $|z|>3$

Ques (7) Expand $f(z) = \frac{z}{(z+1)(z+2)}$ as a Taylor's series about $z=2$.

Ans:(3) $f(z) = 1 + \frac{(z-\pi)^2}{\underline{|3}} - \frac{(z-\pi)^4}{\underline{|5}} + - - -$

4: (I) $f(z) = e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{\lfloor 2} + \frac{z-1}{\lfloor 3} + \cdots\right]$

(II) $f(z) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (z-1)^n - \frac{1}{z-1} - \frac{1}{2}\sum_{n=0}^{\infty}(z-1)^n$

5: (I) $f(z) = \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n + \frac{2-z}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}$

(II) $f(z) = \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n + \frac{2-z}{5z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n$

(III) $f(z) = \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{1}{5}\left(\frac{1}{z} - \frac{2}{z^2}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n$

6: (I) $f(z) = 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

(II) $f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

(III) $f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$

7: $f(z) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z-2}{3}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-2}{4}\right)^n$

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**Ques(1)** For the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

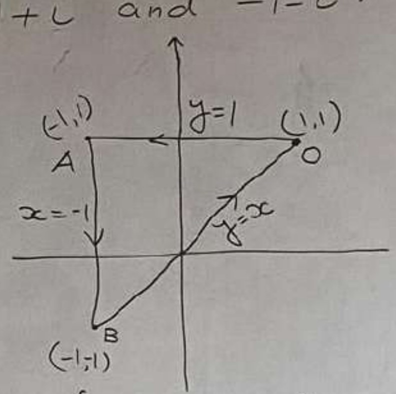
(I) Find the poles of $f(z)$

(II) Find Residue at each pole.

(III) $\int_C f(z)\,dz$ where $C: |z| = 10$

**Ques (2)** Verify Cauchy theorem by integrating $e^{iz}$ along the boundary of triangle with vertices at points $1+i$, $-1+i$ and $-1-i$.

Hint:




$$\int_{\Delta OAB} e^{iz}\,dz = \int_{BO} e^{iz}\,dz + \int_{OA} e^{iz}\,dz + \int_{AB} e^{iz}\,dz$$

BO: Along ($y=x$), $dz = (1+i)\,dx$

OA: Along ($y=1$), $dz = dx$

AB: Along ($x=-1$), $dz = i\,dy$

# Tutorial sheet: 6

## Singularity and Residue

Ques (1) Find out the zero and discuss the nature of singularity for the function $f(z) = \frac{z-2}{z^2} \sin \frac{1}{z-1}$

Ans: zero's are $z = 2$ and $1 + \frac{1}{n\pi}$

$z = 1$ is an Essential Singularity

$z = 0$ is a pole of order 2.

Note: The zero's of $f(z)$ are the solutions of the Eqn: $f(z) = 0$.

Ques (2) Discuss the singularity of the following functions:

(I) $f(z) = \frac{1}{1-e^z}$ at $z = 2\pi i$ (Ans: simple pole)

(II) $f(z) = \frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.

Ans: $z = a$ Pole of order 2
$z = \infty$ Essential

(III) $f(z) = \sin \frac{1}{z-1}$ at $z = 1$.

$z = 1$ Essential Singularity

(IV) $f(z) = \frac{z - \sin z}{z^3}$ at $z = 0$

$z = 0$ Removable Singularity

Ques (3) Determine the poles of the functions and Residue at each pole

(I) $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$ Ans: $z=1$, simple pole

$\operatorname*{Res}_{z=1} f(z) = 1$

$z=2$, double pole, $\operatorname*{Res}_{z=2} f(z) = 0$.

(II) $f(z) = \dfrac{2z+1}{z^2-z-2}$ Ans: $z = -1, 2$ $\operatorname*{Res}_{z=-1} f(z) = \frac{1}{3}$

$\operatorname*{Res}_{z=2} f(z) = \frac{5}{3}$.

(III) $f(z) = \dfrac{z+1}{z^2(z-2)}$ Ans: $z = 0, 0, 2$ $\operatorname*{Res}_{z=0} f(z) = -\frac{3}{4}$

$\operatorname*{Res}_{z=2} f(z) = \frac{3}{4}$.

Ques (4) Evaluate $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and $C : |z| = 10$

Ans: $\operatorname*{Res}_{z=-1} f(z) = -\frac{14}{25}$ $\operatorname*{Res}_{z=2i} f(z) = \dfrac{i-1}{4i-3}$

$\operatorname*{Res}_{z=-2i} f(z) = \dfrac{i+1}{4i+3}$. By CRT $\int_C f(z)\,dz = 0$

Ques (5) Determine the poles of the following function and Residue at each pole $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$ and hence Evaluate $\int_C f(z)\,dz$ where $C$ is the circle $|z-i| = 2$.

Ans: $z=-1$, double pole $\operatorname*{Res}_{z=-1} f(z) = -\frac{1}{9}$

$z=2$, simple pole $\operatorname*{Res}_{z=2} f(z) = \frac{1}{9}$

$\int_c f(z)dz = -\frac{2\pi i}{9}$

Ques (6) Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$

at the poles which lies inside the circle $|z-3| = 1.5$

hence evaluate $\int_c f(z)dz$ where $C: |z-3| = 1.5$.

Ans: $\operatorname*{Res}_{z=2} f(z) = -8$ $\quad \operatorname*{Res}_{z=3} f(z) = \frac{27}{16}$ $\quad \int_c f(z)dz = 2\pi i \times \frac{-101}{16}$

Ques (7) Evaluate $\int_c \dfrac{2z^2+5}{(z+2)^3 (z^2+4)}$ where c is the square

with vertices at $1+i, 2+i, 2+2i, 1+2i$.

Ans: 0

Ques (8) Evaluate $\int_c (5z^4 - z^3 + 2)\, dz$ around

(I) $c: |z| = 1$

(II) Square with vertices $(0,0)$ $(1,0)$, $(1,1)$, $(0,1)$.

(III) $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$.

Ans: (I) 0 (II) 0 (III) 0.

Ques (9): Can Cauchy Integral Theorem be applied for the following integral? Hence evaluate them

(I) $\int_C e^{\sin z^2} dz \quad C: |z|=1$

(II) $\int_C \tan z \, dz \quad C: |z|=1$

Ans: (I) Yes, 0 (II) Yes, 0

Ques (10): Evaluate $\int_C \frac{e^{3iz}\, dz}{(z+\pi)^3}$ where $C: |z-\pi| = 3\cdot 2$

Ans: 0

Ques (11): Evaluate $\int_C \frac{e^{-z} dz}{z+1}$ where $C: |z|=2$.

Ans: $2\pi i e$

Ques (12): Evaluate $\int_C \frac{z^2+5}{z-3} dz$ where $C: |z|=4$

Ans: $28\pi i$

Ques (13): Evaluate $\int_C \frac{e^z dz}{z^2+1}$ where $C: |z|=2$.

Ans: $2\pi i \sin 1$.

(14) Evaluate the following integrals

(I) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ $\underline{a>|b|}$ Ans: $\frac{2\pi}{\sqrt{a^2-b^2}}$

(II) $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$ Ans: $\frac{\pi}{2}$

(III) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ Ans: $\frac{2\pi}{\sqrt{3}}$

(IV) $\int_0^{\pi} \frac{d\theta}{5+4\cos\theta}$ Ans: $\frac{\pi}{3}$

Note: $\int_0^{\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$

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