Analytic function, Harmonic function and Harmonic Conjugate

Q.1. Find the value of the constants 'a' and 'b' such that the following function f(z) is analytic.

$$f(z) = \cos x\,(\cosh y + a \sinh y) + i \sin\,(\cosh y + b \sinh y)$$

Q.2. Discuss the analyticity of the following functions:

(a) $e^{1/z}$ (b) $\cosh z$ (c) $(1-z^2)^4$

(d) $\frac{1}{z}$

Q.3. Construct the analytic function f(z) of which real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

Q.4. Find the analytic function $f(z) = u + iv$ such that the imaginary part is given as

$$v = x^2 - y^2 + \frac{x}{x^2+y^2}.$$

Q.5. Determine the analytic function, whose real part is

$$u(x,y) = \frac{\sin 2x}{(\cosh 2y - \cos 2x)}$$

Q.6. Determine if $u(x,y) = \cos(x).\cos(hy)$ is harmonic. Find its harmonic conjugate.

Q.7 Construct the harmonic conjugate of
$u(x,y) = x^2 - y^2$.

Q.8 Show that $u(x,y) = e^y \sin x$ is harmonic function and find the harmonic conjugate $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

Q.9 Show ~~that~~ $v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$ is harmonic function. Find its harmonic conjugate.

Q.10 Show that the function $f(z) = \dfrac{x^3 y^5 (x+iy)}{x^6 + y^{10}}, z \neq 0,$ $f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin.

Q.11 Show that for the function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^3}{z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the Cauchy-Riemann equations are satisfied or not at the origin. And the complex derivative $f'(0)$ does not exist.

Q.12. Show that the functions $u(x,y) = x^3 - 3xy^2$ is harmonic and find its harmonic conjugate.

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Q.13. Determine wether the given function
$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$
where $(a, b, c, d \in \mathbb{R})$, is harmonic in nature.

Q.14. Find the harmonic conjugate of the function $u(x,y) = 4xy - x^3 + 3xy^2$.

Q.15. Show that the functions

(i) $f(z) = \dfrac{z}{z+1}$ is analytic at $z = \infty$

(ii) $f(z) = z$ is not analytic at $z = \infty$.

Q.16. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of z if
$$u - v = e^x(\cos y - \sin y)$$

Q.17. Construct the analytic function $f(z) = u + iv$, ~~whose~~ where
$$u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$
and
$$f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}.$$

Q.18. Show that $u(x, y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic function. Find the harmonic conjugate $v(x, y)$ such that the function $f(z) = u(x, y) + i v(x, y)$ is analytic. Also express $f(z)$ in terms of z. (4).

Q.19. For the given potential function $\log(x^2+y^2)$, find the flux function and the complex potential function.

Q.20. An electrostatic field in the xy-plane is given by potential function $\phi(x, y) = \log(x^2+y^2)$, find the stream function.

Answers:

1. $a = -1, \quad b = -1$

2. (a) analytic (except $= \neq 0$), $f'(z) = -\dfrac{e^{-1/z}}{z^2}$

(b)

(c) analytic, $f'(z) = -8(1-z^2)^3 z$

(d)

3. $f(z) = z e^{2z} + c$

4. $u(x, y) = -2xy + \dfrac{y}{x^2+y^2} + c$

5. $f(z) = 2 \cot \underline{z} + c$

6. Harmonic function

7. $v(x, y) = 4xy + c$

8. $v(x, y) = -e^{y} \cos x$

9. Harmonic function, $u(r, \theta) = -r^2 \sin 2\theta + r \sin\theta + c.$

12. $v = 3x^2 y - y^3 + c$

14.

16. $f(z) = e^{z} + c$

17. $f(z) = \cot \dfrac{z}{2} + \dfrac{1}{2}(1-i)$

18. $v = x^2 - y^2 + 2xy - 2y - 3x + c$

$f(z) = (1+i)z^2 - (2+3i)z + ic$

19. flux fn $= 2 \tan^{-1}\left(\dfrac{y}{x}\right)$, complex potential fn $= 2 \log z + ic$