MODULE 3

TOPIC: INTERPOLATION

Suppose we are given the following values of $y = f(x)$ for a set of values of x as follows:

$x:\quad x_0 \quad x_1 \quad x_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad x_n$

$y:\quad y_0 \quad y_1 \quad y_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad y_n$

Then, interpolation deals with problems of finding the value of y corresponding to any value of $x = x_i$ not present in the table (non-tabular point).

Interpolation is the technique of estimating the value of a function (dependent variable) for any value of the independent variable lying inside the given range.

Extrapolation is the technique of estimating the value of function (dependent variable) for any value of the independent variable outside the given range. If the function $y = f(x)$ is known, then the value of y corresponding to any value of x can easily be found, but the problem arises when the function $f(x)$ is not known. In this case, it becomes very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i).

In such cases, $f(x)$ is replaced by a simpler function $P(x)$ which assumes the same values as those of $f(x)$ at tabulated set of point's i.e.

$P(x_0) = f(x_0) = y_0,$

$P(x_1) = f(x_1) = y_1$ ………………….. etc.

Here, the points $x_0, x_1, x_2, ...x_n$ are called nodal points, arguments or tabular points. For any other value of x not given in the table, that is non-tabular point $x = x^* \neq x_i (i = 0,1,2,....,n)$, we try to predict $f(x^*) = P(x^*)$. It means $f(x)$ at $x = x^*$, may be determined from $P(x)$, and this $P(x)$ is known as **Interpolating function or Smoothing function.** If $P(x)$ is a polynomial, then it is called **Interpolating polynomial** and the process is called **Polynomial Interpolation.**

Note: *If we have (n+1) data pairs as* $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$, *then these can be represented by a polynomial of at most of nth degree, that is,* $P(x)$ can be a polynomial of degree at most $n(\leq n)$. *The polynomial of degree* n *can be any of the following form:*

i) $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \quad (a_n \neq 0)$

ii) $P(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + ... + a_n(x - c)^n \quad (a_n \neq 0)$

(Shifted Power form)

iii)

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ...a_n(x - x_0)(x - x_1).........(x - x_{n-1})$$

(Newton's form) $(a_n \neq 0)$

iv)

$$P(x) = a_0(x - x_1)(x - x_2)..........(x - x_n) + a_1(x - x_0)(x - x_2)..........(x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)..........(x - x_n) + ... + a_n(x - x_0)(x - x_1).........(x - x_{n-1})$$

(Lagrange form)

CALCULUS OF FINITE DIFFERENCES

In calculus which we have studied so far, deals with the changes of the function which occur when the independent variable changes continuously in a given interval.

e.g. $\frac{dy}{dx} = x + y$, when $x \in (0,1)$

whereas, the calculus of finite differences deals with the changes that takes place in the value of the function (dependent variable), due to finite changes in the value of independent variable e.g. change in y when x is changed from 0 to 1 directly. The finite changes in independent variable are of following two types:

i. **Unequal intervals:** Unequal change in the value of independent variable throughout the given range.

ii. **Equal intervals**: Equal change in the value of independent variable throughout the given range

Suppose we are given the following values of $y = f(x)$ for a set of values of x as follows:

$$\begin{array}{llllll} x: & x_0 & x_1 & x_2 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & x_n \\ y: & y_0 & y_1 & y_2 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & y_n \end{array}$$

LAGRANGE INTERPOLATION FORMULA

This formula is used for **equal as well as for unequal intervals, that is, in both the cases, whether the independent variable** x are equispaced or unequispaced.

If $y = f(x)$ takes the values $y_0, y_1, \ldots\ldots, y_n$ corresponding to value of arguments $x_0, x_1, \ldots\ldots, x_n$, then Lagrange interpolation formula is given by:

$$P(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} y_1 +$$

(1)
$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)} y_2$$

$$+\ldots\ldots+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

Inverse Interpolation: The process of estimating the value of independent variable for the value of given dependent variable not in the table is called Inverse interpolation. In this case, Lagrange's formula (1) can be used by interchanging x and y.

Other form of Lagrange Interpolation Formula

Lagrange formula for (n+1) data pairs can be a polynomial of degree $\le n$, so

$$P(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots(x_2-x_n)} y_2 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

If we take:

$$l_0(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots(x_0-x_n)}, \; l_1(x) = \frac{(x-x_0)(x-x_2)\ldots\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots(x_1-x_n)}$$

……………………………………………………..

$$l_i(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_{i-1})(x-x_{i+1})\ldots\ldots\ldots(x-x_n)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots\ldots(x_i-x_n)}$$

…………………………………………………………

$$l_n(x) = \frac{(x-x_0)(x-x_1)........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)........(x_n-x_{n-1})}$$

These $l_0(x), l_1(x), l_2(x)........l_i(x).....l_n(x)$ are called **Lagrange Fundamental Polynomials or Lagrange Coefficients.** So, with these Lagrange Coefficients, Lagrange Interpolating Polynomial can be written in the form:

$$y = P(x) = l_0(x)y_0 + l_1(x)y_1 + ... + l_n(x)y_n$$

or

$$y = P_n(x) = \sum_{i=0}^{n} l_i(x)y_i$$

Note:

i. Sum of Lagrange coefficients is unity.

ii. $l_i(x_j) = \delta_{ij} = \begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i = j \end{cases}$

e.g. $l_0(x_0) = 1, l_0(x_1) = l_0(x_2) = ... = 0,$

Lagrange Linear Interpolation: The two data pairs (x_0, y_0) and (x_1, y_1) can be approximated by linear (of degree one) interpolating polynomial, that can be given through Lagrange formula as:

$$y = P(x) = \frac{(x-x_1)}{(x_0-x_1)}y_0 + \frac{(x-x_0)}{(x_1-x_0)}y_1 = l_0(x)y_0 + l_1(x)y_1$$

Lagrange Quadratic Interpolation: The three data pairs $(x_0, y_0), (x_1, y_1),$ and (x_2, y_2) can be approximated by quadratic (of degree two) interpolating polynomial, that can be given through Lagrange formula as:

$$y = P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$
$$= l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2$$

NEWTON DIVIDED DIFFERENCE FORMULA (OR NEWTON GENERAL INTERPOLATION FORMULA)

Lagrange formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This can be avoided by using Newton General Interpolation formula which uses "Divided Differences". This formula is used only for **unequal intervals** only, that is, when the independent variables x are unequispaced.

Divided Differences

If $(x_0, y_0),(x_1, y_1),...........,(x_n, y_n)$ be given points, then the First Divided Difference for the arguments x_0, x_1 is defined by:

$$f[x_0,x_1] \text{ or } [x_0,x_1] = \frac{y_1-y_0}{x_1-x_0}$$

Similarly, $f[x_1,x_2] \text{ or } [x_1,x_2] = \dfrac{y_2-y_1}{x_2-x_1}$,

$f[x_2,x_3]$ or $[x_2,x_3] = \dfrac{y_3-y_2}{x_3-x_2}$ …………………….. and so on

Second Divided differences for arguments x_0, x_1, x_2 are defined by:

$$f[x_0,x_1,x_2] \text{or} [x_0,x_1,x_2] = \frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$$

This can also be written as:

$$f\left[x_0, x_1, x_2\right] = \frac{\left[x_1, x_2\right] - \left[x_0, x_1\right]}{x_2 - x_0}$$

$$= \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

Similarly, $f\left[x_1, x_2, x_3\right] = \dfrac{f\left[x_2, x_3\right] - f\left[x_1, x_2\right]}{x_3 - x_1}, \ldots\ldots\ldots\ldots\ldots\ldots$ and so on

As, first divided difference require two arguments, second divided difference requires three arguments, so in general nth divided difference requires (n+1) arguments and it is defined as:

$$f\left[x_0, x_1, x_2, \ldots\ldots\ldots\ldots x_{n-1}, x_n\right] = \frac{\left[x_1, x_2, \ldots\ldots\ldots, x_n\right] - \left[x_0, x_1, \ldots\ldots\ldots, x_{n-1}\right]}{x_n - x_0}$$

n^th^ Divided Difference

Properties of Divided Differences

i. The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.

$$f\left[x_0, x_1\right] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-(y_0 - y_1)}{-(x_0 - x_1)} = f\left[x_1, x_0\right]$$

Similarly, $f\left[x_0, x_1, x_2\right] = f\left[x_2, x_0, x_1\right] = f\left[x_1, x_2, x_0\right]$

ii. Divided Difference Table for four arguments $(x_0, f_0), (x_1, f_1), (x_2, f_2), (x_3, f_3)$ is given as:

x	$f(x)$	First d · d	Second d · d	Third d · d
x_0	f_0			
x_1	f_1	$f[x_0, x_1]$		
x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Newton Divided Difference Formula (or Newton General Interpolation Formula)

If $y = f(x)$ takes the values $y_0, y_1, \ldots\ldots, y_n$ corresponding to value of arguments $x_0, x_1, \ldots\ldots, x_n$, then **Newton divided difference formula** is given by:

$$\boxed{\begin{gathered} y = f(x) = y_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_{n-1}) f[x_0, x_1, x_2 \ldots\ldots\ldots, x_n] \end{gathered}}$$

Problems (Interpolation)

(1) Find the Lagrange interpolating polynomial that fits the following data.

x :	1	2	4
$f(x)$:	1	7	61

Determine the approximate value of $f(3)$.

Solu:- Since, there are 3 data pairs, so the data can be fitted in a polynomial of degree ≤ 2.

So by Lagrange interpolation formula.

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

Take $(x_0, f(x_0)) = (1, 1)$

$(x_1, f(x_1)) = (2, 7)$

$(x_2, f(x_2)) = (4, 61)$

$$P_2(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)} \cdot 1 + \frac{(x-1)(x-4)}{(2-1)(2-4)} (7)$$

$$+ \frac{(x-1)(x-2)}{(4-1)(4-2)} \cdot 61$$

or

$$P_2(x) = \frac{1}{3}\left(x^2 - 6x + 8\right) - \frac{1}{2}\left(x^2 - 5x + 4\right) . 7$$
$$+ \frac{1}{6}\left(x^2 - 3x + 2\right)(61)$$

$$= \left(\frac{1}{3} - \frac{7}{2} + \frac{61}{6}\right)x^2 + \left(-2 + \frac{35}{2} - \frac{61}{2}\right)x$$
$$+ \left(\frac{8}{3} - 14 + \frac{61}{3}\right)$$

$$= 7x^2 - 15x + 9$$ (Lagrange Interpolating polynomial)

at $x = 3$

$$P_2(3) = f(3) = 7(3^2) - 15(3) + 9 = 27.$$

72
45
7

(2) Find the second divided difference of $f(x) = \frac{1}{x}$, using the points x_0, x_1, x_2. Hence, extend it to n^{th} divided difference (3)

Soln:-

First divided difference

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{1}{x_1 - x_0}\left\{\frac{1}{x_1} - \frac{1}{x_0}\right\}$$

$$= -\frac{1}{x_0 x_1}$$

$$f[x_1, x_2] = \frac{f_2 - f_1}{x_2 - x_1} = -\frac{1}{x_1 x_2}$$

Second divided Difference.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left\{-\frac{1}{x_1 x_2} + \frac{1}{x_0 x_1}\right\}$$

$$\boxed{f[x_0, x_1, x_2] = \frac{1}{x_0 x_1 x_2}}$$ Second Divided Difference.

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{1}{x_3 - x_0}\left\{\frac{1}{x_1 x_2 x_3} - \frac{1}{x_0 x_1 x_2}\right\}$$

$$= \frac{1}{(x_3 - x_0)}\frac{1}{(x_1 x_2)}\left\{\frac{x_0 - x_3}{x_0 x_3}\right\} = -\frac{1}{x_0 x_1 x_2 x_3}.$$

Third Divided Difference

$$f[x_0, x_1, x_2, x_3] = \frac{(-1)}{x_0 x_1 x_2 x_3}$$

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n^{th} Divided difference

$$f[x_0, x_1, x_2 \ldots x_n] = \frac{(-1)^n}{x_0 x_1 x_2 \ldots x_n}$$

3:- Using Newton divided difference interpolation, calculate the value of $f(6)$ from the following data

x :	1	2	7	8
$f(x)$:	1	5	5	4

Soln:- Divided Difference Table

x values are not at equal intervals

x	$f(x)$	First dd	Second dd	Third dd
$1\ (x_0)$	$1\ (f(x_0))$			
		$\frac{5-1}{2-1} = 4$		
			$\frac{0-4}{7-1} = -\frac{2}{3}$	
$2\ (x_1)$	$5\ (f(x_1))$			$\frac{-\frac{1}{6}-\left(-\frac{2}{3}\right)}{8-1} = \frac{1}{14}$
		$\frac{5-5}{7-2} = 0$		
			$\frac{-1-0}{8-2} = -\frac{1}{6}$	
$7\ (x_2)$	$5\ (f(x_2))$			
		$\frac{4-5}{8-7} = -1$		
$8\ (x_3)$	$4\ (f(x_3))$			

Using Newton Divided Difference formula.

$$f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

$$= 1 + (x-1)\times 4 + (x-1)(x-2)\left(-\frac{2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

6

$$f(x) = 1 + (x-1)\cdot 4 + (x-1)(x-2)\left(-\frac{2}{3}\right)$$
$$+ (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$\Downarrow$

polynomial can be obtained from here. of degree at most 3.

at $x = 6$

$$f(6) = 1 + (6-1)\cdot 4 + (6-1)(6-2)\left(-\frac{2}{3}\right)$$
$$+ (6-1)(6-2)(6-7)\left(\frac{1}{14}\right)$$

$$= 6.2381$$ #

4. If $f(x) = U(x)\,V(x)$, show that

$$f[x_0, x_1] = U[x_0]\,V[x_0, x_1] + U[x_0, x_1]\,V[x_1]$$

<u>Solu:-</u>

$$f[x_0, x_1] = \frac{U(x_1)\,V(x_1) - U(x_0)\,V(x_0)}{x_1 - x_0}$$

$$= \frac{U(x_1)\,V(x_1) - U(x_0)\,V(x_0) + U(x_0)\,V(x_1) - U(x_0)\,V(x_1)}{x_1 - x_0}$$

$$= \frac{U(x_0)\left\{V(x_1) - V(x_0)\right\}}{x_1 - x_0} + \frac{V(x_1)\left\{U(x_1) - U(x_0)\right\}}{x_1 - x_0}$$

$$= U(x_0)\,V[x_0, x_1] + V(x_1)\,U[x_0, x_1)$$

5:- From the given table (8)

x :	20	25	30	35
$y(x)$:	.342	.423	.5	.65

find the value of x for $y(x) = .390$

Soln:- By using Lagrange interpolation formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$x_0 = \cdot 20 \quad y_0 = .342$

$x_1 = 25, \quad y_1 = .423$

$x_2 = 30, \quad y_2 = .5$

$x_3 = 35, \quad y_3 = .65$

at $y = .390$

$\boxed{x = 22.84058}$ #

6:- Using Lagrange's interpolation formula, find $y(10)$ (9)
from the following table

x :	5	6	9	11
y :	12	13	14	16

Soln:- $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$f(x_0) = y_0 = 12$, $f(x_1) = y_1 = 13$, $f(x_2) = y_2 = 14$, $f(x_3) = y_3 = 16$

Lagrange's Interpolation formula.

$$P(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \cdot 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \cdot 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \cdot 14$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

With $x = 10$

$P(10) = 14.6667$

Problem 7:- Given $f(2) = 4$, $f(2.5) = 5.5$. Find the linear interpolating polynomial using (10)

i) Lagrange interpolation

ii) Newton divided difference Interpolation

Hence, find an approximate value of $f(2.2)$.

Soln:-

$x_0 = 2$, $\quad x_1 = 2.5$

$f(x_0) = f(2) = 4 \quad f(x_1) = f(2.5) = 5.5$.

i) Using Lagrange interpolation.

$$P(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

$$= \frac{(x-2.5)}{(2-2.5)} \cdot 4 + \frac{(x-2)}{(2.5-2)} \cdot 5.5$$

$$= -2(x-2.5) \cdot 4 + 2(x-2)\, 5.5.$$

$$= -8x + 20 + 11x - 22$$

$$= 3x - 2$$

$$f(2.2) = P(2.2) = 3.(2.2) - 2 = 4.6.$$

ii) Using Newton-difference interpolation.

$$P(x) = f(x_0) + (x-x_0) f[x_0, x_1]$$

$$= f(x_0) + (x-x_0) \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= 4 + (x-2) \frac{5.5-4}{.5} = 3x - 2$$

$f(2.2) = P(2.2) = 4.6.$