MODULE 2

TOPIC: SYSTEM OF LINEAR EQUATIONS

Consider a non - homogeneous system of $n-$simultaneous linear algebraic equations in $n-$unknowns as:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \\
(1) \qquad &a_{31}x_1 + a_{32}x_2 + \ldots\ldots\ldots + a_{3n}x_n = b_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots + a_{nn}x_n = b_n
\end{aligned}
$$

Using matrix notation, the above equation (1) can be written as:

$$\mathrm{AX} = B$$

where $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3}\ldots\ldots\ldots & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$

By finding a solution of the system, we mean to obtain the values of n unknowns $x_1, x_2, \ldots\ldots, x_n$ such that they satisfy the given equations. If $B = 0$, then the system is called homogenous. The methods of solution of system of equations (1) can be classified into two types:

1) **Direct Methods:** These methods yield the exact solution after a finite number of steps.

2) **Iterative Methods:** In such methods, we start from an approximation to the true solution and obtain better and better approximation till we get the solution up to desired level of accuracy.

Diagonally Dominant System: When the system of equations can be ordered, so that each diagonal entry of the coefficient matrix is larger in magnitude than the sum of the magnitudes of the other coefficients in that row. Such a system is called diagonally dominant system. If we have following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + .a_{33}x_3 &= b_3 \end{aligned} \tag{I}$$

For diagonal - dominant system, we have:

$$|a_{11}| \geq |a_{12}| + |a_{13}|,\ |a_{22}| \geq |a_{21}| + |a_{23}|, \text{and} |a_{33}| \geq |a_{31}| + |a_{32}|.$$

In a strictly diagonal-dominant system, there would be strict inequality, that is

$$|a_{11}| > |a_{12}| + |a_{13}|,\ |a_{22}| > |a_{21}| + |a_{23}|, \text{and} |a_{33}| > |a_{31}| + |a_{32}|.$$

DIRECT METHODS

GAUSS ELIMINATION

Gauss elimination is the direct method to obtain the solution of system of linear equations in which the given system of equations first reduced into upper triangular matrix and then it can be solved through Back Substitution method. So, if the following system in three unknowns is given:

$$\text{(I)} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + .a_{33}x_3 &= b_3 \end{aligned}$$

Here, in the first stage of elimination, multiply the first equation (or first row) in above system (I) by $\frac{a_{21}}{a_{11}}$ and subtract from the second equation (second row). Similarly, multiply the first equation (or first row) by $\frac{a_{31}}{a_{11}}$ and subtract from the third equation (third row). Hence, the second and third equations transform as:

$$\text{(II)} \qquad \begin{aligned} a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\ a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 &= b_3^{(2)} \end{aligned}$$

where $a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$, $a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$, $a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}$,

$a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}}a_{13}$, $b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}}b_1$, $b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}}b_1$.

In the second stage of elimination, multiply the first equation in set (II) by $\frac{a_{32}^{(2)}}{a_{22}^{(2)}}$ and subtract from the second equation in set (II). Then, the second equation of set (II) transforms as

$$\text{(III)} \qquad a_{33}^{(3)}x_3 = b_3^{(3)}$$

where $a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} a_{23}^{(2)}, \; b_3^{(3)} = b_3^{(2)} -- \frac{a_{32}^{(2)}}{a_{22}^{(2)}} b_2^{(2)}$.

Now, collecting the first equations from each stages, that is, from set of equations (I), (II) and (III), the following system is obtained

(IV)
$$\begin{aligned} a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 &= b_1^{(1)} \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\ a_{33}^{(3)}x_3 &= b_3^{(3)} \end{aligned}$$

where $a_{ij}^{(1)} = a_{ij},\, b_i^{(1)} = b_i$, for $i, j = 1,2,3.$

The obtained system (IV) is an upper triangular system and can be solved by using the Back Substitution method. The Gauss Elimination method gives in matrix form as:

$$[A|b] \xrightarrow{\text{Gauss Eliminati on}} [U|c]$$

where $[A|b]$ is an augmented matrix.

Note: The elements $a_{11}^{(1)}, a_{22}^{(2)}$, and $a_{33}^{(3)}$ which have been assumed to be non-zero are called Pivot elements. In the elimination process of Gauss elimination method if any of the pivot elements vanishes (or becomes very small compared to other elements) in that column, then an attempt should be made to rearrange the remaining equations (or rows) so as to obtain a non-vanishing pivot. This strategy is called Pivoting.

Pivoting

So, the strategy of obtaining a non-zero (or large) value as pivot element by rearranging the equations (or rows) is called Pivoting.

The process of selecting the largest element in magnitude in the column under consideration as the pivot element and interchanging the rows is called **Partial Pivoting.**

Procedure for Partial Pivoting: To execute partial pivoting, in the first step, the numerically (absolute value disregarding the sign) largest coefficient of x_1 is chosen from all the equations and brought it as the first pivot element by interchanging the first equation with the equation having the largest coefficient of x_1.

In the second elimination stage, the numerically largest coefficient of x_2 is chosen from the remaining equations (leaving the first one) and brought as the second equation with the equation having the largest coefficient of x_2.

The process is continued till we arrive at the equation with the single variable.

Note: If the system is diagonally dominant, then there is no need to execute partial pivoting.

GAUSS JORDAN METHOD

Gauss Jordan method is a modification of the Gauss Elimination, the essential difference being that when an unknown is eliminated, it is eliminated from all the equations. So, instead of upper triangular matrix, the diagonal matrix is obtained and the method does not require Back Substitution to obtain the value of unknowns.

$$[A|b] \xrightarrow{\text{Gauss Jordan}} [D|c]$$

Here, depending on situation, partial pivoting can be employed.

LU DECOMPOSITION (OR FACTORISATION) METHOD -

Triangularization method is also known as the LU decomposition or factorization method and Crout's method is a particular case of this. The system of equations in three unknowns (I) in matrix notation can be written in the form:

(2) $$\mathrm{AX} = \mathrm{B}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

In this method, the coefficient matrix A of the system of equations (2) is decomposed or factorized into the product of a lower triangular matrix L and upper triangular matrix U, that is,

(3) $$\mathrm{A} = LU$$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$,

Using the matrix multiplication rule to multiply the matrices L and U and comparing the elements of the resulting matrix with those of A we obtain

(4)

$$l_{11}u_{11}=a_{11}; l_{11}u_{12}=a_{12}; l_{11}u_{13}=a_{13};$$
$$l_{21}u_{11}=a_{21}; l_{21}u_{12}+l_{22}u_{22}=a_{22}; l_{21}u_{13}+l_{21}u_{23}=a_{23};$$
$$l_{31}u_{11}=a_{31}; l_{31}u_{12}+l_{32}u_{22}=a_{32}; l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33}=a_{33};$$

To produce a unique solution it is convenient to choose either $u_{ii}=1$ or $l_{ii}=1$ for $i=1,2,3.$ When we choose

i) $l_{ii}=1,$ the method is called **Doolittle's Method**

ii) $u_{ii}=1,$ the method is called **Crout's Method**

The given system of equations is $AX=B$, which converts into the following form using equation (3), that is,

(5) $$LUX=B$$

Let $UX=Y,$ then equation (8) becomes

(6) $$LY=B$$

The unknowns y_1, y_2, y_3 in (6) are determined by forward substitution and the unknowns x_1, x_2, x_3 in $UX=Y$ are obtained by back - substitution.

The inverse of matrix A can also be determined from $A^{-1}=U^{-1}L^{-1}$.

Note: The Crout's method fails if any of the diagonal elements l_{ii} or u_{ii} is zero. Depending on the situation, partial pivoting can be used.

ITERATIVE METHODS

The iterative methods start from an approximation to the true solution and if convergent, derive a sequence of closer approximations. The cycle of computations is repeated till the desired accuracy is obtained. Thus, in an iterative method, the amount of computation depends on the degree of accuracy required. For large systems, iterative methods may be faster than direct methods; even the errors in iterative methods are smaller.

Two popular iterative methods to solve the system of linear equations are:

1) Jacobi Method

2) Gauss-Seidel Method

Relevance of Diagonal Dominance in Iterative Methods: For strictly diagonal dominant system, we have an advantage in iterative method, that the method will converge for any starting values taken. Thus, the sufficient condition for their use is that the system of equations should be strictly diagonally dominant. In other words, after rearranging the equations, if necessary, the largest coefficient must be along the leading diagonal of the coefficient matrix.

JACOBI (OR GAUSS JACOBI) METHOD:

If the system (I) is diagonally dominant, solve for x_1, x_2 and x_3 in terms of other variables from the three equations as follows:

$$(7) \qquad \begin{aligned} x_1 &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3\right) && \text{(i)} \\ x_2 &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3\right) && \text{(ii)} \\ x_3 &= \frac{1}{a_{33}}\left(b_3 - a_{31}x_1 - a_{32}x_2\right) && \text{(iii)} \end{aligned}$$

Now, iteration method can be started by taking initial values $x_1^{(0)}, x_2^{(0)}$ and $x_3^{(0)}$ for x_1, x_2 and x_3 respectively and put it on RHS of first equation of set 7 (7- (i)),

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

While in the second equation of set 7 ((7) - ii)), we use $x_3^{(0)}$ for x_3 and $x_1^{(0)}$ for x_1 to get new approximation $x_2^{(1)}$ that is

$$x_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$$

Similarly

$$x_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$$

This completes the first iteration of Jacobi Method. In finding the values of unknowns, **the values of the previous iterations are used on the RHS**, that is, if $x_1^{(k)}, x_2^{(k)}$ and $x_3^{(k)}$ are k^{th} iterates, then the iteration scheme can be written as:

$$(*) \qquad \begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right) \\ x_2^{(k+1)} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right) \quad \text{for } k = 0,1,2\ldots\ldots \\ x_3^{(k+1)} &= \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right) \end{aligned}$$

The process is continued till the two consecutive values of corresponding variables are same up to desired level of accuracy. In general, if not specified, the initial approximations $x_1^{(0)}$, $x_2^{(0)}$ and $x_3^{(0)}$ can be taken to be zero.

GAUSS - SIEDEL METHOD

If the system (I) is assumed to be diagonally dominant, solve for x_1, x_2 and x_3 in terms of other variables from the three equations as follows:

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3\right) \qquad \text{(i)}$$

(8) $$x_2 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3\right) \qquad \text{(ii)}$$

$$x_3 = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1 - a_{32}x_2\right) \qquad \text{(iii)}$$

Now, iteration method can be started by taking initial values $x_2^{(0)}$ and $x_3^{(0)}$ for x_2 and x_3 respectively and put it on RHS of first equation of set 8 (8- (i)),

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

While second equation of set 8 ((8) - ii)), we use $x_3^{(0)}$ for x_3 and $x_1^{(1)}$ for x_1 (most recent values of unknown) to get $x_2^{(1)}$ that is

$$x_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right)$$

Now, we know $x_1^{(1)}$ and $x_2^{(1)}$, so we use $x_1^{(1)}$ for x_1 and $x_2^{(1)}$ for x_2 to get $x_3^{(1)}$ from third equation of set (8) ((8) - (iii)), that is,

$$x_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right)$$

This completes the first iteration of Gauss - Seidel Method. In finding the values of unknowns, **the latest available values are used on the RHS**, that is, if ${x_1}^{(k)}, {x_2}^{(k)}$ and ${x_3}^{(k)}$ are k^{th} iterates, then the iteration scheme can be written as:

(**)

$$\begin{aligned} {x_1}^{(k+1)} &= \frac{1}{a_{11}}\left(b_1 - a_{12}{x_2}^{(k)} - a_{13}{x_3}^{(k)}\right) \\ {x_2}^{(k+1)} &= \frac{1}{a_{22}}\left(b_2 - a_{21}{x_1}^{(k+1)} - a_{23}{x_3}^{(k)}\right) \quad \text{for } k = 0,1,2\ldots\ldots \\ {x_3}^{(k+1)} &= \frac{1}{a_{33}}\left(b_3 - a_{31}{x_1}^{(k+1)} - a_{32}{x_2}^{(k+1)}\right) \end{aligned}$$

The process is continued till the two consecutive values of corresponding variables are same up to desired level of accuracy. In general, if not specified, we take initial approximations ${x_2}^{(0)}$ and ${x_3}^{(0)}$ can be taken to be zero.

Note:

i. In Gauss Seidel methods, latest available values of variables are used at each step, therefore, it converges faster than Jacobi method.

ii. If any of the diagonal elements becomes zero, then interchange the rows to obtain non-zero diagonal element (partial pivoting can be employed). If the system is diagonally dominant, then there is no need of partial pivoting.

iii. If the accuracy that is to be achieved in any of the iterative methods is not mentioned, then the two consecutive values should be matched at least up to three decimal places (calculations up to at least four decimal places).

iv. **(Optional)** In matrix form, the system of linear equations can be written as:

$$\mathrm{AX} = \mathrm{B}$$

Let $\mathrm{A} = \bar{L} + \bar{D} + \bar{U}$, where D is the diagonal matrix, $\bar{L}$ and $\bar{U}$ are **strictly** lower and upper triangular matrices respectively

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{21} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \bar{L} + \bar{D} + \bar{U}$$

In Jacobi method

$$(L + D + U)\mathrm{X} = \mathrm{B},$$

can be reduced as

$$\bar{D}\mathrm{X} = -(\bar{L} + \bar{U})\mathrm{X} + \mathrm{B},$$

In Jacobi method, it can be written as

$$\bar{D}\mathrm{X}^{(k+1)} = -(\bar{L} + \bar{U})\mathrm{X}^{(k)} + \mathrm{B}$$

or

$$\mathrm{X}^{(k+1)} = -\bar{D}^{-1}(\bar{L} + \bar{U})\mathrm{X}^{(k)} + \bar{D}^{-1}\mathrm{B} \qquad k = 0,1,2......$$

If the iteration method is written in the form

$$\mathrm{X}^{(k+1)} = Hx^{(k)} + C$$

where $H = -\bar{D}^{-1}(\bar{L} + \bar{U})$ is the iteration matrix and $C = \bar{D}^{-1}\mathrm{B}$ for the Jacobi method.

Similarly, in the case of Gauss -Seidel method

$$\mathrm{AX} = \mathrm{B}$$

can be written as:

$$(L+D+U)X = B,$$

or

$$(\overline{L}+\overline{D})X = -\overline{U}X + B,$$

or

$$(\overline{L}+\overline{D})X^{(k+1)} = -\overline{U}X^{(k)} + B$$

or

$$X^{(k+1)} = -(\overline{L}+\overline{D})^{-1}\overline{U}X^{(k)} + (\overline{L}+\overline{D})^{-1}B \qquad k = 0,1,2......$$

If the iteration method is written in the form

$$X^{(k+1)} = HX^{(k)} + C$$

where $H = -(\overline{L}+\overline{D})^{-1}\overline{U}$ is the iteration matrix and $C = (\overline{L}+\overline{D})^{-1}B$ for the Gauss Seidel method.

Problem:- ① Solve the eq^{ns}

$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

using Gauss - ellimination method.

Soln:- System is diagonally dominant, no pivoting is necessary.

$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

Writing in matrix form as

$$AX = B$$

$$A = \begin{pmatrix} 10 & -1 & 2 \\ 1 & 10 & -1 \\ 2 & 3 & 20 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$$

$$[A|b] = \left(\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 \to R_2 - \frac{1}{10}R_1$, $R_3 \to R_3 - \frac{1}{5}R_1$

$$\sim \left(\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & \frac{101}{10} & -\frac{6}{5} & \frac{13}{5} \\ 0 & \frac{16}{5} & \frac{98}{5} & \frac{31}{5} \end{array}\right)$$

2

$$R_3 \to R_3 - \frac{32}{101} R_2$$

$$\sim \left(\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & \frac{101}{10} & -\frac{6}{5} & \frac{13}{5} \\ 0 & 0 & \frac{20180}{1010} & \frac{5430}{1010} \end{array}\right)$$

$$\sim [U|c]$$

Writing in eqⁿ form.

$$10x_1 - x_2 + 2x_3 = 4$$

$$\frac{101}{10} x_2 - \frac{6}{5} x_3 = \frac{13}{5}$$

$$\frac{20180}{1010} x_3 = \frac{5430}{101}$$

Using Back-substⁿ

$$x_3 = \frac{5430}{\cancel{1010}} \times \frac{\cancel{1010}}{2018\cancel{0}} = \frac{5430}{20180}$$

$$\boxed{x_3 = .269}$$

$$\frac{101}{10} x_2 = \frac{13}{5} + \frac{6}{5} \times \frac{543\cancel{0}}{2018\cancel{0}}$$

$$\boxed{x_2 = .289}$$

$$\boxed{x_1 = .375}$$

Q:- Solve the system of eqns ③

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

using the Gauss - elimination method with partial pivoting.

Soln:-

$$AX = B$$

$$[A|B] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right]$$

$R_2 \leftrightarrow R_1$ ⟶ Pivoting

$$\sim \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 2 & 1 & 1 & -2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{2} R_1$, $R_3 \to R_3 - \frac{3}{4} R_1$

$R_4 \to R_4 - \frac{1}{4} R_1$

$$\sim \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -\frac{5}{2} & -14 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & 1 \\ 0 & 3 & \frac{3}{2} & -\frac{5}{4} & -7 \end{array}\right]$$

Now, again pivoting $R_2 \leftrightarrow R_4$.

(4)

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{5}{4} & -7 \\ 0 & 2 & \frac{1}{2} & -\frac{3}{4} & 1 \\ 0 & 1 & \frac{0}{2} & -\frac{5}{2} & -14 \end{array}\right]$$

$R_3 \to R_3 - \frac{2}{3} R_2$, $R_4 \to R_4 - \frac{1}{3} R_2$

$$\sim \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{5}{4} & -7 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{12} & \frac{17}{3} \\ 0 & 0 & -\frac{1}{2} & -\frac{25}{12} & -\frac{35}{3} \end{array}\right]$$

$R_4 \to R_4 - R_3$

$$\sim \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & \frac{3}{2} & -\frac{5}{4} & -7 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{12} & \frac{17}{3} \\ 0 & 0 & 0 & -\frac{13}{6} & -\frac{52}{3} \end{array}\right]$$

Writing the eq^{ns}

$$4x_1 + 2x_3 + x_4 = 8$$
$$3x_2 + \frac{3}{2}x_3 - \frac{5}{4}x_4 = -7$$
$$-\frac{1}{2}x_3 + \frac{1}{12}x_4 = \frac{17}{3}$$
$$-\frac{13}{6}x_4 = -\frac{52}{3}.$$

gives

$x_4 = 8$, $x_3 = -10$, $x_2 = 6$, $x_1 = 5$.

(3) Solve the system 5

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

by Gauss – Jordan method without using partial pivoting.

Soln:-

$$AX = B$$

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

$$R_2 \to R_2 - \frac{3}{2}R_1, \quad R_3 \to R_3 - \frac{1}{2}R_1$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & \frac{7}{2} & \frac{17}{2} & 11 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2, \quad R_3 \to R_3 - 7R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -2 & 4 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$$R_1 \to R_1 - R_3, \quad R_2 \to R_2 + \frac{3}{4}R_3$$

⑥

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 14 \\ 0 & \frac{1}{2} & 0 & -\frac{9}{2} \\ 0 & 0 & -2 & -10 \end{array}\right] \sim [D|c].$$

Now the

$$\left.\begin{array}{r} 2x = 14 \\ \frac{1}{2}y = -\frac{9}{2} \\ -2z = -10 \end{array}\right\} \left.\begin{array}{l} x = 7 \\ y = -9 \\ z = 5 \end{array}\right\} \#$$

CROUT's METHOD

Problem 5:30 :- Consider the equations. $x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_2 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$

Use the decomposition (or Crout's method) to solve the system.

Soln:-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Now $A = LU$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$

So we have $u_{11} = u_{22} = u_{33} = 1$ (Crout's)

First Column:- $\left.\begin{aligned} l_{11}u_{11} &= 1 \\ l_{21}u_{11} &= 4 \\ l_{31}u_{11} &= 3 \end{aligned}\right\}$ $l_{11} = 1$ — (1)

$l_{21} = 4$ — (2)

$l_{31} = 3$ — (3)

First row:- $l_{11}u_{12} = 1 \Rightarrow u_{12} = 1$ — (4)

$l_{11}u_{13} = 1 \Rightarrow u_{13} = 1$ — (5)

Second column: $l_{21}u_{12} + l_{22}u_{22} = 3$ — (6)

$4 \times 1 + l_{22} = 3 \Rightarrow l_{22} = -1$

$l_{31}u_{12} + l_{32}u_{22} = 5 \Rightarrow 3 \cdot \times 1 + l_{32} = 5$ — (7)

$l_{32} = 2$

Second row:- $l_{21} u_{13} + l_{22} u_{23} = -1$

$$4 \times 1 + (-1) u_{23} = -1$$

$$-u_{23} + 4 = -1$$

$$\boxed{u_{23} = 5} \quad -(8)$$

Third Column:-

$$l_{31} u_{13} + l_{32} u_{23} + l_{33} u_{33} = 3$$

$$3 \times 1 + 2 \times 5 + l_{33} = 3$$

$$l_{33} = 3 - 10 - 3$$

$$l_{33} = -10$$

Sow get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$AX = B$$

$$LUX = B$$

Taking $UX = Y$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$y_1 = 1$$

$$4y_1 - y_2 = 6 \Rightarrow -y_2 = 6 - 4y_1$$
$$= 6 - 4$$
$$y_2 = 2$$
$$\boxed{y_2 = -2}$$

and

$$3y_1 + 2y_2 - 10y_3 = 4$$
$$3 \times 1 - 4 - 10y_3 = 4$$
$$-1 - 10y_3 = 4 \Rightarrow y_3 = -\frac{1}{2}$$

so

$$UX = Y$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{1}{2} \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 1$$
$$x_2 + 5x_3 = -2$$
$$x_3 = -\frac{1}{2}$$
$$x_2 = -2 + \frac{5}{2} = \frac{1}{2}$$
$$x_1 = 1 - \frac{1}{2} + \frac{1}{2} = 1$$

$$X = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$ Ans

Problem:- Solve the following system of eqns by Gauss - Seidel method

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Compare the results with Jacobi method.

Soln:- First we check the system for diagonal dominance

In the first eqn, $|20| > |1| + |-2|$

" " second eqn, $|20| > |3| + |-1|$

" " third eqn $|20| > |2| + |-3|$

So, the system is diagonal dominant, therefore, both Gauss - Seidel & Jacobi converge for any initial approx.

Solving the given eqns for x, y, z from first, second & third eqns respectively

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{—— (A)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{—— (B)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{—— (C)}$$

Choosing initial approx. $x^{(0)} = y^{(0)} = z^{(0)} = 0$.

since nothing is given about accuracy, so we do calculations upto 4 decimal places. That is, terminate the process when two consecutive values of corresponding variables are same upto 3 decimal places.

For comparison, the above pts will be adopted for both Gauss - Seidel & Jacobi

I) Gauss - Seidel

1st Iteration

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = \frac{17}{20} = \cdot 8500.$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(0)}\right) =$$

$$= \frac{1}{20}\left(-18 - 3 \times \cdot 85 + 0\right) = -1 \cdot 0275$$

$$z^{(1)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times \cdot 85 + 3(-1\cdot 0275)\right) = 1\cdot 0109$$

After 1st Iteration

$$\left.\begin{aligned} x^{(1)} &= \cdot 8500, \\ y^{(1)} &= -1\cdot 0275 \\ z^{(1)} &= 1\cdot 0109 \end{aligned}\right\}$$

2nd Iteration

(3)

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right)$$
$$= \frac{1}{20}\left(17 + 1.0275 + 2 \times 1.0109\right)$$
$$= 1.0025$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(1)}\right)$$
$$= \frac{1}{20}\left(-18 - 3 \times 1.0025 + 1.0109\right)$$
$$= -.9998.$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2 \times x^{(2)} + 3y^{(2)}\right)$$
$$= \frac{1}{20}\left(25 - 2 \times 1.0025 + 3 \times (-.9998)\right)$$
$$= .9999$$

3rd Iteration

$$x(3) = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right)$$
$$= \frac{1}{20}\left(17 + .9998 + 2 \times .9999\right) = .99998 \simeq 1.0000$$

$$y(3) = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(2)}\right)$$
$$= \frac{1}{20}\left(-18 - 3 \times 1 + .9999\right) = 1.0000.$$

$$z(3) = \frac{1}{20}\left(25 - 2 \times x^{(3)} + 3y^{(3)}\right)$$
$$= \frac{1}{20}\left(25 - 2 \times 1 + 3(-1)\right) = 1.0000.$$

4th Iteration

4

$$x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right)$$

$$= \frac{1}{20}\left(17 + 1 + 2\right) = 1.0000$$

$$y^{(4)} = \frac{1}{20}\left(-18 - 3x^{(4)} + z^{(3)}\right)$$

$$= \frac{1}{20}\left(-18 - 3(1) + 1\right) = -1.0000$$

$$z^{(4)} = \frac{1}{20}\left(25 - 2x^{(4)} + 3y^{(4)}\right)$$

$$= \frac{1}{20}\left(25 - 2 - 3\right) = 1.0000$$

$$x^{(3)} \simeq x^{(4)}$$
$$y^{(3)} \simeq y^{(4)}$$
$$z^{(3)} \simeq z^{(4)}$$

Hence, $x = 1,\ y = -1,\ z = 1.$

is the soln. by Gauss–Seidel method in 4 iterations.

II) Jacobi Method $x^{(0)} = y^{(0)} = z^{(0)} = 0$ 5

1st iteration

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = .8500$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(0)} + z^{(0)}\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times 0 + 0\right) = -.9000$$

$$z^{(1)} = \frac{1}{20}\left(25 - 2x^{(0)} + 3y^{(0)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times 0 + 3 \times 0\right) = 1.2500$$

2nd iteration

$$x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right)$$

$$= \frac{1}{20}\left(17 + .9000 + 2 \times 1.25\right) = 1.0200$$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(1)}\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times .85 + 1.25\right)$$

$$= -.9650$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times .85 + 3(-.9)\right)$$

$$= 1.0300.$$

3rd iteration

$$x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right)$$

$$= \frac{1}{20}\left(17 + .9650 + 2 \times 1.0300\right) = 1.0012$$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(2)}\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times 1.0200 + 1.0300\right)$$

$$= -1.0015$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2x^{(2)} + 3y^{(2)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times 1.0200 + 3 \times (-.9650)\right)$$

$$= 1.0032$$

4th iteration

$$x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right)$$

$$= \frac{1}{20}\left(17 + 1.0015 + 2 \times 1.0032\right) = 1.0004$$

$$y^{(4)} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(3)}\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times 1.0012 + 1.0032\right) = -1.0000$$

$$z^{(4)} = \frac{1}{20}\left(25 - 2x^{(3)} + 3y^{(3)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times 1.0012 - 3 \times 1.0015\right)$$

$$= .9996$$

5 th iteration

7

$$x^{(5)} = \frac{1}{20}\left(17 - y^{(4)} + 2z^{(4)}\right)$$

$$= \frac{1}{20}\left(17 + 1 + 2 \times .9996\right) = .99996..$$

$$\simeq 1.0000.$$

$$y^{(5)} = \frac{1}{20}\left(-18 - 3x^{(4)} + z^{(4)}\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times (1.0004) + .9996\right)$$

$$= -1.0000$$

$$z^{(5)} = \frac{1}{20}\left(25 - 2x^{(4)} + 3y^{(4)}\right)$$

$$= \frac{1}{20}\left(25 - 2 \times 1.0004 - 3 \times .1\right)$$

$$= .99996.. \simeq 1.0000.$$

6 th iteration

$$x(6) = \frac{1}{20}\left(17 - y(5) + 2z(5)\right)$$

$$= \frac{1}{20}\left(17 + 1 + 2 \times 1\right) = 1.0000$$

$$y(6) = \frac{1}{20}\left(-18 - 3x(5) + z(5)\right)$$

$$= \frac{1}{20}\left(-18 - 3 \times 1 + 1\right) = -1.0000$$

$$z(6) = \frac{1}{20}\left(25 - 2x(5) + 3y(5)\right)$$

$$= \frac{1}{20}\left(25 - 2 \times 1 - 3 \times 1\right) = 1.000.$$

$$x^{(5)} = x^{(6)}$$
$$y^{(5)} = y^{(6)}$$
$$z^{(5)} = z^{(6)}$$

Hence $x = 1$, $y = -1$, $z = 1$

is soln of given system by Jacobi method in

6 iterations.

Comparison shows, Gauss-Seidel converges faster than Jacobi method.

Problem:- Use Gauss – Seidel iteration method to solve the following system (9)

$$9x + 4y + z = 17$$
$$x - 2y - 6z = 14$$
$$x + 6y = 4$$

Perform four iterations.

Soln:- Checking the system for diagonal dominance

First eqn $\quad 9 > 4 + 1$

Second eqn $\quad |-2| \ngtr |1| + |-6|$

Third eqn $\quad 0 \ngtr |1| + |6|$

Hence, the system is not diagonally dominant.

Now, rearranging the eqns. Interchanging second & third eqn

$$\left.\begin{aligned} 9x + 4y + z &= 17 \\ x + 6y &= 4. \\ x - 2y - 6z &= 14. \end{aligned}\right\} \text{—— (A)}$$

Now, it becomes diagonal dominance system. Gauss Seidel method can be applied for any initial approx. Hence, obtain x, y, z from set (A)

$$x = \frac{1}{9}(17 - 4y - z) \quad \text{—— (1)}$$

$$y = \frac{1}{6}(4 - x) \quad \text{—— (2)}$$

$$z = \frac{1}{6}(x - 2y - 14) \quad \text{—— (3)}$$

Jdiking: $x^{(0)} = y^{(0)} = z^{(0)} = 0$ as initial approx. Calculation should be done upto 4 decimal places (nothing is mentioned)

(10)

1st Iteration

$$x^{(1)} = \frac{1}{9}\left(-17 - 4y^{(0)} - z^{(0)}\right)$$

$$= \frac{1}{9}\left(-17 - 4 \times 0 - 0\right) =$$

$$= -1.8889$$

$$y^{(1)} = \frac{1}{6}\left(4 - x^{(1)}\right) = \frac{1}{9}\left(4 + 1.8889\right)$$

$$= .9815$$

$$z^{(1)} = \frac{1}{6}\left(x^{(1)} - 2y^{(1)} - 14\right)$$

$$= \frac{1}{6}\left(-1.8889 - 2 \times .9815 - 14\right)$$

$$= -2.9753.$$

2nd iteration

$$x^{(2)} = \frac{1}{9}\left(-17 - 4y^{(1)} - z^{(1)}\right)$$

$$= \frac{1}{9}\left(-17 - 4 \times .9815 + 2.9753\right)$$

$$= -1.9945$$

$$y^{(2)} = \frac{1}{6}\left(4 - x^{(2)}\right) = \frac{1}{6}\left(4 + 1.9945\right) = .9991$$

$$z^{(2)} = \frac{1}{6}\left(x^{(2)} - 2y^{(2)} - 14\right) = \frac{1}{6}\left(-1.9945 - 2 \times .9991 - 14\right)$$

$$= -2.9988.$$

3rd iteration

$$x^{(3)} = \frac{1}{9}\left(-17 - 4y^{(2)} - z^{(2)}\right)$$
$$= \frac{1}{9}\left(-17 - 4 \times .9991 + 2.9968\right) = -1.9997$$

$$y^{(3)} = \frac{1}{6}\left(4 - x^{(3)}\right) = \frac{1}{6}\left(4 + 1.9997\right) = .99995$$
$$\simeq 1.0000.$$

$$z^{(3)} = \frac{1}{6}\left(x^{(3)} - 2y^{(3)} - 14\right) = \frac{1}{6}\left(-1.9997 - 2 \times 1 - 14\right)$$
$$= -2.99995$$
$$\simeq -3.0000.$$

4th iteration.

$$x^{(4)} = \frac{1}{9}\left(-17 - 4y^{(3)} - z^{(3)}\right)$$
$$= \frac{1}{9}\left(-17 - 4 \times 1 + 3\right) = -2.0000$$

$$y^{(4)} = \frac{1}{6}\left(4 - x^{(4)}\right) = \frac{1}{6}\left(4 + 2\right) = 1.0000.$$

$$z^{(4)} = \frac{1}{6}\left(x^{(4)} - 2y^{(4)} - 14\right) = \frac{1}{6}\left(-2 - 2 - 14\right)$$
$$= -3.000.$$

After four iterations.

$x = -2, \; y = 1, \; z = -3.$