MODULE -1

TOPIC: SOLUTIONS OF NONLINEAR EQUATIONS

PRELIMINARIES

Nonlinear equations can be classified into following two categories:

i. **Algebraic Equation:** An expression of the form:

$$f_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1} x + a_n,$$

where $a's$ are constants $(a_0 \neq 0)$ and n is a positive integer is called a polynomial in x of degree n. The polynomial $f_n(x) = 0$ is called an algebraic equation of degree n.

ii. **Transcendental Equation:** The equation of the form $f_n(x) = 0$ is called transcendental equation, if $f_n(x)$ contains some other functions like exponential, logarithmic, trigonometric etc.

Examples:

$$p_3(x) = 2.75x^3 - 2.95x^2 + 3.16x - 4.67 = 0 \longrightarrow \text{Algebraic}$$
$$xe^x - 3 = 0 \longrightarrow \text{Transcendental}$$
$$2^x - x - 3 = 0 \longrightarrow \text{Transcendental}$$

The most important problem of non-linear equations is to find the roots of a non-linear equation i.e. to solve the equation $f(x) = 0$, where $f(x)$ is any non-linear equation.

<u>DEFINITION</u> (Root or Zero of f(x)): A number ξ is a solution of $f(x) = 0$ if $f(\xi) \equiv 0$, and ξ is said to be a root or zero of $f(x) = 0$. Geometrically, a root of the equation $f(x) = 0$ is the value of x at which the graph of $y = f(x)$ intersects or crosses the x-axis.

DEFINITION (Multiplicity of root): Taking the nonlinear equation

$$f(x) = (x-2)^2(x-1),$$

Here, $x = 2$ and $x = 1$ are the roots of $f(x) = 0$. But here $x = 2$ is a root of multiplicity 2 (i.e. double root) and $x = 1$ is a root of multiplicity 1(i.e. simple root).

So in general for a root ξ of multiplicity m,

$$f(\xi) = f^{'}(\xi) = f^{''}(\xi) = \ldots\ldots\ldots\ldots\ldots\ldots = f^{m-1}(\xi) = 0, \text{but } f^{m}(\xi) \neq 0.$$

Methods for solving Nonlinear equations

1. **Direct Methods:** These methods give the exact value of roots in a finite no. of steps and in addition capable of giving all the roots at the same time.

   e.g. Roots of a quadratic equation: $ax^2 + bx + c = 0, a \neq 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2. **Iterative Methods:** These methods are based on the idea of successive approximations in which we start with one or more initial approximations to the root and go on repeating the whole process till we get the desired solution with reasonable accuracy e.g.. To solve the quadratic equation: $ax^2 + bx + c = 0, a \neq 0$, we may choose one of the following iterative methods:

   i) $x_{k+1} = -\dfrac{c + ax_k^2}{b}, k = 0,1\ldots\ldots$

   ii) $x_{k+1} = -\dfrac{c}{ax_k + b}, k = 0,1\ldots\ldots$

   iii) $x_{k+1} = -\dfrac{c + bx_k^2}{ax_k}, k = 0,1\ldots\ldots$

i), ii), and iii) are iteration formula for solving the same quadratic equation. In iterative methods one or more initial guesses are required to start the process and need one iteration formula to carry out the whole process.

Convergence of an iterative method: Convergence of an iterative method simply means that starting with initial approximations, the method approaches towards the true value of the root. To measure how fast the initial approximations approach towards the root is defined through the rate or order of convergence of an iterative method,

Rate of convergence of an iterative method depends on the following two factors:

a) Rearrangement of the iterative formula

b) Choice of starting or initial approximation

Methods for obtaining initial approximation in Iterative methods:

i) **Graphical method:** Since the value of x at which the graph of the equation $y = f(x)$ intersects the x-axis, gives the root of $f(x) = 0$. Any value in the neighbourhood may be taken as initial approximation to the root.

ii) **Intermediate Value Theorem:** If $f(x)$ is continuous on some interval and $f(a), f(b)$ have different signs i.e. $f(a).f(b) < 0$, then the equation $f(x) = 0$ has at least one root between $x = a$ and $x = b$. e.g. $f(x) = 2^x - x - 3$, root lies in the interval (-3, -2) and (2, 3).

Iterative methods can be divided into two categories:

i. Bracketing Methods

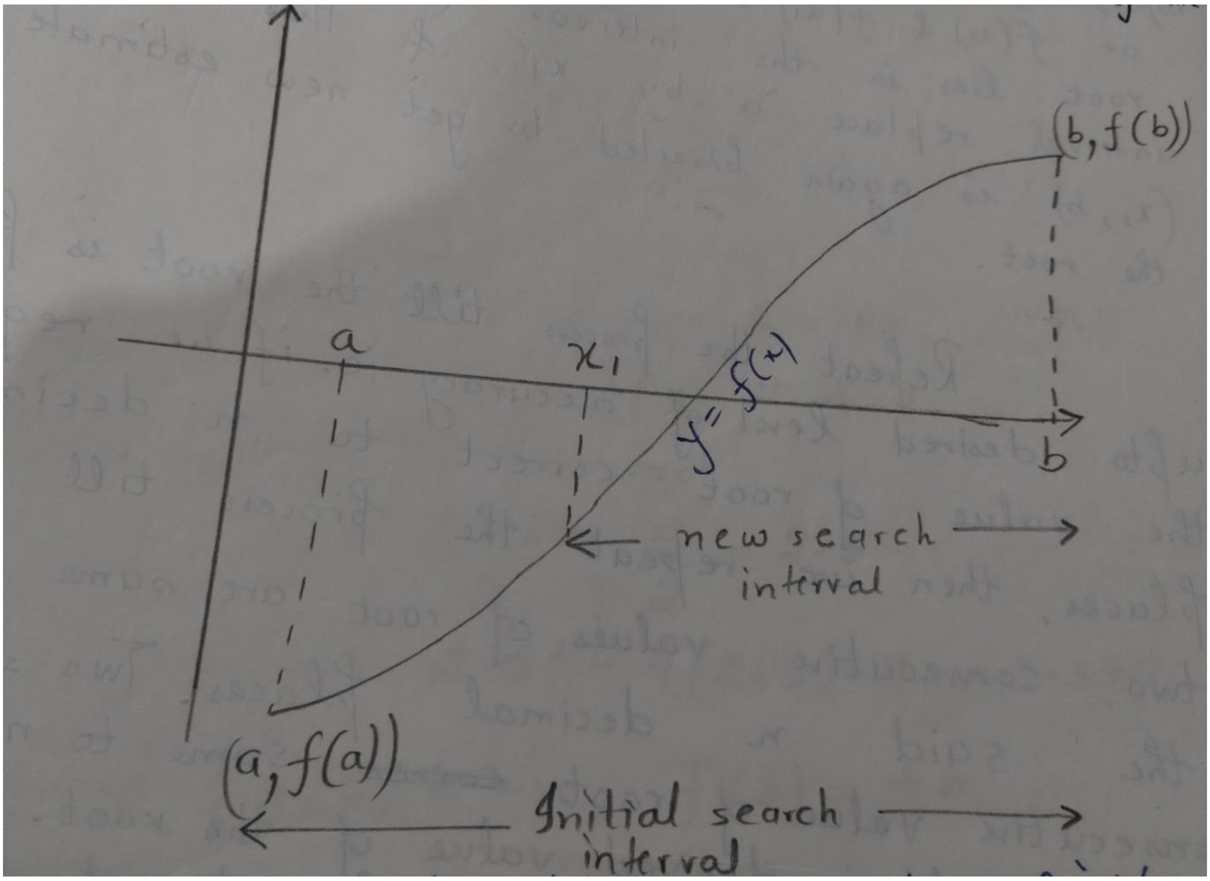
ii. Open Methods

BRACKETING METHODS

In these types of methods, two initial guesses for the root are required and these two guesses must "bracket" or be on the either side of the root. The particular methods described herein employ different strategies to systematically reduce the width of the bracket and hence get towards the correct value. Since the root is confined within the bracket so these methods always converges. Bracketing methods are absolutely (always) convergent methods. Two important bracketing methods are:

I) Bisection Method II) Regula Falsi Method

I) BISECTION METHOD (Bolzano method or Interval Halving or Binary Chopping method):



Procedure:

1. Choose two initial approximations a and b such that $f(a).f(b)<0$ (i.e. $f(a)$ and $f(b)$ are of opposite signs) ensuring that the root lies between a and b.
2. The first estimate of the root, say x_1, is determined by taking the mid-point of the interval (a,b) i.e. $x_1=\dfrac{a+b}{2}$
3. After this, there are three possibilities:
   - **a.** If $f(x_1)=0$, then there is a root at x_1.
   - **b.** If $f(a).f(x_1)<0$, then the root lies in the interval (a,x_1). So replace b by x_1 and this new interval (a,x_1) is again bisected to get next estimate of the root.
   - **c.** If $f(a).f(x_1)>0$, then the root lies in the interval (x_1,b). So replace a by x_1 and this new interval (x_1,b) is again bisected to get next estimate of the root.

Repeat the process, with new approximations; till the root is found up to desired level of accuracy i.e. if we require the value of root correct to n decimal places, then we repeat the process till two consecutive values of root are same up to n decimal places. Two same consecutive values of root same up to n decimal places is the desired value of the root.

STAR RULE

If we require the value of root correct to n decimal places, then following points should be kept in mind while performing calculations:

- All calculations should be at least up to $n+1$ decimal places and always rounding is used.
- Calculations should be up to same no. of decimal places i.e. if we are doing calculations up to 4 decimal places, all calculations should be up to 4 decimal places.

- Repeat the process till two consecutive values of root are same up to the said n decimal places.
- Two consecutive same values up to n decimal places is the required value of the root.

Problem 1: Find a real root of the equation $x^3 - 4x - 9 = 0$, using the Bisection method correct to three decimal places.

<u>Solution</u>: $f(x) = x^3 - 4x - 9 = 0.$ First we find two initial approximations, such that $f(x)$ has opposite signs at these initial approximations. For this we use hit and trial method.

$f(1) = 1 - 4 - 9 = -12(-ve)$, $f(2) = 8 - 8 - 9 = -9(-ve)$,
$f(3) = 27 - 12 - 9 = 6(+ve)$,

So, root lies b/w 2 & 3 and we take 2 and 3 as initial approximations. Since we require root correct to 3 decimal places, so, here, we do all calculations up to 4 decimal places (one more than required places) and same for all values and repeat the process till the two consecutive values are same up to 3 decimal places.

Iteration-01: $x_1 = \frac{2+3}{2} = 2.5000$ }, $f(2.5) = 2.5^3 - 4(2.5) - 9 = -3.3750$

$$f(2.5) = -ve \left| \begin{array}{l} f(2) = -ve, \\ f(3) = +ve \end{array} \right.$$ So, new interval is **(2.5, 3)**

Iteration-02: $x_2 = \frac{2.5+3}{2} = 2.7500$, $f(2.75) = 2.75^3 - 4(2.75) - 9 = +0.7969$

$$f(2.75) = +ve \left| \begin{array}{l} f(2.5) = -ve \\ f(3) = +ve \end{array} \right.,$$ So, new interval is **(2.5, 2.75)**

Iteration-03: $x_3 = \dfrac{2.5+2.75}{2} = 2.6250$, $f(2.625) = -1.4121$

$$f(2.625) = -ve \left| \begin{array}{l} f(2.5) = -ve \\ f(2.75) = +ve \end{array} \right.$$, So, new interval is **(2.625, 2.75)**

Iteration-04: $x_4 = \dfrac{2.625+2.75}{2} = 2.6875$, $f(2.6875) = -0.3391$

$$f(2.6875) = -ve \left| \begin{array}{l} f(2.625) = -ve \\ f(2.75) = +ve \end{array} \right.$$, So, new interval is **(2.6875, 2.75)**

Iteration-05: $x_5 = \dfrac{2.6875+2.75}{2} = 2.7188$, $f(2.7188) = 0.2218$

$$f(2.7188) = +ve \left| \begin{array}{l} f(2.6875) = -ve \\ f(2.75) = +ve \end{array} \right.$$, So, new interval is **(2.6875, 2.7188)**

Iteration-06: $x_6 = \dfrac{2.6875+2.7188}{2} = 2.7032$, $f(2.7032) = -0.0597$

$$f(2.7032) = -ve \left| \begin{array}{l} f(2.6875) = -ve \\ f(2.7188) = +ve \end{array} \right.$$, So, new interval is **(2.7032, 2.7188)**

Iteration-07: $x_7 = \dfrac{2.7032+2.7188}{2} = 2.7110$, $f(2.7110) = .0806$

$$f(2.7110) = +ve \left| \begin{array}{l} f(2.7032) = -ve \\ f(2.7188) = +ve \end{array} \right.$$, So, new interval is **(2.7032, 2.7188)**

Iteration-08: $x_8 = \dfrac{2.7032+2.7110}{2} = 2.7071$, $f(2.7071) = 0.0103$

$$f(2.7071) = +ve \left| \begin{array}{l} f(2.7032) = -ve \\ f(2.7110) = +ve \end{array} \right.$$, So, new interval is **(2.7032, 2.7071)**

Iteration-09: $x_9 = \dfrac{2.7032 + 2.7071}{2} = 2.7052$, $f(2.7052) = -0.0239$

$$f(2.7052) = -ve \left| \begin{array}{l} f(2.7032) = -ve \\ f(2.7071) = +ve \end{array} \right.$$, So, new interval is **(2.7052, 2.7071)**

Iteration-10: $x_{10} = \dfrac{2.7052 + 2.7071}{2} = 2.\underline{706}2$, $f(2.7062) = -0.0059$

$$f(2.7062) = -ve \left| \begin{array}{l} f(2.7052) = -ve \\ f(2.7071) = +ve \end{array} \right.$$, So, new interval is **(2.7062, 2.7071)**

Iteration-11: $x_{11} = \dfrac{2.7062 + 2.7071}{2} = 2.\underline{706}6$, $f(2.7066) = 0.0013$

$$f(2.7066) = +ve \left| \begin{array}{l} f(2.7062) = -ve \\ f(2.7071) = +ve \end{array} \right.$$, So, new interval is **(2.7062, 2.7066)**

Iteration-12: $x_{12} = \dfrac{2.7062 + 2.7066}{2} = 2.\underline{706}4$, $f(2.7064) = -0.3391$

Now, $x_{10} \approx x_{11}$ (Since both are same up to 3 decimal places), hence the value of the root is **2.<u>706.</u>**

***<u>Special Property Of Bisection Method</u>*:**

If a root of f(x) = 0 lies in the interval (a, b), then in the Bisection method only(because no function values are involved), the number of iterations n, required when the permissible error is given to be E, by the formula

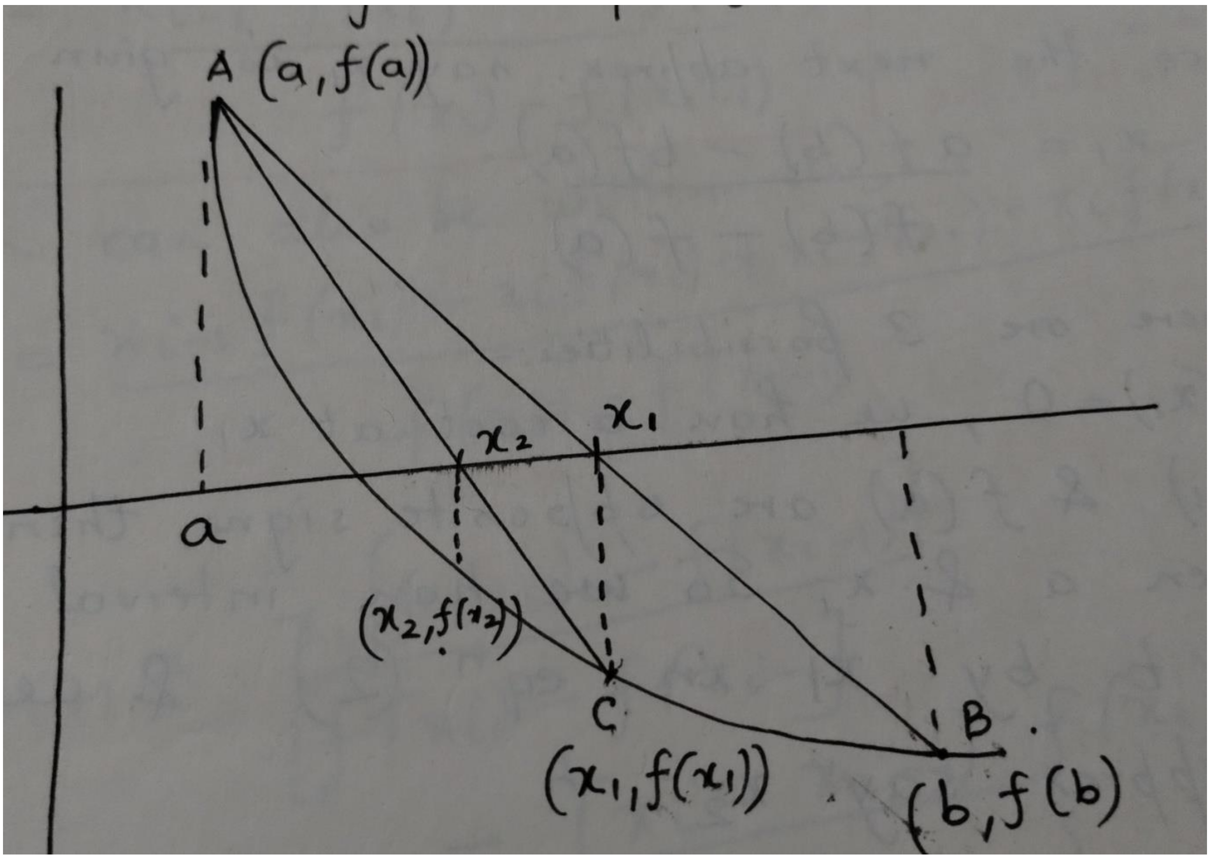
$$\boxed{n \geq \frac{\log(b-a) - \log E}{\log 2}}$$

e.g. ***(1, 2) and E = 10*** $^{-3}$

$$n \geq \frac{\log(2-1) - \log 10^{-3}}{\log 2} = \frac{3}{\log 2} \approx 10 \; i.e. \; n \geq 10$$

II) METHOD OF FALSE POSITION (Regula Falsi Method or Linear Interpolation or Method of Chords)

A shortcoming of the Bisection method is that in dividing the interval from x_l (lower value of approximation) and x_u (upper value of approximation) into two equal halves, no account is taken of the magnitudes of $f(x_l)$ and $f(x_u)$ i.e. if $f(x_u)$ is much closer to zero than $f(x_l)$, it is likely that the root is closer x_u to than to x_l. This is the main reason why the Bisection method converges very slowly. An alternative method that takes the magnitude of function values into account is Regula Falsi method, in which $f(x_l)$ and $f(x_u)$ are joined by a straight line. The intersection of this line with the $x-$ axis represents an improved estimate of the root. The fact that the replacement of the curve $y = f(x)$ by a straight line gives a false position of the root and which leads to the origin of the name, **Method of False position, or in Latin, Regula Falsi.**



Procedure:

1. Choose two initial approximations a and b such that $f(a).f(b)<0$ (i.e. $f(a)$ and $f(b)$ are of opposite signs) ensuring that the root lies between a and b.
2. Then join the points $\mathbf{A}(a, f(a))$ and $\mathbf{B}(b, f(b))$ by a straight line and the equation of the straight line(chord) joining these points is

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a) \qquad (1)$$

3. Now we find the point where this straight line **AB** intersects the x-axis, this will give first approximation to the root. This x coordinate (where the chord cuts the x-axis) is computed by putting $y=0$ in equation (1) and let this point be x_1.

$$-f(a) = \frac{f(b)-f(a)}{b-a}(x-a).$$

or

$$x_1 = a - \frac{b-a}{f(b)-f(a)} f(a)$$

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} \qquad (2)$$

4. After this, there are 3 possibilities:
   (a) If $f(x_1)=0$, then there is a root at x_1.
   (b) If $f(a).f(x_1)<0$, then the root lies in the interval (a, x_1). So replace b by x_1 in equation (2) above and we get the new approximation say, x_2.
   (c) If $f(a).f(x_1)>0$, then the root lies in the interval (x_1, b). So replace a by x_1 in equation (2) above and we get the next approximation say, x_2.

Repeat the process, with new approximations; till the root is found up to desired level of accuracy i.e. if we require the value of root correct to n decimal places, then we repeat the process till two consecutive values of root are same up to n

decimal places. Two same consecutive values of root same to n decimal places is the desired value of the root.

In general iterative formula of Regula Falsi method can be written as:

(A)
$$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}, i = 1,2.......}$$

where $x_0 = a$ and $x_1 = b$ are initial approximations. The above formula (A) can also be written in the form:

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1}) + \{x_i f(x_i) - x_i f(x_i)\}}{f(x_i) - f(x_{i-1})}$$

$$or, x_{i+1} = \frac{x_i [f(x_i) - f(x_{i-1})]}{f(x_i) - f(x_{i-1})} - \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} f(x_i)$$

(B)
$$\boxed{x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} f(x_i)}$$

Note: *Method of false position approach towards the root only from one side i.e. one approximation taken at initial stage remains fixed throughout and only other approximation gets changing with each iteration. Because of this reason, at certain instances* $(f(x) = x^n - 1 = 0)$ *(sometimes) Bisection method converges faster than Regula- Falsi method.*

OPEN METHODS

In bracketing methods, the root is located within an interval prescribed by a lower and upper bound i.e. root is confined in a bracket and so it is sure to find the root by these methods. In contrast, open methods are based on the formulae which require only a single starting values or two starting values, that do not necessarily bracket the root. As such, they sometimes diverge or move away from the true root as the computation progresses. So, open methods are conditionally convergent (may or may not converge) methods. However, when the open method converges they do more quickly than bracketing methods. Some open methods are:

I) Secant method

II) Newton Raphson method

II) Direct iteration method (General Iterative Method)

SECANT METHOD

It is similar to the Regula-Falsi method. The only difference is that it is an open method, hence, at any stage of the iterations, it is not required to test whether the root lies in the interval (x_{i-1}, x_i). Instead, last two consecutive approximations are used to obtain the next approximation to the root. The Secant method can also work on the same iterative formula of Regula Falsi as:

$$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}, i = 1,2........}$$

or

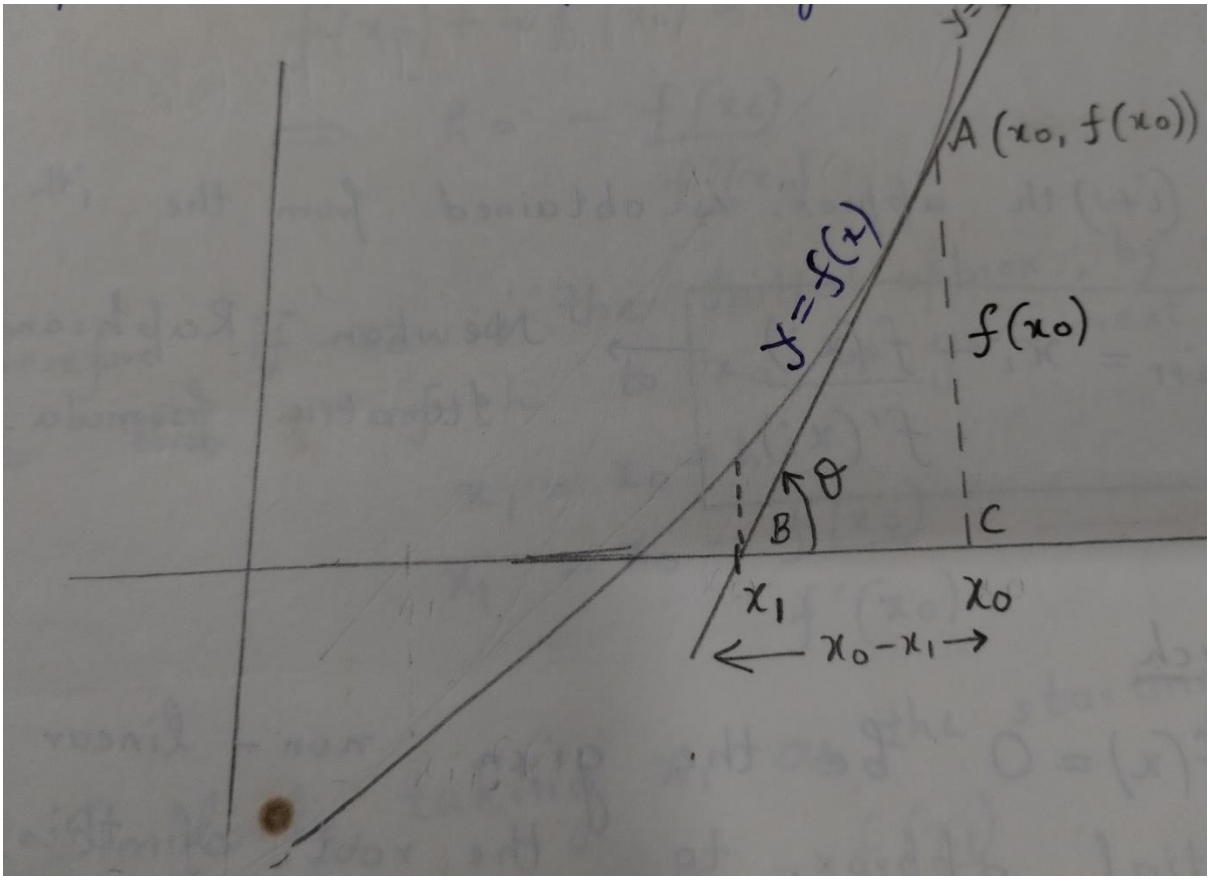
$$\boxed{x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} f(x_i)}$$

So, starting with two initial approximations, we can get the successive approximations to the root, without bracketing the root at every step.

Note: In Secant method, it is not necessary that the initial approximations should bracketed the root. Hence, Secant method can sometimes fail if initial approximations are very far away from the root.

NEWTON RAPHSON METHOD AND ITS VARIANTS (Newton's method or Method of tangents):

This method is based on a linear approximation of the function but does so by using a tangent to the curve. In this we start with a single initial approximation, say x_0, that is not far away from the root. Now then we draw the tangent to the curve at this point x_0 and then take the intersection of this tangent with the x-axis as the next approximation to the root. This process is continued till the difference b/w the two consecutive values of the root are same up to desired level of accuracy.(refer figure)



Geometrical Proof: The slope of the curve $y = f(x)$ at point $(x_0, f(x_0))$ is given by:

(1) $$\tan\theta = f'(x_0)$$

and in triangle ABC

(2) $$\tan\theta = \frac{f(x_0)}{x_0 - x_1}$$

Equating (1) and (2), we get: $$f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

and therefore, $$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

or, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

(*)

Similarly, next approximation would be:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

……………………..

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In general, (i+1) th approximation is obtained from the ith approximation as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This equation represents **Newton Raphson's Iterative formula**.

<u>Analytical Proof:</u> If $f(x) = 0$ be the given nonlinear equation and x_0 be the initial approximation to the root of this equation. If α be the exact value of the root, then

$$\alpha = x_0 + h$$

where h is correction to x_0 and it is very small quantity. Since $f(x)=0$ and α is exact root, hence $f(\alpha)=0$ i.e. $f(x_0+h)=0$

Expanding by Taylor series about x_0 we get,

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots\ldots\ldots\ldots=0$$

As h is very small quantity, so neglecting h^2 and higher power terms i.e. $O(h^2)$, we get:

$$f(x_0)+hf'(x_0)=0$$

or, $$h=-\frac{f(x_0)}{f'(x_0)}$$

Therefore, if x_0 is the initial approximation then by adding this value of h to x_0, we get next approximation x_1 as: $x_1=x_0+h$

or, $$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

Similarly, taking x_1 as the starting value we get: $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$

In general,

$$\boxed{x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}}$$

Note:

i. Newton-Raphson method may not converge if the initial approximation is very far away from the root.

ii. If $f'(x)=0$ (or very small) for any initial approximation or for any intermediate approximation of the root, then the method is not convergent, since then we are at the maxima or minima(Figure below) of the curve i.e. tangent is parallel to the x axis, and we don't get the intersection point with the x-axis. So, always check slope for all approximations to the root. Due to this reason, this is called conditionally convergent method and called Pathological Situation in this method.

Modified Newton Raphson Formula (for multiple roots)

Let's consider the Newton Raphson formula in the form:

$$\boxed{x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)}}$$

This is Modified Newton Raphson Formula for determining root of multiplicity m. Original Newton Raphson Formula gives linear rate of convergence for $m\geq 2$, so Newton modified the original one into the above form for again getting quadratic convergence.

GENERAL ITERATIVE METHOD (DIRECT ITERATION METHOD or FIXED ITERATION METHOD)

In this method, the nonlinear equation $f(x)=0$ is written in an equivalent form:

$$x=\phi(x)$$

It is called fixed point equation. So, the method provides a convenient form for predicting the value of x as a function of x. If x_0 is the initial guess to the root, then the next approximation to the root is given by:

$$x_1=\phi(x_0)$$

Further approximation is given by :

$$x_1=\phi(x_1)$$

This iteration process can be expressed in general form as:

$$x_{i+1} = \phi(x_i), \text{ where } \quad i = 1,2,3,.....$$

This is called the fixed point iteration formula. This method of solution is also known as **method of successive approximations or method of direct substitution**. Repeat the process till the consecutive values of root are same up to desired level of accuracy.

For example, consider the equation $x^3 - 5x + 1 = 0$, which has a root in the interval (0, 1). We write it in the form $x = \phi(x)$ and the corresponding iteration method in the following ways:

(i) $\quad x = \frac{1}{5}(x^3 + 1)$, and $x_{k+1} = \frac{1}{5}(x_k^3 + 1)$, $\quad k = 0, 1, 2, \ldots$ (18.17)

(ii) $\quad x = (5x - 1)^{1/3}$, and $x_{k+1} = (5x_k - 1)^{1/3}$, $\quad k = 0, 1, 2, \ldots$ (18.18)

(iii) $\quad x = x^3 - 4x + 1$, and $x_{k+1} = x_k^3 - 4x_k + 1$, $\quad k = 0, 1, 2, \ldots$ (18.19)

If we take $x_0 = 1.0$, we obtain from

Method (18.17): $x_1 = 0.4$, $x_2 = 0.2128$, $x_3 = 0.2019$, ..., which is converging to the root in (0, 1).

Method (18.18): $x_1 = 1.5874$, $x_2 = 1.9072$, $x_3 = 2.0437$, ..., which is not converging to the root in (0, 1).

Method (18.19): $x_1 = -2$, $x_2 = 1$, $x_3 = -2$, ..., which is not converging to any root and the iteration values oscillate.

Hence, the convergence of the method of the form (18.16) depends on the suitable choice of the iteration function $\phi(x)$ and the initial approximation x_0.

The condition for convergence of direct iteration method is

$$|\phi'(x_k)| < 1 \text{ for } k = 0,1,2,....$$

Method (18.17):

$$\phi(x) = \frac{1}{5}(x^3+1), \quad \phi'(x) = \frac{3x^2}{5}, \quad |\phi'(x)| < 1, \quad 0 < x < 1.$$

Therefore, the method converges to the root in (0, 1).

Method (18.18):

$$\phi(x) = (5x-1)^{1/3}, \quad \phi'(x) = \frac{5}{3(5x-1)^{2/3}}, \quad \max|\phi'(x)| = \frac{5}{3} > 1, \quad 0 < x < 1.$$

The iterations do not converge to the root in (0, 1).

Method (18.19):

$$\phi(x) = x^3 - 4x + 1, \quad \phi'(x) = 3x^2 - 4, \quad |\phi'(x)| > 1, \quad 0 < x < 1.$$

The iterations do not converge to the root in (0, 1).

COMPARISON OF DIFFERENT ITERATIVE METHODS

	Convergence	**Rate of Convergence**	**Complex roots**	**Function evaluation per iteration**
Bisection	Absolute	1 (linear)	Cannot find	01
Regula Falsi	Absolute	1 (linear)	Cannot find	01
Newton Raphson	Conditional	2 (quadratic)	Capable to find but only when initial approx. is complex	02

Secant	Conditional	1.618	---	01

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