① Using $a = 0$ & $b = 1$ as initial approx. to the root, execute four iterations to find the ① the real root of $x - \cos x = 0$ correct to four decimal places

Soln:- $f(x) = x - \cos x = 0$ (*Calculator in Radian Mode)

$f(0) = 0 - \cos 0 = -1$ (-ve)

$f(1) = 1 - \cos 1 = .45970$ (five decimal places as four decimal places are required)

Initial Interval $(0, 1)$

I Iteration $x_1 = \frac{0+1}{2} = .5$

$f(x_1) = f(.5) = .5 - \cos(.5) < 0$ (-ve)

$f(x_1)$ ⊖ — $f(0)$ (-ve), $f(1)$ (+ve)

New interval $(x_1, 1)$ or $(.5, 1)$

II Iteration $x_2 = \frac{(.5+1)}{2} = .75$

$f(x_2) = .75 - \cos(.75) > 0$ (+ve)

$f(x_2) = f(.75)$ (+ve) — $f(.5)$ (-ve), $f(1)$ +ve)

New Interval $(.5, .75)$

III Iteration

$$x_3 = \frac{.5 + .75}{2} = .625$$

$f(x_3) = f(.625) < 0 \; (-ve)$

$f(.625) < 0 \; (-ve)$ $\begin{cases} f(.5) & -ve \\ f(.75) & +ve. \end{cases}$

New Interval $(.625, .75)$

IV Iteration

$$x_4 = \frac{.625 + .75}{2} = .6875$$

Root after 4th Iteration is $.6875$.

(2) The nonlinear eq^n $x \log_{10} x = 1.2$ is given (3)

a) Obtain an interval of unit length which contains the positive real root of the eq^n

b) Using end points of the above obtained interval as initial approx., compute the value of root correct to four decimal places using Regula - Falsi method.

Solu:-

$$f(x) = x \log_{10} x - 1.2 = 0$$

i) Interval of unit length is required

$$f(1) = 1 \log_{10} 1 - 1.2 = -1.2 \ (-ve)$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -.59794 \ (-ve)$$

$$f(3) = 3 \log_{10} 3 - 1.2 = .23136 \ (+ve)$$

Root lies in an interval of unit length, that is $(2, 3)$ [Length of interval = 1]

ii) Taking Regula Falsi formula

$a = 2 \qquad f(a) = f(2) = -.59794$

$b = 3 \qquad f(b) = f(3) = .23136$.

Calculations upto five decimal places as root is required upto four decimal places

I Iteration
$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2 \times (-.23136) + 3 \times .59794}{.23136 + .59794}$$

$$= 2.72102$$

$$f(x_1) = f(2.72102) = -.017086 \ (-ve)$$

$f(x_1)$ $(-ve)$ $\begin{cases} f(a) = -ve \\ f(b) = +ve \end{cases}$

New Interval (x_1, b)

II Iteration
$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{2.72102 \times .23136 + 3 \times .017086}{.23136 + .017086}$$

$$= 2.74020$$

$$f(x_2) = f(2.74020) = -.0003890$$

$f(x_2)$ $(-ve)$ — $f(x_1)$ $(-ve)$, $f(b)$ $+ve$

New Interval (x_2, b)

III Iteration
$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$

$$= \frac{2.74020 \times .23136 + 3 \times .0003890}{.23136 + .0003890}$$

$$= 2.74064$$

$f(2.74064) = -.00000532$ 5

$f(x_3)(-ve)$ — $f(x_3) - ve$, $f(b) = +ve$

New Interval (x_3, b)

IV Iteration

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$= \frac{2.74064 \times .23136 + 3 \times .0000532}{.23136 + .00000532}$$

$$= 2.74064$$

$x_3 \simeq x_4$ upto four decimal places

Hence root is $\underline{\underline{2.7406}}$ upto four decimal places

Problem:- A real root of the eqⁿ

$$f(x) = x^3 - 5x + 1 = 0$$

lies in the interval $(0,1)$. Perform four iterations of the secant method & Regula-falsi method to obtain the root.

1) $x_0 = 0,\ x_1 = 1$

$f(x_0) = 1,\ f(x_1) = f(1) = -3.$

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

x_2 $x_0 \ x_1 \downarrow$

x_3 $x_1 \ x_2 \downarrow$

$x_4.$ $x_2 \ x_3 \downarrow$

I Iteration

$i = 1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} =$$

$$= \frac{0 \times (-3) - 1 \times 1}{-3 - 1} = \frac{1}{4} = .25$$

$(0, 1)$

$($

$f(.25) = -.234375.$

$x_1 \longrightarrow x_2$

$x_2 \longrightarrow x$

$i=2$

II Iteration

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1 \times (-.234375) - (.25)(-3)}{-.234375 + 3}$$

$$= .186441.$$

$f(x_3) = .074276$

<u>III Iteration</u>

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{.25 \times .074276 - .186441 \times (-.234375)}{.074276 + .234375}$$

$$= .201736 \qquad f(.201736) = -.000470.$$

<u>IV Iteration</u>

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{.186441 \times (-.000470) - .201736 \times .074276}{-.000470 - .074276}$$

$$= .201640$$

Regula - falsi method

I Iteration

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = .25$$

$$f(x_2) = -.234375$$

$f(0) = 1$ $+$

$f(1) = -3$ $-$

$\left.\right\}$ $\cancel{(0, 1)}$ $\longrightarrow$ $(0, .25)$

x_0 $\quad$ $x_1 \to x_0$

II Iteration

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{0 \times f(x_2) - .25 \times 1}{-.234375 - 1} = \frac{-.25}{} = .202532$$

$$f(x_3) = -.004352$$

$(0, .25)$ — $(0, x_3)$, $(0, .202532)$

$x_0 \quad x_2$

III Iteration

$$x_4 = \frac{x_0 f(x_3) - x_3 f(x_0)}{f(x_3) - f(x_0)} =$$

$$= \frac{0 - .202532 \times 1}{-.004352 - 1} = .201654$$

$$f(x_4) = -.000070.$$

$(0, .202532)$ — $(0, .201654)$

IV Iteration

$$x_5 = \frac{x_0 f(x_4) - x_4 f(x_0)}{f(x_4) - f(x_0)}$$

$$= \frac{-.201654 \times 1}{-.000070 - 1} = .201640.$$

Problem 2.6 :- Find the Iterative methods based on the Newton Raphson method for finding $\sqrt{N}$, $1/N$, $N^{1/3}$, when N is a positive real no. Apply the methods to $N = 15$ for obtaining results correct to 2 decimal places.

Soln:-

I) :- for finding $\sqrt{N}$

let $x = \sqrt{N}$

$$x^2 = N$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{(x_i^2 - N)}{2x_i}$$

$$= \frac{2x_i^2 - x_i^2 + N}{2x_i} = \frac{x_i^2 + N}{2x_i}$$

$$\boxed{x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)}$$ Iteration formula for $\sqrt{N}$.

for $N = 15$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{15}{x_i}\right)$$

As square root of 15 lies b/w 3 & 4

let $x_0 = 3.5$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{15}{x_0}\right) = \frac{1}{2}\left(3.5 + \frac{15}{3.5}\right)$$

$$= 3.893.$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{15}{x_1}\right) = \frac{1}{2}\left(3.893 + \frac{15}{3.893}\right)$$

$$= 3.873.$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{15}{x_2}\right) = \frac{1}{2}\left(3.873 + \frac{15}{3.873}\right)$$ (11)

$$= 3.873$$

square root of 15 correct to 2 decimal places $= 3.87$

II) for evaluating $1/N$

$$x = \frac{1}{N}$$

$$f(x) = N - \frac{1}{x}$$

$$f'(x) = \frac{1}{x^2}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(N - \frac{1}{x_i}\right)}{\frac{1}{x_i^2}}$$

$$= x_i - \left(N - \frac{1}{x_i}\right) x_i^2$$

$$= x_i - N x_i^2 + x_i$$

$$= 2x_i - N x_i^2$$

$$\boxed{x_{i+1} = x_i(2 - N x_i)}$$

Now $N = 15$, so $\frac{1}{15}$ lies b/w 0 & 1.

$x_0 = .05$

$$x_1 = x_0(2 - 15x_0) = .05(2 - 15 \times .05)$$
$$= .0625$$

$$x_2 = x_1(2 - 15x_1) = .0625(2 - 15 \times .0625)$$
$$= .0664$$

$$x_3 = x_2(2 - 15x_2) = .0664(2 - 15 \times .0664)$$
$$= .0667$$

$$x_4 = x_3(2 - 15x_3) = .0667(2 - 15 \times .0667)$$
$$= .0667$$

Ans: $.067$.

iii). for evaluating $N^{1/3}$

$$x = N^{1/3}$$

$$x^3 = N$$

$$f(x) = x^3 - N$$

$$f'(x) = 3x^2$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{(x_i^3 - N)}{3x_i^2}$$

$$= \frac{3x_i^3 - x_i^3 + N}{3x_i^2} = \frac{2x_i^3 + N}{3x_i^2}$$

$$x_{i+1}. = \frac{1}{3}\left(2x_i + \frac{N}{x_i^2}\right)$$

root lies b/w for N = 15 b/w 2 & 3

$$x_0 = 2.5$$

$$x_1 = \frac{1}{3}\left(2x_0 + \frac{15}{x_0^2}\right) = 2.\underline{467}$$

$$x_2 = \frac{1}{3}\left(2x_1 + \frac{15}{x_1^2}\right) = 2.\underline{466}.$$

Ans:- 2.47.

Ques: Find by Newton's method, the real root of the eqⁿ $3x = \cos x + 1$ correct to four decimal places. Also, obtain the root through direct iteration method. (13)

Solu:-

$$f(x) = 3x - \cos x - 1$$

$$f(0) = 0 - \cos 0 - 1 = -1 - 1 = -2 = -ve.$$

$$f(1) = 3 - \cos 1 - 1 = 1.4597 = +ve.$$

So root lies b/w 0 & 1

Take $x_0 = .5$

$$f(x) = 3x - \cos x - 1.$$

$$f'(x) = 3 + \sin x$$

Newton's Iterative formula gives.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(3x_n - \cos x_n - 1)}{(3 + \sin x_n)}$$

$$x_{n+1} = \frac{\cancel{3x_n} + x_n \sin x_n - \cancel{3x_n} + \cos x_n + 1}{3 + \sin x_n}$$

$$\boxed{x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}}$$

for $n = 0$, $x_0 = .5$

$$x_1 = \frac{.5 \sin .5 + \cos .5 + 1}{3 + \sin .5} = .60852$$

$$x_2 = \frac{.60852 \sin(.60852) + \cos(.60852) + 1}{3 + \sin .60852}$$

$$= .60710$$

$$x_3 = \frac{.60710 \sin(.60710) + \cos(.60710) + 1}{3 + \sin(.60710)}$$ (14)

$$= .60710.$$

so $x_2 \simeq x_3$

Ans .6071

ii) Using direct iteration method.

$$f(x) = 3x - \cos x - 1 = 0$$

$$f(0) = 3 \times 0 - 1 - 1 = -2 \ (-ve)$$

$$f(1) = 3 \times 1 - \cos 1 - 1 = 1.4597 \ (+ve)$$

Hence root lies b/w $(0, 1)$

Rewriting the eq^n in the form.

$$x = \frac{1}{3}(\cos x + 1) = \phi(x)$$

when

$$\phi(x) = \frac{1}{3}(\cos x + 1)$$

$$\phi'(x) = \frac{1}{3}(-\sin x)$$

$$|\phi'(x)| = \frac{1}{3}|\sin x| < 1 \quad \text{for } x \in (0,1)$$

Hence, the method is convgt. for

$$\phi(x) = \frac{1}{3}(\cos x + 1) \quad \text{in } (0,1)$$

Taking initial approx. as. $x_0 = \frac{0+1}{2} = .5$

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos .5 + 1) = .62586$$

$$x_2 = \phi(x_1) = .60349$$

$$x_3 = \phi(x_2) = .60779$$

$x_4 = \phi(x_3) = .60697$

$x_5 = \phi(x_4) = .60713$

$x_6 = \phi(x_5) = .60710$

$x_5 \simeq x_6$ upto four decimal places

Ans $\boxed{\text{root} = .6071}$

Problem - 2.4: Apply False Position Method to find the smallest positive root of the equation $x - e^{-x} = 0$ correct to three decimal places. (17)

Solu:-

$$f(x) = x - e^{-x}$$
$$f(0) = 0 - 1 = -1 = -ve$$
$$f(1) = 1 - e^{-1} = .6321 = +ve.$$

root lies b/w 0 & 1 = (0, 1)

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times .6321 + 1 \times 1}{.6321 + 1} = \frac{1}{1.6321}$$
$$\cong .6127$$

$$f(x_1) = f(.6127) = .6127 - e^{-.6127}$$
$$= .0708 = +ve.$$

$f(.6127) = +ve$ | $f(0) = -ve$, $f(1) = +ve.$ | $(.6127, 1)$

new Interval $(0, .6127)$

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)} = \frac{.6127 \times 1}{1.0708} = .5722$$

$$f(x_2) = f(.5722) = .5722 - e^{-.5722}$$
$$= .0079 = +ve.$$

$f(.5722) = +ve$ | $f(0) = -ve$, $f(.6127) = +ve$

new Interval $(0, .5722)$

so $$x_3 = \frac{af(x_2) - x_2 f(a)}{f(x_2) - f(a)} = \frac{.5722}{1.0079} = .5677$$

$$f(x_3) = f(.5677) = .5677 - e^{-.5677}$$
$$= .0009 = +ve.$$

new Interval $(0, .5677)$

$$x_4 = \frac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)} = \frac{.5677}{1.0009} = \underline{.5672}$$

$f(.5672) = .0001$

new Interval $(0, .5672)$

$$x_5 = \frac{af(x_4) - x_4 f(a)}{f(x_4) - f(a)} = \underline{.5672}$$

so $x_4 \simeq x_5$ correct to three decimal places

hence, Ans $\simeq .567$ Ans