Quiz 5

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Problem 1 Use the definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}(s)$ if:

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$$

Solution:

$$\mathcal{L}\{f\}(s) = \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt$$

LHS integral:

$$\begin{aligned} &= \int_0^1 te^{-st}\,dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\Big|_0^1 \\ &= \left(-\frac{1}{s} - \frac{1}{s^2}\right)e^{-s} - \left(0 - \frac{1}{s^2}\right) \\ &= \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}, \text{ for } s > 0 \end{aligned}$$

RHS integral:

$$\begin{aligned} &= \int_1^\infty e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_1^\infty \\ &= 0 - \left(-\frac{1}{s}e^{-s}\right) \\ &= \frac{1}{s}e^{-s}, \text{ for } s > 0 \end{aligned}$$

Answer:

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s} \\ &= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} \text{ for } s > 0 \end{aligned}$$

Problem 2 Use the definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}(s)$ if:

$$f(t) = \begin{cases} \cos 6t & \text{if } 0 < t < \frac{\pi}{2} \\ 0 & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution:

$$\mathcal{L}\{f\}(s) = \int_0^{\frac{\pi}{2}} e^{-st} \cos 6t \, dt + \int_{\frac{\pi}{2}}^{\infty} 0 \times e^{-st} \, dt$$

The second integral is zero since it is a product of 0 and an exponential function. We only need to solve the first integral:

$$\mathcal{L}\{f\}(s) = \int_0^{\frac{\pi}{2}} e^{-st} \cos 6t \, dt$$

Now, we integrate by parts again. Let $u = \cos 6t$ and $dv = e^{-st}\,dt$.
Then, $du = -6 \sin 6t \, dt$ and $v = -\frac{1}{s} e^{-st}$.

$$\mathcal{L}\{f\}(s) = \left[-\frac{1}{s} e^{-st} \cos 6t \right]_0^{\frac{\pi}{2}} - \frac{6}{s} \int_0^{\frac{\pi}{2}} \frac{1}{s} e^{-st} \sin 6t \, dt$$

Now, we integrate by parts again. Let $u = \sin 6t$ and $dv = e^{-st}\,dt$.
Then, $du = 6 \cos 6t \, dt$ and $v = -\frac{1}{s} e^{-st}$.

$$\mathcal{L}\{f\}(s) = \left[-\frac{1}{s} e^{-st} \cos 6t \right]_0^{\frac{\pi}{2}} - \left[-\frac{6}{s^2} e^{-st} \sin 6t \right]_0^{\frac{\pi}{2}} - \frac{36}{s^2} \mathcal{L}\{f\}(s)$$

Now, we add $\frac{36}{s^2}\mathcal{L}\{f\}(s)$ to both sides of the equation.
Then we factor out $\mathcal{L}\{f\}(s)$.

$$\begin{aligned} \mathcal{L}\{f\}(s) \left(\frac{s^2 + 36}{s^2} \right) &= \left[-\frac{1}{s} e^{-st} \cos 6t \right]_0^{\frac{\pi}{2}} - \left[-\frac{6}{s^2} e^{-st} \sin 6t \right]_0^{\frac{\pi}{2}} \\ &= \left(\left(-\frac{1}{s} e^{-s\frac{\pi}{2}} + \frac{1}{s} \right) + (0) \right) \end{aligned}$$

Now, we multiply both sides by $\frac{s^2}{s^2+36}$ and simplify.

$$
\begin{aligned}
\mathcal{L}\{f\}(s) &= \frac{s^2}{s^2+36}\left(-\frac{1}{s}e^{-s\frac{\pi}{2}}+\frac{1}{s}\right) \\
&= \frac{s}{s^2+36}(-e^{-s\frac{\pi}{2}}+1) \\
&= \frac{s(1-e^{-s\frac{\pi}{2}})}{s^2+36},\ \text{for } s>0
\end{aligned}
$$

Problem 3 Use the Laplace Transform Table (LTT) to find F(s) = $\mathcal{L}\{f\}(s)$ if:

$$f(t) = (\sin t + \cos t)^2$$

Solution:

$$\begin{aligned} f(t) &= \sin^2 t + 2\sin t\cos t + \cos^2 t \\ &= \frac{1}{2} - \frac{1}{2}\cos 2t + \sin 2t + \frac{1}{2} + \frac{1}{2}\cos 2t \\ &= 1 + \sin 2t \end{aligned}$$

Now, we use the LTT to find the Laplace transform of $1 + \sin 2t$.

$$\begin{aligned} \mathcal{L}\{f\}(s) &= \mathcal{L}\{1\} + \mathcal{L}\{\sin 2t\} \\ &= \frac{1}{s} + \frac{1}{s^2+4}, \text{ for } s > 0 \end{aligned}$$

Problem 4 Use the Laplace Transform Table (LTT) to find F(s) = $\mathcal{L}\{f\}(s)$ if:

$$f(t) = t^4 + t^2 e^{-t} + \sin^2 t - e^{3t} \sin \sqrt{3}t$$

Solution:

$$\begin{aligned}
&= \mathcal{L}\{t^4\} + \mathcal{L}\{t^2 e^{-t}\} + \mathcal{L}\{\sin^2 t\} + \mathcal{L}\{e^{3t} \sin \sqrt{3}t\} \\
&= \frac{4!}{s^5} + \frac{2!}{(s+1)^3} + \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} - \frac{3\sqrt{3}}{(s-3)^2+3} \\
&= \frac{24}{s^5} + \frac{2}{(s+1)^3} + \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} - \frac{3\sqrt{3}}{(s-3)^2+3}
\end{aligned}$$