

## Quiz 7

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**Problem 1** Solve the given initial value problem using the method of Laplace transforms.

$$y''(t) - y(t) = t - 2$$

where  $y(2) = 3$  and  $y'(2) = 0$ .

**Solution:**

*Step 1: We first move the initial condition to  $t = 0$ .*

$$y''(t+2) - y(t+2) = (t+2) - 2$$

$$y''(t+2) - y(t+2) = t$$

$$w(t) := y(t+2)$$

Hence,

$$y'(t+2) = w'(t)$$

$$y''(t+2) = w''(t)$$

*Step 2: We replace  $y(t+2)$  by  $w(t)$ .*

$$w''(t) - w(t) = (t+2) - 2$$

$$w''(t) - w(t) = t$$

*Step 3: We apply the Laplace transform.*

To the RHD,

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad \text{for } s > 0$$

To the LHD,

$$\begin{aligned} \mathcal{L}\{w''(t) - w(t)\} &= (s^2Y - s(3) - 0) - Y \\ &= s^2Y - 3s - Y \\ &= Y(s^2 - 1) - 3s \end{aligned}$$

Step 4: We solve for  $Y(s)$ .

$$\begin{aligned} Y(s^2 - 1) - 3s &= \frac{1}{s^2} \\ &= \frac{1}{s^2} + 3s \\ &= \frac{1 + 3s^3}{s^2} \end{aligned}$$

$$Y = \frac{1 + 3s^3}{s^2(s^2 - 1)}$$

Step 5: We perform partial fraction decomposition to the function.

$$\begin{aligned} \frac{1 + 3s^3}{s^2(s^2 - 1)} &= \frac{1 + 3s^3}{s^2(s - 1)(s + 1)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 1} + \frac{D}{s + 1} \end{aligned}$$

Step 6: We solve for  $A$ ,  $B$ ,  $C$ , and  $D$ .

$$= As(s - 1)(s + 1) + B(s - 1)(s + 1) + Cs^2(s + 1) +Ds^2(s - 1)$$

$s = 0$  :

$$-B = 1$$

$$B = -1$$

$s = 1$  :

$$2C = 4$$

$$C = 2$$

$s = -1$  :

$$-2D = -2$$

$$D = 1$$

We solve for  $A$  by equating the function.

$$\begin{aligned} &= As(s^2 - 1) + B(s^2 - 1) + Cs^2(s + 1) +Ds^2(s - 1) \\ &= A(s^3 - s) + B(s^2 - 1) + C(s^3 + s^2) + D(s^3 - s^2) \\ &= As^3 - As + Bs^2 - B + Cs^3 + Cs^2 +Ds^3 -Ds^2 \\ &\quad s^3(A + D) + s^2(B + C - D) + s(-A) - B \end{aligned}$$

$$A + C + D = 3; \quad B + C - D = 0; \quad -A = 0; \quad -B = 1$$

$$A = 0; \quad B = -1; \quad C = 2; \quad D = 1$$

Hence,

$$\frac{1 + 3s^2}{s^2(s+1)(s-1)} = -\frac{1}{s^2} + \frac{2}{s-1} + \frac{1}{s+1}$$

*Step 7: We take the inverse Laplace transform.*

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1 + 3s^2}{s^2(s+1)(s-1)} \right\} &= \mathcal{L}^{-1} \left\{ -\frac{1}{s^2} + \frac{2}{s-1} + \frac{1}{s+1} \right\} \\ &= -\mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} + 2\mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} \\ &= -t + 2e^t + e^{-t} \end{aligned}$$

*Step 8: We move the initial condition to  $t = 2$ .*

$$= -(t-2) + 2e^{t-2} + e^{-(t-2)}$$

$$= 2 - t + 2e^{t-2} + e^{2-t}$$

**Problem 2** Solve the given third-order initial value problem for  $y(t)$  using the method of Laplace transforms.

$$y'''(t) + 3y''(t) + 3y'(t) + y(t) = 0$$

where,  $y(0) = -4, \quad y'(0) = 4, \quad y''(0) = -2$

**Solution:**

*Step 1: We first take the LT of the LHS:*

$$\begin{aligned} &= \mathcal{L}\{y''' + 3y'' + 3y' + y\} \\ &= \mathcal{L}\{y'''\} + \mathcal{L}\{3y''\} + \mathcal{L}\{3y'\} + \mathcal{L}\{y\} \\ &= (s^3Y - s^2(-4) - s(4) - (-2)) + 3(s^2Y - s(-4) - (4)) + 3(sY - (-4)) + Y \\ &= Y(s^3 + 3s^2 + 3s + 1) + 4s^2 + 8s + 2 \end{aligned}$$

*Step 2: We solve for  $Y(s)$ :*

$$\begin{aligned} Y(s^3 + 3s^2 + 3s + 1) + 4s^2 + 8s + 2 &= 0 \\ Y(s^3 + 3s^2 + 3s + 1) &= -(4s^2 + 8s + 2) \\ Y &= -\frac{4s^2 + 8s + 2}{s^3 + 3s^2 + 3s + 1} \end{aligned}$$

*Step 3: We factor the denominator:*

$$\begin{aligned} (s^3 + 3s^2 + 3s + 1) &= (s + 1)(s^2 + 2s + 1) \\ &= (s + 1)(s + 1)(s + 1) \\ &= (s + 1)^3 \end{aligned}$$

*Step 4: We break the fraction into partial fractions:*

$$\begin{aligned} -\frac{4s^2 + 8s + 2}{(s + 1)^3} &= \frac{A}{(s + 1)} + \frac{B}{(s + 1)^2} + \frac{C}{(s + 1)^3} \\ &= A(s + 1)^2 + B(s + 1) + C \\ &= A(s^2 + 2s + 1) + Bs + B + C \\ &= As^2 + 2As + A + Bs + B + C \\ &= s^2(A) + s(2A + B) + A + B + C \end{aligned}$$

*Step 5: We solve for the coefficients:*

$$s = -1 :$$

$$C = -(4 - 8 + 2)$$

$$C = 2$$

$$A = -4$$

$$B = -2(-4) - 8$$

$$B = 0$$

Hence,

$$A = -4$$

$$B = 0$$

$$C = 2$$

$$-\frac{4s^2 + 8s + 2}{(s + 1)^3} = -\frac{4}{(s + 1)} + \frac{2}{(s + 1)^3}$$

*Step 6: We take the ILT:*

$$= -4\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^3}\right\}$$

$$= -4e^{-t} + \frac{2}{2}t^2e^{-t}$$

$$= -4e^{-t} + t^2e^{-t}$$

$$= e^{-t}(t^2 - 4)$$