Quiz 2

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January 30, 2022

Q.1 Solve the following differential equation:

$$\left(3e^{3x}\ln\left(1+y^2\right)-\frac{x}{\sqrt[2]{x^2+1}}\right)dx=\left(2\sin\left(y\right)cos(y)-2e^{3x}\frac{y}{1+y^2}\right)dy \tag{1}$$

where $y(0)=0$

Exactness Condition:

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \tag{2}$$

$$M=3e^{3x}\ln\left(1+y^2\right)-\frac{x}{\sqrt[2]{x^2+1}} \tag{3}$$

$$N=-2\sin\left(y\right)cos(y)+2e^{3x}\frac{y}{1+y^2} \tag{4}$$

$$\frac{\partial M}{\partial y}=\frac{6ye^{3x}}{1+y^2} \qquad \frac{\partial N}{\partial x}=\frac{6ye^{3x}}{1+y^2} \tag{5}$$

Therefore, the differential equation is exact.

Solution:

$$\left(3e^{3x}\ln\left(1+y^2\right)-\frac{x}{\sqrt[2]{x^2+1}}\right)dx-\left(2\sin\left(y\right)cos(y)-2e^{3x}\frac{y}{1+y^2}\right)dy=0 \tag{6}$$

$$\left(3e^{3x}\ln\left(1+y^2\right)-\frac{x}{\sqrt[2]{x^2+1}}\right)dx+\left(-2\sin\left(y\right)cos(y)+2e^{3x}\frac{y}{1+y^2}\right)dy=0 \tag{7}$$

$$\frac{\partial F}{\partial x}=M \qquad \frac{\partial F}{\partial y}=N \tag{8}$$

$$\int\frac{\partial F}{\partial x}dx=\int\left(3e^{3x}\ln\left(1+y^2\right)-\frac{x}{\sqrt[2]{x^2+1}}\right)dx \tag{9}$$

$$= \int \left(3e^{3x}\ln(1+y^2)\right)dx - \int \left(\frac{x}{\sqrt[2]{x^2+1}}\right)dx$$
$$= e^{3x}\ln(1+y^2) - \int \left(\frac{x}{\sqrt[2]{x^2+1}}\right)dx$$

$u = x^2 + 1$ so that $du = 2xdx$ and $dx = \frac{du}{2x}$

$$= e^{3x}\ln(1+y^2) - \int \left(\frac{1}{2\sqrt[2]{u}}\right)du$$
$$= e^{3x}\ln(1+y^2) - \sqrt[2]{u} + f'(y)$$
$$= e^{3x}\ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y)$$

Hence, F equals:

$$F(x,y) = e^{3x}\ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y) \tag{10}$$

Now, we solve for $f(y)$:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(e^{3x}\ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y)\right)$$
$$= \frac{2ye^{3x}}{1+y^2} + f'(y)$$

Therefore, $f'(y)$ equals:

$$\frac{2ye^{3x}}{1+y^2} + f'(y) = \left(-2\sin(y)\cos(y) + \frac{2ye^{3x}}{1+y^2}\right)$$
$$f'(y) = -2\sin(y)\cos(y)$$

We integrate $f'(y)$:

$$\int f'(y)dy = \int -2\sin(y)\cos(y)dy$$
$$= -\int \sin(2y)dy$$
$$= \frac{\cos(2y)}{2}$$

Therefore, $f(y)$ equals:

$$f(y) = \frac{\cos(2y)}{2}$$

Finally, $F(x, y)$ equals:

$$F(x, y) = e^{3x} \ln(1 + y^2) - \sqrt[2]{x^2 + 1} + \frac{\cos(2y)}{2} = C_1$$

Q.2 Find the general solution to

$$(y - x)\,dx - (y + x)\,dy = 0 \tag{11}$$

Solution to homogeneous differential equation:

$$\begin{aligned}(y + x)\,dy &= (y - x)\,dx \\ \frac{dy}{dx} &= \frac{y - x}{y + x}\end{aligned}$$

Setting up the equation in the form $u = \frac{y}{x}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y - x}{y + x} * \frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \frac{\frac{y}{x} - 1}{\frac{y}{x} + 1} \\ &= \frac{u - 1}{u + 1}\end{aligned}$$

Therefore, $g(u) = \frac{u-1}{u+1}$ and now we set up $g(u) - u$:

$$\begin{aligned}g(u) - u &= \frac{u - 1}{u + 1} - u \\ &= \frac{u - 1}{u + 1} - u\frac{u - 1}{u + 1} \\ &= \frac{u - 1}{u + 1} - \frac{u^2 - u}{u + 1} \\ &= \frac{u - 1 - u^2 + u}{u + 1} \\ &= \frac{-u^2 - 1}{u + 1}\end{aligned}$$

Now, we set up $g(u) - u = x\frac{du}{dx}$ and integrate $\frac{dx}{x} = \frac{du}{g(u)-u}$:

$$\begin{aligned}\frac{dx}{x} &= \frac{du}{g(u) - u} \\ \int \frac{dx}{x} &= \int \frac{du}{g(u) - u}\end{aligned}$$

$$
\begin{aligned}
\ln(x) &= \int \frac{du}{g(u) - u} \\
\ln(x) &= \int \frac{du}{\frac{-u^2-1}{u+1}} \\
\ln(x) &= \int -\frac{u+1}{u^2+1} du \\
\ln(x) &= -\int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du
\end{aligned}
$$

Now, we solve the integral $\int \frac{u}{u^2+1} du$:

$$
\int \frac{u}{u^2-1} du = \int \frac{u}{u^2+1} du
$$

We substitute $t = u^2 + 1$ and $dt = 2udu$:

$$
\begin{aligned}
&\int \frac{u}{u^2+1} du = \int \frac{u}{t} \frac{dt}{2u} \\
&= \frac{1}{2} \int \frac{dt}{t} \\
&= \frac{1}{2} \ln(t) \\
&= \frac{1}{2} \ln |u^2 + 1|
\end{aligned}
$$

Now, we solve the integral $\int \frac{1}{u^2+1} du$:

$$
\int \frac{1}{u^2+1} du = \arctan(u)
$$

Therefore:

$$
\begin{aligned}
\ln|x| &= -\left(\frac{1}{2} \ln |u^2 + 1| + \arctan(u) \right) \\
\ln|x| &= -\frac{1}{2} \ln |\frac{y^2}{x^2} + 1| - \arctan\left(\frac{y}{x}\right)
\end{aligned}
$$