Quiz 4

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February 28, 2023

1 Problem 1

Find the general solution to the differential equation.

$$y''(x) + y(x) = \sec^3(x)$$

$Charasteristic\ Equation:$

$$\begin{aligned} r^2 + 1 &= 0 \\ r^2 &= -1 \\ r &= \pm i \end{aligned}$$

$Homogenuous\ Solution:$

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

$Variation\ of\ Parameters:$

$$y_p(x) = \nu_1 \cos(x) + \nu_2 \sin(x)$$

$$\begin{cases} -\nu_1' \sin(x) + \nu_2' \cos(x) = \sec^3(x) \\ \nu_1' \cos(x) + \nu_2' \sin(x) = 0 \end{cases}$$

$Solving\ for\ \nu_1:$

$$\begin{aligned} &\nu_1' \sin(x) + \nu_1' \cos(x) = -\sec^3(x)\sin(x) \\ &\nu_1' = -\sec^3(x)\sin(x) = -\frac{\sin(x)}{\cos^3(x)} \\ &\nu_1 = -\int \frac{\sin(x)}{\cos^3(x)} dx, \quad \text{u = cos(x), du = -sin(x)dx} \\ &\nu_1 = -\int \frac{1}{u^3} du = -\frac{1}{2u^2} \\ &\nu_1 = -\frac{1}{2}\sec^2(x) \end{aligned}$$

$Solving\ for\ \nu_2:$

$$\begin{aligned} &\nu_2' \cos^2(x) + \nu_2' \sin^2(x) = \sec^3(x)\cos(x) = \sec^2(x) \\ &\nu_2 = \int \sec^2(x) dx = \tan(x) \end{aligned}$$

Generel Solution :

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2}\sec(x) + \frac{\sin^2(x)}{\cos(x)}$$

2 Problem 2

Solve the initial value problem.

$$x^2y''(x) + 7xy'(x) + 5y(x) = 0$$

Initial Conditions:

$$\begin{aligned} y(1) &= -1 \\ y'(1) &= 13 \end{aligned}$$

$Cauchy - Euler\ Equation:$

$$(ar^2) + (b-a)r + c = 0$$

$$\begin{aligned} r^2 + (7-1)r + 5 &= 0 \\ r^2 + 6r + 5 &= 0 \\ (r+1)(r+5) &= 0 \\ r &= -1, -5 \end{aligned}$$

$Homogenuous\ Solution:$

$$y_h = c_1t^{-1} + c_2t^{-5}$$

Solving for the constants:

$$\begin{aligned} &y(1) = -1 = c_1 + c_2 \\ &y'(1) = 13 = -c_1 - 5c_2 \\ &\begin{cases} c_1 + c_2 = -1 \\ -c_1 - 5c_2 = 13 \end{cases} \\ &\begin{cases} c_1 = 2 \\ c_2 = -3 \end{cases} \end{aligned}$$

Solution:

$$y = 2x^{-1} - 3x^{-5}$$

3 Problem 3

Find the general solution to the differential equation.

$$y^{(6)}(x) - 7y^{(5)}(x) + 48y^{(4)}(x) - 94y'''(x) + 157y''(x) + 777y' - 882y(x) = 0$$

Hints:

$$e^{2x}\cos(\sqrt{17}x) \text{ } and \text{ } xe^{2x}\sin\sqrt{17}x$$

Hence: $2 \pm i\sqrt{17}$ is a root of the characteristic equation.

Then:

$$(r - (2 + i\sqrt{17}))(r - (2 - i\sqrt{17}))$$

is a factor of the characteristic equation.

Solving for the other roots:

$$\begin{aligned}
&(r - (2 + i\sqrt{17}))(r - (2 - i\sqrt{17})) \\
&((r-2) - i\sqrt{17})((r-2) + i\sqrt{17}) \\
&(r-2)^2 - i^2 17 \\
&r^2 - 4r + 4 + 17 \\
&r^2 - 4r + 21
\end{aligned}$$

Long division:

$$\begin{array}{r}
r^4 - 3r^3 + 15r^2 + 29r - 42 \\
r^2 - 4r + 21 \overline{\big) r^6 - 7r^5 + 48r^4 - 94r^3 + 157r^2 + 777r - 882} \\
-\ (r^6 - 4r^5 + 21r^4) \\
\hline
-3r^5 + 27r^4 - 94r^3 + 157r^2 + 777r - 882 \\
-\ (-3r^5 + 12r^4 - 63r^3) \\
\hline
15r^4 - 31r^3 + 157r^2 + 777r - 882 \\
-\ (15r^4 - 60r^3 + 315r^2) \\
\hline
29r^3 - 158r^2 + 777r - 882 \\
-\ (29r^3 - 116r^2 + 609r) \\
\hline
-42r^2 + 168r - 882 \\
-\ (-42r^2 + 168r - 882) \\
\hline
0
\end{array}$$

Hence:

$$(r^2 - 4 + 21)(r^4 - 3r^3 + 15r^2 + 29r - 42)$$

are the factors of the differential equation.

According to the rational root theorem, the possible roots are:

$$\pm 1, \pm 2, \pm 3, ...$$

1 is root thus: $(r-1)$ is a factor of the differential equation.

Hence, using the synthetic division:

1	1	-3	15	29	-42
	1	1	-2	14	42
	1	-2	13	42	0

Hence, $r^3 - 2r^2 + 13r + 42$ is a factor of the differential equation.

Another root is -2, thus:

-2	1	-2	13	42
	1	-2	8	-42
	1	-4	21	0

Hence, $r^2 - 4r + 21$ is a factor of the differential equation.

The factorized form of the differential equation is:

$$(r+2)(r-1)(r^2 - 4r + 21)^2$$

Hence, the roots are:

$$r = -2$$
$$r = 1$$
$$r_{12} = 2 \pm i\sqrt{17}$$

The general solution is:

$$y = C_1 e^{-2x} + C_2 e^{x} + e^{2x}\left((C_3 x + C_4)\cos(\sqrt{17}x) + (C_5 x + C_4)\sin(\sqrt{17}x)\right)$$