Quiz 1

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Differential
Equations

Q1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y(y^2-4)}{2}$$

where $y(0) = 3$

Solution:

$$\frac{dy}{y(y^2-4)} = \frac{dx}{2}$$

$$\int \frac{1}{y(y^2-4)} dy = \frac{1}{2}\int dx$$

Solving for the left side:

$$\int \frac{1}{y(y^2-4)} dy = \int \left(\frac{1}{8}\frac{1}{(y\ +2)} + \frac{1}{8}\frac{1}{(y-2)} + \left(-\frac{1}{4}\right)\frac{1}{y}\right) dy$$

Solving for the partial fractions:

$$\frac{1}{y(y^2-4)} = \frac{A}{y+2} + \frac{B}{y-2} + \frac{C}{y}$$

$$1 = A\big(y(y-2)\big) + B\big(y(y+2)\big) + C\big((y-2)(y+2)\big)$$

$$1 = A(y^2-2y) + B(y^2+2y) + C(y^2-4)$$

$$1 = Ay^2 - 2Ay + By^2 + 2By + Cy^2 - 4C$$

$$1 = y^2(A+B+C) + y(-2A+2B) + (-4C)$$

Solving for A, B and C:

$$A + B + C = 0$$

$$-2A + 2B = 0$$

$$-4C = 1$$

$$C = -\frac{1}{4}$$

$$A + B = \frac{1}{4}$$

$$-2\left(\frac{1}{4} - B\right) + 2B = 0$$

$$-\frac{1}{2} + 4B = 0$$

$$B = \frac{1}{8}$$

$$A = \frac{1}{8}$$

Simplifying the left side:

$$= \frac{1}{8} ln|y+2| + \frac{1}{8} ln|y-2| - \frac{1}{4} ln|y|$$

$$= \frac{1}{8}\left(ln(|y^2-4|)\right) - \frac{1}{4} ln|y|$$

(Simplified the equation log(a) + log(b) = log(a*b))

Solving for the right side:

$$\frac{1}{2}\int dx = \frac{1}{2}x$$

Solving for y:

$$\frac{1}{8}\left(ln(|y^2-4|)\right) - \frac{1}{4} ln|y| = \frac{1}{2}x + c_1$$

(In a neighborhood around y, y is positive)
(Hence, the absolute value of (y^2 -4) and y are not needed)

$ln(y^2-4) - 2ln(y) = 4x + c_1$ (Multiplied both sides by 8)

$ln\left(\frac{(y^2-4)}{y^2}\right) = 4x + c_1$ (log(a) - log(b)) = log(a/b))

$$ln\left(1 - \frac{4}{y^2}\right) = 4x + c_1$$

$$1 - \frac{4}{y^2} = e^{4x+c_1}$$

$$\frac{4}{y^2} = 1 - e^{4x+c_1}$$

$$y^2 = \frac{4}{1 - e^{4x+c_1}}$$

$$y = \pm 2\sqrt{\frac{1}{1 - e^{4x+c_1}}}$$

Solving for c:

$$3 = 2\sqrt{\frac{1}{1 - e^{c_1}}}$$

(Substituted y for 3 and x for 0)

$$\left(\frac{3}{2}\right)^2 = \frac{1}{1 - e^{c_1}}$$

(Took the reciprocal of both sides)

$$\frac{4}{9} = 1 - e^{c_1}$$

$$e^{c_1} = \frac{5}{9}$$

$$c_1 = ln\left(\frac{5}{9}\right)$$

Answer:

$$y = \pm 2\sqrt{\frac{1}{1 - \frac{5}{9}e^{4x}}}$$

Q2. Find the explicit solution to the initial value problem

$$\frac{e^{y^2}}{2y} + \frac{e^x}{x+1}\frac{dy}{dx} = 0$$

where $y(0) = \sqrt{\ln(2)}$

Solution:

$$\frac{dy}{dx} = \left(\frac{x+1}{e^x}\right)\left(-\frac{e^{y^2}}{2y}\right)$$

$$\left(-\frac{2y}{e^{y^2}}\right)dy = \left(\frac{x+1}{e^x}\right)dx$$

(Separated the functions)

$$\int\left(-\frac{2y}{e^{y^2}}\right)dy = \int\left(\frac{x+1}{e^x}\right)dx$$

Solving for the left side:

$$\int\left(-\frac{2y}{e^{y^2}}\right)\mathrm{dy} = -\int\frac{\mathrm{du}}{\mathrm{e^u}}$$

$(u = y^2 \ and \ du = 2ydy)$

$$= \frac{1}{e^u} = \frac{1}{e^{y^2}}$$

Solving for the right side:

$$\int\left(\frac{x+1}{e^x}\right)dx = -\left(\frac{x+1}{e^x}\right) + \int\frac{dx}{e^x}$$

$\left(Integration\ by\ parts;\ u\ =\ x+1, dv = \frac{dx}{e^x}\right)$

$(du = dx,\ and\ v = -\frac{1}{e^x})$

$$\int\frac{dx}{e^x} = -\frac{1}{e^x} + c_1$$

(Adding the other part: $-\left(\frac{x+1}{e^x}\right)$)

$$\frac{-x-1}{e^x} - \frac{1}{e^x} + c_1 = \frac{-x-2}{e^x} + c_1$$

Solving for y:

$$\frac{1}{e^{y^2}} = \frac{-(x+2)}{e^x} + c_1$$

$$e^{y^2} = \left(\frac{-(x+2)}{e^x} + c_1\right)^{-1}$$

$$y^2 = ln\left(\left(\frac{-(x+2)}{e^x} + c_1\right)^{-1}\right)$$

$$y = \pm\sqrt{ln\left(\left(\frac{-(x+2)}{e^x} + c_1\right)^{-1}\right)}$$

Solving for c:

$$\sqrt{ln(2)} = \sqrt{ln((c_1 - 2)^{-1})}$$

(Substituted y for $\sqrt{ln(2)}$ and x for 0)

$$2 = \frac{1}{c_1 - 2}$$

(Squared both sides and took both sides to power of e)

$$c_1 = \frac{5}{2}$$

Answer:

$$y = \pm\sqrt{ln\left(\left(\frac{-(x+2)}{e^x} + \frac{5}{2}\right)^{-1}\right)}$$

Q3. Find the general explicit solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2}{x^2}y = x^3 \sin x$$

Solution:

$$\frac{dy}{dx} - \frac{2}{x}y = x^4 \sin x$$

(Converting to standard form: $y' + p(x)y = q(x)$)

Solving $\mu(x)$:

$$\mu(x) = e^{\int -\frac{2}{x}dx}$$

$$\left(\mu(x) = e^{\int p(x)dx}\right)$$

$$= e^{-2ln(x)} = e^{ln\left(\frac{1}{x^2}\right)}$$

$$= \frac{1}{x^2}$$

Setting up the problem:

$$\left(\frac{1}{x^2}\right)y = \int \frac{1}{x^2}x^4 \sin x \, dx$$

$$\left(\mu(x)y = \int \mu(x)q(x)dx\right)$$

$$\frac{1}{x^2}y = \int x^2 \sin x \, dx$$

Solving for the right side:

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx$$

$$(Integration\ by\ parts{:}\ u = x^2, dv = \sin x \, dx)$$
$$(du = 2x, v = -\cos x)$$

$$= -x^2 \cos x + 2x \sin x - \int 2\sin x \, dx$$

$$(Integration\ by\ parts{:}\ t = 2x, dk = \cos x \, dx)$$
$$(dt = 2dx, k = \sin x)$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c_1$$

Solving for y:

$$\frac{1}{x^2}y = -x^2 \cos x + 2x \sin x + 2 \cos x + c_1$$

$$y = -x^4 \cos x + 2x^3 \sin x + 2x^2 \cos x + c_1 x^2$$

Q4. Find the general explicit solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2}{x^2}y = x^2e^{-x}$$

Solution:

$$\frac{dy}{dx} + \left(-\frac{2}{x}y\right) = x^3e^{-x}$$

(Converting to standard form: $y' + p(x)y = q(x)$)

Solving for $\mu(x)$:

$$\mu(x) = e^{\int\left(-\frac{2}{x}\right)dx}$$

$$\left(\mu(x) = e^{\int p(x)dx}\right)$$

$$= e^{-2ln(x)} = e^{ln\left(\frac{1}{x^2}\right)}$$

$$= \frac{1}{x^2}$$

Setting up the problem:

$$\frac{1}{x^2}y = \int\left(\frac{1}{x^2}x^3e^{-x}\right)dx$$

$$\left(\mu(x)y = \int \mu(x)q(x)dx\right)$$

$$\frac{1}{x^2}y = \int \frac{x}{e^x}dx$$

Solving for the right side:

$$\int \frac{x}{e^x}dx = -\frac{x}{e^x} + \int \frac{dx}{e^x}$$

$$\left(Integartion\ by\ parts: u = x, dv = \frac{1}{e^x}\right)$$

$$\left(du = dx, v = -\frac{1}{e^x}\right)$$

$$= -\frac{x}{e^x} + \int \frac{dx}{e^x} = -\frac{x}{e^x} - \frac{1}{e^x} + c_1$$

$$= \frac{-(x+1)}{e^x} + c_1$$

Solving for y:

$$\frac{1}{x^2}y = \frac{-(x+1)}{e^x} + c_1$$

$$y = \frac{-x^3 - x^2}{e^x} + c_1x^2$$