

Quiz 4

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1 Problem 1

Find the general solution to the differential equation.

$$y''(x) + y(x) = \sec^3(x)$$

Characteristic Equation :

$$\begin{aligned}r^2 + 1 &= 0 \\r^2 &= -1 \\r &= \pm i\end{aligned}$$

Homogenous Solution :

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

Variation of Parameters :

$$\begin{aligned}y_p(x) &= \nu_1 \cos(x) + \nu_2 \sin(x) \\ \begin{cases} -\nu_1' \sin(x) + \nu_2' \cos(x) = \sec^3(x) \\ \nu_1' \cos(x) + \nu_2' \sin(x) = 0 \end{cases}\end{aligned}$$

Solving for ν_1 :

$$\begin{aligned}\nu_1' \sin(x) + \nu_1' \cos(x) &= -\sec^3(x) \sin(x) \\\nu_1' &= -\sec^3(x) \sin(x) = -\frac{\sin(x)}{\cos^3(x)} \\\nu_1 &= -\int \frac{\sin(x)}{\cos^3(x)} dx, \quad u = \cos(x), \quad du = -\sin(x) dx \\\nu_1 &= -\int \frac{1}{u^3} du = -\frac{1}{2u^2} \\\nu_1 &= -\frac{1}{2} \sec^2(x)\end{aligned}$$

Solving for ν_2 :

$$\begin{aligned}\nu_2' \cos^2(x) + \nu_2' \sin^2(x) &= \sec^3(x) \cos(x) = \sec^2(x) \\\nu_2 &= \int \sec^2(x) dx = \tan(x)\end{aligned}$$

General Solution :

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{1}{2} \sec(x) + \frac{\sin^2(x)}{\cos(x)}$$

2 Problem 2

Solve the initial value problem.

$$x^2 y''(x) + 7xy'(x) + 5y(x) = 0$$

Initial Conditions:

$$\begin{aligned}y(1) &= -1 \\ y'(1) &= 13\end{aligned}$$

Cauchy – Euler Equation :

$$(ar^2) + (b - a)r + c = 0$$

$$r^2 + (7 - 1)r + 5 = 0$$

$$r^2 + 6r + 5 = 0$$

$$(r + 1)(r + 5) = 0$$

$$r = -1, -5$$

Homogeneous Solution :

$$y_h = c_1 t^{-1} + c_2 t^{-5}$$

Solving for the constants:

$$y(1) = -1 = c_1 + c_2$$

$$y'(1) = 13 = -c_1 - 5c_2$$

$$\begin{cases} c_1 + c_2 = -1 \\ -c_1 - 5c_2 = 13 \end{cases}$$

$$\begin{cases} c_1 = 2 \\ c_2 = -3 \end{cases}$$

Solution:

$$y = 2x^{-1} - 3x^{-5}$$

3 Problem 3

Find the general solution to the differential equation.

$$y^{(6)}(x) - 7y^{(5)}(x) + 48y^{(4)}(x) - 94y'''(x) + 157y''(x) + 777y' - 882y(x) = 0$$

Hints:

$$e^{2x} \cos(\sqrt{17}x) \text{ and } xe^{2x} \sin(\sqrt{17}x)$$

Hence: $2 \pm i\sqrt{17}$ is a root of the characteristic equation.

Then:

$$(r - (2 + i\sqrt{17}))(r - (2 - i\sqrt{17}))$$

is a factor of the characteristic equation.

Solving for the other roots:

$$\begin{aligned} &(r - (2 + i\sqrt{17}))(r - (2 - i\sqrt{17})) \\ &((r - 2) - i\sqrt{17})((r - 2) + i\sqrt{17}) \\ &(r - 2)^2 - i^2 17 \\ &r^2 - 4r + 4 + 17 \\ &r^2 - 4r + 21 \end{aligned}$$

Long division:

$$\begin{array}{r} r^4 - 3r^3 + 15r^2 + 29r - 42 \\ r^2 - 4r + 21 \overline{) r^6 - 7r^5 + 48r^4 - 94r^3 + 157r^2 + 777r - 882} \\ \underline{-(r^6 - 4r^5 + 21r^4)} \\ -3r^5 + 27r^4 - 94r^3 + 157r^2 + 777r - 882 \\ \underline{-(-3r^5 + 12r^4 - 63r^3)} \\ 15r^4 - 31r^3 + 157r^2 + 777r - 882 \\ \underline{-(15r^4 - 60r^3 + 315r^2)} \\ 29r^3 - 158r^2 + 777r - 882 \\ \underline{-(29r^3 - 116r^2 + 609r)} \\ -42r^2 + 168r - 882 \\ \underline{-(-42r^2 + 168r - 882)} \\ 0 \end{array}$$

Hence:

$$(r^2 - 4 + 21)(r^4 - 3r^3 + 15r^2 + 29r - 42)$$

are the factors of the differential equation.

According to the rational root theorem, the possible roots are:

$$\pm 1, \pm 2, \pm 3, \dots$$

1 is root thus: $(r - 1)$ is a factor of the differential equation.

Hence, using the synthetic division:

1	1	-3	15	29	-42
	1	1	-2	14	42
	1	-2	13	42	0

Hence, $r^3 - 2r^2 + 13r + 42$ is a factor of the differential equation.

Another root is -2, thus:

-2	1	-2	13	42
	1	-2	8	-42
	1	-4	21	0

Hence, $r^2 - 4r + 21$ is a factor of the differential equation.

The factorized form of the differential equation is:

$$(r + 2)(r - 1)(r^2 - 4r + 21)^2$$

Hence, the roots are:

$$r = -2$$

$$r = 1$$

$$r_{12} = 2 \pm i\sqrt{17}, \quad \text{Double root}$$

The general solution is:

$$y = C_1 e^{-2x} + C_2 e^x + e^{2x} \left((C_3 x + C_4) \cos(\sqrt{17}x) + (C_5 x + C_6) \sin(\sqrt{17}x) \right)$$