

Quiz 6

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Problem 1 Find the inverse Laplace transform of the given function.

$$\frac{2s + 16}{s^2 + 4s + 13}$$

Solution:

Step 1: Simplify the given rational function.

$$\begin{aligned}\frac{2s + 16}{s^2 + 4s + 13} &= \frac{2(s + 8)}{(s + 2)^2 + 9} \\ &= \frac{2(s + 2 + 6)}{(s + 2)^2 + (3)^2} = \frac{2(s + 2)}{(s + 2)^2 + (3)^2} + \frac{12}{(s + 2)^2 + (3)^2} \\ &= 2 \left(\frac{s + 2}{(s + 2)^2 + (3)^2} \right) + 4 \left(\frac{3}{(s + 2)^2 + (3)^2} \right)\end{aligned}$$

Step 2: Find the inverse Laplace transform.

$$\begin{aligned}&= 2\mathcal{L}^{-1} \left\{ \frac{s + 2}{(s + 2)^2 + (3)^2} \right\} + 4\mathcal{L}^{-1} \left\{ \frac{3}{(s + 2)^2 + (3)^2} \right\} \\ &= 2(e^{-2t} \cos 3t) + 4(e^{-2t} \sin 3t) \\ &= 2e^{-2t} \cos 3t + 4e^{-2t} \sin 3t\end{aligned}$$

Problem 2 Determine the partial fraction expansion for the given rational function.

$$\frac{4s^2 - 21s + 16}{s(s-2)^2}$$

Solution:

We first simplify the given rational function.

$$\begin{aligned} &= \frac{4s^2 - 21s + 16}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2} \\ &= 4s^2 - 21s + 16 = A(s-2)^2 + Bs(s-2) + Cs \end{aligned}$$

Now we find the coefficients.

$$\begin{aligned} s &= 2 : \\ 4(2)^2 - 21(2) + 16 &= A(0)^2 + B(0) + 2C \\ 16 - 42 + 16 &= 2C \\ 2C &= -10 \\ C &= -5 \end{aligned}$$

$$\begin{aligned} s &= 0 : \\ 4(0)^2 - 21(0) + 16 &= A(2)^2 \\ 16 &= A(4) \\ A &= 4 \end{aligned}$$

Solving by equating coefficients.

$$\begin{aligned} &= A(s^2 - 4s + 4) + B(s^2 - 2s) + Cs \\ &= As^2 - 4As + 4A + Bs^2 - 2Bs + Cs \\ &= s^2(A + B) + s(-4A - 2B + C) + 4A \end{aligned}$$

Hence:

$$\begin{aligned}A + B &= 4 \\ -4A - 2B + C &= -21\end{aligned}$$

$$4A = 16$$

Substitute $A = 4$

$$4 + B = 4$$

$$B = 0$$

Therefore,

$$\begin{aligned}\frac{4s^2 - 21s + 16}{s(s-2)^2} &= \frac{4}{s} + \frac{0}{s-2} + \frac{-5}{(s-2)^2} \\ &= \frac{4}{s} - \frac{5}{(s-2)^2}\end{aligned}$$

Problem 3 Determine the partial fraction expansion for the given rational function.

$$\frac{3s + 5}{s(s^2 + s - 6)}$$

Solution:

We first simplify the given rational function.

$$\frac{3s + 5}{s(s^2 + s - 6)} = \frac{3s + 5}{s(s + 3)(s - 2)}$$

Now we find the partial fraction expansion.

$$\begin{aligned}\frac{3s + 5}{s(s + 3)(s - 2)} &= \frac{A}{s} + \frac{B}{s + 3} + \frac{C}{s - 2} \\ &= A(s + 3)(s - 2) + Bs(s - 2) + Cs(s + 3)\end{aligned}$$

Solving by equating coefficients.

$$\begin{aligned}s = 0 : \\ 3(0) + 5 &= A(3)(-2) + B(0)(0) + C(0)(0) \\ 5 &= -6A \\ A &= -\frac{5}{6}\end{aligned}$$

$$\begin{aligned}s = -3 : \\ 3(-3) + 5 &= A(0) + B(-3)(-3 - 2) + C(0) \\ -4 &= B(15) \\ B &= -\frac{4}{15}\end{aligned}$$

$$\begin{aligned}s = 2 : \\ 3(2) + 5 &= A(0) + B(0) + C(2)(2 + 3) \\ 11 &= C(10) \\ C &= \frac{11}{10}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{3s+5}{s(s^2+s-6)} &= \frac{-\frac{5}{6}}{s} + \frac{-\frac{4}{15}}{s+3} + \frac{\frac{11}{10}}{s-2} \\ &= -\frac{5}{6s} - \frac{4}{15(s+3)} + \frac{11}{10(s-2)}\end{aligned}$$

Problem 4 Determine $\mathcal{L}^{-1}\{F\}$

$$F(s) = \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)}$$

Solution:

We first simplify the given rational function.

$$\begin{aligned} \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} &= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1} \\ &= 5s^2 + 34s + 53 = A(s+3)(s+1) + B(s+1) + C(s+3)^2 \end{aligned}$$

Now we find the coefficients.

$$\begin{aligned} s &= -3 : \\ 5(-3)^2 + 34(-3) + 53 &= -2B \\ -4 &= -2B \\ B &= 2 \end{aligned}$$

$$\begin{aligned} s &= -1 : \\ 5(-1)^2 + 34(-1) + 53 &= C(2)^2 \\ 24 &= 4C \\ C &= 6 \end{aligned}$$

$$s = 0 :$$

Solving by equating coefficients.

$$\begin{aligned}
&= A(s+3)(s+1) + B(s+1) + C(s+3)^2 \\
&= A(s^2 + 4s + 3) + B(s+1) + C(s^2 + 6s + 9) \\
&= s^2(A+C) + s(4A+6C+B) + 3A+B+9C
\end{aligned}$$

$$s^2 :$$

$$A+C=5$$

$$A=5-C$$

$$A=5-6=-1$$

$$A=-1$$

Therefore,

$$\frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} = -\frac{1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{(s+1)}$$

Now we find the inverse Laplace transform.

$$\begin{aligned}
\mathcal{L}^{-1}\{F\} &= -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} \\
&= -e^{-3t} + 2te^{-3t} + 6e^{-t}
\end{aligned}$$