

Quiz 2

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Solve the following differential equation:

$$\left(3e^{3x} \ln(1+y^2) - \frac{x}{\sqrt[2]{x^2+1}}\right) dx = \left(2 \sin(y) \cos(y) - 2e^{3x} \frac{y}{1+y^2}\right) dy \quad (1)$$

where $y(0) = 0$

Exactness Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2)$$

$$M = 3e^{3x} \ln(1+y^2) - \frac{x}{\sqrt[2]{x^2+1}} \quad (3)$$

$$N = -2 \sin(y) \cos(y) + 2e^{3x} \frac{y}{1+y^2} \quad (4)$$

$$\frac{\partial M}{\partial y} = \frac{6ye^{3x}}{1+y^2} \quad \frac{\partial N}{\partial x} = \frac{6ye^{3x}}{1+y^2} \quad (5)$$

Therefore, the differential equation is exact.

Solution:

$$\left(3e^{3x} \ln(1+y^2) - \frac{x}{\sqrt[2]{x^2+1}}\right) dx - \left(2 \sin(y) \cos(y) - 2e^{3x} \frac{y}{1+y^2}\right) dy = 0 \quad (6)$$

$$\left(3e^{3x} \ln(1+y^2) - \frac{x}{\sqrt[2]{x^2+1}}\right) dx + \left(-2 \sin(y) \cos(y) + 2e^{3x} \frac{y}{1+y^2}\right) dy = 0 \quad (7)$$

$$\frac{\partial F}{\partial x} = M \quad \frac{\partial F}{\partial y} = N \quad (8)$$

$$\int \frac{\partial F}{\partial x} dx = \int \left(3e^{3x} \ln(1+y^2) - \frac{x}{\sqrt[2]{x^2+1}}\right) dx \quad (9)$$

$$\begin{aligned}
&= \int (3e^{3x} \ln(1+y^2) dx - \int \left(\frac{x}{\sqrt[2]{x^2+1}} \right) dx \\
&= e^{3x} \ln(1+y^2) - \int \left(\frac{x}{\sqrt[2]{x^2+1}} \right) dx
\end{aligned}$$

$u = x^2 + 1$ so that $du = 2x dx$ and $dx = \frac{du}{2x}$

$$\begin{aligned}
&= e^{3x} \ln(1+y^2) - \int \left(\frac{1}{2\sqrt[2]{u}} \right) du \\
&= e^{3x} \ln(1+y^2) - \sqrt[2]{u} + f'(y) \\
&= e^{3x} \ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y)
\end{aligned}$$

Hence, F equals:

$$F(x, y) = e^{3x} \ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y) \quad (10)$$

Now, we solve for $f(y)$:

$$\begin{aligned}
\frac{\partial F}{\partial y} &= \frac{\partial}{\partial y} \left(e^{3x} \ln(1+y^2) - \sqrt[2]{x^2+1} + f'(y) \right) \\
&= \frac{2ye^{3x}}{1+y^2} + f'(y)
\end{aligned}$$

Therefore, $f'(y)$ equals:

$$\begin{aligned}
\frac{2ye^{3x}}{1+y^2} + f'(y) &= \left(-2 \sin(y) \cos(y) + \frac{2ye^{3x}}{1+y^2} \right) \\
f'(y) &= -2 \sin(y) \cos(y)
\end{aligned}$$

We integrate $f'(y)$:

$$\begin{aligned}
\int f'(y) dy &= \int -2 \sin(y) \cos(y) dy \\
&= - \int \sin(2y) dy \\
&= \frac{\cos(2y)}{2}
\end{aligned}$$

Therefore, $f(y)$ equals:

$$f(y) = \frac{\cos(2y)}{2}$$

Finally, $F(x, y)$ equals:

$$F(x, y) = e^{3x} \ln(1 + y^2) - \sqrt[2]{x^2 + 1} + \frac{\cos(2y)}{2} = C_1$$

Find the general solution to

$$(y - x) dx - (y + x) dy = 0 \quad (11)$$

Solution to homogeneous equation:

$$\begin{aligned} (y + x) dy &= (y - x) dx \\ \frac{dy}{dx} &= \frac{y - x}{y + x} \end{aligned}$$

Setting up the equation in the form $u = \frac{y}{x}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y - x}{y + x} * \frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \frac{\frac{y}{x} - 1}{\frac{y}{x} + 1} \\ &= \frac{u - 1}{u + 1} \end{aligned}$$

Therefore, $g(u) = \frac{u-1}{u+1}$ and now we set up $g(u) - u$:

$$\begin{aligned} g(u) - u &= \frac{u - 1}{u + 1} - u \\ &= \frac{u - 1}{u + 1} - u \frac{u + 1}{u + 1} \\ &= \frac{u - 1}{u + 1} - \frac{u^2 + u}{u + 1} \\ &= \frac{u - 1 - u^2 + u}{u + 1} \\ &= \frac{-u^2 - 1}{u + 1} \end{aligned}$$

Now, we set up $g(u) - u = x \frac{du}{dx}$ and integrate $\frac{dx}{x} = \frac{du}{g(u)-u}$:

$$\begin{aligned} \frac{dx}{x} &= \frac{du}{g(u) - u} \\ \int \frac{dx}{x} &= \int \frac{du}{g(u) - u} \end{aligned}$$

$$\begin{aligned}
\ln(x) &= \int \frac{du}{g(u) - u} \\
\ln(x) &= \int \frac{du}{\frac{-u^2-1}{u+1}} \\
\ln(x) &= \int -\frac{u+1}{u^2+1} du \\
\ln(x) &= - \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du
\end{aligned}$$

Now, we solve the integral $\int \frac{u}{u^2+1} du$:

$$\int \frac{u}{u^2-1} du = \int \frac{u}{u^2+1} du$$

We substitute $t = u^2 + 1$ and $dt = 2u du$:

$$\begin{aligned}
\int \frac{u}{u^2+1} du &= \int \frac{u}{t} \frac{dt}{2u} \\
&= \frac{1}{2} \int \frac{dt}{t} \\
&= \frac{1}{2} \ln(t) \\
&= \frac{1}{2} \ln(u^2 + 1)
\end{aligned}$$

Now, we solve the integral $\int \frac{1}{u^2+1} du$:

$$\int \frac{1}{u^2+1} du = \arctan(u)$$

Therefore:

$$\begin{aligned}
\ln|x| &= - \left(\frac{1}{2} \ln|u^2 + 1| + \arctan(u) \right) \\
\ln|x| &= -\frac{1}{2} \ln\left|\frac{y^2}{x^2} + 1\right| - \arctan\left(\frac{y}{x}\right)
\end{aligned}$$