Quiz 3

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1 Problem 1

1. Find the general solution to the differential equation

$$y'' + 6y' + 9y = x^2 + \cos(x) - xe^{-3x} \tag{1}$$

Homogeneous Equation :

$$y'' + 6y' + 9y = 0$$

Characteristic Equation :

$$\begin{aligned} r^2 + 6r + 9 &= 0 \\ (r+3)(r+3) &= 0 \\ r &= -3 \end{aligned}$$

Homogeneous Solution :

$$y_h = \frac{(A + Bx)}{e^{3x}}$$

Particular Solution 1 :

$$y'' + 6y' + 9y = \cos x$$

$$m = 0, \alpha = 0, k = 0, s = 0, \beta = 1, C = 1$$

$$y_{p1} = A\cos x + B\sin x$$

Solving for the constants:

$$\begin{cases} [9] & y_{p1} = A\cos x + B\sin x \\ [6] & y'_{p1} = -A\sin x + B\cos x \\ [1] & y''_{p1} = -A\cos x - B\sin x \end{cases}$$

$$
\begin{aligned}
\cos x: \quad & 9A + 6B - A = 1 \\
& 8A + 6B = 1 \\
& A = \frac{1-6B}{8} \\
\sin x: \quad & 9B - A6 - B = 0 \\
& 8B - 6A = 0 \\
& 8B - 6\left(\frac{1-6B}{8}\right) = 0 \\
& 100B = 6 \\
& B = \frac{6}{100} = \frac{3}{50} \\
& A = \frac{2}{25}
\end{aligned}
$$

$$y_{p1} = \frac{2}{25}\cos x + \frac{3}{50}\sin x$$

$Particular\ Solution\ 2:$

$$y'' + 6y' + 9y = x^2$$

$$m = 2, \alpha = 0, s = 0, C = 1$$

$$y_{p2} = Ax^2 + Bx + C$$

Solving for the constants:

$$
\begin{cases}
[9] \quad y_{p2} = Ax^2 + Bx + C \\
[6] \quad y'_{p2} = 2Ax + B \\
[1] \quad y''_{p2} = 2A
\end{cases}
$$

$$
\begin{aligned}
x^2: \quad & 9A = 1 \\
& A = \frac{1}{9} \\
x: \quad & 9B + 12A = 0 \\
& 9B = -12A \\
& 9B = -\frac{12}{9} \\
& 9B = -\frac{4}{3} \\
& B = -\frac{4}{27} \\
1: \quad & 9C + 6B + 2A = 0 \\
& 9C + 6\left(-\frac{4}{27}\right) + 9\left(\frac{1}{9}\right) = 0 \\
& 9C - \frac{8}{9} + \frac{2}{9} = 0 \\
& 9C - \frac{8}{9} = -\frac{2}{9} \\
& 9C = \frac{6}{9} \\
& C = \frac{2}{27}
\end{aligned}
$$

$$y_{p2} = \frac{1}{9}x^2 - \frac{4}{27}x + \frac{2}{27}$$

$Particular\ Solution\ 3:$

$$y'' + 6y' + 9y = xe^{-3x}$$

$$m = 0, \alpha = 0, k = 0, s = 0, \beta = 1, C = 1$$

$$y_{p3} = \frac{(Ax^3 + Bx^2)}{e^{3x}}$$

Solving for the constants:

$$
\begin{cases}
[9] \quad y_{p3} = \frac{(Ax^3+Bx^2)}{e^{3x}} \\
[6] \quad y'_{p3} = \frac{(3Ax^2+2Bx-3Ax^3-3Bx^2)}{e^{3x}} \\
[1] \quad y''_{p3} = \frac{(6Ax+2B-9Ax^2-6Bx-9Ax^2-6Bx+9Ax^3+9Bx^2)}{e^{3x}}
\end{cases}
$$

$$x^3 : 9A - 18A + 9A = 0$$
$$x^2 : 9B - 18B + 9B - 9A + 18A - 9A = 0$$
$$x : 12B + 6A - 12B = -1$$
$$6A = -1$$
$$A = -\frac{1}{6}$$
$$B = 0$$

$$y_{p3} = \frac{(-\frac{1}{6})x^3}{e^{3x}}$$

$$y = y_h + y_{p1} + y_{p2} + y_{p3}$$

2 Problem 2

2. Find the general solution to the differential equation

$$y'' - 2y' + 5y = xe^x + xe^x \sin(2x) \tag{2}$$

Homogeneous Equation :

$$y'' - 2y' + 5y = 0$$

$$\begin{aligned} r &= x^2 - 2x + 5 \\ r &= \frac{2 \pm i4}{2} \\ \alpha &= 1, \beta = 2 \end{aligned}$$

Homogeneous Solution :

$$y_h = e^x (A\cos(2x) + B\sin(2x))$$

Particular Solution 1 :

$$y'' - 2y' + 5y = xe^x$$

$$m = 1, \alpha = 1, s = 0, C = 1$$

$$y_{p1} = (Ax + B)e^x$$

Solving for the constants:

$$\begin{cases} [5] \quad y_{p1} = (Ax + B)e^x \\ [-2] \quad y'_{p1} = (Ax + B + A)e^x \\ [1] \quad y''_{p1} = (Ax + B + 2A)e^x \end{cases}$$

$$\begin{aligned} x: \quad & 5A - 2A + A = 1 \\ & A = \frac{1}{4} \\ & B = 0 \end{aligned}$$

$$y_{p1} = \frac{1}{4} xe^x$$

$Particular\ Solution\ 2:$

$$y'' - 2y' + 5y = xe^x \sin 2x$$

$$m = 1, \alpha = 1, s = 0, C = 1, \beta = 2$$

$$y_{p1} = xe^x((Ax + B)\cos 2x + (Cx + D)\sin(2x))$$

Solving for the constants:

$$\begin{cases} [5] \quad y_{p2} = xe^x((Ax + B)\cos 2x + (Cx + D)\sin(2x)) \\ [-2] \quad y'_{p2} = e^x((C - 2A)x^2 + (D + 2C - 3B)x + D)\sin 2x + e^x((2C + A)x^2 + (2D + B + 2A)x + B)\cos 2x \\ [1] \quad y''_{p2} = ((-3C - 4A)x^2 + (-3D + 4C - 4B - 8A)x + 2D + 2C - 4B)e^x \sin 2x + \\ ((4C - 3A)x^2 + (4D + 8C - 3B + 4A)x + 4D + 2B + 2A)e^x \cos 2x \end{cases}$$

$$x^2 : 5A + 5C - 2C + 4A - 4A - 2C + -2A - 3C - 4A + 4C - 3A = 0$$
$$x : 5B + 5D + -2D - 4C + 6B - 4D - 2B - 4A - 3D + 4C - 4B - 8A + 4D + 8C - 3B + 4A = 1$$
$$A = -\frac{1}{8}$$
$$B = 0$$
$$C = 0$$
$$D = \frac{1}{16}$$

$$y_{p2} = xe^x\left(\frac{\sin(2x)}{16} - \frac{x\cos(2x)}{8}\right)$$

$$y = y_h + y_{p1} + y_{p2}$$