Quiz 7

Adrian Lozada, U71130053

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Problem 1 Solve the given initial value problem using the method of Laplace transforms.

$$y''(t) - y(t) = t - 2$$

where $y(2) = 3$ and $y'(2) = 0$.

Solution:

Step 1: We first move the initial condition to t = 0.

$$y''(t+2) - y(t+2) = (t+2) - 2$$
$$y''(t+2) - y(t+2) = t$$
$$w(t) := y(t+2)$$

Hence,

$$y'(t+2) = w'(t)$$
$$y''(t+2) = w''(t)$$

Step 2: We replace y(t + 2) by w(t).

$$w''(t) - w(t) = (t+2) - 2$$
$$w''(t) - w(t) = t$$

Step 3: We apply the Laplace transform.

To the RHD,

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad \text{for} \quad s > 0$$

To the LHD,

$$\begin{aligned} \mathcal{L}\{w''(t) - w(t)\} &= (s^2 Y - s(3) - 0) - Y \\ &= s^2 Y - 3s - Y \\ &= Y(s^2 - 1) - 3s \end{aligned}$$

Step 4: We solve for Y(s).

$$
\begin{aligned}
Y(s^2-1) - 3s &= \frac{1}{s^2} \\
&= \frac{1}{s^2} + 3s \\
&= \frac{1+3s^3}{s^2}
\end{aligned}
$$

$$
Y = \frac{1+3s^3}{s^2(s^2-1)}
$$

Step 5: We perform partial fraction decomposition to the function.

$$
\begin{aligned}
\frac{1+3s^3}{s^2(s^2-1)} &= \frac{1+3s^3}{s^2(s-1)(s+1)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}
\end{aligned}
$$

Step 6: We solve for A, B, C, and D.

$$
= As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1)
$$

$s = 0:$

$$
-B = 1
$$

$$
B = -1
$$

$s = 1:$

$$
2C = 4
$$

$$
C = 2
$$

$s = -1:$

$$
-2D = -2
$$

$$
D = 1
$$

We solve for A by equating the function.

$$
\begin{gathered}
= As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1) \\
= A(s^3 - s) + B(s^2-1) + C(s^3+s^2) + D(s^3-s^2) \\
= As^3 - As + Bs^2 - B + Cs^3 + Cs^2 + Ds^3 - Ds^2 \\
s^3(A+D) + s^2(B+C-D) + s(-A) - B
\end{gathered}
$$

$$A + C + D = 3; \quad B + C - D = 0; \quad -A = 0; \quad -B = 1$$

$$A = 0; \quad B = -1; \quad C = 2; \quad D = 1$$

Hence,

$$\frac{1 + 3s^2}{s^2(s+1)(s-1)} = -\frac{1}{s^2} + \frac{2}{s-1} + \frac{1}{s+1}$$

Step 7: We take the inverse Laplace transform.

$$\mathcal{L}^{-1}\left\{\frac{1 + 3s^3}{s^2(s+1)(s-1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2} + \frac{2}{s-1} + \frac{1}{s+1}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -t + 2e^t + e^{-t}$$

Step 8: We move the initial condition to $t = 2$.

$$= -(t-2) + 2e^{t-2} + e^{-(t-2)}$$

$$= 2 - t + 2e^{t-2} + e^{2-t}$$

Problem 2 Solve the given third-order initial value problem for y(t) using the method of Laplace transforms.

$$y'''(t) + 3y''(t) + 3y'(t) + y(t) = 0$$

where, $\quad y(0) = -4, \quad y'(0) = 4, \quad y''(0) = -2$

Solution:

Step 1: We first take the LT of the LHS:

$$\begin{aligned}
&= \mathcal{L}\{y''' + 3y'' + 3y' + y\} \\
&= \mathcal{L}\{y'''\} + 3\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + \mathcal{L}\{y\} \\
&= (s^3Y - s^2(-4) - s(4) - (-2)) + 3(s^2Y - s(-4) - (4)) + 3(sY - (-4)) + Y \\
&= Y(s^3 + 3s^2 + 3s + 1) + 4s^2 + 8s + 2
\end{aligned}$$

Step 2: We solve for Y(s):

$$\begin{aligned}
Y(s^3 + 3s^2 + 3s + 1) + 4s^2 + 8s + 2 = 0 \\
Y(s^3 + 3s^2 + 3s + 1) = -(4s^2 + 8s + 2) \\
Y = -\frac{4s^2 + 8s + 2}{s^3 + 3s^2 + 3s + 1}
\end{aligned}$$

Step 3: We factor the denominator:

$$\begin{aligned}
(s^3 + 3s^2 + 3s + 1) &= (s + 1)(s^2 + 2s + 1) \\
&= (s + 1)(s + 1)(s + 1) \\
&= (s + 1)^3
\end{aligned}$$

Step 4: We break the fraction into partial fractions:

$$\begin{aligned}
-\frac{4s^2 + 8s + 2}{(s + 1)^3} &= \frac{A}{(s + 1)} + \frac{B}{(s + 1)^2} + \frac{C}{(s + 1)^3} \\
&= A(s + 1)^2 + B(s + 1) + C \\
&= A(s^2 + 2s + 1) + Bs + B + C \\
&= As^2 + 2As + A + Bs + B + C \\
&= s^2(A) + s(2A + B) + A + B + C
\end{aligned}$$

Step 5: We solve for the coefficients:

$$s = -1:$$

$$C = -(4 - 8 + 2)$$

$$C = 2$$
$$A = -4$$
$$B = -2(-4) - 8$$
$$B = 0$$

Hence,

$$A = -4$$
$$B = 0$$
$$C = 2$$

$$-\frac{4s^2 + 8s + 2}{(s+1)^3} = -\frac{4}{(s+1)} + \frac{2}{(s+1)^3}$$

Step 6: We take the ILT:

$$= -4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= -4e^{-t} + \frac{2}{2}t^2e^{-t}$$

$$= -4e^{-t} + t^2e^{-t}$$

$$= e^{-t}(t^2 - 4)$$