Quiz 6

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Problem 1 Find the inverse Laplace transform of the given function.

$$\frac{2s+16}{s^2+4s+13}$$

Solution:

We first simplify the given rational function.

$$\begin{aligned}
\frac{2s+16}{s^2+4s+13} &= \frac{2(s+8)}{(s+2)^2+9} \\
&= \frac{2(s+2+6)}{(s+2)^2+(3)^2} = \frac{2(s+2)}{(s+2)^2+(3)^2} + \frac{12}{(s+2)^2+(3)^2} \\
&= 2\left(\frac{s+2}{(s+2)^2+(3)^2}\right) + 4\left(\frac{3}{(s+2)^2+(3)^2}\right)
\end{aligned}$$

Now we find the inverse Laplace transform.

$$= 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(3)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+(3)^2}\right\}$$

$$= 2(e^{-2t}\cos 3t) + 4(e^{-2t}\sin 3t)$$

$$= 2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t$$

Problem 2 Determine the partial fraction expansion for the given rational function.

$$\frac{4s^2 - 21s + 16}{s(s-2)^2}$$

Solution:

We first simplify the given rational function.

$$= \frac{4s^2 - 21s + 16}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= 4s^2 - 21s + 16 = A(s-2)^2 + Bs(s-2) + Cs$$

Now we find the coefficients.

$$\begin{aligned}
s &= 2 : \\
4(2)^2 - 21(2) + 16 &= A(0)^2 + B(0) + 2C \\
16 - 42 + 16 &= 2C \\
2C &= -10 \\
C &= -5
\end{aligned}$$

$$\begin{aligned}
s &= 0 : \\
4(0)^2 - 21(0) + 16 &= A(2)^2 \\
16 &= A(4) \\
A &= 4
\end{aligned}$$

Solving by equation coefficients.

$$\begin{aligned}
&= A(s^2 - 4s + 4) + B(s^2 - 2s) + Cs \\
&= As^2 - 4As + 4A + Bs^2 - 2Bs + Cs \\
&= s^2(A + B) + s(-4A - 2B + C) + 4A
\end{aligned}$$

Hence:

$$\begin{aligned}
A + B &= 4 \\
-4A - 2B + C &= -21 \\
4A &= 16 \\
\text{Subsitute } A = 4 \text{ and } C &= -5 \\
4 + B &= 4 \\
-16 - 2B + (-5) &= -21 \\
-2B &= -21 + 16 + 5 \\
-2B &= 0 \\
B &= 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{4s^2 - 21s + 16}{s(s-2)^2} &= \frac{4}{s} + \frac{0}{s-2} + \frac{-5}{(s-2)^2} \\
&= \frac{4}{s} - \frac{5}{(s-2)^2}
\end{aligned}$$

Problem 3 Determine the partial fraction expansion for the given rational function.

$$\frac{3s+5}{s(s^2+s-6)}$$

Solution:

We first simplify the given rational function.

$$\frac{3s+5}{s(s^2+s-6)} = \frac{3s+5}{s(s+3)(s-2)}$$

Now we find the partial fraction expansion.

$$\begin{aligned}\frac{3s+5}{s(s+3)(s-2)} &= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2} \\ &= A(s+3)(s-2) + Bs(s-2) + Cs(s+3)\end{aligned}$$

Solving by equating coefficients.

$$\begin{aligned}s &= 0: \\ 3(0)+5 &= A(3)(-2) + B(0)(0) + C(0)(0) \\ 5 &= -6A \\ A &= -\frac{5}{6}\end{aligned}$$

$$\begin{aligned}s &= -3: \\ 3(-3)+5 &= A(0) + B(-3)(-3-2) + C(0) \\ -4 &= B(15) \\ B &= -\frac{4}{15}\end{aligned}$$

$$\begin{aligned}s &= 2: \\ 3(2)+5 &= A(0) + B(0) + C(2)(2+3) \\ 11 &= C(10) \\ C &= \frac{11}{10}\end{aligned}$$

Therefore,

$$\frac{3s+5}{s(s^2+s-6)} = \frac{-\frac{5}{6}}{s} + \frac{-\frac{4}{15}}{s+3} + \frac{\frac{11}{10}}{s-2}$$

$$= -\frac{5}{6s} - \frac{4}{15(s+3)} + \frac{11}{10(s-2)}$$

Problem 4 Determine $\mathcal{L}^{-1}\{F\}$

$$F(s) = \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)}$$

Solution:

We first simplify the given rational function.

$$= \frac{5s^2 + 34s + 53}{(s+3)^2(s+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1}$$

$$= 5s^2 + 34s + 53 = A(s+3)(s+1) + B(s+1) + C(s+3)^2$$

Now we find the coefficients.

$$\begin{aligned} s &= -3: \\ 5(-3)^2 + 34(-3) + 53 &= -2B \\ -4 &= -2B \\ B &= 2 \end{aligned}$$

$$\begin{aligned} s &= -1: \\ 5(-1)^2 + 34(-1) + 53 &= C(2)^2 \\ 24 &= 4C \\ C &= 6 \end{aligned}$$

$$s = 0:$$

Solving by equating coefficients.

$$
\begin{aligned}
&= A(s+3)(s+1) + B(s+1) + C(s+3)^2 \\
&= A(s^2+4s+3) + B(s+1) + C(s^2+6s+9) \\
&= s^2(A+C) + s(4A+6C+B) + 3A+B+9C
\end{aligned}
$$

$$
\begin{aligned}
s^2 &: \\
A + C &= 5 \\
A &= 5 - C \\
A &= 5 - 6 = -1
\end{aligned}
$$

Therefore,

$$
\frac{5s^2+34s+53}{(s+3)^2(s+1)} = -\frac{1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{(s+1)}
$$

Now we find the inverse Laplace transform.

$$
\mathcal{L}^{-1}\{F\} = -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\}
$$

$$
= -e^{-3t} + 2te^{-3t} + 6e^{-t}
$$