Method of Undetermined Coefficients (Revisited)

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(t)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}\,; \tag{13}$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t\,, \quad \beta \neq 0\,,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$\begin{aligned} y_p(t) &= t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t \\ &\quad + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t\,, \end{aligned} \tag{14}$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Variation of Parameters

Theorem 7. If y_1 and y_2 are two linearly independent solutions to the homogeneous equation (10) on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution to (11) is given by $y_p = v_1 y_1 + v_2 y_2$, where v_1 and v_2 are determined up to a constant by the pair of equations

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0\,, \\ y_1' v_1' + y_2' v_2' &= g\,, \end{aligned}$$

which have the solution

$$v_1(t) = \int \frac{-g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt\,, \quad v_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt\,. \tag{12}$$

Note the formulation (12) presumes that the differential equation has been put into standard form [that is, divided by $a_2(t)$].

Method of Undetermined Coefficients

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}, \tag{14}$$

with

(i) $s = 0$ if r is not a root of the associated auxiliary equation;
(ii) $s = 1$ if r is a simple root of the associated auxiliary equation; and
(iii) $s = 2$ if r is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ \quad\text{or} \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$$

for $\beta \neq 0$, use the form

$$\begin{aligned} y_p(t) &= t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t \\ &\quad + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t\,, \end{aligned} \tag{15}$$

with

(iv) $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
(v) $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.