I. DEFINITIONS

Sensitivities

$$\bar{\boldsymbol{x}}_\theta = \frac{d\boldsymbol{x}}{d(\log(\theta))} = A\frac{d\boldsymbol{x}}{d\theta}$$

Sensitivity equations

$$\frac{d\bar{\boldsymbol{x}}_\theta}{dt} = \theta\frac{\partial \boldsymbol{f}}{\partial\theta} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\bar{\boldsymbol{x}}_\theta, \quad \text{where } \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\theta}, t)$$

Fisher information matrix

$$\frac{d}{dt}I_{ij} = \bar{\boldsymbol{x}}^T_{\theta_i}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{x}}_{\theta_j}, \qquad \boldsymbol{I}(0) = \boldsymbol{0}$$

In calculations I take $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1} = \boldsymbol{1}$ (zero noise approximation).

II. EXAMPLE: SIMPLE HARMONIC OSCILLATOR

The dynamics for a simple harmonic oscillator with amplitude, A, and time-varying frequency, $u(t)$, is given by

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & -(\dot{u}t+u)/A \\ A(\dot{u}t+u) & 0 \end{pmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

which has the solution

$$\boldsymbol{x} = \begin{pmatrix} \cos(ut) \\ A\sin(ut) \end{pmatrix} \tag{2}$$

Then the sensitivity wrt A is

$$\bar{\boldsymbol{x}}_A = \begin{pmatrix} 0 \\ A\sin(ut) \end{pmatrix} \tag{3}$$

So that the change in Fisher information is

$$\frac{dI}{dt} = A^2\sin^2(ut) \tag{4}$$

Then

$$I(u,t) = \int_0^t A^2\sin^2(ut')dt' \tag{5}$$

Taking $L(t,u,\dot{u}) = A^2\sin^2(ut)$ as the Lagrangian, extrema of I are found by solving the Euler-Lagrange equation

$$\begin{aligned} &\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} = 0 \\ \implies\ & 2A^2 t\sin(ut)\cos(ut) = 0 \\ \implies\ & \boxed{u = n\frac{\pi}{2t}, \quad n \in Z^+} \end{aligned}$$

even values of n give minima of I whereas odd values of n give maxima.

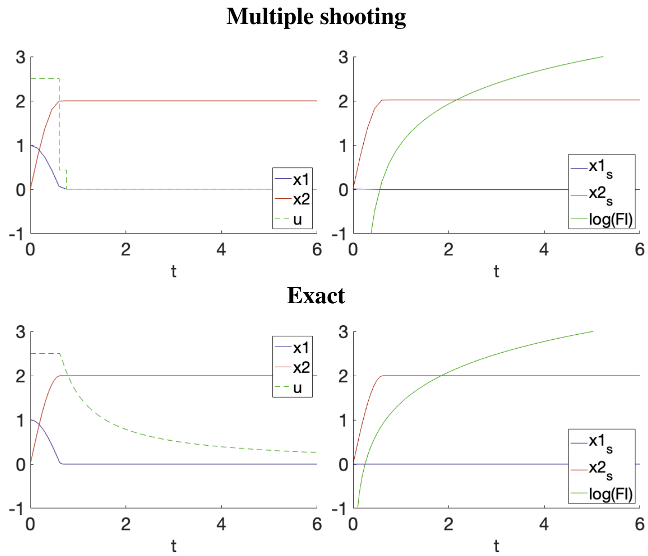

FIG. 1. The parameter value is $A = 2$. The constraint on u is $0 < u(t) < 2.5$. (The multiple shooting is actually the optimal solution for (1) if we set $\dot{u}t = 0$).