I. CLASSICAL COMPASS MODEL

The classical compass model on a square lattice has the Hamiltonian,

$$\mathcal{H} = -\sum_{\boldsymbol{r}} \left(J_x S^x_{r_x} S^x_{r_x+1} + J_y S^y_{r_y} S^y_{r_y+1} \right) \tag{1}$$

which can also be written as

$$\mathcal{H} = -\sum_{\langle ij \rangle} \boldsymbol{S_i} \cdot \boldsymbol{J}(\boldsymbol{r_{ij}}) \cdot \boldsymbol{S_j} \tag{2}$$

where the coupling strength is

$$\boldsymbol{J}(\boldsymbol{r}) = \begin{pmatrix} \begin{pmatrix} J_x & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{r} = \pm\hat{x} \\ \begin{pmatrix} 0 & 0 \\ 0 & J_y \end{pmatrix} & \text{if } \boldsymbol{r} = \pm\hat{y} \end{pmatrix} \tag{3}$$

II. THE GROUND STATE

The Fourier transform of the coupling matrix is

$$\tilde{\boldsymbol{J}}(\boldsymbol{k}) = \sum_{\boldsymbol{r}} \tilde{\boldsymbol{J}}(\boldsymbol{r}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{4}$$

$$= \begin{pmatrix} \begin{pmatrix} J_x \cos k_x & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \boldsymbol{k}_n = \pm\frac{\pi n \hat{x}}{Na} \\ \begin{pmatrix} 0 & 0 \\ 0 & J_y \cos k_y \end{pmatrix} & \text{if } \boldsymbol{k}_n = \pm\frac{\pi n \hat{y}}{Na} \end{pmatrix} \tag{5}$$

The spin variables have the Fourier series decomposition

$$\boldsymbol{S}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \tilde{\boldsymbol{S}}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{6}$$

So the Hamiltonian may be written as,

$$\mathcal{H} = -\sum_{\boldsymbol{k}} \tilde{\boldsymbol{S}}_{-\boldsymbol{k}} \cdot \tilde{\boldsymbol{J}}_{\boldsymbol{k}} \cdot \tilde{\boldsymbol{S}}_{\boldsymbol{k}} \tag{7}$$

In particular, the energy of a configuration defined by a single ordering wave vector, $\boldsymbol{k}$, is

$$E_{\boldsymbol{k}} = -J_x \cos k_x \left| S_{k_x} \right|^2 + -J_y \cos k_y \left| S_{k_y} \right|^2 \tag{8}$$

The spin length constraint in Fourier space is by Parseval's theorem

$$|S_{k_x}|^2 + |S_{k_y}|^2 = 1 \tag{9}$$

So we have

$$E_{\boldsymbol{k}} = -J_y \cos k_y - (J_x \cos k_x - J_y \cos k_y)|S_{k_x}|^2 \tag{10}$$

Taking $\boldsymbol{S_k} = (\cos\theta, \sin\theta)$ we have

$$\nabla_{\boldsymbol{k}} E_{\boldsymbol{k}} = J_x \begin{pmatrix} 0 \\ \sin k_x \end{pmatrix} + J_y \begin{pmatrix} \sin k_y \\ -\sin k_y \end{pmatrix} \cos^2\theta \tag{11}$$

So we see that the ferromagnetic state fulfills $\nabla_{\boldsymbol{k}} E_{\boldsymbol{k}} = 0$ and that θ is a degeneracy parameter for this state. In the limits of $\theta \to 0$ (x-aligned) and $\theta \to \pi/2$ (y-aligned) of colinear configurations the ground state degeneracy becomes 2^L. This is due to the fact that entire lines of spins may be flipped at zero energy cost (subsystem symmetry).

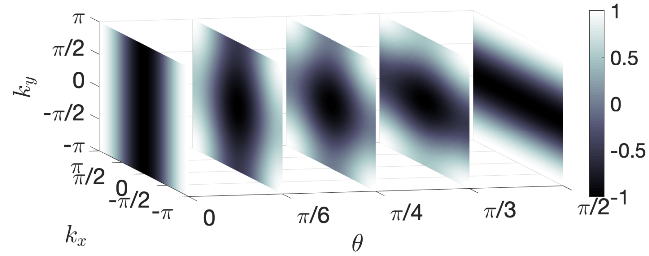

FIG. 1. Energy of configurations with ordering wave vector, $\boldsymbol{k}$, in the isotropic model as a function of the spin orientation parameter, θ.

III. SPIN-WAVE EXPANSION

A. Free Energy

The ferromagnetic ground state of the isotropic model ($J_x = J_y = J$) is taken to be

$$\boldsymbol{S_i} = [0, 1] \tag{12}$$

Spin-wave perturbations about the ferromagnetic ground state are defined as

$$\boldsymbol{S_i} \to \boldsymbol{S_i} + \boldsymbol{\delta S_i} \tag{13}$$

$$\boldsymbol{S_i} + \boldsymbol{\delta S_i} = \left(\delta\theta_i, \sqrt{1 - \delta\theta_i^2} \right) \tag{14}$$

$$\approx (\delta\theta_i, 1 - \delta\theta_i^2/2) \tag{15}$$

The energy due to spin waves is

$$\mathcal{H}_{\text{FM}} \to \mathcal{H}_{\text{FM}} + \mathcal{H}_{\text{SW}} \tag{16}$$

$$\mathcal{H}_{\text{SW}} = -J \sum_{\boldsymbol{r}} \left(\delta\theta_x \delta\theta_{x+1} - \frac{1}{2} \left(\delta\theta_y^2 + \delta\theta_{y+1}^2 \right) \right) \tag{17}$$

$$= -J \sum_{\boldsymbol{k}} (\cos k_x - 1) |\delta\theta_{\boldsymbol{k}}|^2 \tag{18}$$

The partition function is

$$Z = \sum_{\text{c}} \exp(-\beta\mathcal{H}_{\text{c}}) \tag{19}$$

$$= \sum_{\text{c}} \exp\left(-\beta\mathcal{H}_{\text{FM}} + \beta \sum_{\boldsymbol{k}} J (\cos k_x - 1) |\delta\theta_{\boldsymbol{k}}|^2 \right) \tag{20}$$

$$\propto \prod_{\boldsymbol{k}} \int d(\delta\theta_{\boldsymbol{k}}) \exp\left(-\beta J (1 - \cos k_x) |\delta\theta_{\boldsymbol{k}}|^2 \right) \tag{21}$$

$$= \prod_{\boldsymbol{k}} \sqrt{\frac{\pi}{\beta J (1 - \cos k_x)}} \tag{22}$$

So the free energy becomes

$$F \to F_0 + F_{SW} \tag{23}$$

where

$$F_{SW} = -T\log\left(\prod_{\boldsymbol{k}}\sqrt{\frac{1}{1-\cos k_x}}\right) \tag{24}$$

$$= \frac{T}{2}\sum_{\boldsymbol{k}}\log\left(1-\cos k_x\right) \tag{25}$$

$$= \frac{TL}{4\pi}\sum_{k_x}\log\left(1-\cos k_x\right) \tag{26}$$

To explore the full (ferromagnetic) ground state manifold the global transformation is applied

$$\boldsymbol{S_i}+\boldsymbol{\delta S_i}\to\boldsymbol{R}(\theta)\left(\boldsymbol{S_i}+\boldsymbol{\delta S_i}\right), \quad \boldsymbol{R}(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{27}$$

Then the Hamiltonian is

$$\mathcal{H} = -\sum_{\langle ij\rangle}\boldsymbol{R}(\theta)\left(\boldsymbol{S_i}+\boldsymbol{\delta S_i}\right)\cdot\boldsymbol{J}(\boldsymbol{r_{ij}})\cdot\boldsymbol{R}(\theta)\left(\boldsymbol{S_j}+\boldsymbol{\delta S_j}\right) \tag{28}$$

$$= \mathcal{H}_{FM}-\sum_{\langle ij\rangle}\boldsymbol{\delta S_i}\cdot\boldsymbol{J}(\boldsymbol{r_{ij}},\theta)\cdot\boldsymbol{\delta S_j} \tag{29}$$

where,

$$\boldsymbol{J}(\boldsymbol{r},\theta) = \boldsymbol{R}(\theta)^T\boldsymbol{J}(\boldsymbol{r})\boldsymbol{R}(\theta) \tag{30}$$

$$= \begin{pmatrix} J_x\begin{pmatrix}\cos^2\theta & -\cos\theta\sin\theta\\ -\cos\theta\sin\theta & \sin^2\theta\end{pmatrix} & \text{if } \boldsymbol{r}=\pm\hat{x}\\ J_y\begin{pmatrix}\sin^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \cos^2\theta\end{pmatrix} & \text{if } \boldsymbol{r}=\pm\hat{y}\end{pmatrix} \tag{31}$$

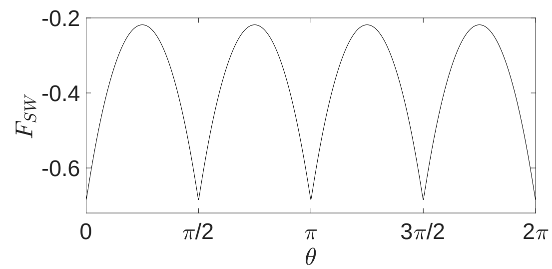

FIG. 2. Free energy of spin waves in the isotropic model as a function of the spin orientation parameter, θ. The free energy is minimized for the ferromagnetic configurations aligned with the x or y directions.

is the rotated coupling matrix. Then we have

$$\mathcal{H}_{SW} = -J\sum_{\boldsymbol{r}}\delta\theta_x\delta\theta_{x+1}\cos^2\theta-\frac{1}{2}\left(\delta\theta_x^2+\delta\theta_{x+1}^2\right)\sin^2\theta \tag{32}$$

$$+\delta\theta_y\delta\theta_{y+1}\sin^2\theta-\frac{1}{2}\left(\delta\theta_y^2+\delta\theta_{y+1}^2\right)\cos^2\theta \tag{33}$$

$$= -J\sum_{\boldsymbol{k}}\left(\cos k_x\cos^2\theta+\cos k_y\sin^2\theta-1\right) \tag{34}$$

So that the spin-wave correction to the free energy in the rotated basis is

$$F_{SW} = \frac{T}{2}\sum_{\boldsymbol{k}}\log\left(1-\cos k_x\cos^2\theta-\cos k_y\sin^2\theta\right) \tag{35}$$

B. Spin Correlations

[1] J. Dorier, F. Becca, and F. Mila, Phys. Rev. B **72**, 024448 (2005).

[2] S. Prakash and C. L. Henley, Phys. Rev. B **42**, 6574 (1990).