I. THE MODEL

The classical nearest neighbor Ising model on a 2D square lattice is,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{1}$$

where $\sigma = \pm 1$, $J > 0$, and $\langle ij \rangle$ means only sum the nearest neighbors.

II. TRANSFER MATRIX DECOMPOSITION

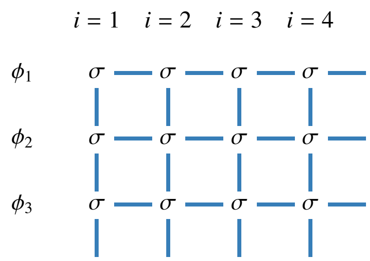

FIG. 1. Ising spins on a finite lattice with periodic boundary conditions. Each row is in some configuration labeled by ϕ, and each lattice site in each row is indexed by i.

We may represent the $2D$ ising model as coupled $1D$ systems (FIG 1.) each of which has the same set of configurations.

The general form of the partition function for a finite system with periodic boundary conditions and nearest neighbor interactions is then

$$Z = \sum_{\phi_1} \sum_{\phi_2} \cdots \sum_{\phi_L} T(\phi_1, \phi_2) T(\phi_2, \phi_3) \ldots T(\phi_L, \phi_1) \tag{2}$$

where we have

$$T(\phi, \phi') = \exp\left(\frac{\beta J}{2} \sum_i \left(\sigma_i \sigma_{i+1} + \sigma'_i \sigma'_{i+1} + 2\sigma_i \sigma'_i \right) \right) \tag{3}$$

which carries all of the Boltzmann weights for coupling two rows - given the procedure is carried out iteratively under periodic boundary conditions.

So we may write the partition function as

$$Z = \mathrm{Tr}\left(\boldsymbol{T}^L \right) \tag{4}$$

where $\boldsymbol{T}$ is called the transfer matrix and has the matrix elements

$$\boldsymbol{T}_{\phi\phi'} = T(\phi, \phi') \tag{5}$$