Complicating the RHS of the Linear Model

Fundamental Techniques in Data Science with R

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Outline

Categorical Predictors
- Dummy Coding
- Significance Testing for Dummy Codes

Moderation
- Categorical Moderators

Polynomial Regression

Categorical Predictors

Most of the predictors we've considered thus far have been *quantitative*.

- Continuous variables that can take any real value in their range
- Interval or Ratio scaling

We often want to include grouping factors as predictors.

- These variables are *qualitative*.
  - Their values are simply labels.
  - There is no ordering of the categories.
  - Nominal scaling

How to Model Categorical Predictors

We need to be careful when we include categorical predictors into a regression model.

- The variables need to be coded before entering the model

Consider the following indicator of major:
$X_{maj} = \{1 = \textit{Law}, 2 = \textit{Economics}, 3 = \textit{Data Science}\}$

- What would happen if we naïvely used this variable to predict program satisfaction?

How to Model Categorical Predictors

```
mDat <- readRDS("../data/major_data.rds")
mDat[seq(25, 150, 25), ]

    sat majF majN
25  1.9  law    1
50  1.4  law    1
75  4.3 econ    2
100 4.1 econ    2
125 5.7   ds    3
150 5.1   ds    3

out1 <- lm(sat ~ majN, data = mDat)
```

How to Model Categorical Predictors

```
partSummary(out1, -1)

Residuals:
   Min     1Q Median     3Q    Max
-1.303 -0.313 -0.113  0.342  1.342

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.33200    0.12060  -2.753  0.00664
majN         2.04500    0.05582  36.632  < 2e-16

Residual standard error: 0.5582 on 148 degrees of freedom
Multiple R-squared:  0.9007,Adjusted R-squared:    0.9
F-statistic:  1342 on 1 and 148 DF,  p-value: < 2.2e-16
```

Dummy Coding

The most common way to code categorical predictors is *dummy coding*.

- A G-level factor must be converted into a set of $G - 1$ dummy codes.
- Each code is a variable on the dataset that equals 1 for observations corresponding to the code's group and equals 0, otherwise.
- The group without a code is called the *reference group*.

Example Dummy Code

Let's look at the simple example of coding biological sex:

	sex	male
1	female	0
2	male	1
3	male	1
4	female	0
5	male	1
6	female	0
7	female	0
8	male	1
9	female	0
10	female	0

Example Dummy Codes

Now, a slightly more complex example:

	drink	juice	tea
1	juice	1	0
2	coffee	0	0
3	tea	0	1
4	tea	0	1
5	tea	0	1
6	tea	0	1
7	juice	1	0
8	tea	0	1
9	coffee	0	0
10	juice	1	0

Using Dummy Codes

To use the dummy codes, we simply include the $G - 1$ codes as $G - 1$ predictor variables in our regression model.

$$Y = \beta_0 + \beta_1 X_{male} + \varepsilon$$
$$Y = \beta_0 + \beta_1 X_{juice} + \beta_2 X_{tea} + \varepsilon$$

- The intercept corresponds to the mean of Y for the reference group.
- Each slope represents the difference between the mean of Y in the coded group and the mean of Y in the reference group.

Example

First, an example with a single, binary dummy code:

```r
## Read in some data:
cDat <- readRDS("../data/cars_data.rds")

## Fit and summarize the model:
out2 <- lm(price ~ mtOpt, data = cDat)
```

Example

```
partSummary(out2, -1)

Residuals:
    Min      1Q  Median      3Q     Max 
-10.341  -6.338  -3.141   2.662  38.059 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   23.841      1.623  14.691   <2e-16
mtOpt         -6.603      2.004  -3.295   0.0014

Residual standard error: 9.18 on 91 degrees of freedom
Multiple R-squared:  0.1066,Adjusted R-squared:  0.09679 
F-statistic: 10.86 on 1 and 91 DF,  p-value: 0.001403
```

Interpretations

- The average price of a car without the option for a manual transmission is $\hat{\beta}_0 = 23.84$ thousand dollars.
- The average difference in price between cars that have manual transmissions as an option and those that do not is $\hat{\beta}_1 = -6.6$ thousand dollars.

Example

Fit a more complex model:

```
out3 <- lm(price ~ front + rear, data = cDat)
partSummary(out3, -1)

Residuals:
    Min      1Q  Median      3Q     Max
-14.050  -6.250  -1.236   3.264  32.950

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.63000    2.76119   6.385 7.33e-09
front       -0.09418    2.96008  -0.032  0.97469
rear        11.32000    3.51984   3.216  0.00181

Residual standard error: 8.732 on 90 degrees of freedom
Multiple R-squared:  0.2006,Adjusted R-squared:  0.1829
F-statistic: 11.29 on 2 and 90 DF,  p-value: 4.202e-05
```

Interpretations

- The average price of a four-wheel-drive car is $\hat{\beta}_0 = 17.63$ thousand dollars.
- The average difference in price between front-wheel-drive cars and four-wheel-drive cars is $\hat{\beta}_1 = -0.09$ thousand dollars.
- The average difference in price between rear-wheel-drive cars and four-wheel-drive cars is $\hat{\beta}_2 = 11.32$ thousand dollars.

Example

Include two sets of dummy codes:

```
out4 <- lm(price ~ mtOpt + front + rear, data = cDat)
partSummary(out4, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  21.7187     2.9222   7.432 6.25e-11
mtOpt        -5.8410     1.8223  -3.205  0.00187
front        -0.2598     2.8189  -0.092  0.92677
rear         10.5169     3.3608   3.129  0.00237

Residual standard error: 8.314 on 89 degrees of freedom
Multiple R-squared:  0.2834,Adjusted R-squared:  0.2592
F-statistic: 11.73 on 3 and 89 DF,  p-value: 1.51e-06
```

Interpretations

- The average price of a four-wheel-drive car that does not have a manual transmission option is $\hat{\beta}_0 = 21.72$ thousand dollars.
- After controlling for drive type, the average difference in price between cars that have manual transmissions as an option and those that do not is $\hat{\beta}_1 = -5.84$ thousand dollars.
- After controlling for transmission options, the average difference in price between front-wheel-drive cars and four-wheel-drive cars is $\hat{\beta}_2 = -0.26$ thousand dollars.
- After controlling for transmission options, the average difference in price between rear-wheel-drive cars and four-wheel-drive cars is $\hat{\beta}_3 = 10.52$ thousand dollars.

Significance Testing

For variables with only two levels, we can test the overall factor's significance by evaluating the significance of a single dummy code.

```
partSummary(out2, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   23.841      1.623  14.691   <2e-16
mtOpt         -6.603      2.004  -3.295   0.0014

Residual standard error: 9.18 on 91 degrees of freedom
Multiple R-squared:  0.1066,Adjusted R-squared:  0.09679
F-statistic: 10.86 on 1 and 91 DF,  p-value: 0.001403
```

Significance Testing

For variables with more than two levels, we need to simultaneously evaluate the significance of each of the variable's dummy codes.

```
partSummary(out4, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  21.7187     2.9222   7.432 6.25e-11
mtOpt        -5.8410     1.8223  -3.205  0.00187
front        -0.2598     2.8189  -0.092  0.92677
rear         10.5169     3.3608   3.129  0.00237

Residual standard error: 8.314 on 89 degrees of freedom
Multiple R-squared:  0.2834,Adjusted R-squared:  0.2592
F-statistic: 11.73 on 3 and 89 DF,  p-value: 1.51e-06
```

Significance Testing

```
summary(out4)$r.squared - summary(out2)$r.squared

[1] 0.1767569

anova(out2, out4)

Analysis of Variance Table

Model 1: price ~ mtOpt
Model 2: price ~ mtOpt + front + rear
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     91 7668.9
2     89 6151.6  2    1517.3 10.976 5.488e-05 ***
---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Significance Testing

For models with a single nominal factor is the only predictor, we use the omnibus F-test.

```
partSummary(out3, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.63000    2.76119   6.385 7.33e-09
front       -0.09418    2.96008  -0.032  0.97469
rear        11.32000    3.51984   3.216  0.00181

Residual standard error: 8.732 on 90 degrees of freedom
Multiple R-squared:  0.2006,Adjusted R-squared:  0.1829
F-statistic: 11.29 on 2 and 90 DF,  p-value: 4.202e-05
```

Moderation

Moderation

So far we've been discussing *additive models*.

- Additive models allow us to examine the partial effects of several predictors on some outcome.
  - The effect of one predictor does not change based on the values of other predictors.

Now, we'll discuss *moderation*.

- Moderation allows us to ask *when* one variable, X, affects another variable, Y.
  - We're considering the conditional effects of X on Y given certain levels of a third variable Z.

Equations

In additive MLR, we might have the following equation:

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \varepsilon$$

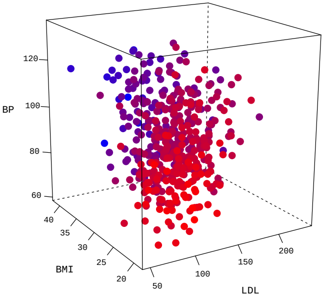
This additive equation assumes that X and Z are independent predictors of Y.

When X and Z are independent predictors, the following are true:

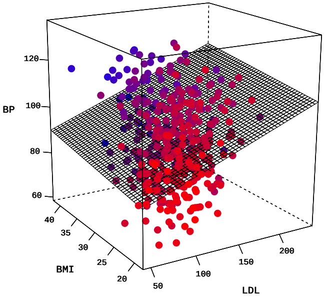
- X and Z *can* be correlated.
- β_1 and β_2 are *partial* regression coefficients.
- The effect of X on Y is the same at **all levels** of Z, and the effect of Z on Y is the same at **all levels** of X.

Additive Regression

The effect of X on Y is the same at **all levels** of Z.

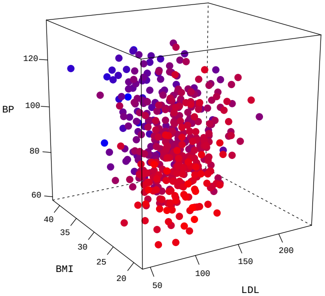


$\longrightarrow$

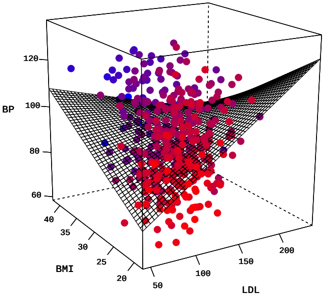


Moderated Regression

The effect of X on Y varies **as a function** of Z.



$\longrightarrow$



Equations

The following derivation is adapted from Hayes (2017).

- When testing moderation, we hypothesize that the effect of X on Y varies as a function of Z.
- We can represent this concept with the following equation:

$$Y = \beta_0 + f(Z)X + \beta_2 Z + \varepsilon \tag{1}$$

Equations

The following derivation is adapted from Hayes (2017).

- When testing moderation, we hypothesize that the effect of X on Y varies as a function of Z.
- We can represent this concept with the following equation:

$$Y = \beta_0 + f(Z)X + \beta_2 Z + \varepsilon \tag{1}$$

- If we assume that Z linearly (and deterministically) affects the relationship between X and Y, then we can take:

$$f(Z) = \beta_1 + \beta_3 Z \tag{2}$$

Equations

- Substituting Equation 2 into Equation 1 leads to:

$$Y = \beta_0 + (\beta_1 + \beta_3 Z)X + \beta_2 Z + \varepsilon$$

Equations

- Substituting Equation 2 into Equation 1 leads to:

$$Y = \beta_0 + (\beta_1 + \beta_3 Z)X + \beta_2 Z + \varepsilon$$

- Which, after distributing X and reordering terms, becomes:

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + \varepsilon$$

Testing Moderation

Now, we have an estimable regression model that quantifies the linear moderation we hypothesized.

$$\boxed{Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + \varepsilon}$$

- To test for significant moderation, we simply need to test the significance of the interaction term, XZ.
    - Check if $\hat{\beta}_3$ is significantly different from zero.

Interpretation

Given the following equation:

$$Y = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 Z + \hat{\beta}_3 XZ + \hat{\varepsilon}$$

- $\hat{\beta}_3$ quantifies the effect of Z on the focal effect (the $X \rightarrow Y$ effect).
  - For a unit change in Z, $\hat{\beta}_3$ is the expected change in the effect of X on Y.
- $\hat{\beta}_1$ and $\hat{\beta}_2$ are *conditional effects*.
  - Interpreted where the other predictor is zero.
  - For a unit change in X, $\hat{\beta}_1$ is the expected change in Y, when $Z = 0$.
  - For a unit change in Z, $\hat{\beta}_2$ is the expected change in Y, when $X = 0$.

Example

Still looking at the *diabetes* dataset.

- We suspect that patients' BMIs are predictive of their average blood pressure.
- We further suspect that this effect may be differentially expressed depending on the patients' LDL levels.

Example

```
## Focal Effect:
out0 <- lm(bp ~ bmi, data = dDat)
partSummary(out0, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  61.9973     3.6659   16.91   <2e-16
bmi           1.2379     0.1371    9.03   <2e-16

Residual standard error: 12.72 on 440 degrees of freedom
Multiple R-squared:  0.1563,Adjusted R-squared:  0.1544
F-statistic: 81.54 on 1 and 440 DF,  p-value: < 2.2e-16
```

Example

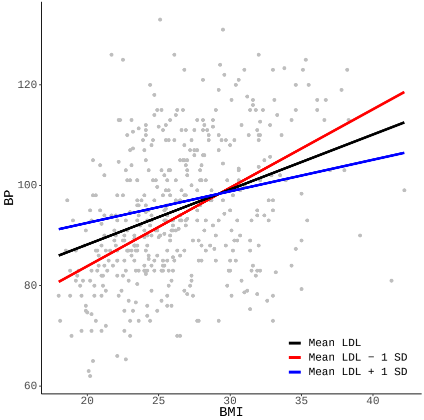
```
## Additive Model:
out1 <- lm(bp ~ bmi + ldl, data = dDat)
partSummary(out1, -c(1, 2))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 59.26577    3.91281  15.147  < 2e-16
bmi          1.16567    0.14156   8.235 2.08e-15
ldl          0.04016    0.02056   1.953   0.0515

Residual standard error: 12.68 on 439 degrees of freedom
Multiple R-squared:  0.1636,Adjusted R-squared:  0.1598
F-statistic: 42.94 on 2 and 439 DF,  p-value: < 2.2e-16
```

Example

```
## Moderated Model:
out2 <- lm(bp ~ bmi * ldl, data = dDat)
partSummary(out2, -c(1, 2))

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 14.480616  14.291677   1.013 0.311514
bmi          2.867825   0.541312   5.298 1.86e-07
ldl          0.448771   0.127160   3.529 0.000461
bmi:ldl     -0.015352   0.004716  -3.255 0.001221

Residual standard error: 12.54 on 438 degrees of freedom
Multiple R-squared:  0.1834,Adjusted R-squared:  0.1778
F-statistic: 32.78 on 3 and 438 DF,  p-value: < 2.2e-16
```

Visualizing the Interaction

We can get a better idea of the patterns of moderation by plotting the focal effect at conditional values of the moderator.



Categorical Moderators

Categorical moderators encode *group-specific* effects.

- E.g., if we include *sex* as a moderator, we are modeling separate focal effects for males and females.

Given a set of codes representing our moderator, we specify the interactions as before:

$$Y_{total} = \beta_0 + \beta_1 X_{inten} + \beta_2 Z_{male} + \beta_3 X_{inten} Z_{male} + \varepsilon$$

$$\begin{aligned} Y_{total} = \beta_0 &+ \beta_1 X_{inten} + \beta_2 Z_{lo} + \beta_3 Z_{mid} + \beta_4 Z_{hi} \\ &+ \beta_5 X_{inten} Z_{lo} + \beta_6 X_{inten} Z_{mid} + \beta_7 X_{inten} Z_{hi} + \varepsilon \end{aligned}$$

Example

```r
## Load data:
socSup <- readRDS(paste0(dataDir, "social_support.rds"))

## Focal effect:
out3 <- lm(bdi ~ tanSat, data = socSup)
partSummary(out3, -c(1, 2))
```

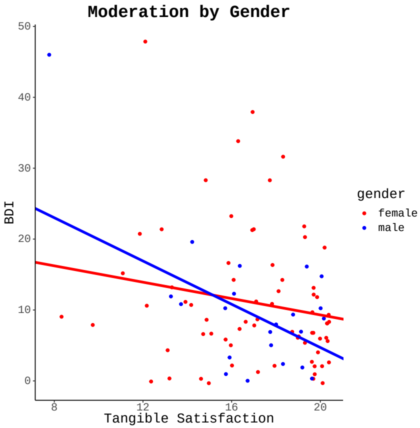
```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  24.4089     5.3502   4.562 1.54e-05
tanSat       -0.8100     0.3124  -2.593   0.0111

Residual standard error: 9.278 on 93 degrees of freedom
Multiple R-squared:  0.06742,Adjusted R-squared:  0.05739
F-statistic: 6.723 on 1 and 93 DF,  p-value: 0.01105
```

Example

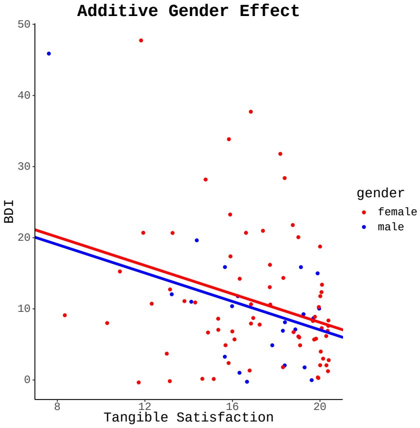
```
## Estimate the interaction:
out4 <- lm(bdi ~ tanSat * sex, data = socSup)
partSummary(out4, -c(1, 2))

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)     20.8478     6.2114   3.356  0.00115
tanSat          -0.5772     0.3614  -1.597  0.11372
sexmale         14.3667    12.2054   1.177  0.24223
tanSat:sexmale  -0.9482     0.7177  -1.321  0.18978

Residual standard error: 9.267 on 91 degrees of freedom
Multiple R-squared:  0.08955,Adjusted R-squared:  0.05954
F-statistic: 2.984 on 3 and 91 DF,  p-value: 0.03537
```

Visualizing Categorical Moderation

$$\hat{Y}_{BDI} = 20.85 - 0.58X_{tsat} + 14.37Z_{male} - 0.95X_{tsat}Z_{male}$$



$$\hat{Y}_{BDI} = 28.10 - 1.00X_{tsat} - 1.05Z_{male}$$



Polynomial Regression

Polynomial Regression

Polynomial regression simply incorporates powered transformations of the predictors into the model.

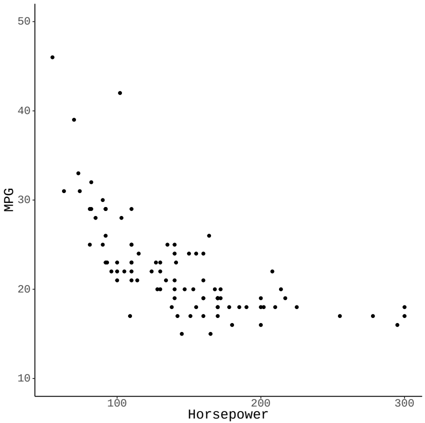
- Polynomial terms (i.e., power terms) model curvature in the relationships.

We can think about polynomial terms as interactions between a predictor and itself.

- Many of the rules that apply to interactions transfer directly to polynomials.

Polynomial Visualization

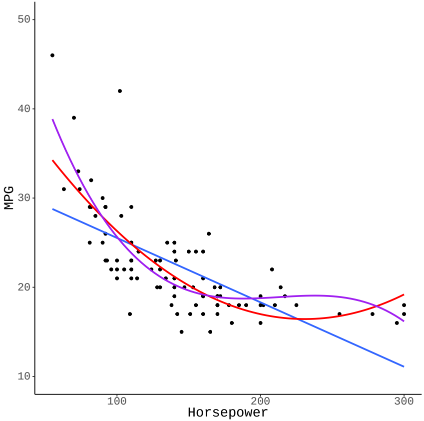
We may hypothesize a curvilinear relationship between X and Y.



Polynomial Visualization

Polynomials are one way to model curvilinear relationships.

- $\hat{Y}_{mpg} = \hat{\beta}_0 + \hat{\beta}_1 X_{hp}$
- $\hat{Y}_{mpg} = \hat{\beta}_0 + \hat{\beta}_1 X_{hp} + \hat{\beta}_2 X_{hp}^2$
- $\hat{Y}_{mpg} = \hat{\beta}_0 + \hat{\beta}_1 X_{hp} + \hat{\beta}_2 X_{hp}^2 + \hat{\beta}_3 X_{hp}^3$



Example

```r
## Attach the data:
data(Cars93)

## Fit the linear model:
out6 <- lm(MPG.city ~ Horsepower, data = Cars93)

## Fit the quadratic model:
out7 <- lm(MPG.city ~ Horsepower + I(Horsepower^2),
           data = Cars93)
```

Example

```
partSummary(out6, -c(1, 2))

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 32.746279   1.273229  25.719  < 2e-16
Horsepower  -0.072174   0.008323  -8.671 1.54e-13

Residual standard error: 4.181 on 91 degrees of freedom
Multiple R-squared:  0.4524,Adjusted R-squared:  0.4464
F-statistic: 75.19 on 1 and 91 DF,  p-value: 1.537e-13
```

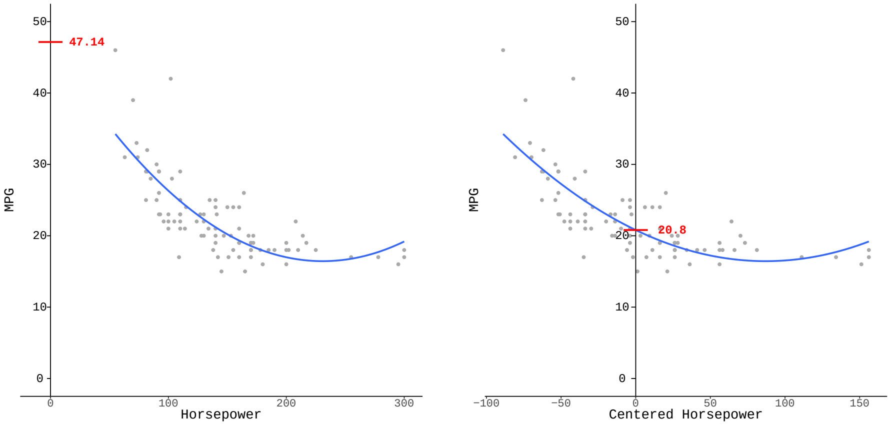
- For each unit increase in horsepower, the expected change in fuel economy is $\hat{\beta}_1 = -0.0722$ units.

Example

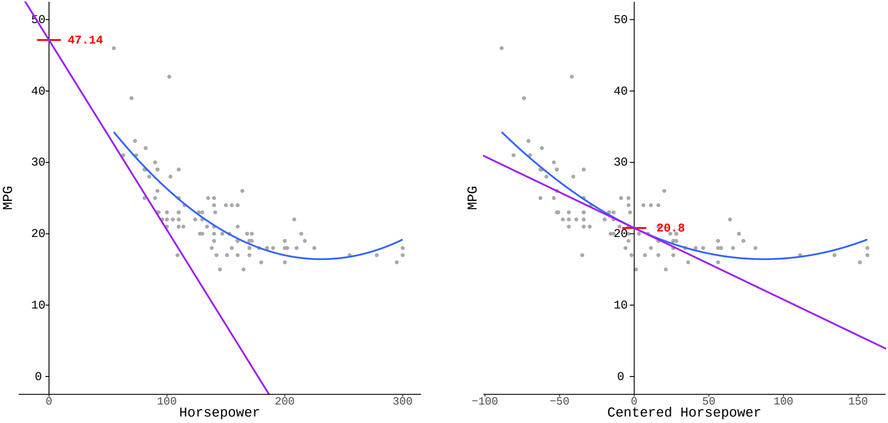
```
partSummary(out7, -c(1, 2))

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)       4.714e+01  2.544e+00  18.528  < 2e-16
Horsepower       -2.660e-01  3.186e-02  -8.350 7.71e-13
I(Horsepower^2)   5.762e-04  9.239e-05   6.237 1.42e-08

Residual standard error: 3.513 on 90 degrees of freedom
Multiple R-squared:  0.6177,Adjusted R-squared:  0.6092
F-statistic:  72.7 on 2 and 90 DF,  p-value: < 2.2e-16
```

- Extrapolating from powerless cars, each unit increase in horsepower, is expected to change fuel economy by $\hat{\beta}_1 = -0.266$ units.
- For a unit increase in horsepower, the effect of horsepower on fuel economy is expected to increase by $\hat{\beta}_2 = 5.76 \times 10^{-4}$ units.

Effects of Centering



Effects of Centering



Example

```r
## Mean center horsepower:
Cars93 <- mutate(Cars93, HorsepowerMC = Horsepower - mean(Horsepower))

## Fit the quadratic model:
out8 <- lm(MPG.city ~ HorsepowerMC + I(HorsepowerMC^2), data = Cars93)
```

Example

```
partSummary(out8, -c(1, 2))

Coefficients:
                     Estimate Std. Error t value Pr(>|t|)
(Intercept)         2.080e+01  4.422e-01  47.038  < 2e-16
HorsepowerMC       -1.003e-01  8.320e-03 -12.053  < 2e-16
I(HorsepowerMC^2)   5.762e-04  9.239e-05   6.237 1.42e-08

Residual standard error: 3.513 on 90 degrees of freedom
Multiple R-squared:  0.6177,Adjusted R-squared:  0.6092
F-statistic:  72.7 on 2 and 90 DF,  p-value: < 2.2e-16
```

- Averaging over cars, each unit increase in horsepower, is expected to change fuel economy by $\hat{\beta}_1 = -0.1003$ units.
- For a unit increase in horsepower, the effect of horsepower on fuel economy is expected to increase by $\hat{\beta}_2 = 5.76 \times 10^{-4}$ units.

References

Hayes, A. F. (2017). *Introduction to mediation, moderation, and conditional process analysis: A regression-based approach*. New York: Guilford Press.