Lower Bounds for Possibly Divergent Probabilistic Programs

Shenghua Feng, **Mingshuai Chen**, Han Su, Benjamin L. Kaminski,
Joost-Pieter Katoen, Naijun Zhan

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A Fun Fact

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— Shizuo Kakutani

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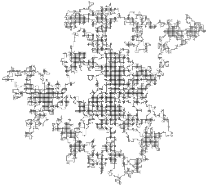

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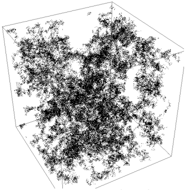

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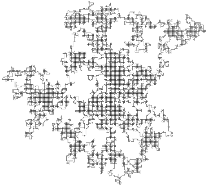

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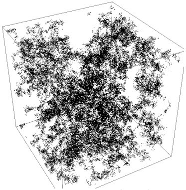

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Question : How to compute *sound approx.* of the returning probability of the bird?

Probabilistic Programs

$$C_{\mathsf{brw}}: \quad \texttt{while}\,(\,n > 0\,)\,\{\,n \coloneqq n - 1\ [1/3]\ n \coloneqq n + 1\,\}$$

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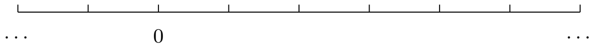

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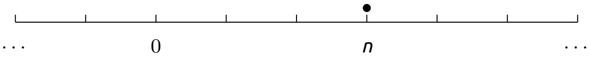

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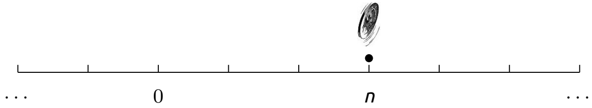

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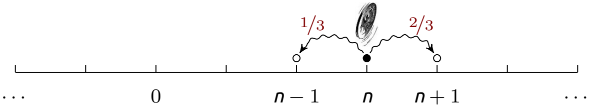

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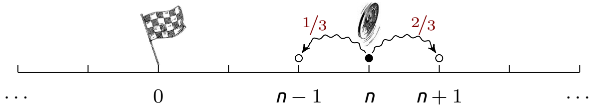

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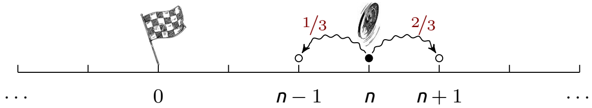
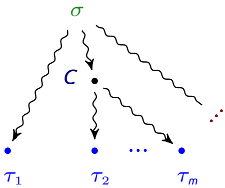


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]



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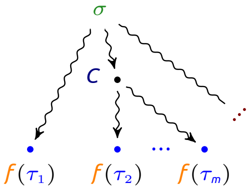

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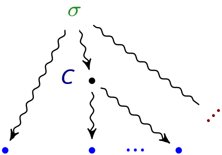

$$\mathrm{wp}[\![C]\!](f)(\sigma) \triangleq \mathrm{Exp}\Big[\, f(\tau_1) \qquad f(\tau_2) \qquad f(\tau_m) \,\Big]$$

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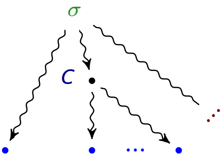

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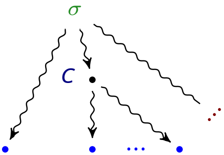

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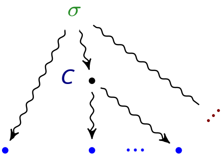

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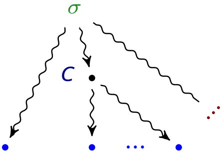

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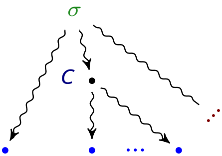

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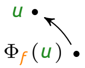

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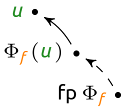

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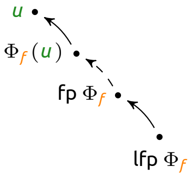

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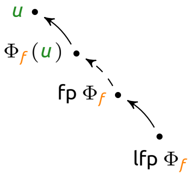

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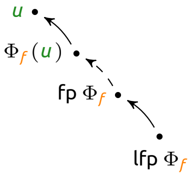

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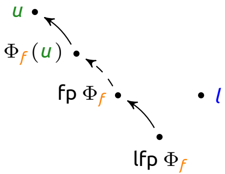

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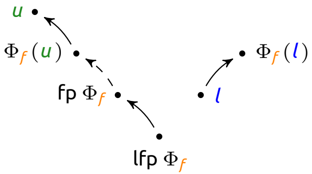

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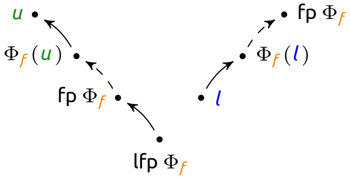

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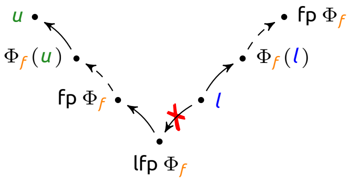

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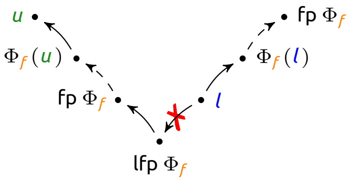

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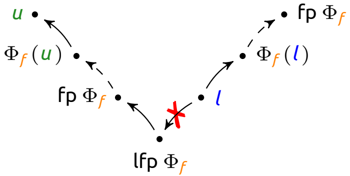
almost-sure termination (AST)
bounded expectations
...



A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

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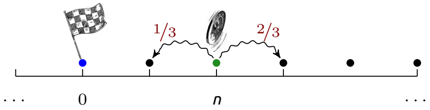

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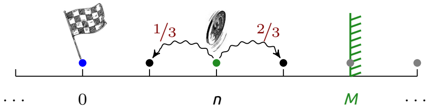

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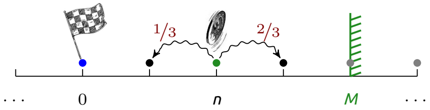

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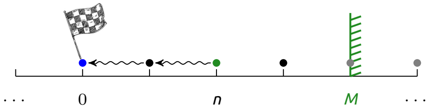

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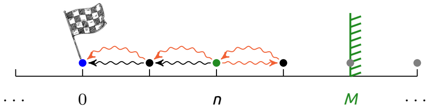
Theorem (Guard-Strengthening Rule)

$$C_{\text{loop}}: \quad \texttt{while}\,(\varphi)\,\{\,C\,\} \quad \rightsquigarrow \quad C'_{\text{loop}}: \quad \texttt{while}\,(\varphi')\,\{\,C\,\}$$

$$\frac{\varphi' \implies \varphi \qquad l \preceq \text{wp}[\![C'_{\text{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \text{wp}[\![C_{\text{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

$$C_{\text{brw}}: \quad \texttt{while}\,(0 < n)\,\{\; n := n-1\;[1/3]\; n := n+1\,\} \quad \rightsquigarrow \quad C^{M}_{\text{brw}}: \quad \texttt{while}\,(0 < n < M)\,\{\; n := n-1\;[1/3]\; n := n+1\,\}$$

$$\text{wp}[\![C^{M}_{\text{brw}}]\!]\,([n \leq 0]\cdot 1) \;\preceq\; \text{wp}[\![C_{\text{brw}}]\!]\,(1)$$



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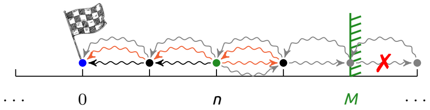

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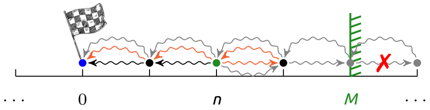

- C'_{loop} features a **stronger** termination property (e.g., becoming AST).
- **Easier** to verify the uni. int. of l and the boundedness of expectations.

Behind the Proof Rule

Theorem (wp-Difference)

$$
\begin{aligned}
\mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f) - \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,(f) \;=\;& \\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\neg\varphi \wedge \varphi'] \cdot f) + \lambda\sigma\mathbf{.}\int_A f_{C_{\mathsf{loop}}}\,\mathrm{d}\,({}^{\sigma}\mathbb{P}) - \\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\varphi \wedge \neg\varphi'] \cdot f) - \lambda\sigma\mathbf{.}\int_B f_{C'_{\mathsf{loop}}}\,\mathrm{d}\,({}^{\sigma}\mathbb{P})
\end{aligned}
$$

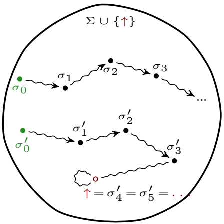

Figure – Infinite prog. traces.

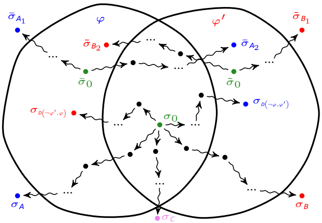

Figure – Illustration of wp-Difference.

- Potentially applicable to *sensitivity analysis* and *model repair*.

Properties of the Proof Rule

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

- (Trivially) **complete :** where there's an l, there's a φ' (albeit not "good" enough).

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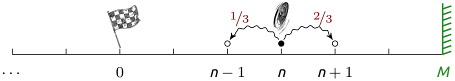
- **General :** applicable to *possibly divergent* C_{loop} and unbounded expectations f, l:



$$l_M \;=\; [n<0] + [0\leq n\leq M]\cdot\left(\left(1/2\right)^n - \left(1/2\right)^M\right)$$

$$\forall M\in\mathbb{N}:\quad l_M \overset{\text{Hark}}{\preceq} \mathrm{wp}[\![C^M_{\mathrm{brw}}]\!]\left([n\leq 0]\cdot 1\right) \overset{\text{Stren.}}{\preceq} \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\left(1\right)$$

Properties of the Proof Rule

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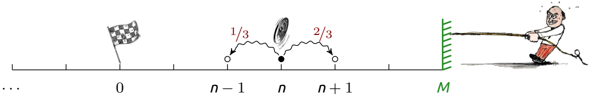

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- **Tight :** the underapproximation error approaches 0 as $\varphi' \to \varphi$:

$$[n < 0] + [n \geq 0] \cdot (1/2)^n \;=\; \lim_{M\to\infty} l_M \;\preceq\; \mathsf{wp}[\![C_{\mathsf{brw}}]\!]\,(1)$$

Properties of the Proof Rule

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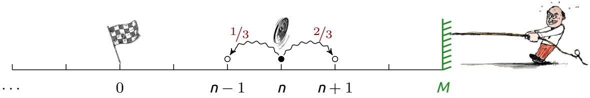

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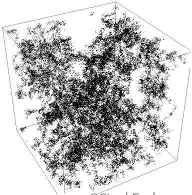
- **Automatable :** reducible to *probabilistic model checking* for finite-state C'_{loop} :

$\texttt{while}\,(\,x \neq 0 \vee y \neq 0 \vee z \neq 0\,)\,\{$

$\quad x \coloneqq x-1 \,\oplus\, x \coloneqq x+1 \,\oplus\, y \coloneqq y-1 \,\oplus\, y \coloneqq y+1 \,\oplus\, z \coloneqq z-1 \,\oplus\, z \coloneqq z+1\,\}$



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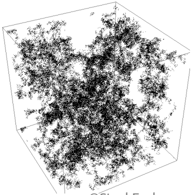

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$$\mathcal{P} = 1 - \left(\frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x-\cos y-\cos z}\right)^{-1} = 0.3405373296\ldots$$

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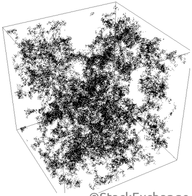

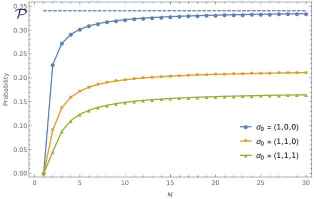

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A “Real” Application : Zeroconf Protocol [Bohnenkamp et al. 2003]

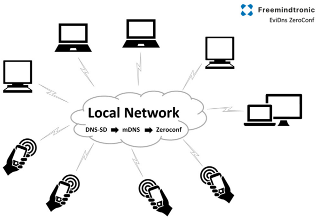

Figure – Self-configuring IP network.

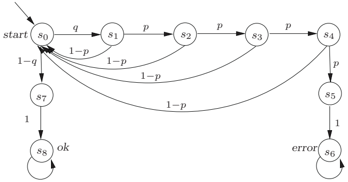

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Figure – Markov-chain snippet ($N = 4$).

C_{zc} :

$$
\begin{array}{l}
\mathit{start} = 1 \,⨟\, \mathit{established} = 0 \,⨟\, \mathit{probe} = 0 \,⨟ \\
\texttt{while}\ (\, \mathit{start} \leq 1 \wedge \mathit{established} \leq 0 \wedge \mathit{probe} < N \wedge N \geq 4 \,)\ \{ \\
\quad \texttt{if}\ (\, \mathit{start} = 1 \,)\ \{ \\
\qquad \{\, \mathit{start} := 0 \,\}\ [0.5]\ \{\, \mathit{start} := 0 \,⨟\, \mathit{established} := 1 \,\}\ \} \\
\quad \texttt{else}\ \{\ \{\, \mathit{probe} := \mathit{probe} + 1 \,\}\ [0.001]\ \{\, \mathit{start} := 1 \,⨟\, \mathit{probe} := 0 \,\}\ \}\ \}
\end{array}
$$

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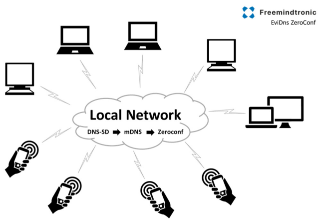

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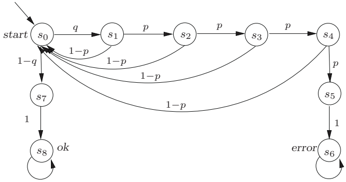

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Figure – Markov-chain snippet ($N = 4$).

C_{zc} : $\quad$ *start* $= 1 \fatsemi$ *established* $= 0 \fatsemi$ *probe* $= 0 \fatsemi$

while (*start* $\leq 1 \wedge$ *established* $\leq 0 \wedge$ *probe* $< N \wedge N \geq 4$) {

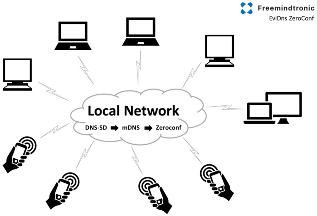
if (*start* $= 1$) {

{ *start* $:= 0$ } $[0.5]$ { *start* $:= 0 \fatsemi$ *established* $:= 1$ } }

else { { *probe* $:=$ *probe* $+ 1$ } $[0.001]$ { *start* $:= 1 \fatsemi$ *probe* $:= 0$ } } }

Pr("starting within the loop guard, C_{zc} terminates with *established* $= 1$") ≥ 0.99999999999

A “Real” Application : Zeroconf Protocol [Bohnenkamp et al. 2003]



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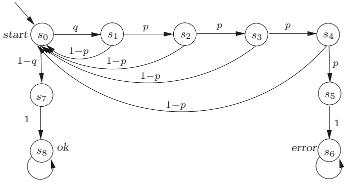

Figure – Markov-chain snippet ($N = 4$).

C_{zc} :
$$\begin{array}{l} \mathit{start} = 1 ⨾ \mathit{established} = 0 ⨾ \mathit{probe} = 0 ⨾ \\ \texttt{while}\ (\ \mathit{start} \leq 1 \wedge \mathit{established} \leq 0 \wedge \mathit{probe} < N \wedge N \geq 4\ )\ \{ \\ \quad \texttt{if}\ (\ \mathit{start} = 1\ )\ \{ \\ \quad\quad \{\ \mathit{start} := 0\ \}\ [0.5]\ \{\ \mathit{start} := 0 ⨾ \mathit{established} := 1\ \}\ \} \\ \quad \texttt{else}\ \{\ \{\ \mathit{probe} := \mathit{probe} + 1\ \}\ [0.001]\ \{\ \mathit{start} := 1 ⨾ \mathit{probe} := 0\ \}\ \}\ \} \end{array}$$

Pr(“starting within the loop guard, C_{zc} terminates with *established* $= 1$”) $\overset{?}{\geq} 0.99999999999$

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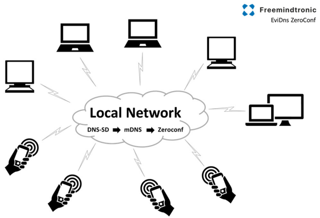

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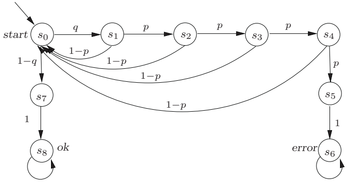

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Figure – Markov-chain snippet ($N = 4$).

C_{zc} : $\quad$ *start* $= 1 \,⨟\,$ *established* $= 0 \,⨟\,$ *probe* $= 0 \,⨟$

while (*start* $\leq 1 \wedge$ *established* $\leq 0 \wedge$ *probe* $< N \wedge N \geq 4 \wedge N \leq 10$) {

if (*start* $= 1$) {

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Pr("starting within the loop guard, C_{zc} terminates with *established* $= 1$") $\overset{\checkmark}{\geq} 0.99999999999$

Summary

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

- a new lower bound rule based on wp-difference and guard-strengthening;

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan : *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA '23.

Summary

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More in the paper :

- how to find a "good" strengthening $\varphi' \implies \varphi$?
- how to generate a non-trivial lower bound for C'_{loop}?
- corner cases where guard strengthening is insufficient;
- ...

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan : *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA '23.