Lower Bounds for Possibly Divergent Probabilistic Programs

Shenghua Feng, **Mingshuai Chen**, Han Su, Benjamin L. Kaminski,
Joost-Pieter Katoen, Naijun Zhan

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A Fun Fact

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— Shizuo Kakutani

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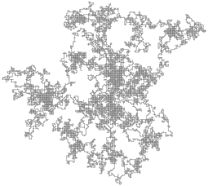

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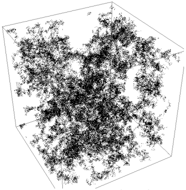

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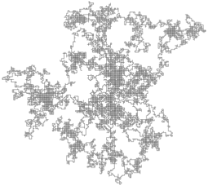

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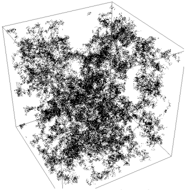

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Question : How to compute *sound approx.* of the returning probability of the bird ?

Probabilistic Programs

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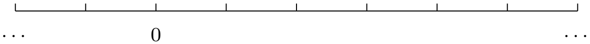

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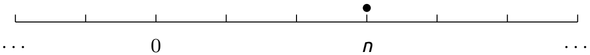

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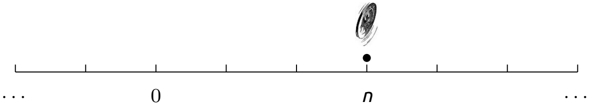

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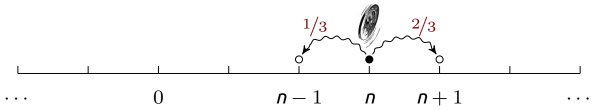

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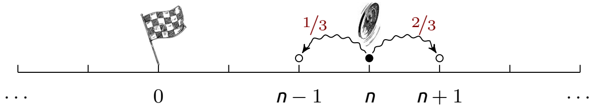

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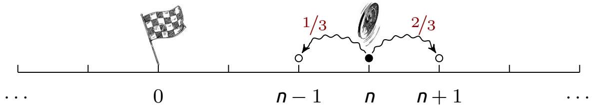
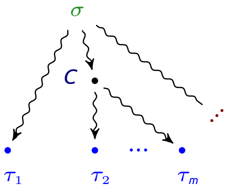


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]



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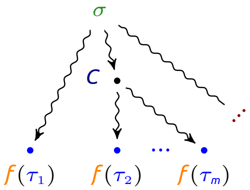

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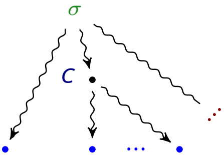

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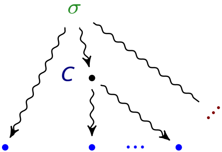

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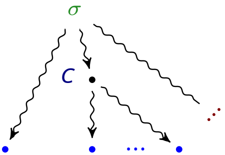

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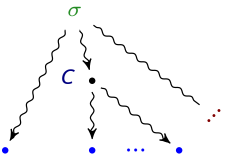

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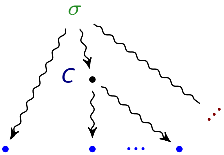

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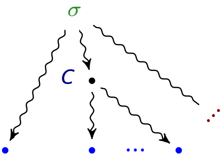

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$$l \;\preceq\; \mathsf{lfp}\,\Phi_f \;\preceq\; u$$

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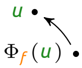

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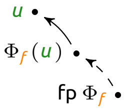

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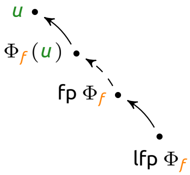

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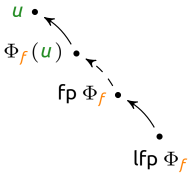

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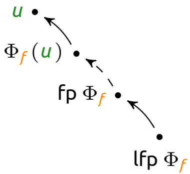

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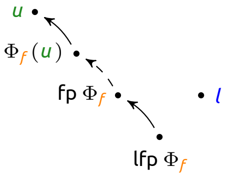

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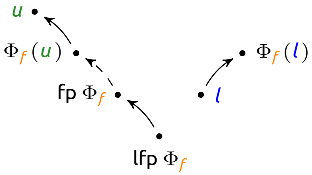

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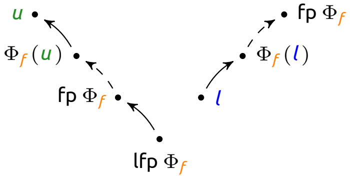

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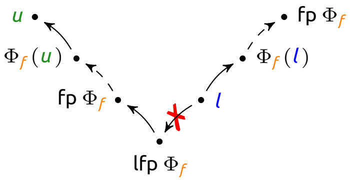

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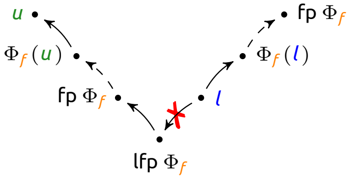

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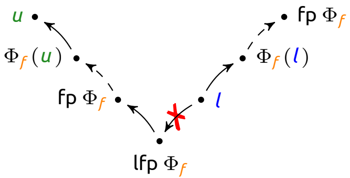
almost-sure termination (AST)
bounded expectations
...



A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

$$C_{\text{loop}} : \quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \qquad \rightsquigarrow \qquad C'_{\text{loop}} : \quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

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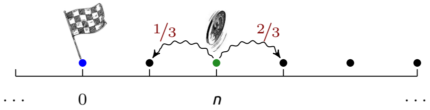

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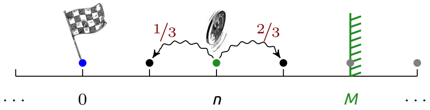

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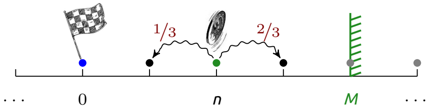

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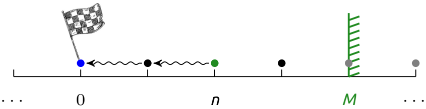

A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

$$C_{\text{loop}} : \quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \quad \rightsquigarrow \quad C'_{\text{loop}} : \quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

$$\frac{\varphi' \implies \varphi \qquad l \preceq \text{wp}[\![C'_{\text{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \text{wp}[\![C_{\text{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

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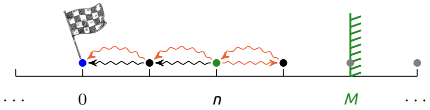

A New Proof Rule for Lower Bounds

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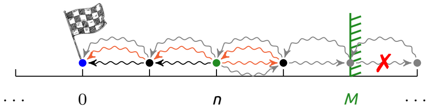

A New Proof Rule for Lower Bounds

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$$C_{\text{loop}}: \quad \texttt{while}\,(\varphi)\,\{\,C\,\} \quad \rightsquigarrow \quad C'_{\text{loop}}: \quad \texttt{while}\,(\varphi')\,\{\,C\,\}$$

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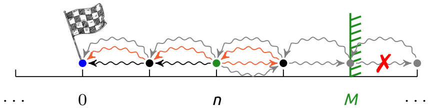

- C'_{loop} features a **stronger** termination property (e.g., becoming AST).
- **Easier** to verify the uni. int. of l and the boundedness of expectations.

Behind the Proof Rule

Theorem (wp-Difference)

$$
\begin{aligned}
\mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f) - \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,(f) \;=\;& \\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\neg\varphi \wedge \varphi'] \cdot f) + \lambda\sigma\text{.}\; \int_A f_{C_{\mathsf{loop}}}\; \mathrm{d}\,({}^{\sigma}\mathbb{P}) - \\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\varphi \wedge \neg\varphi'] \cdot f) - \lambda\sigma\text{.}\; \int_B f_{C'_{\mathsf{loop}}}\; \mathrm{d}\,({}^{\sigma}\mathbb{P})
\end{aligned}
$$

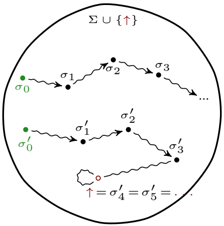

Figure – Infinite prog. traces.

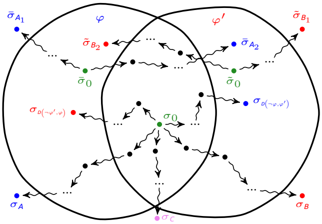

Figure – Illustration of wp-Difference.

- Potentially applicable to *sensitivity analysis* and *model repair*.

Properties of the Proof Rule

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\left([\neg\varphi] \cdot f\right)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\left(f\right)} \quad \text{(Guard-Strengthening)}$$

- (Trivially) **complete :** where there's an l, there's a φ' (albeit not "good" enough).

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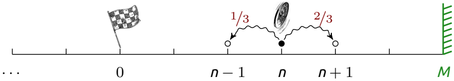
- **General :** applicable to *possibly divergent* C_{loop} and unbounded expectations f, l:



$$l_M \;=\; [n<0] + [0 \le n \le M]\cdot\left(\left(1/2\right)^n - \left(1/2\right)^M\right)$$

$$\forall M \in \mathbb{N}\colon \quad l_M \overset{\text{Hark}}{\preceq} \mathrm{wp}[\![C^M_{\mathrm{brw}}]\!]\left([n \le 0]\cdot 1\right) \overset{\text{Stren.}}{\preceq} \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\left(1\right)$$

Properties of the Proof Rule

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

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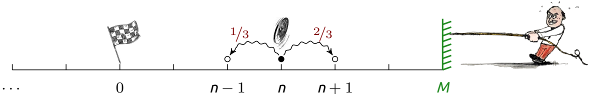

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- **Tight :** the underapproximation error approaches 0 as $\varphi' \to \varphi$:

$$[n<0] + [n \geq 0]\cdot(1/2)^n \;=\; \lim_{M\to\infty} l_M \;\preceq\; \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\,(1)$$

Properties of the Proof Rule

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

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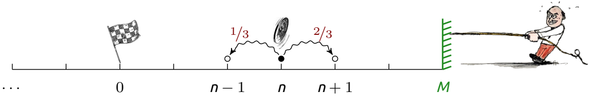

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Properties of the Proof Rule

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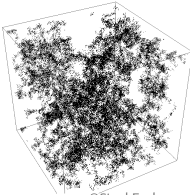
- **Automatable :** reducible to *probabilistic model checking* for finite-state C'_{loop} :

$$\texttt{while}\,(\, x \neq 0 \vee y \neq 0 \vee z \neq 0\,)\,\{$$
$$x \coloneqq x-1 \,\oplus\, x \coloneqq x+1 \,\oplus\, y \coloneqq y-1 \,\oplus\, y \coloneqq y+1 \,\oplus\, z \coloneqq z-1 \,\oplus\, z \coloneqq z+1\,\}$$



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Properties of the Proof Rule

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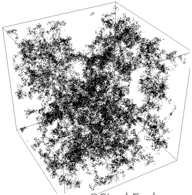

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$$\mathcal{P} = 1 - \left(\frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x-\cos y-\cos z}\right)^{-1} = 0.3405373296\ldots$$

Properties of the Proof Rule

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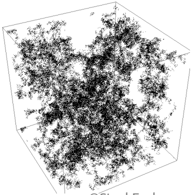

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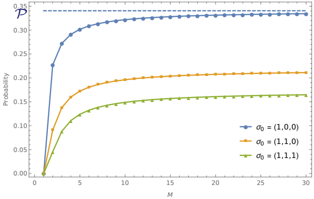

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A “Real” Application : Zeroconf Protocol [Bohnenkamp et al. 2003]

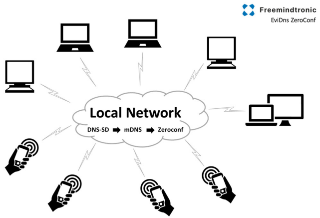

Figure – Self-configuring IP network.

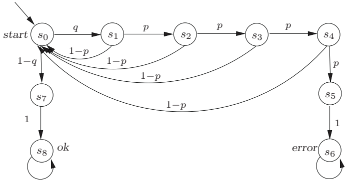

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Figure – Markov-chain snippet ($N = 4$).

$$
\begin{aligned}
C_{zc}: \quad & \mathit{start} = 1 \,;\, \mathit{established} = 0 \,;\, \mathit{probe} = 0 \,; \\
& \texttt{while}\ (\, \mathit{start} \leq 1 \wedge \mathit{established} \leq 0 \wedge \mathit{probe} < N \wedge N \geq 4 \,)\ \{ \\
& \quad \texttt{if}\ (\, \mathit{start} = 1 \,)\ \{ \\
& \qquad \{\, \mathit{start} := 0 \,\}\ [0.5]\ \{\, \mathit{start} := 0 \,;\, \mathit{established} := 1 \,\}\ \} \\
& \quad \texttt{else}\ \{\ \{\, \mathit{probe} := \mathit{probe} + 1 \,\}\ [0.001]\ \{\, \mathit{start} := 1 \,;\, \mathit{probe} := 0 \,\}\ \}\ \}
\end{aligned}
$$

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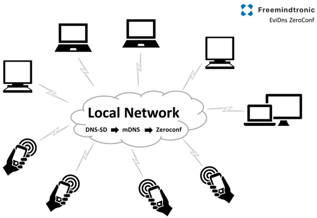

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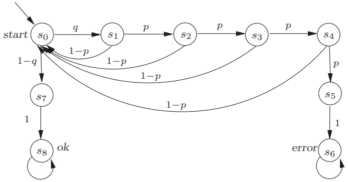


Figure – Markov-chain snippet ($N = 4$).

C_{zc} : $\quad$ $\mathit{start} = 1 \fatsemi \mathit{established} = 0 \fatsemi \mathit{probe} = 0 \fatsemi$

while ($\mathit{start} \leq 1 \wedge \mathit{established} \leq 0 \wedge \mathit{probe} < N \wedge N \geq 4$) {

if ($\mathit{start} = 1$) {

$\{\, \mathit{start} := 0 \,\}\ [0.5]\ \{\, \mathit{start} := 0 \fatsemi \mathit{established} := 1 \,\}\ \}$

else $\{\ \{\, \mathit{probe} := \mathit{probe} + 1 \,\}\ [0.001]\ \{\, \mathit{start} := 1 \fatsemi \mathit{probe} := 0 \,\}\ \}\ \}$

Pr(“starting within the loop guard, C_{zc} terminates with $\mathit{established} = 1$”) ≥ 0.99999999999

A "Real" Application : Zeroconf Protocol [Bohnenkamp et al. 2003]

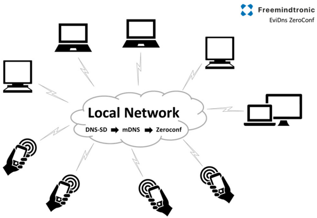
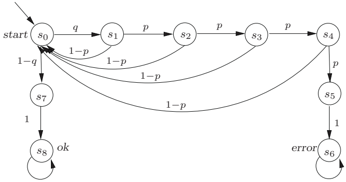


Figure – Self-configuring IP network.



©[Baier & Katoen 2008]

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$$
\begin{aligned}
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& \texttt{while}\,(\, \mathit{start} \leq 1 \wedge \mathit{established} \leq 0 \wedge \mathit{probe} < N \wedge N \geq 4 \,)\,\{ \\
& \quad \texttt{if}\,(\, \mathit{start} = 1 \,)\,\{ \\
& \quad\quad \{\, \mathit{start} := 0 \,\}\,[0.5]\,\{\, \mathit{start} := 0 \fatsemi \mathit{established} := 1 \,\}\,\} \\
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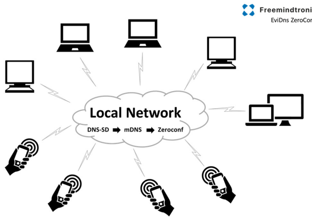
\end{aligned}
$$

$$\Pr(\text{"starting within the loop guard, } C_{zc} \text{ terminates with } \mathit{established} = 1\text{"}) \overset{?}{\geq} 0.99999999999$$

A "Real" Application : Zeroconf Protocol [Bohnenkamp et al. 2003]



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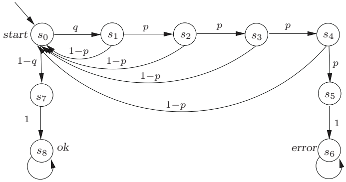

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$$
\begin{aligned}
C_{zc}: \quad & start = 1 \fatsemi established = 0 \fatsemi probe = 0 \fatsemi \\
& \texttt{while}\ (\, start \leq 1 \wedge established \leq 0 \wedge probe < N \wedge N \geq 4 \wedge N \leq 10\,)\ \{ \\
& \quad \texttt{if}\ (\, start = 1\,)\ \{ \\
& \qquad \{\, start := 0 \,\}\ [0.5]\ \{\, start := 0 \fatsemi established := 1 \,\}\ \} \\
& \quad \texttt{else}\ \{\ \{\, probe := probe + 1 \,\}\ [0.001]\ \{\, start := 1 \fatsemi probe := 0 \,\}\ \}\ \}
\end{aligned}
$$

Pr("starting within the loop guard, C_{zc} terminates with $established = 1$") $\overset{\checkmark}{\geq} 0.99999999999$

Summary

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

- a new lower bound rule based on wp-difference and guard-strengthening;

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan : *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA '23.

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- a new lower bound rule based on wp-difference and guard-strengthening;
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- tight lower bounds for 3-D random walks on $\mathbb{Z}^3$ and the Zeroconf protocol.

More in the paper:

- how to find a "good" strengthening $\varphi' \implies \varphi$?
- how to generate a non-trivial lower bound for C'_{loop}?
- corner cases where guard strengthening is insufficient;
- ...

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan: *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA '23.