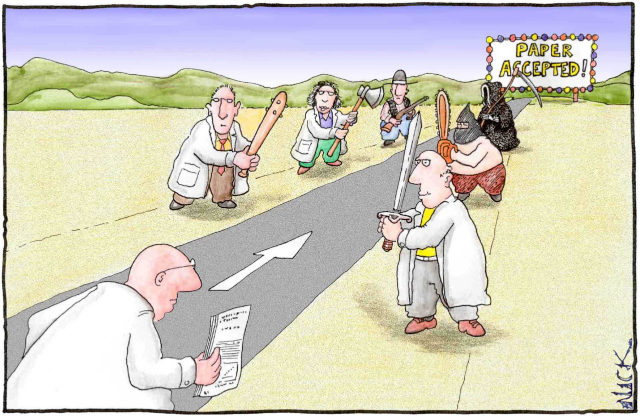
Lower Bounds for Possibly Divergent Probabilistic Programs

Mingshuai Chen

—Joint work with S. Feng, B. L. Kaminski, J.-P. Katoen, H. Su, and N. Zhan—

ROCKS · Nijmegen · May 2022

Work in Progress …



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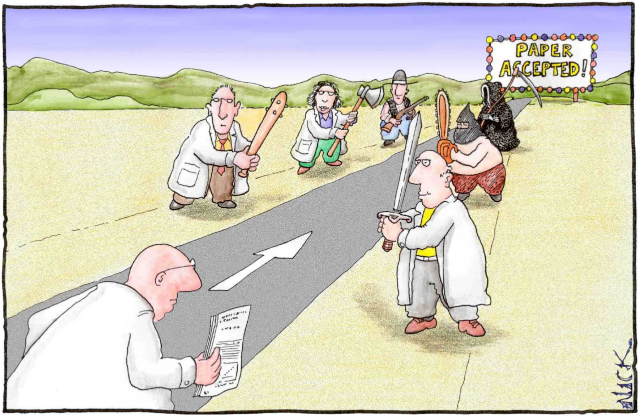

Your Program Diverges? Follow My Rule!

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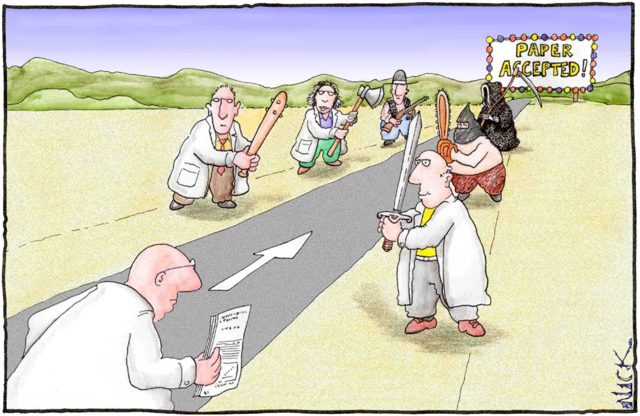

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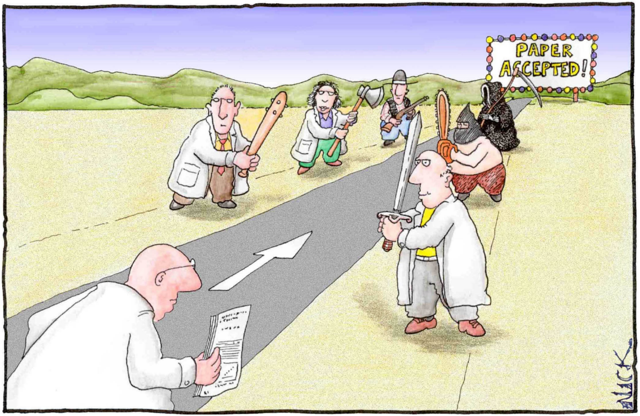

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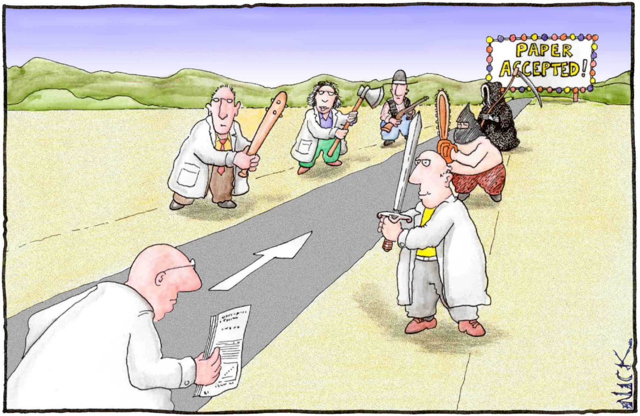

Lower Bounds for Possibly Divergent Probabilistic Programs

Probabilistic Programs

C_{brw}: $\quad \texttt{while}\,(\,n > 0\,)\,\{\,n \coloneqq n - 1\ [0.3]\ n \coloneqq n + 1\,\}$

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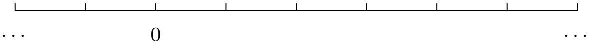

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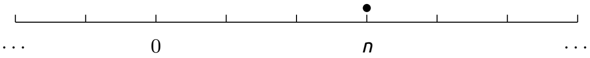

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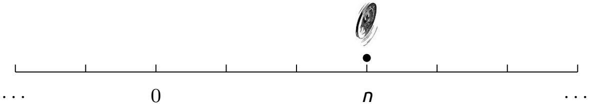

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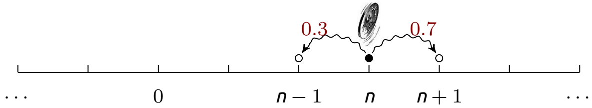

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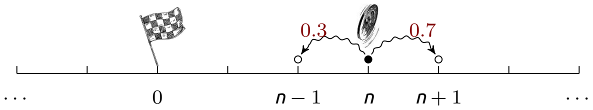

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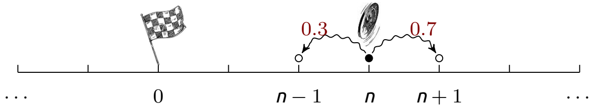
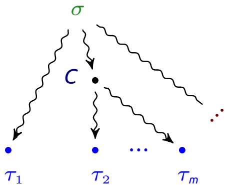


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]



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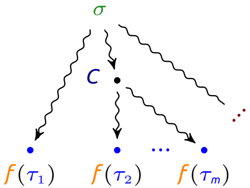
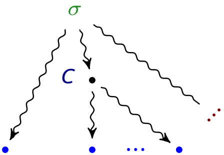
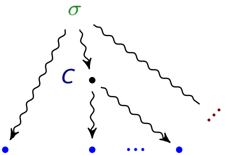




$$\mathsf{wp}[\![C]\!](f)(\sigma) \triangleq \mathsf{Exp}\Big[f(\tau_1) \quad f(\tau_2) \quad \cdots \quad f(\tau_m)\Big]$$



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$$\mathsf{wp}[\![\mathtt{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(f) \;=\; \mathsf{lfp}\;\Phi_f \;=\; ?$$

Bounding the Least Fixed Point

$$l \;\preceq\; \mathsf{lfp}\,\Phi_f \;\preceq\; u$$

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$$\Phi_f(u) \;\preceq\; u \quad \text{implies} \quad \mathsf{lfp}\,\Phi_f \;\preceq\; u\,.$$

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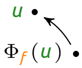

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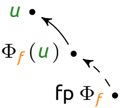

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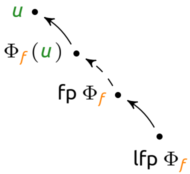

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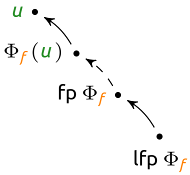

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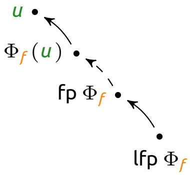

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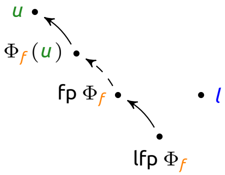

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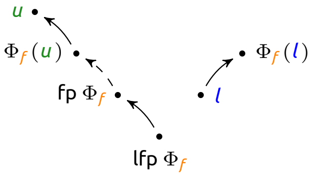

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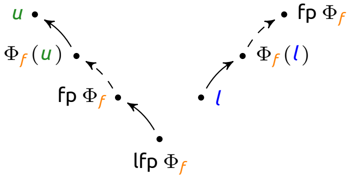

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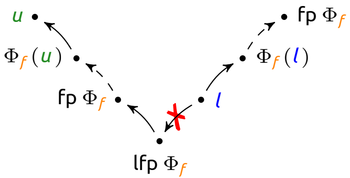

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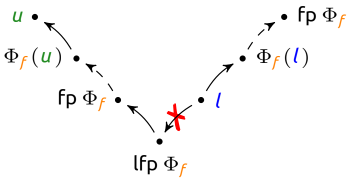

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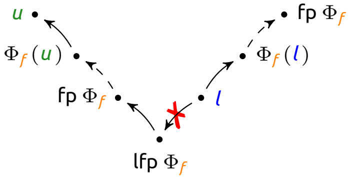
almost-sure termination
bounded expectations
...



A New Proof Rule for Lower Bounds

$$\mathit{Loop}\colon \quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \qquad \rightsquigarrow \qquad \mathit{Loop}'\colon \quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

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- Applicable to *possibly divergent Loop*.
- l can be *arbitrarily tight*.
- Reducible to *probabilistic BMC*.
- Easier to ensure *uni. int.* for Loop'.
- ...

Example : Biased Random Walk

C_{brw}: $\texttt{while}\,(0 < n)\,\{$
$\quad n := n - 1\;[0.3]\; n := n + 1$
$\}$

C_{brw}^{M}: $\texttt{while}\,(0 < n < M)\,\{$
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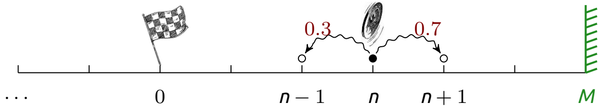

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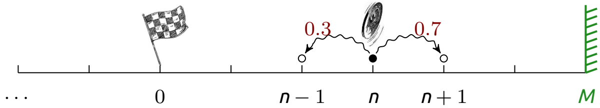

$$l_M(n) \quad = \quad (3/7)^{n} - (3/7)^{M} \;.$$

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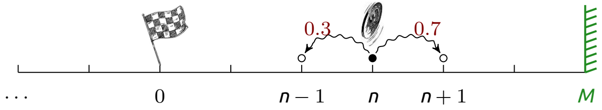

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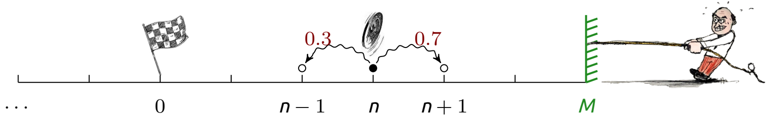

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$$(3/7)^n \;=\; \lim_{M \to \infty} l_M \;\preceq\; \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\,(1)\ .$$

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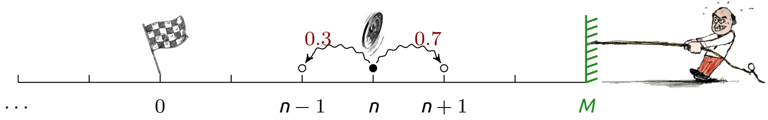

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