Harvard University
Physics 141 Pset 1

Due September 12, 2025

1. (From *Physical Biology of the Cell*) Consider 8 particles, four of which are black and four of which are white. Four particles can fit left of a permeable membrane and four can fit to the right of the membrane. Imagine that due to random motion of the particles, every arrangement of the eight particles is equally likely. Some possible arrangements are given by $BBBB|WWWW$, $BBWW|BBWW$, and $BWBW|BWBW$ where | represents the membrane.

   (a) How many different arrangements are there? Hint: the idea of a binomial coefficient may be useful here.

   (b) Calculate the probability that all four black particles are on the left side of the membrane. Next, calculate the probability that exactly three black particles and one white particle are on the left side. Then calculate the probability that exactly two black and two white particles are on the left side. Which outcome is most likely?

2. Let X be a continuous random variable, and let $f(x)$ be a probability density function, with $x \in X$. When defined, the mean $\mu \equiv \langle X \rangle$ is given by

$$\mu = \int x f(x)\, dx$$

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   (a) Show that the variance $\sigma^2 \equiv \langle (X - \langle x \rangle)^2 \rangle$ can be written as

$$\sigma^2 = \left\langle X^2 \right\rangle - \left\langle X \right\rangle^2$$

   (b) Recall from class that the probability of a thermal fluctuation with energy E_c is exponentially distributed according to the Boltzmann distribution

$$P(E_c) \propto e^{-E_c/k_B T}$$

   Let us write $\lambda = 1/k_B T$ such that $P(E_c) \propto e^{-\lambda E_c}$. We will first determine the coefficient that normalizes this distribution. Assume $P(E_c) = K e^{-E_c/k_B T}$ for some coefficient K. Find K using the fact that the integral of $P(E_c)$ over all positive values of energy is equal to 1.

(c) Now that you have found $P(E_c)$, calculate its mean and variance in terms of λ.

3. Consider a gas (ex: O_2) in Earth's atmosphere. Assume that these gas molecules are in thermal equilibrium near 300K (note that temperature has no dependence on height). This is a somewhat poor assumption, but correct to within an order of magnitude. Let $n(h)$ denote the number density of these gas molecules as a function of height above sea level h, and let n_0 denote the number density of the gas molecules at sea level.

   (a) Assuming a Bolztmann distribution, write an equation for the relative density of this gas molecule $n(h)/n_0$.

   (b) Using realistic values, how much less O_2 gas should you expect at the top of Mount Everest relative to sea level? Google the real answer and compare.

4. As mentioned in class, for any sensory system to work it must be filtered to separate true signals from noise. The rhodopsin molecule, for example, must distinguish energy from a photon from that of random thermal fluctuations. Let us model both the baseline energy distribution and the energy distribution from the signal as two gaussians:

$$\text{baseline} = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$
$$\text{signal} = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2}$$

   where $\mu_2 > \mu_1$ and $\sigma_1, \sigma_2 > 0$. Prove that the optimal cutoff c such that the false negative rate (probability of the signal being incorrectly misidentified) plus the false positive rate (probability of noise being identified as signal) is minimized at the intersection of the two gaussians. Solve for an exact value of c; your answer will not be pretty, but it should only take a few lines of work.

5. As mentioned in lecture, John Hopfield's 1982 paper laid the foundation for much of the modern field of artificial neural networks. In the following problem, we will walk you through the (quite simple!) linear algebra that makes his network so useful.

   First, read the first part of the paper (until the **information storage algorithm** section) to see how Hopfield defines his terms. The next parts of the paper might be helpful for the problem, but are not strictly necessary.

   In order to store a set of n stable states V^s, $s = 1 \ldots n$, Hopfield defines a matrix T as

$$T_{ij} = \sum_s (2V_i^s - 1)(2V_j^s - 1)$$

   except $T_{ii} = 0$. The update rule is given by $V_i = \sum_j T_{ij} V_j^s$

   (a) First, transform the problem into matrix and vector notation. Do not include any sums other than over s (i.e. $\sum_s$ is OK but $\sum_j$ is not) and do not use any subscripts. Rewrite T and the update rule in this way.

(b) Given a state $V^{s'}$, show that an update to $V^{s'}$, on average, adds $(2V^{s'} - 1)(N/2)$ (i.e. $\langle TV^{s'} \rangle = (2V^{s'} - 1)(N/2)$). Assume that V^s are identically independently distributed, and use only vector/matrix notation (no indexing!).

(c) Is $V^{s'}$ therefore stable? Explain why, using the previous result.

(d) Now, let us prove that the model will always tend towards a stable state. Define the "energy" of the system to be

$$E = -\frac{1}{2} \sum_{i \neq j} T_{ij} V_i V_j$$

Think of this energy as the potential energy of a real life physical system: it will always evolve towards the steepest gradient of potential energy (as in $F = -\nabla U$). Consider some perturbation ΔV_ℓ to the system, and show that the change in E due to that perturbation V_ℓ is

$$\Delta E = -\Delta V_\ell \sum_j T_{\ell j} V_j$$

Use the simplified case where $V_{ij} = V_{ji}$. It may help to stay in index notation.

(e) Use the previous result to argue that E is monotonically decreasing over time.