Problem Set #2

Physics/Neuro 141: Physics of Sensory Systems

October 3, 2023

Problem 1

Using realistic numbers, estimate the mean separation between rhodopsin molecules in the disks of the human rod cell. How does this distance compare with the diffraction limit of visible light?

Problem 2

Use binomial expansion to show that the exponential function can be written in terms of the following limit:

$$\lim_{N\to\infty}\left(1-\frac{x}{N}\right)^N = e^{-x}$$

You will have to use the fact that exp(x) is defined as the infinite sum $\sum_{k=0}^{\infty}\frac{x^k}{k!}$.

Problem 3

For a Poisson process where the expectation value of successes is $\langle k\rangle = \mu$, the probability distribution of any number of successes is:

$$P(k,\mu) = \frac{\mu^k}{k!}e^{-\mu}. \tag{1}$$

(a) Show that the variance is equal to the mean:

$$\left\langle k^2\right\rangle - \langle k\rangle^2 = \langle k\rangle$$

(b) Consider a rod cell with N=1000 disks. If 30% of the photons that pass through the rod are not absorbed, what is the absorption probability at any individual disk?

Problem 4 from Bialek's *Biophysics*

In lecture we discussed how optimizing the signal-to-noise ratio (SNR) in a rod cell gives the counterintuitive result that 30% of photons should pass through undetected. The SNR can be expressed as

$$\mathrm{SNR} = \frac{n_{\mathrm{flash}}(1-e^{-\sigma\rho L})}{\sqrt{\rho L A r_{\mathrm{dark}}\tau}},$$

where σ is the absorption cross section of rhodopsin, ρ is the concentration of rhodopsin, L and A are respectively the length and cross-sectional area of the rod cell, n_{flash} is the number of incident photons, r_{dark} is the rate of spontaneous thermal isomerization, and τ is the effective integration time.

(a) Explain why $1 - e^{-\sigma\rho L}$ is just the probability for a photon to be absorbed.

(b) Show that the SNR is maximized when $\rho L \approx 1.26/\sigma$.

(c) Cats, dogs, and many other animals have a reflective layer of tissue in their eye known as the *tapetum lucidum*, which gives them superior night vision to humans. Assuming that the *tapetum lucidum* acts as a mirror refelcting the incoming photons with a ratio $R(0 \leq R \leq 1)$ at the end of the rod cell, repeat the SNR optimization. What fraction of photons now pass through undetected and How does the *tapetum lucidum* improve the SNR, as a function of R? Discuss.

Problem 5

Simulate the central limit theorem in Matlab or a programming language of your choice. Draw a sample of N points from a given probability distribution (see below). Plot a histogram of the drawn sample. Now, repeat the experiment M times, and plot a histogram for the mean values of each experiment. Start with $N = 100$, and $M = 1000$. Play with the N, and M values. Sample from the following distributions: uniform, normal, and the distribution given below. Discuss the results.

$$p(x) = \begin{cases} 0 & \text{if } x < -2 \\ 0.5 & \text{if } -2 \leq x < -1 \\ 0 & \text{if } -1 \leq x < 1 \\ 0.5 & \text{if } 1 \leq x < 2 \\ 0 & \text{if } 2 \leq x \end{cases}$$

Hint. Matlab has a **rand** function that returns a random scalar drawn from the uniform distribution in the interval (0,1). Similarly, **randn** returns a random scalar drawn from the normal distribution in the interval (0,1).

Problem 6

As discussed in lecture, the probability of seeing in the Hecht, Schlaer, and Pirenne experiment is $P_{see} = \sum_{k=\theta}^{\infty} \frac{a^k}{k!} e^{-a}$; a is the mean number of photons absorbed by the retina, and θ represents the threshold number of photons for a particular individual. $a = \alpha N$, N is the number of photons that arrive at the cornea, α is a constant for a particular individual.

(a) Using a programming language of your choice, plot P_{see} as a function of $logN$, for individuals with the following values of α and θ. Comment on the positions along the x-axis and the slopes of the curves.

Individual	α value	θ values
Individual 1	0.1	2
Individual 2	0.1	3
Individual 3	0.1	4
Individual 4	0.1	5
Individual 5	0.5	2
Individual 6	0.5	3
Individual 7	0.5	4
Individual 8	0.5	5

(b) Consider now another individual (Individual 9) with $\alpha = 0.5$ which does not have a constant threshold number during the experimental trials. Assume that θ can be either 2, 3, 4, or 5, with equal probability for this individual. Plot his P_{see} as a function of $logN$. Comment on the slope and position of this curve relative to the curves of Individuals 5-9 with same α, but constant threshold.