Problem Set #1

Physics 141: Physics of Sensory Systems

September 12, 2023

Problem 1 (from *Physical Biology of the Cell*)

Consider eight particles, four of which are black and four white. Four particles can fit left of a permeable membrane and four can fit to the right of the membrane. Imagine that due to random motion of the particles, every arrangement of the eight particles is equally likely. Some possible arrangements are $BBBB|WWWW$, $BBWW|BBWW$, and $BWBW|BWBW$ where membrane position is given by $|$.

(a) How many different arrangements are there?

(b) Calculate the probability of having all four black particles on the left of the membrane. What is the probability of having one white particle and three black particles on the left of the membrane? Finally, calculate the probability that two white and two black particles are left of the membrane. Compare these three probabilities. Which arrangement is most likely?

(c) Imagine that, in one time instant, a random particle from the left side exchanges places with a random particle on the right side. Starting with three black particles and one white particle on the left of the membrane, compute the probability that after one time instant there are four black particles on the left. What is the probability that there are two black and two white particles on the left, after the same time instant? Which is the more likely scenario?

Problem 2

Consider a generalized Monty Hall Problem, with 1 prize, N doors, and k ($0 \leq k \leq N-2$) opened doors. Use Bayes' theorem to determine how much more likely are you to win the prize by switching than by not switching. How does this vary as a function of k?

Problem 3

When defined, the mean $\mu \coloneqq \langle X \rangle$ of a continuous random variable X is given by

$$\mu = \int x f(x) d^3x, \tag{1}$$

where the integral is taken over all possible values of X.

(a) Show that the variance $\sigma^2 \coloneqq \langle (X - \langle X \rangle)^2 \rangle$ can be written (when defined) as

$$\sigma^2 = \int x^2 f(x) d^3x - \left(\int x f(x) d^3x \right)^2 = \langle X^2 \rangle - \langle X \rangle^2.$$

(b) Calculate the mean and variance of the exponential distribution $f(x) = \lambda e^{-\lambda x}$.

Problem 4

Assume that the gas molecules in Earth's atmosphere are in thermal equilibrium near 300K. This is a somewhat poor assumption, but correct to within an order of magnitude. Using realistic values, estimate the relative density of different gas molecules (O_2, N_2,H_2) at different altitudes. How much less O_2 should you expect at the top of Mt. Everest with respect to sea level? Google the answer, and see if knowing the Boltzmann distribution led to a decent estimate.

Problem 5 from *Physical Biology of the Cell*, courtesy of Daniel Fisher

The rate of a molecule passing over a free energy barrier of height B is proportional to the Boltzmann factor $e^{-B/kT}$, with T the absolute temperature and k the Boltzmann constant. The exponential dependence on B can be understood heuristically as follows. Thermal fluctuations typically give the molecule an energy of about kT. To go up in energy by kT is somewhat less likely than to go down. If the probability to go up is q ($q < 0.5$), make a crude estimate of the probability p of getting to the top of the barrier in one try.

Problem 6 from *Physical Biology of the Cell*

Roll a fair 6-sided die. It will have equal probability of rolling 1, 2, 3, 4, 5, or 6. Thus, the probability distribution of possible outcomes is a flat: P(n) = 1/6 for any value of n. The mean value of a roll of a fair die is thus n = 3.5. Using information theory, calculate the 'maximum entropy' probability distribution, P(n), for an unfair die. All you know about the unfair die is that the mean value of a roll is n = 2.5.

Problem 7

(a) In this problem, you will use numerical software to solve systems of linear equations. To warm up, use a computer to solve the following three equations:

$$\begin{bmatrix} x+y \\ x-2y \\ 2z \end{bmatrix} = \begin{bmatrix} 1 \\ 3.14 \\ 2.71 \end{bmatrix}$$

Check your result with pencil and paper and make sure your results agree.

(b) Download a file containing an array of λs (a list of wavelengths in nm) and arrays of sensitivity functions $\mathcal{S}_i$ for i = S, M, and L for different cone cells for monochromatic light delivered at each wavelength.

Consider a color=matching experiment with three basis lights that are monochromatic with wavelengths λ=640, 520, and 440 nm. Each basis light has the same mean photon arrival rate (Φ), and the target light has the same mean photon arrival rate (Φ).

Estimate the amount of each basis light needed to match monochromatic light of wavelength $\lambda^* = 560$ nm.

(c) Extend the solution in (b) to many wavelengths 400 nm $<\lambda^*<$650 nm and make a graph showing the color-matching functions ζ_i.

(d) Repeat (b) and (c) for three other basis lights of your choosing.

Problem 8

In this problem, you will plot real data on the eyes' height and ommatidia size for different insects to confirm H.B. Barlow's findings from his paper, "The size of ommatidia in apposition eyes". Using the table below, plot the diameter of the ommatidia with respect to the square root of the height of the eye and show that the relationship is linear. What is the line of best fit, and how does it relate to Barlow's theoretical prediction?

Species	Ommatidia diameter (μm)	Height of the eye (mm)
Apis florea	24.9	2.2
Apis andreniformis	21.6	1.6
Apis dorsata	46.3	3.7
Apis mellifera	26.3	2.5
Apis cerana	35.8	2.9
Sirex noctilio	21.4	0.4
Vespa crabro	35.2	3.7
Vespula vulgaris	27.0	2.2
Apoica Pallens	26	3.6