Problem Set #3

Physics/Neuro 141

November 2023

Problem 1: Current of a Patchy Cell

In lecture, we found that for a spherical cell of radius a with N disk-shaped receptors each of radius s, the current is:

$$J = J_{\max} \frac{Ns}{Ns + \pi a}. \tag{1}$$

Take $s = 1$ nm and $a = 1$ μm.

(a) At what value of N is J half maximal? Find the mean distance on the surface of the cell between receptors at this value of J, and determine the fraction of the cell surface occupied by receptors.

(b) What happens to the flux if you pack all the receptors together into one patch with an area equal to the sum of all the individual receptor areas?

(c) What are the pros and cons to having receptors concentrated on one area of the cell surface?

Problem 2: Fourier Transforms

Determine the Fourier Transforms of:

(a) $f(t) = e^{-|t|}$

(b) $f(t) = e^{-|t|} \cos(\omega_0 t)$

Problem 3: Combination Tones

(a) Using the Boltzmann distribution, re-derive the probability p of a channel in a hair cell being open:

$$p = [1 + e^{-zx/kT}]^{-1} \tag{2}$$

$$x = X - X_o \tag{3}$$

(b) Setting $\beta = z/KT$ show that $\frac{dp}{dx} = \beta p(1-p)$

(c) Rearrange the force displacement relationship derived in class in the form:

$$F = Ax - Bp + C \tag{4}$$

Plot

$$F' = F - C = Ax - Bp \tag{5}$$

and K, the bundle stiffness,

$$K = \frac{dF}{dx} \tag{6}$$

for $x \in [-100 \times 10^{-9} 100 \times 10^{-9}]$ (nanometer size displacements), given that $\kappa_G = 500 \frac{\mu N}{m}$, $\kappa_S = 500 \frac{\mu N}{m}$, $\gamma = 0.14$; N (number of gated channels in a bundle) and d (the swing of the channel's gate) values are given in the table below:

Possibility	N	$d(nm)$
1	50	4
2	50	8
3	50	10
4	100	4
5	100	8

(d) Given the force displacement curves plotted above and sinusoidal displacement oscillations of the form $s = 10^{-8}[\sin(2\pi \times 500t) + \sin(2\pi \times 850t)]$m use a programming language of your choice to determine:
(1)the power spectrum of the sinusoidal displacement.
(2)the power spectrum of the sinusoidal forces exerted by the hair bundles. Can you detect combination tones at all of the values of N and d given in the table above? Discuss.

In Python, the numpy fft.fft function could be useful.

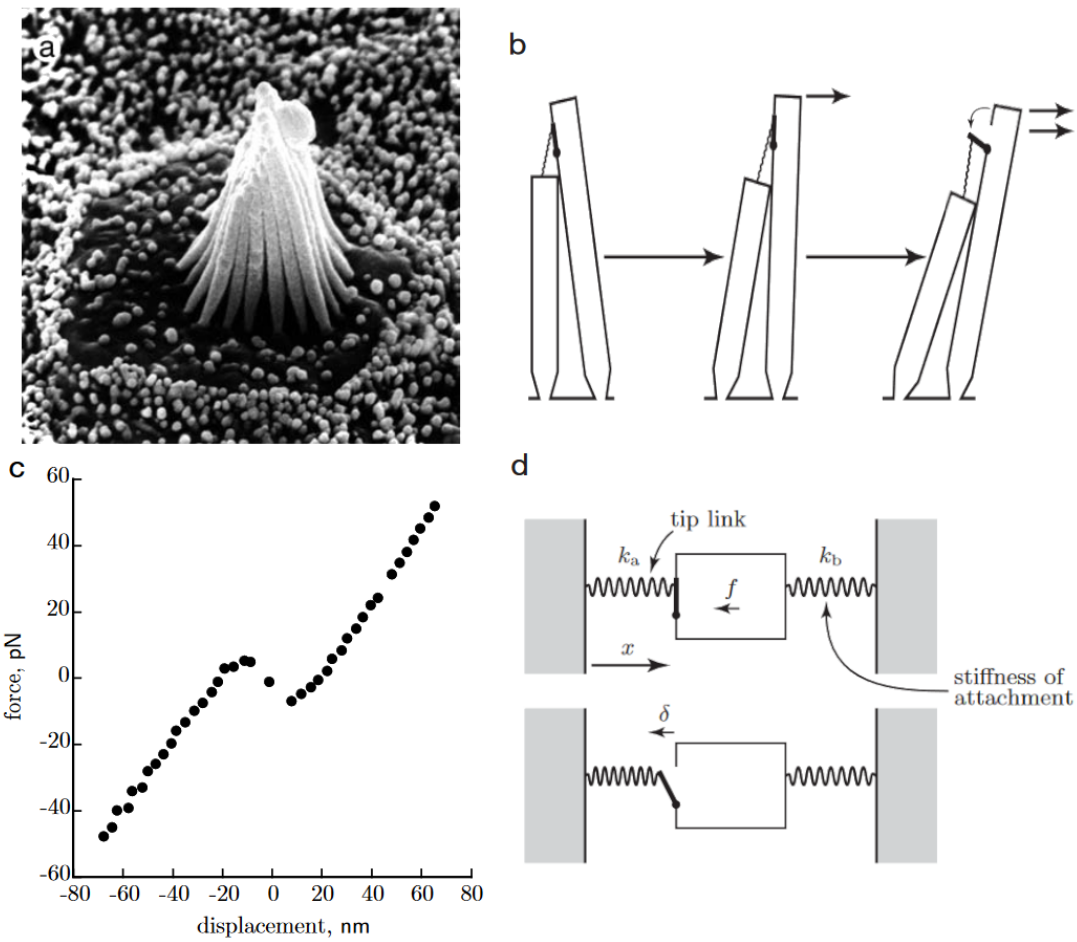
Problem 4: Gating Compliance

We can think of the bundle of stereocilia projecting from an auditory hair cell as a collection of N elastic units in parallel. Each element consists of two springs representing the tip link (stiffness k_a, equilibrium position x_a) and the actin filaments at the attachment point (stiffness k_b, equilibrium position x_b), respectively. The first spring attaches via a "trap door" which, when open, extends a distance δ from the body of the stereocilium. We may regard the trap door as an equilibrium system with two states (open and closed) separated by an energy difference ΔE_0.

(a) Derive the formula $f_{\text{closed}} = k_a(x - x_a) + k_b(x - x_b)$ for the net force on the stereocilium in the closed state. Rewrite this in the more compact form $f_{\text{closed}} = k(x - x_1)$, and find the effective parameters k, x_1 in terms of the original quantities. Then obtain the analogous formula for the net force $f(x, y)$ when the trap door is open a distance $y < \delta$.

(b) The total force f_{tot} is the sum of N copies of the formula you just found. In NP_{open} of these terms the trap door is open, while in the remainder it is closed. To obtain P_{open} from the Boltzmann factor, you will need to compute $\Delta E = \Delta E_0 + \int_0^\delta f(x, y)\, dy$, where $f(x, y)$ is your answer from (a).

(c) Assemble the pieces of your answer to get the force $f_{\text{tot}}(x)$ in terms of the parameters N, k, x_1, δ, and $\Delta E_1 = \Delta E_0 + \frac{1}{2} k_a \delta^2$. Plot your solution with $N = 65$, $k = 0.017\,\text{nm}^{-1}$, $x_1 = 23\,\text{nm}$, and $\Delta E_1 = 15\,\text{nm}$ for various values of $\delta > 0$. What choice of δ yields a curve resembling the observed nonlinearity? Is this a reasonable number?



a
b
c
force, pN
displacement, nm
60
40
20
0
-20
-40
-60
-80
-60
-40
-20
0
20
40
60
80
d
tip link
k_a
k_b
f
x
stiffness of attachment
δ