Editors' Suggestion

Scaling and Localization in Multipole-Conserving Diffusion

Jung Hoon Han,[1,*] Ethan Lake,[2,3,†] and Sunghan Ro[2,‡]

[1]*Department of Physics, Sungkyunkwan University, Suwon 16419, South Korea*
[2]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3]*Department of Physics, University of California Berkeley, Berkeley, California 94720, USA*

(Received 11 May 2023; revised 3 September 2023; accepted 30 January 2024; published 28 March 2024)

We study diffusion in systems of classical particles whose dynamics conserves the total center of mass. This conservation law leads to several interesting consequences. In finite systems, it allows for equilibrium distributions that are exponentially localized near system boundaries. It also yields an unusual approach to equilibrium, which in d dimensions exhibits scaling with dynamical exponent $z = 4 + d$. Similar phenomena occur for dynamics that conserves higher moments of the density, which we systematically classify using a family of nonlinear diffusion equations. In the quantum setting, analogous fermionic systems are shown to form real-space Fermi surfaces, while bosonic versions display a real-space analog of Bose-Einstein condensation.

DOI: 10.1103/PhysRevLett.132.137102

Introduction.—The phenomenon of diffusion is ubiquitous in physics, capturing the near-universal tendency of many-body systems to relax at long times toward a uniform steady state. Diffusion is typically modeled using Fick's law by the equation $\partial_t \rho = \nabla \cdot (D\nabla\rho)$, with ρ the particle density and D the diffusion constant. The assumptions going into the derivation of this equation are often very minimal, and the results are applicable to a broad range of physical phenomena. It is therefore important to understand situations in which conventional diffusive behavior breaks down.

An interesting question to ask in this direction is how diffusion is modified in systems with *constraints*, where restrictions are placed on how particles can move. A natural way of doing so is by the imposition of conservation laws that constrain particle motion. An example that has attracted much interest in the quantum dynamics [1–10] and many-body physics [11–26] communities is dynamics that conserves both the total particle number $N = \int dx \rho(x,t)$ and the total dipole moment $Q_x = \int dx x\rho(x,t)$, or equivalently the total center of mass $x_{\rm cm} = Q_x/N$ (working in 1D for simplicity). The requirement that both N and Q_x be time-independent is well-known [11] to mandate a continuity equation with two derivatives, viz.

$$\partial_t \rho = -\partial_x^2 J. \tag{1}$$

An expression for the current explored in recent hydrodynamically motivated studies [5,6,14,15,27,28] is

$$J = \tilde{D}\partial_x^2 \rho, \tag{2}$$

with the coefficient $\tilde{D}$ having dimensions of [length]^4 [time]$^{-1}$. This expression for J then leads to subdiffusive behavior with dynamical exponent $z = 4$.

In this Letter, we show that Eq. (2) only describes small fluctuations about a nonzero background density, and the full physics of dipole-conserving diffusion is much richer. By deriving an explicit lattice master equation for a natural class of dipole-conserving dynamics [see Fig. 1(a)], we find a current that has the nonlinear form

$$J = D[\rho\partial_x^2\rho - (\partial_x\rho)^2] = D\rho^2\partial_x^2 \ln(\rho), \tag{3}$$

with D having dimensions of [length]^5 [time]$^{-1}$ [29].

A scaling analysis of (3) yields a dynamic exponent of $z = d + 4$ in d dimensions, with the dimension dependence arising from the fact that J is a nonlinear function of ρ. This scaling governs the relaxation that occurs around a zero-density background. On the other hand, the relaxation of *small* fluctuations around a *nonzero* background density $\bar{\rho}$ evolves with $z = 4$, as can be seen by noting that (3) reduces to (2) with respect to the density difference $\delta\rho \equiv \rho - \bar{\rho}$ after linearizing in $\delta\rho$. A further notable feature of the expression (3) is that $\rho(x) \propto e^{-x/\ell}$ is a steady-state solution for all ℓ [see Fig. 1(d)]. This reflects the principle of entropy maximization, suggesting that the form of (3) should be general regardless of the specifics of the underlying microscopic dynamics.

Our analysis extends to dynamics that conserves higher multipole moments of density. As an example, in quadrupole-conserving dynamics that preserves both $x_{\rm cm}$ and the standard deviation $\sigma \equiv [N^{-1}\int dx x^2\rho(x,t) - x_{\rm cm}]^{1/2}$, the resulting steady-state distributions are Gaussians, with the conservation of σ preventing the spreading of the density field.

Finally, the principles identified in our study are also applicable to particles obeying quantum statistics. Multipole-conserving fermions turn out to form real-space

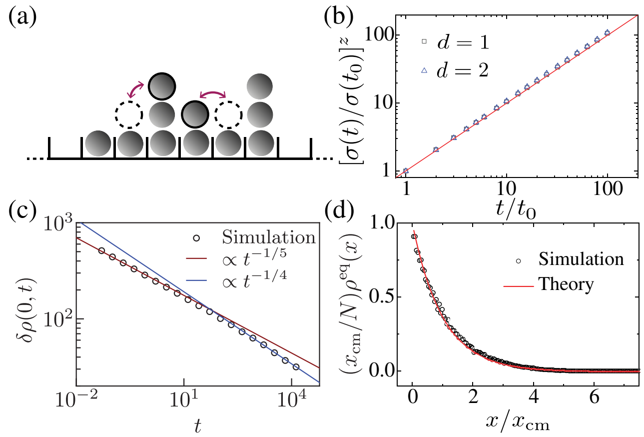

FIG. 1. (a) Schematic of our dipole-conserving hopping process, where particles hop toward opposite directions in pairs. (b) Evolution of the standard deviation of the density $\sigma(t)$ starting from initial Gaussian distribution. In both 1D (circles) and 2D (triangles), $\sigma(t)$ exhibits an excellent scaling collapse with dynamical exponent $z = 4 + d$ [the straight line has slope 1, demonstrating $\sigma(t) \propto t^{1/z}$]. (c) Relaxation of a large number $N_0 = 6000$ of particles localized at $x = 0$ in the presence of a small constant-density background $\rho_0 = 10$, showing a crossover from $z = 5$ at short times to $z = 4$ at long times. (d) Exponentially localized equilibrium density profile for 1D dipole-conserving diffusion on the half-space $x > 0$.

Fermi surfaces, while bosons exhibit a real-space analog of Bose-Einstein condensation, with a macroscopic number of bosons condensing at a single site.

Steady-state distribution.—We begin by describing the origin of the exponentially localized steady state shown in Fig. 1(d). In the absence of the potential energy due to interactions, the steady-state density $\rho^{\rm eq}(x)$ can be derived by maximizing the entropy $S = -\int dx \rho(x) \ln \rho(x)$ subject to the conservation of the particle number $N = \int dx \rho(x)$ and the center of mass $x_{\rm cm} = N^{-1} \int dx x \rho(x)$. Carrying out the extremization using Lagrange multipliers directly yields an exponentially localized density profile. In particular, if the particles are confined to the half-line $x > 0$, the equilibrium density is

$$\rho^{\rm eq}(x) = \frac{N}{x_{\rm cm}} e^{-x/x_{\rm cm}}. \tag{4}$$

The distribution of dipole-conserving particles is thus analogous to the Boltzmann distribution in the canonical ensemble, with $x_{\rm cm}$ playing the role of an effective temperature controlling the real-space width of the particle distribution.

It is important to note that this approach of maximizing S tacitly assumes ergodic dynamics, an assumption that can break down in certain conditions. As an extreme example, a single isolated particle is immobile due to the dipole conservation, leading to completely nonergodic behavior. More broadly, at low particle densities, the dynamics can be nonergodic when only the strictly local hopping processes are considered [2–4]. In what follows we will avoid this issue by always working with a large number of particles per site, far above [30,31] the critical density for the onset of nonergodicity.

Derivation of the diffusion equation.—We now consider a microscopic model within which (3) can be derived explicitly. To proceed, consider a continuous-time lattice gas where a pair of particles close to each other are randomly chosen and then displaced locally in a way that conserves $x_{\rm cm}$. Denoting the number of particles at site i as ρ_i, the particles can "diffuse out" as

$$(\Delta\rho_i, \Delta\rho_{i+1}, \Delta\rho_{i+2}, \Delta\rho_{i+3}) = (+1, -1, -1, +1) \tag{5}$$

with a rate taken to be $r dt \rho_{i+1}\rho_{i+2}$ (here $\Delta\rho$ gives the change in ρ_i between $t + dt$ and t). Similarly, the particles may "diffuse in" as

$$(\Delta\rho_i, \Delta\rho_{i+1}, \Delta\rho_{i+2}, \Delta\rho_{i+3}) = (-1, +1, +1, -1) \tag{6}$$

with a rate $r dt \rho_i \rho_{i+3}$ (these rates can be readily checked to satisfy detailed balance [32]). The fact that these rates are density-dependent is crucial for obtaining the nonlinear diffusion equation to follow. This should be contrasted with a distinct class of models inspired by quantum circuits [1–10], where the dynamics is driven by randomly implementing dipole-conserving hops at a density-*in*dependent rate.

Using the rates computed above, we obtain a master equation for the evolution of ρ_i of the form

$$\begin{aligned} \partial_t \rho_i = {} & r(\rho_{i-1}\rho_{i+2} + \rho_{i-2}\rho_{i+1} + \rho_{i+1}\rho_{i+2} + \rho_{i-2}\rho_{i-1}) \\ & - r\rho_i(\rho_{i-1} + \rho_{i+1} + \rho_{i-3} + \rho_{i+3}). \end{aligned} \tag{7}$$

We now perform a derivative expansion of the above equation, yielding

$$\partial_t \rho = -D\partial_x^2[\rho \partial_x^2 \rho - (\partial_x \rho)^2 + \mathcal{O}(a^2 \partial_x^2)], \tag{8}$$

where we have defined $x \equiv ia$ and $D \equiv 4ra^4$ with a the lattice spacing. For distributions with a characteristic length scale ℓ [e.g., $\ell = x_{\rm cm}$ in (4)], these terms are suppressed in powers of a/ℓ, and (3) is then reproduced in the limit $a/\ell \ll 1$. More details on the validity of this limit, as well as what happens when different types of elementary hopping processes are considered, can be found in the Supplemental Material [32]. The nonlinear dependence of J on ρ is a simple consequence of the fact that dipole-conserving motion always involves *pairs* of particles.

The unusual scaling observed in the relaxation toward equilibrium follows from the nonlinearity of the new diffusion Eq. (8). Indeed, the fact that ρ has dimensions of inverse length immediately yields a dynamic exponent of $z = 4 + 1$, which we confirm in numerics by measuring the scaling of the standard deviation σ of the density evolving from a Gaussian initial state [Fig. 1(b)].

Nevertheless, *small* fluctuations around a nonzero background density adhere to a dynamic scaling with $z = 4$, aligning with the hydrodynamic expectation (2). This is seen by setting $\rho(x,t) = \rho^{\mathrm{eq}}(x) + \delta\rho(x,t)$, linearizing in $\delta\rho$ given that $\delta\rho \ll \rho^{\mathrm{eq}}(x)$, and dropping all derivatives of $\rho^{\mathrm{eq}}(x)$, under which (3) reduces to

$$\partial_t \delta\rho \approx -D\rho^{\mathrm{eq}}(x)\partial_x^4 \delta\rho, \tag{9}$$

with $\tilde{D} \equiv D\rho^{\mathrm{eq}}(x)$ constituting an effective generalized diffusion constant. This means that a quench from a strongly inhomogeneous density distribution will exhibit a dynamical crossover, with $\delta\rho(t) \sim t^{-1/5}$ at short times (where the fluctuations about the background are large) switching to $\delta\rho(t) \sim t^{-1/4}$ at long times (where fluctuations are small). It is especially remarkable that the density relaxes *faster* at longer times, since in a situation where multiple modes with distinct z contribute to the relaxation, one would normally expect the long-time behavior to be dominated by the *slower* modes. To verify this crossover numerically, we consider an initial configuration where a localized peak of N_0 particles are placed on top of a uniform background of density ρ_0 [see Fig. 1(c)]. Initially the background density is negligible in comparison to the localized peak, and the relaxation of $\delta\rho$ at the peak center follows $\delta\rho(0,t) \propto t^{-1/5}$. However, at later times when $\delta\rho$ becomes comparable to ρ_0, the scaling exhibits dynamic exponent showing a clear crossover to $\delta\rho(0,t) \propto t^{-1/4}$ [36].

We briefly note two extensions of this analysis. The first is to incorporate the effect of noise into the deterministic Eq. (3). An application of standard techniques [37,38] to the dipole-conserving case gives the Langevin equation [32] (the structure of which can also be inferred on symmetry grounds alone [39])

$$\partial_t \rho(x,t) = -D\partial_x^2[\rho^2 \partial_x^2 \ln(\rho)] + \sqrt{2D}\partial_x^2 \eta(x,t), \tag{10}$$

where the noise field η satisfies the fluctuation-dissipation theorem,

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(t-t')\delta(x-x')\rho^2(x,t), \tag{11}$$

relating the strength of the noise on the lhs to the "mobility", which as shown in (3) is proportional to ρ^2 [40]. Linearizing the Langevin equation in $\delta\rho$ allows us to derive the structure factor [32]

$$S(k) \sim \langle \delta\rho(k,t)\delta\rho(-k,t)\rangle = \rho^{\mathrm{eq}}, \tag{12}$$

which is interestingly the same result as for conventional diffusion.

The second extension is the generalization of (3) to d dimensions, where the dipole-conserving diffusion equation reads $\partial_t \rho = \partial_a \partial_b J^{ab}$, with the current

$$J^{ab} = -D\rho^2 \partial^a \partial^b \ln(\rho). \tag{13}$$

The resulting dynamical exponent is then

$$z = 4 + d. \tag{14}$$

We verify this numerically by repeating the scaling analysis for a system of dipole-conserving particles on a 2D square lattice, with the results of Fig. 1(b) indeed demonstrating excellent $z = 6$ scaling.

Quadrupole-conserving diffusion.—We now move on to considering dynamics that conserves the quadrupole moment $Q_{xx} = \int dx x^2 \rho(x,t)$ (related to the standard derivation as $\sigma = \sqrt{Q_{xx}/N - x_{\mathrm{cm}}^2}$) in addition to the center of mass. Any quadrupole-conserving processes must involve the coordinated motion of at least three particles [32], with the simplest allowed three-particle process given by $(\Delta\rho_i, \Delta\rho_{i+1}, \ldots, \Delta\rho_{i+4}) = (\pm 1, \mp 2, 0, \pm 2, \mp 1)$, as depicted in Fig. 2(a). By writing down a master equation via manipulations similar to those that led to (7), we obtain the quadrupole-conserving diffusion equation

$$\partial_t \rho = \partial_x^3 J, \quad J = D\rho^3 \partial_x^3 \ln(\rho). \tag{15}$$

Maximizing the entropy subject to the conservation of both dipole and quadrupole moments yields Gaussian steady states of the form

$$\rho^{\mathrm{eq}}(x) = \frac{N}{\sqrt{2\pi\sigma^2}} e^{-(x-x_{\mathrm{cm}})^2/2\sigma^2}, \tag{16}$$

which are also static zero-current solutions to (15). This thus predicts the scenario whereby a generic initial distribution eventually "congeals" into a Gaussian, and then remains that way for all further times. The simulation results shown in Fig. 2(b) indeed confirm this expectation.

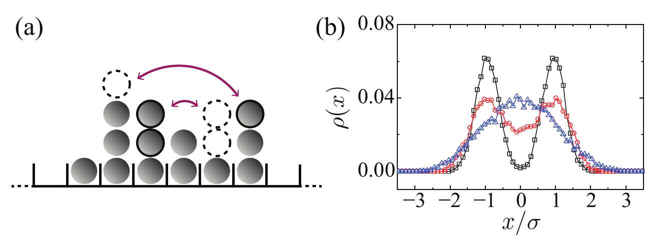

FIG. 2. (a) A schematic illustration of a 1D quadrupole-conserving diffusion process: two particles move two sites in one direction, while another particle moves three sites in the opposite direction. (b) The time evolution for quadrupole-conserving diffusion. The density is initially given as two widely separated Gaussian packets, which merge over time (black → red → blue) into a single Gaussian profile, with the standard deviation σ remaining conserved (solid curves are guides to the eye).

Because of the presence of the nonzero conserved length scale σ, no single-parameter scaling ansatz $\rho(x^z/t)$ with finite z is possible in the present case. Small deviations about ρ^{eq} will however relax with $z=6$, as seen by linearizing (15).

General multipole moments.—We further generalize the discussion to a d-dimensional system that conserves $\int d^d r p_n(\mathbf{r})\rho(\mathbf{r})$ for all degree-n polynomials $p_n(\mathbf{r})$. Maximizing the entropy yields equilibrium states formed as exponentials of degree-n polynomials in the coordinates, which leads to generalizations of (1), (3), and (15) as

$$\partial_t\rho = -\partial_A J^A, \qquad J^A = (-1)^{n+1} D\rho^m \partial^A \ln\rho, \quad (17)$$

where $A=\{a_1,\ldots,a_{n+1}\}$ is a composite index and $\partial_A \equiv \partial_{a_1}\cdots\partial_{a_{n+1}}$. The minimum value for m in the definition of the current J^A (which determines the mobility $D\rho^m$) is equal to the minimum number of particles needed to form a microscopic n-pole conserving hopping process. In [32] we show that the minimum of m satisfies $m=n+1$ for $n\le 10$ and $n=12$; determining whether or not $m=n+1$ for all n turns out to be equivalent to a longstanding open problem in number theory [41]. As a further consistency check, the two relations in (17) imply that the entropy $S=-\int d^d r\rho\ln\rho$ is monotonically increasing with time, as they yield

$$\frac{dS}{dt} = D\int dx \rho^m (\partial^A[\ln\rho + 1])^2 \ge 0. \quad (18)$$

Quantum multipolar diffusion.—So far, we have treated the objects undergoing diffusion as classical (distinguishable) particles. We now generalize by letting the particles obey quantum statistics. This is particularly important in light of the fact that a natural experimental realization of multipole-conserving dynamics is found in quantum particles in strongly tilted optical lattices [8,42,43].

We begin by generalizing the expression (3) obtained for dipole-conserving dynamics. This can be done by suitably modifying the lattice master equation. In the classical case, we took the rate for a pair at locations i_1, i_2 to hop to i_3, i_4 to be proportional to $\rho_{i_1}\rho_{i_2}$. To generalize, we simply modify this to $\rho_{i_1}\rho_{i_2}(1+\zeta\rho_{i_3})(1+\zeta\rho_{i_4})$, where $\zeta=+1,-1$ for bosons and fermions, respectively. These factors account for the "statistical interactions" (attractive or repulsive) experienced by indistinguishable particles, and follow from demanding that the dynamics satisfy detailed balance [32].

With these modifications to the hopping rates, the same approach used previously produces [32]

$$\begin{aligned} J &= D\{\rho\partial_x^2\rho - (\partial_x\rho)^2 + \zeta[\rho^2\partial_x^2\rho - 2\rho(\partial_x\rho)^2]\} \\ &= D\rho^2(1+\zeta\rho)^2\partial_x^2 \ln\left(\frac{1}{1/\rho+\zeta}\right). \end{aligned} \quad (19)$$

From the second line, it is easy to see that the zero-current steady states are real-space Bose-Einstein ($\zeta=1$) or Fermi-Dirac ($\zeta=-1$) distributions:

$$\rho^{\mathrm{eq}}(x) = [e^{(x-\mu)/T} - \zeta]^{-1}, \quad (20)$$

as could be expected on entropy-maximization grounds (here T,μ are determined by N, Q_x). Note that (19) reduces to (3) in the (classical) low-density limit ($\zeta=0$). However, for bosons at high densities, J becomes *cubic* in ρ, giving a crossover from $z=5$ to $z=6$ dynamics as the density is increased.

We now point out a few curious features of the steady-state distributions that arise in this case. For illustration, first consider fermions in a 1D box $x\in[-L/2, L/2]$. At half-filling $N=L/(2a)$ one finds $\mu=0$, with T fixed by the center of mass: $T=\infty$ corresponds to $x_{\mathrm{cm}}=0$, while $T=0^{\pm}$ gives $x_{\mathrm{cm}}=\mp L/4$ [here, $\mathrm{sgn}(T)=-\mathrm{sgn}(x_{\mathrm{cm}})$]. The fermions thus form a Fermi surface in *real space*, with the Fermi "radius" roughly fixed by N, and the "temperature" roughly fixed by x_{cm} [44].

Now consider N dipole-conserving bosons in the positive-coordinate space $\mathbb{P}^d \equiv [0,\infty)^d$, and suppose for simplicity that all components of the dipole moment Q_a are equal. The new physics afforded by bosonic statistics is the possibility of forming a Bose-Einstein condensate (BEC). The BECs that form in the present case are distinguished by the fact that the ordering occurs in real space, with a macroscopic number of bosons condensing onto a single lattice site.

The transition "temperature" T_* at which a real-space BEC forms is determined by the point at which the chemical potential vanishes, calculated by solving

$$N = \int_{\mathbb{P}^d} \frac{d^d r}{e^{\sum_a r_a/T_*} - 1} \quad (21)$$

for T_*. For $d=1$ this integral has a *short*-distance divergence, which precludes ordering by mandating that $T_*\to 0$ (cf. the *long*-distance divergences that destroy conventional 1D BECs). When $d>1$, a BEC can form at $T_*\propto N^{1/d}$ [32]. By calculating the dipole moment Q_x at $T=T_*$, we find that a BEC forms when $N>N_*$, with

$$N_* \propto (Q_x/a)^{\frac{d}{d+1}}, \quad (22)$$

where we have restored the lattice constant a. To understand this, imagine increasing the number of particles in the system while keeping Q_x fixed. For $N<N_*$, the system will smoothly adjust its overall density profile as particles are added. When $N>N_*$, all of the particles subsequently added to the system will join the condensate at $\mathbf{r}=\mathbf{0}$, where they do not contribute to the dipole moment. In this regime the system will thus possess a condensate at the origin, together with a surrounding "normal fluid" localized nearby.

Bosons with higher moment conservation behave similarly. As an example, quadrupole-conserving bosons in infinite space have $\rho^{\mathrm{eq}}(r) = [e^{(r^2-\mu)/T} - 1]^{-1}$ (assuming rotational symmetry), rendering the problem equivalent to the nonrelativistic ideal Bose gas after the interchange of coordinates and momenta. A short-distance divergence prevents a BEC from forming when $d < 3$ [32]. For $d \geq 3$, the critical boson number is determined in terms of the quadrupole moment Q_{xx} as $N_* \propto (Q_{xx}/a^2)^{(d/d+2)}$. Understanding other physical properties of these unusual BECs, especially the nature of their condensation transitions, are questions that we leave to future work.

Discussion.—We have demonstrated that multipole-conservation laws can radically alter how diffusion occurs. Even in the absence of interactions, these conservation laws lead to exponentially localized steady states, and unconventional approaches to equilibrium. In the case of dipole conservation, we have explicitly demonstrated that relaxation occurs with an anomalously large scale-dependent dynamical exponent, with large-scale features relaxing with $z = 4 + d$, and short-scale ones with $z = 4$.

It would be of great interest to pursue experiments realizing multipole-conserving dynamics in the lab. A natural platform in this regard is found in strongly tilted optical lattices [2,4,8,42,43], where the lattice tilt enforces emergent dipole conservation over a long prethermal timescale. The breakdown of diffusion due to ergodicity breaking can be avoided by working at sufficiently high densities, with the envisioned experiment rather similar to the one performed in [43]. As an example, we predict that sufficiently high-density and sufficiently weakly interacting bosons in a strongly tilted 1D lattice will relax with a dynamical exponent $z = 6$, and will admit equilibrium distributions that condense on system boundaries. For fermions, the equilibrium density profile will instead follow the Fermi-Dirac distribution.

There are many natural extensions of our work. Instead of full multipole moment conservation, one could examine *nonmaximal* multipole groups [21,45] in $d > 1$, where the dynamics conserves $\sum_{\mathbf{r}} P(\mathbf{r})\rho_{\mathbf{r}}$ for only a subset of degree-n polynomials $P(\mathbf{r})$, e.g., a 2D system conserving $P(\mathbf{r}) = x^2, y^2$, but not $P(\mathbf{r}) = xy$. It may also be interesting to examine the importance of the lattice geometry, with more complicated lattices potentially leading to emergent subsystem symmetries *á la* the mechanism of Ref. [46]. For pursuing the optical lattice experiments mentioned above, it will also be crucial to understand to what extent the physics studied here is modified by the presence of interparticle interactions.

Another intriguing direction is to break the time-reversal symmetry of the diffusion process, for example by allowing only "inward" pair hopping processes while disallowing "outward" ones. Such a modification would make the system *active* and is known to result in non-equilibrium phenomena that cannot be captured by entropy maximization [47–49].

We are grateful to Soonwon Choi, Johannes Feldmeier, Byungmin Kang, Andy Lucas, Francisco Machado, Seth Musser, Rahul Nandkishore, Marvin Qi, Brian Skinner, Charles Stahl, Daniel Swartz, and Hongzheng Zhao for helpful discussions and feedback. J. H. H. was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2023R1A2C1002644). He also acknowledges financial support from EPIQS Moore theory centers at MIT and Harvard, where this work was initiated. E. L. was supported by a Miller research fellowship.

*hanjemme@gmail.com
†elake@berkeley.edu
‡sunghan@mit.edu

[1] S. Pai, M. Pretko, and R. M. Nandkishore, Localization in fractonic random circuits, Phys. Rev. X **9**, 021003 (2019).

[2] V. Khemani, M. Hermele, and R. Nandkishore, Localization from hilbert space shattering: From theory to physical realizations, Phys. Rev. B **101**, 174204 (2020).

[3] T. Rakovszky, P. Sala, R. Verresen, M. Knap, and F. Pollmann, Statistical localization: From strong fragmentation to strong edge modes, Phys. Rev. B **101**, 125126 (2020).

[4] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity breaking arising from hilbert space fragmentation in dipole-conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).

[5] J. Iaconis, A. Lucas, and R. Nandkishore, Multipole conservation laws and subdiffusion in any dimension, Phys. Rev. E **103**, 022142 (2021).

[6] J. Feldmeier, P. Sala, G. De Tomasi, F. Pollmann, and M. Knap, Anomalous diffusion in dipole-and higher-moment-conserving systems, Phys. Rev. Lett. **125**, 245303 (2020).

[7] M. Schulz, C. A. Hooley, R. Moessner, and F. Pollmann, Stark many-body localization, Phys. Rev. Lett. **122**, 040606 (2019).

[8] S. Scherg, T. Kohlert, P. Sala, F. Pollmann, B. H. Madhusudhana, I. Bloch, and M. Aidelsburger, Observing non-ergodicity due to kinetic constraints in tilted Fermi-Hubbard chains, Nat. Commun. **12**, 1 (2021).

[9] X. Feng and B. Skinner, Hilbert space fragmentation produces an effective attraction between fractons, Phys. Rev. Res. **4**, 013053 (2022).

[10] S. Moudgalya, A. Prem, D. A. Huse, and A. Chan, Spectral statistics in constrained many-body quantum chaotic systems, Phys. Rev. Res. **3**, 023176 (2021).

[11] M. Pretko, Subdimensional particle structure of higher rank U(1) spin liquids, Phys. Rev. B **95**, 115139 (2017).

[12] A. Prem, M. Pretko, and R. M. Nandkishore, Emergent phases of fractonic matter, Phys. Rev. B **97**, 085116 (2018).

[13] M. Pretko, The fracton gauge principle, Phys. Rev. B **98**, 115134 (2018).

[14] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Res. **2**, 033124 (2020).

[15] P. Glorioso, J. Guo, J. F. Rodriguez-Nieva, and A. Lucas, Breakdown of hydrodynamics below four dimensions in a fracton fluid, Nat. Phys. **18**, 912 (2022).

[16] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Global dipole symmetry, compact lifshitz theory, tensor gauge theory, and fractons, Phys. Rev. B **106**, 045112 (2022).

[17] S. Pielawa, T. Kitagawa, E. Berg, and S. Sachdev, Correlated phases of bosons in tilted frustrated lattices, Phys. Rev. B **83**, 205135 (2011).

[18] S. Sachdev, K. Sengupta, and S. M. Girvin, Mott insulators in strong electric fields, Phys. Rev. B **66**, 075128 (2002).

[19] E. Lake, M. Hermele, and T. Senthil, Dipolar Bose-Hubbard model, Phys. Rev. B **106**, 064511 (2022).

[20] E. Lake, H.-Y. Lee, J. H. Han, and T. Senthil, Dipole condensates in tilted Bose-Hubbard chains, Phys. Rev. B **107**, 195132 (2023).

[21] C. Stahl, E. Lake, and R. Nandkishore, Spontaneous breaking of multipole symmetries, Phys. Rev. B **105**, 155107 (2022).

[22] P. Zechmann, E. Altman, M. Knap, and J. Feldmeier, Fractonic Luttinger liquids and supersolids in a constrained Bose-Hubbard model, Phys. Rev. B **107**, 195131 (2023).

[23] E. Lake and T. Senthil, Non-Fermi liquids from kinetic constraints in tilted optical lattices, Phys. Rev. Lett. **131**, 043403 (2023).

[24] A. Anakru and Z. Bi, Non-Fermi liquids from dipolar symmetry breaking, Phys. Rev. B **108**, 165112 (2023).

[25] S. A. Chen, J.-K. Yuan, and P. Ye, Fractonic superfluids. II. Condensing subdimensional particles, Phys. Rev. Res. **3**, 013226 (2021).

[26] J.-K. Yuan, S. A. Chen, and P. Ye, Fractonic superfluids, Phys. Rev. Res. **2**, 023267 (2020).

[27] A. G. Burchards, J. Feldmeier, A. Schuckert, and M. Knap, Coupled hydrodynamics in dipole-conserving quantum systems, Phys. Rev. B **105**, 205127 (2022).

[28] A. Glodkowski, F. Pena-Benitez, and P. Surowka, Hydrodynamics of dipole-conserving fluids, Phys. Rev. E **107**, 034142 (2023).

[29] Note that ordinary diffusion equation can be rewritten in a similar form as $\partial_t \rho = -\partial_x J$ with $J = -D\rho \partial_x \ln(\rho)$.

[30] C. Pozderac, S. Speck, X. Feng, D. A. Huse, and B. Skinner, Exact solution for the filling-induced thermalization transition in a one-dimensional fracton system, Phys. Rev. B **107**, 045137 (2023).

[31] A. Morningstar, V. Khemani, and D. A. Huse, Kinetically constrained freezing transition in a dipole-conserving system, Phys. Rev. B **101**, 214205 (2020).

[32] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.132.137102 for details, which includes Refs. [33–35].

[33] G. E. Crooks, On thermodynamic and microscopic reversibility, J. Stat. Mech. (2011) P07008.

[34] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Low-energy limit of some exotic lattice theories and UV/IR mixing, Phys. Rev. B **104**, 235116 (2021).

[35] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford University Press, New York, 1998).

[36] The results of direct scaling of density fields with the dynamic exponents are also presented in Supplemental Material [32].

[37] A. Lefèvre and G. Biroli, Dynamics of interacting particle systems: stochastic process and field theory, J. Stat. Mech. (2007) P07024.

[38] D. Hexner and D. Levine, Noise, diffusion, and hyperuniformity, Phys. Rev. Lett. **118**, 020601 (2017).

[39] J. Guo, P. Glorioso, and A. Lucas, Fracton hydrodynamics without time-reversal symmetry, Phys. Rev. Lett. **129**, 150603 (2022).

[40] The fluctuation-dissipation theorem holds for multipole-conserving diffusive system as well, being as it is applicable to any near-equilibrium situation.

[41] P. Borwein, *Computational Excursions in Analysis and Number Theory* (Springer Science & Business Media, New York, 2002).

[42] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and heat transport in a tilted two-dimensional Fermi-Hubbard system, Phys. Rev. X **10**, 011042 (2020).

[43] H. P. Zahn, V. P. Singh, M. N. Kosch, L. Asteria, L. Freystatzky, K. Sengstock, L. Mathey, and C. Weitenberg, Formation of spontaneous density-wave patterns in dc driven lattices, Phys. Rev. X **12**, 021014 (2022).

[44] Since the dynamics freezes out below a critical (hopping-range-dependent) density, we expect that the density will follow a Fermi-Dirac distribution only in regions where $\rho^{\mathrm{eq}}(x) \gtrsim 1/\ell_{\max}$, where $\ell_{\max}$ is the range of the longest-ranged hopping process [30,31].

[45] A. Gromov, Towards classification of fracton phases: The multipole algebra, Phys. Rev. X **9**, 031035 (2019).

[46] D. Bulmash, O. Hart, and R. Nandkishore, Multipole groups and fracton phenomena on arbitrary crystalline lattices, SciPost Phys. **15**, 235 (2023).

[47] R. Soto and R. Golestanian, Run-and-tumble dynamics in a crowded environment: Persistent exclusion process for swimmers, Phys. Rev. E **89**, 012706 (2014).

[48] A. G. Thompson, J. Tailleur, M. E. Cates, and R. A. Blythe, Lattice models of nonequilibrium bacterial dynamics, J. Stat. Mech. (2011) P02029.

[49] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur, Exact hydrodynamic description of active lattice gases, Phys. Rev. Lett. **120**, 268003 (2018).