2020 Online Physics Olympiad (OPhO): Invitational Contest

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Instructions

If you wish to request a clarification, please use this form. To see all clarifications, see this document.

- This test contains 10 long answer questions.

- The team leader should submit their final solution document in this here. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.

- You can write on paper, type it up online, or a mix of both. If you wish to use a pre-made LaTeX template, we made one which you can choose to use here.

- Since this is a long answer response, you will be judged on the quality of your work. To receive full points, you need to show your work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in this equation sheet) can be cited without proof.

- You may leave all final answers in symbolic form (in terms of variables) unless otherwise specified.2

- Be sure to state all assumptions.

Sponsors

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg

- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg

- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg

- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}

- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)

- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K

- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$

- 1 electron volt, $1\text{ eV} = 1.60 \cdot 10^{-19}$ J

- Speed of light, $c = 3.00 \cdot 10^8$ m/s

- Universal Gravitational constant,

$$G = 6.67 \cdot 10^{-11} \text{ (N} \cdot \text{m}^2)/\text{kg}^2$$

- Acceleration due to gravity, $g = 9.81$ m/s^2

- 1 unified atomic mass unit,

$$1\text{ u} = 1.66 \cdot 10^{-27} \text{ kg} = 931 \text{ MeV}/c^2$$

- Planck's constant,

$$h = 6.63 \cdot 10^{-34} \text{ J} \cdot \text{s} = 4.41 \cdot 10^{-15} \text{ eV} \cdot \text{s}$$

- Permittivity of free space,

$$\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2/(\text{N} \cdot \text{m}^2)$$

- Coulomb's law constant,

$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9 \text{ (N} \cdot \text{m}^2)/\text{C}^2$$

- Permeability of free space, $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A

- Magnetic constant,

$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \text{ (T} \cdot \text{m)}/\text{A}$$

- 1 atmospheric pressure,

$$1 \text{ atm} = 1.01 \cdot 10^5 \text{ N/m}^2 = 1.01 \cdot 10^5 \text{ Pa}$$

- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K

- Stefan-Boltzmann constant,

$$\sigma = 5.67 \cdot 10^{-8} \text{ W/m}^2/\text{K}^4$$

Problem 1: A Party Trick

It is a well known party trick that by pulling the table-cloth very quickly and suddenly, the plates on top of the table can stay nearly in place.

(a) **(1 pt)** Start with a circular table $D = 1$ m in diameter with a very flat and small (and dimensionless for now) plate in the centre. How fast must you pull the tablecloth so the plate remains on the table? Assume the static friction coefficient is $\mu_s = 0.5$ and kinetic friction is $\mu_k = 0.3$ (for any contacts with the tablecloth) and that the tablecloth accelerates instantaneously. The tablecloth does not overhang the table.

(b) **(2 pts)** Mythbusters famously attempted to replicate the same trick with a giant tablecloth and a motorbike. We will simplify the experiment by assuming that the table was only set with plates placed $d = 0.5$ m apart from each other and from both ends. Assuming a $\ell = 7$ m long table, how fast must the bike be travelling to successfully carry out this experiment? The mass of the cloth and plates are negligible compared to the mass of the motorbike. Assume small plates and that the tablecloth does not overhang the table.

(c) **(3 pts)** A young boy places his toy car in the cen-

tre on the table from part (b). He believes that since it has wheels, it will stay on the table easier. Confirm or deny this statement and find the significant speeds that the tablecloth is pulled at that would cause the car to stay on the table. Assume the car (including wheels) has mass $2m$, and that each wheel is a uniform disk of mass $m/4$. The car is small in comparison to the table.

(d) **(4 pts)** We run the experiment one last time with the glass, with same mass as a plate, placed on top of each plate on the tablecloth, which in turn, is on a frictionless table. The static and kinetic coefficient between the glass and the plate are $\mu'_s = 0.30$ and $\mu'_k = 0.15$ respectively. How fast must the tablecloth be pulled so that the glasses stay completely on the plate and the plates stay completely on the table? Do not assume that the either plate or glass is dimensionless this time. A diagram is provided below:
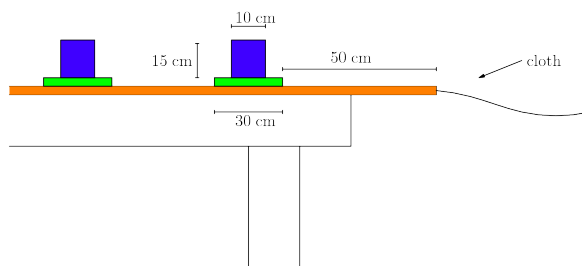
Problem 2: Solar Sails

Much research is being done on the possibility of using solar sails to reach far away reaches in out galaxy. This is a method of propulsion that use light from the sun to exert a pressure on what is usually a large mirror. The sails themselves are often made of a thin reflective film. Compared to traditional spacecraft, while these sails have very limited payloads, they offer long operational lifetimes and are relatively low cost. The most significant advantage is speed: since solar sails do not depend on onboard propellant, they can travel much faster than a standard rocket, with the possibility of reaching a significant percentage of the speed of light. For all parts, assume the solar sails are only under the influence of gravity from the sun only.

(a) **(1 pt)** Assume the solar sails are not revolving around the sun. Assume they are very reflective thin discs and that all of the mass is in the sail. What is the maximum area density so that the sail does not fall towards the sun? Does the distance matter (find the general density to distance equation if it does matter)?

(b) **(1.5 pts)** Assume the solar sails are a thin spherical shell and made of a perfectly absorbent material instead. The area density is exactly half the maximum area density so that it does not fall towards

the sun. What is its final speed if it starts at a stationary position 1 Au from the sun?

(c) **(2.5 pts)** Assume the solar sails are a thin spherical shell and made of material having reflectance r. With the same area density as in part (b), what is the final speed (it can reach this speed either when crashing into the sun, escaping the solar system, or remain in orbit)? This time, assume it starts in circular orbit 1 Au away from the sun as the starting condition. What is the orbital shape?

(d) **(3 pts)** Simply not falling into the sun is insufficient for solar sails. Most are planned to reach relativistic speeds. A way to achieve this is to fire large Earthbound lasers at the sail. How powerful must the lasers be to accelerate the sail up to $v = 0.2c$ in 50 days? Since this far exceeds the value obtained in part 3, neglect the effects of the sun.

(e) **(5 pts)** Suppose a perfectly reflecting solar sail in the shape of a thin disk orbiting around the star in a circular orbit. It is also spinning around itself, such that the spin angular velocity is in the same direction as the orbital angular velocity, and its axis of symmetry always remains parallel to the plane of the orbit. If its initial spin velocity is ω_0 find its spin velocity after it revolves an angle θ around the star.

Problem 3: Electron Escape

An infinite wire with current I has a radius a. The wire is made out of a material with resistivity ρ and heat conductivity κ. The temperature outside the wire is a constant T_0.

(a) **(2 pts)** After a long time, determine the temperature $T(r)$ at a distance r from the center of the wire. Assume that the current in the wire is uniformly distributed.

(b) **(2 pts)** Now, the outer surface of the wire is maintained at a potential of $V \geq 0$. The wire is surrounded by an infinite cylindrical shell with radius $b > a$ that is grounded. Somehow, an electron is able to escape from the wire. Assume that it is at

rest just as it escapes. You can neglect radiation from the electron.

Draw a qualitative graph of the physical path that the electron takes, along with a diagram of the wire.

(c) **(3 pts)** Find the maximum distance $r_{\max}$ of the electron from the center of the wire in the subsequent motion as a function of V, and also in terms of I, a, and b. Ignore all relativistic effects in this part only.

(d) **(6 pts)** Redo the calculation in the previous part with relativistic effects.

(e) **(1 pts)** Graph the maximum radius $r_{\max}$ as a function of V.

Problem 4: Bullet in Cylinder

A hollow cylinder of mass M and radius R rests on a rough horizontal surface. A projectile of mass $m < M$ having a velocity u directed horizontally exactly towards the middle of the cylinder as shown in the figure. The shell gets stuck in the cylinder wall, after which the shell starts to move, slipping on the surface.
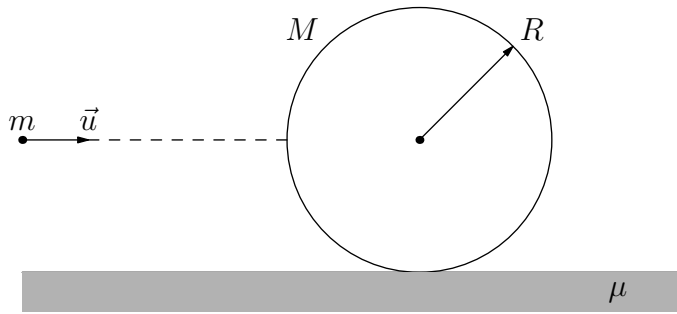
(a) (2 pt) In which direction does the cylinder rotate? State your answers for different values of μ.

(b) (3 pts) Find its angular acceleration at the first moment after the impact.

(c) (5 pts) It is known that some time after the impact, the horizontal projection of the velocity of the center of mass of the system is equal to v, and the angular velocity of the cylinder is Ω. At this point, the cylinder has rotated through an angle ϕ.

How much heat was released in the system at this point, if the cylinder all the time after the impact moved with slipping, rotating in one direction? Coefficient of friction between cylinder and horizontal surface equal to $\mu < 2$.

Problem 5: Mathematical Physics

The study of mathematics has almost always paved the way for the development of new ideas in physics. Newtonian mechanics could not be possible without first inventing calculus, and general relativity could not have existed without heavy development in tensors. However, there are numerous cases where physical insight have paved the way for mathematics.

   Perhaps the most notable would be the Brachistochrone problem, which asks for the path that leads to the fastest descent influenced by gravity between two given points. While it is solvable through the calculus of variations, Newton proposed an easier solution by modelling the path of light through a medium with a variable index of refraction. You may read about this problem and the fascinating history behind it here.

   We will not be dealing with this specific problem, but rather multiple short mathematics problems that can be can be represented with a physical analog. To receive points, you must use the suggested physical set-up.

(a) (2 pts) Show that for small values of x, we have

$$\cos(x) = 1 - \frac{x^2}{2}.$$

Physical Setup: Consider a small object moving in a circle that starts from rest.

(b) (4 pts) A ladder of length ℓ with a thickness of 0.3 m is transported around a right angled corner where the two hallways leading up to it have a width of 3 m and 5 m. What is the maximum length of the ladder such that it can be successfully transported across?

Physical Setup: Consider the ladder as a compressed spring that can freely expand in only its longitudinal direction. Do not explicitly take derivatives. Instead, consider a force balance.

(c) (3 pts) Prove the AM-QM inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

Physical Setup: Design two circuits and compare one of their measurable quantities with each other qualitatively.

Problem 6: Flat Earth

In this problem, we will explore the true gravitational model of the earth, not the one that is claimed in most textbooks. Contrary to popular belief, the Earth is a flat circle of radius R and has a uniform mass per unit area σ. The Earth rotates with angular velocity ω.

(a) **(2.5 pts)** A pendulum of length ℓ that is constrained to only move in one plane is placed on the ground at the center of the Earth. The pendulum has more than one angular frequency of small oscillations. Find the value of each angular frequency of small oscillations $\Omega(0), \Omega_1(0), ...$ in terms of σ, ω, and physical constants and the equilibrium angle $\theta, \theta_1, ...$ that the frequency occurs at. Assume for all parts that $\ell \ll R$.

(b) **(1 pt)** Find the maximum angular velocity that the Earth rotates at such that the oscillations are stable at each angular frequency.

(c) **(6 pts)** The entire pendulum is moved a distance $r \ll R$ away from the center of the Earth. It is oriented so that it is constrained to only move in the radial direction. Now, find the new angular frequency $\Omega(r)$ of small oscillations about the lowest equilibrium point in terms of the given parameters, assuming that $\omega^2 r$ is much less than the local gravitational acceleration.

(d) **(1.5 pts)** The angular frequencies $\Omega(0)$ and $\Omega(r)$ are both measured and the difference is found to be $\Delta\Omega$. Assuming that $\Delta\Omega \ll \Omega(0)$ and $\omega^2 \ll \frac{g}{\ell}$, determine σ in terms of $\omega, r, \Omega(0), \Delta\Omega$, and physical constants.

Problem 7: Boltzmann Statistics

In this problem, we will explore Boltzmann Statistics and using it to build similar models for quantum particles such as bosons and fermions.

(a) **(4 pts)** Consider a energy of a gaseous molecule in space is given by

$$E = E_0(x^r + y^r + z^r)$$

where the coordinates of the molecule are represented by (x, y, z), E_0 is a constant with appropriate units, and r is a non-negative real number. The system is in thermal equilibrium with a reservoir of temperature T. Calculate explicitly using appropriate statistical methods, the average energy of a thermodynamic system consisting of such gaseous molecules, considering the Maxwell-Boltzmann distribution. Analyse your result and provide a qualitative argument to support it.

Hint: If you are not familiar with how to solve this part, try part (b) first.

(b) **(3 pts)** Consider a specific case in which

$$E = E_0(x^2 + y^2 + z^2)$$

except where $|x|, |y|, |z| < 2$ and particles can only exist at integer values of x, y, z. If the system is still in thermal equilibrium at a temperature T, calculate the average energy.

(c) **(3 pts)** In this system, there are two particles. What is the probability that at least one of these particles will be in the ground state (energy is zero) if:

   (i) The two particles are distinct.

   (ii) The two particles are identical bosons.

   (iii) The two particles are identical fermions (fermions follow Pauli exclusion principle).

For each of the parts, assume that there are no other interactions (e.g. electromagnetism) and focus mainly on a statistical argument.

(d) **(2 pts)** Rank the probabilities in part (c) from highest to lowest when the temperature is

   (i) high

   (ii) low

For each, explain qualitatively why this must be true.

Problem 8: Radiation

The Nobel Prize in Physics 2019 was awarded for providing a new understanding of the universe's structure and history, and the first discovery of a planet orbiting a solar-type star outside our solar system.
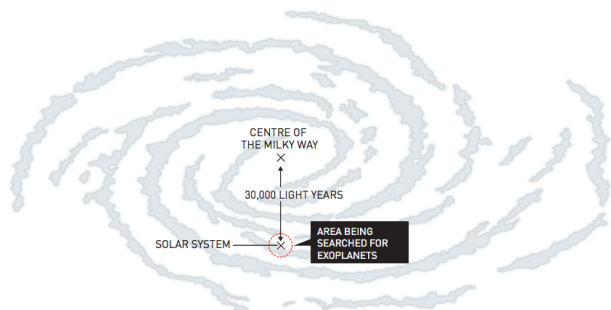
The Sun is one of several hundred billion stars in our home galaxy, the Milky Way, and there should be planets orbiting most of those stars. So far, astronomers have discovered over 4,000 planets around other stars and they are continuing their search in the area of space closest to us.

Since ancient times, humans have speculated whether there are worlds like our own, with points of views at the extremes expressed thousands of years ago. In modern times, the possibility of observing planets orbiting stars other than the Sun was proposed more than 50 years ago, and has grown into a vast and ever-expanding theory to make the evolution of the universe more clear to us than ever before. In 1995, the very first discovery of a planet outside our solar system, an exoplanet, orbiting a solar-type star was made. This discovery challenged our ideas about these strange worlds and led to a revolution in astronomy. The more than 4,000 known exoplanets are surprising in their richness of forms, as most of these planetary systems look nothing like our own, with the Sun and its planets. These discoveries have led researchers to develop new theories about the physical processes responsible for the birth of planets.

(Taken from the Nobel Prize in Physics 2019 summary, and the Laureates' popular science and scientific views.)

In this problem, we analyse and create a model for a system of two fictitious celestial bodies: an exoplanet and a solar-type star. Unless specified otherwise, consider the two bodies to be solely in each other's gravitational influence and rotate about their barycentre. In the three parts that follow, we will model the physics of a star, of the star-planet model, and the planet respectively.

Part A

The star, with mass $M_s = 2M_\odot$ (twice the mass of our Sun) and radius $R_\odot$ uses nuclear fusion reactions to provide pressure against gravity and electron degeneracy pressure, so as to maintain hydrostatic equilibrium in the star. As long as the hydrostatic equilibrium is preserved, the star is said to be in "main sequence". However, once the energy from the reactions taking place in its core start running out, the star's outer layers swell out to form a red giant. The core of the star (having a radius R_c) starts to shrink, becoming hot and dense; the temperature of the core rises to over a 100 billion degrees, and the pressure from the proton-proton interactions in the core exceeds that of gravity, causing the core to recoil out from the heart of the star in an explosive shock wave. In one of the most spectacular events in the Universe, the shock propels the material away from the star in a tremendous explosion called a supernova. The material spews off into interstellar space.

Being solar-type, this star has the same proton-proton nuclear fusion chain reaction as our Sun: essentially, this is conversion of four protons into 1 He nucleus. The star is said to have a "stable lifetime" as long as it is in its "main sequence". The temperature of the star's outermost surface is given by $T_\odot$, and the energy emitted by the sun passing a sphere of radius r per unit time is a constant $P(r)$. Neglect any convection effects in the star.

(a) **(3 pts)** Determine an expression for, during the main sequence of the star:

   (i) the number of protons being fused together per second. What quantities (may or may not be given) will this rate depend on?

   (ii) the stable lifetime of the star, assuming 100% mass of the star can undergo nuclear fusion. Assume the only fusion is between protons.

(b) **(3 pts)** Find the temperature gradient dT/dr of the star as a function of the radial distance r from the center, such that $R_c \leq r \leq R_\odot$ if the star is in its main sequence, or a hydrostatic equilibrium. Neglect any quantum-mechanical pressure effects such as electron degeneracy pressure. State all assumptions.

(c) **(0.5 pts)** What is the temperature at $r = R_c$, the outermost layer of the core?

(d) **(2 pts)** Treating the solar-type star as a perfect black-body, confirm that the temperature of the surface of the star is $T_\odot$ by integrating over all wavelengths.

(e) **(1 pt)** From experiments, it was found that the temperature gradient of the star is actually

$$\frac{dT}{dr} = -\frac{9kGM_\odot cP(r)}{256\pi^2\sigma r^2 T^3}$$

where the modulus of k is one, and has appropriate dimensions. For what value of $P(r)$ will the star's main sequence end, leading to the formation of a supernova?

Part B

In this part, we will analyse the radiation effects from the star onto the exoplanet. Assume only black body radiation from the star on the exoplanet. The density of the planet is ρ.

(f) **(2 pts)** The distance between the star and the exoplanet is r_{SE}. If the surface of the exoplanet has a constant and uniform reflectance R. What is the force exerted by the radiation from the star on the exoplanet? For the exoplanet's gravitational force to completely balance out the radiation force, how large must the radius of the exoplanet r_E be? Comment on your results and their feasibility.

Part C

(g) **(1 pt)** Find the temperature T_0 of the outermost surface of the planet. Assume a perfect black body.

(h) **(7 pts)** Model the exoplanet to be made up of N concentric shells equally spaced across the volume of the planet. Between the shells is a peculiar kind of thick type of tectonic rocks which only allow certain transmissions of energy. The emissivity of all the shells are the same, and is equal to ε. Reflection and absorption of any energy due to radiation, however, might take place. Assume all conduction and convection effects to also be negligible. The temperature gradient of the exoplanet is represented by

$$T(r) = T_0 \left(1 - \frac{n}{10N}\right)$$

where n is the nth shell from the centre of the planet and T_0 is an appropriate constant as calculated from the previous part. Calculate the total thermal energy due to radiation falling on the outermost shell per unit time. Neglect the effects of the star.

Problem 9: Piston Gun

In this problem, we examine a model for a certain type of gun that works by using the expansion of a gas to propel a bullet. We can model the bullet as a piston. Since we are assuming atmospheric pressure is negligible, we can assume that the whole setup is in a vacuum. Also, the gun is insulated.

An ideal monatomic gas of initial temperature T_0 is inside a long cylindrical container of cross-section area A. One side of the container is a wall, while the other side is a piston of mass M that can slide freely along the container without friction. The total mass of the gas is m, and it is made up of N particles. Initially, the piston is at rest and a distance L_0 away from the opposite wall. Then, the piston is released. After a time t, the piston moves at a speed v. Assume that throughout the process, the particles on average move very fast.

(a) **(2.5 pts)** Assume that m is negligible ($m \ll M$). Find v.

(b) **(3 pts)** From now on, do not assume $m \ll M$.

Find the time at which the pressure at the wall opposite the piston changes. Also, does it increase or decrease? State all assumptions.

(c) **(7 pts)** From now on, assume t is much smaller than the mean free time of the particles of the gas, and L_0 is much smaller than the mean free path. Find v.

(d) **(3 pts)** Find the recoil impulse of the gun over the time t.

(e) **(3.5 pts)** Let $r > 1$ be a dimensionless parameter. Suppose at time t, the piston is a distance rL_0 away from the wall; then the piston is stopped, and the gas is allowed to come to equilibrium (after a time much greater than the mean free time). Find the total entropy change (throughout the whole process) of the gas, and verify the Second Law of Thermodynamics.

Problem 10: Magnetostatics

Part A

In 3-D space, a permeable medium covers the region $x > 0$, while the rest of the space is vacuum. The medium's relative magnetic permeability is $\mu_r > 1$. A magnetic dipole with dipole moment m is placed a distance d away from the permeable medium, at position $(-d, 0, 0)$. The dipole is pointed towards the +x direction. Treat the dipole as ideal (point-sized).

(a) **(5 pts)** Find the force required to keep the dipole in place.

(b) **(1.5 pts)** How much work does it take to slowly pull the dipole from its original position to infinity (at $x = -\infty$)?

(c) **(2.5 pts)** How much work does it take to slowly rotate the dipole from its original orientation to one that makes an angle θ with the +x-axis?

After the dipole is rotated an angle θ, a superconducting ring with radius R and self-inductance L is brought in from infinity (with initially no current). It is placed so that the dipole is located at its center and its axis is the x-axis. Assume that $R \gg d$.

(d) **(8 pts)** Find the current I in the ring.

(e) **(4 pts)** Find the force required to hold the dipole in place (not the torque).

Part B

From now on, there is no permeable medium.

(f) **(5 pts)** The dipole (mass M), starts at a distance h from the centre of the ring and pointed towards the centre, and is projected with a small velocity v_0 towards the centre. Find its speed v as a function of h. Ignore any radiation loss and gravity.

(g) **(5 pts)** Consider another scenario, in which the dipole is placed on the axis of a thin infinite magnetic tube with surface conductivity σ, placed at an arbitrary location inside it. (You may neglect the self inductance of the solenoid for the sake of this part.) We find that the motion of the dipole in this case is damped. Find the damping parameter of this motion. (Damping parameter is defined as the ratio of the resistive force to the speed.)

(h) **(3 pts)** Let the magnetic tube have radius R. Determine the terminal velocity of the magnet, assuming that it now falls under gravity. The tube may be considered infinitely long for all calculation purposes in this part.