Solutions to Jaan Kalda's Problems in Waves and Optics

With detailed diagrams and walkthroughs

Edition 1.2.1

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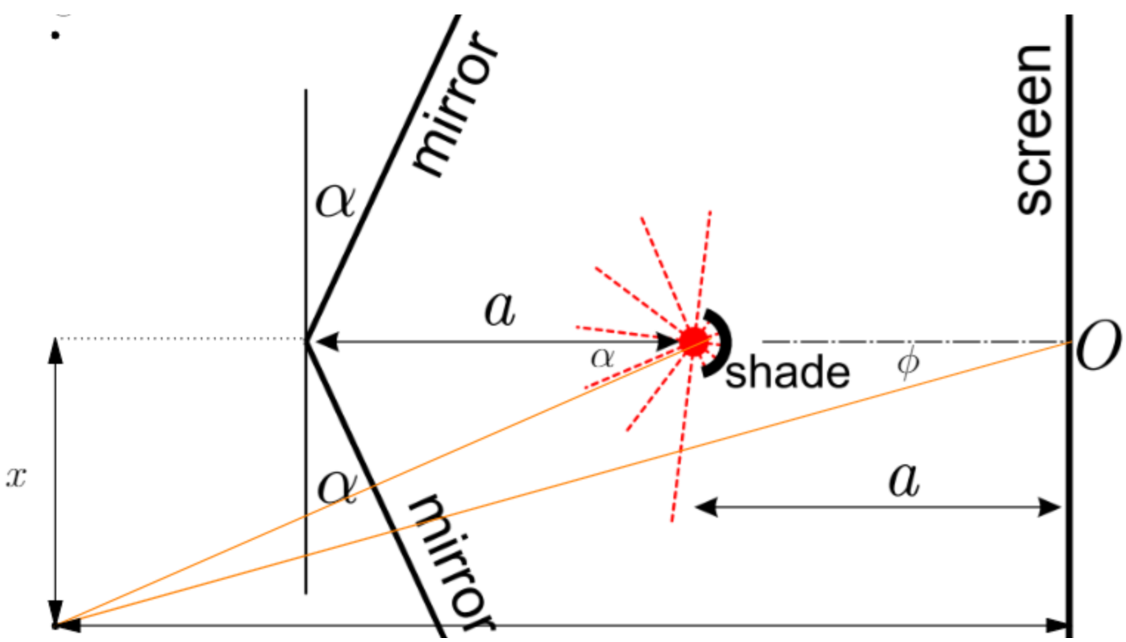

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Preface

Jaan Kalda's handouts are beloved by physics students both in for a quick challenge, to students preparing for international Olympiads. As of writing, the current waves and optics handout has 19 unique problems and 2 main 'ideas'. **Many of the problems had solutions at the back of the handout. This solutions manual attempts to solve the problems that don't have solutions**.

This solutions manual came as a pilot project from the online community at artofproblemsolving.com. Although there were detailed hints provided, full solutions have never been written. The majority of the solutions seen here were written on a private forum given to those who wanted to participate in making solutions. In an amazing show of an online collaboration, students from around the world came together to discuss ideas and methods and created what we see today.

Structure of The Solutions Manual

Each chapter in this solutions manual will be directed towards a section given in Kalda's mechanics handout. There are three major chapters: statics, dynamics, and revision problems. If you are stuck on a problem, cannot make progress even with the hint, and come here for reference, look at only the start of the solution, then try again. Looking at the entire solution wastes the problem for you and ruins an opportunity for yourself to improve.

Contact Us

Despite editing, there is almost zero probability that there are *no* mistakes inside this book. If there are any mistakes, you want to add a remark, have a unique solution, or know the source of a specific problem, then please contact us at hello@physoly.tech. The most current and updated version can be found on our website physoly.tech

Please feel free to contact us at the same email if you are confused on a solution. Chances are that many others will have the same question as you.

1 Solutions to Unsolved Problems Problems

This section will consist of the solutions to unsolved problems which do not have solutions at the back of the handout. These consist of problems 2, 3, 9, 15, 16, 17, 18, and 19. These problems vary from easy high school to hard physics olympiad problems.

Problem 2: The wall blocks almost all the wave front of the original wave, leaving only two points in a cross-section perpendicular to the slits (see figure below). To be precise, these are actually segments, but their size is much smaller than the wavelength; so, from the point of view of wave propagation, the segments can be considered as points. According to the Huygens principle, two point sources of electromagnetic waves of wavelength λ will be positioned into these two points (A and B). The point sources radiate waves in all the directions, and we need to study the interference of this radiation. Let us study, what will be observed at a far-away screen where two parallel rays (drawn in figure) meet.

To begin with, it is quite easy to figure out, where are the intensity maxima and minima. Indeed, as it can be seen from the figure above, the optical path difference between the two rays is $\Delta l = a \sin\varphi$. The two rays add up constructively (giving rise to an intensity maximum) if the two waves arrive to the screen at the same phase, i.e. an integer number of wavelengths fits into the interval: $\Delta l = n\lambda$. Similarly, there is a minimum if the waves arrive in an opposite phase:

$$\sin\varphi_{\text{max}} = \frac{n\lambda}{a}, \quad \sin\varphi_{\text{min}} = \left(n + \frac{1}{3}\right)\lambda/a$$

Problem 3: If the light is coherent, then the amplitude of the light emerging from each slit can individually be written as:

$$E = E_0 \cos(\omega t + \phi)$$

where ω is the frequency and ϕ is the phase shift. An alternative way of writing this is:

$$E = E_0 e^{i\omega t} e^{i\phi}$$

Its real component corresponds with the magnitude of the field that we measure at a certain time t. The complex number $e^{i\omega t}e^{i\phi}$ can be represented as a phasor which is essentially a vector with a constant magnitude of one that rotates in the complex plane at an angular frequency of ω (that is, it makes an angle $\omega t + \phi$ with the real axis). We want the sum of the three different amplitudes to sum up to zero, or:

$$E_1 + E_2 + E_3 = E_0 \cdot \left[e^{i\omega t}e^{i\phi_1} + e^{i\omega t}e^{i\phi_2} + e^{i\omega t}e^{i\phi_3}\right] = 0$$

Although this may look complicated, we can simplify it by treating them geometrically as phasors. To achieve an intensity minima of zero, we need three phasors such that their vector sum equals zero, which is equivalent to making a closed shape. Since they have the same magnitude, this shape must be an equilateral triangle. Without loss of generality, let $\phi_1 = 0$. This means $\phi_2 = \phi_3 = 2\pi k + \frac{2\pi}{3}$ where k is an integer.
The phase shift is defined as:

$$\frac{\phi}{2\pi} = \frac{\text{(path difference)}}{\lambda} \implies k + \frac{1}{3} = \frac{d\sin\theta}{\lambda}$$

where d is the separation between two arbitrary slits and θ is the angle the light makes with the horizontal. Applying this for the slit separation distances in this problem, we have:

$$a = \left(k_1 + \frac{1}{3}\right)\frac{\lambda}{\sin\theta_1}$$

$$b = \left(k_2 + \frac{1}{3}\right)\frac{\lambda}{\sin\theta_2}$$

If we assume that $a, b \ll L$ where L is the distance between the slits and the screen, then $\sin\theta_1 = \sin\theta_2$. Taking the ratio, we get:

$$\frac{a}{b} = \frac{n}{m} \equiv \frac{3k_1+1}{3k_2+1}$$

This produces a minima of zero for any integer combinations of k_1 and k_2. We will prove that for each combination, $n-m$ is a multiple of three. We have:

$$3k_2 + 1 - (3k_2+1) = 3r \implies 3(k_2 - k_1) = 3r \implies k_2 - k_1 = r$$

where r is an integer. Since k_1 and k_2 have to be integers, then r must also be an integer, proving the statement.

Problem 9: After passing through the first polarizer, the intensity of the light is

$$I_1 = \frac{1}{2}I_0$$

After passing through the second polarizer, the intensity is (by Malus' law)

$$I_2 = \frac{1}{2}I_0\cos^2(\alpha)$$

After passing through the third polarizer, the intensity is

$$\begin{aligned} I_3 &= \frac{1}{2}I_0\cos^2(\alpha)\cos^2\left(\frac{\pi}{2}-\alpha\right) \\ &= \frac{1}{2}I_0\cos^2(\alpha)\sin^2(\alpha) \\ &= \boxed{\frac{1}{8}I_0\sin^2(2\alpha)} \end{aligned}$$

Problem 15: Let d be the thickness of the film. Note that the difference in the length of the rays path is maximum when the beam is perpendicular to the surface of the film and is equal to $\Delta l_{\text{max}} = 2n_0 d$.

The minimum path is held when the beam is horizontal to the film. The difference in lengths of the paths is then

$$\Delta l_{\text{min}} = \frac{2n_0 d}{\cos\alpha} - 2d\tan\alpha$$

where α is the angle to the vertical of the film $\sin\alpha = \frac{1}{n_1}$. Therefore,

$$\Delta l_{\text{min}} = \frac{2n_0 d(1-n_0^{-2})}{\sqrt{1-n_0^{-2}}} = 2n_0 d\sqrt{1-n_0^{-2}}.$$

Changing the view direction from vertical to horizontal changes the optical path length difference by $N\lambda$ (because during this process, N interference maxima can be recorded, when the optical path length difference is an integer multiple of wavelength). Therefore,

$$2n_0 d(1-\sqrt{1-n_0^{-2}}) = N\lambda \implies d = \frac{N\lambda}{2n_0(1-\sqrt{1-n_0^{-2}})}$$

Further simplifying this result gives us

$$d = \boxed{\frac{N\lambda}{2(n_0 - \sqrt{n_0^2 - 1})}}.$$

Problem 16:

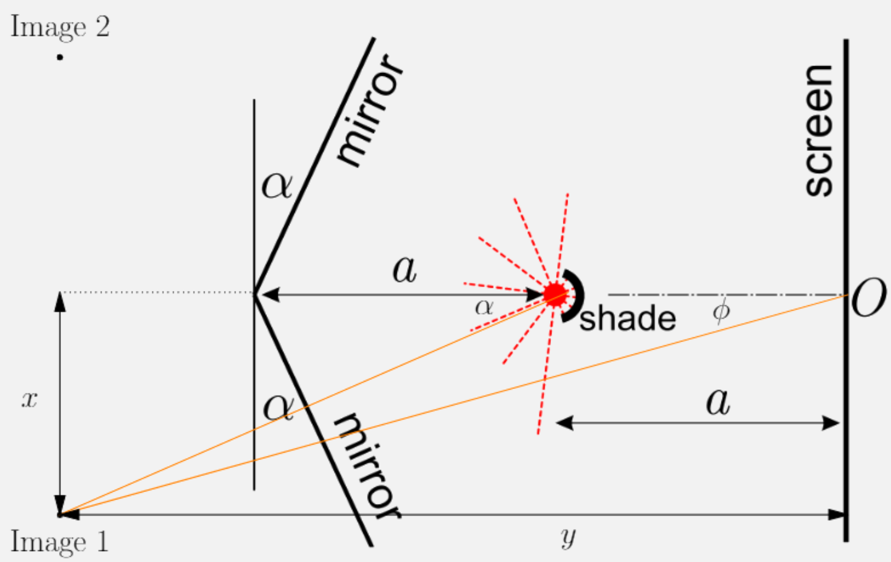
1. The direction C_2 does not go out at all because the contact area generates a fiber C direction in the same direction as the fiber B (Huygens principle can be recalled to confirm this). Anything above the wave A_2 must go in direction C_1, because the energy is preserved. The result is a mirror image (relative to the horizontal axis) of the graph in the problem text that touches $I = 0$ at the bottom and $I = I_0$ at the top.

2. At this wavelength, all light I_0 goes to fiber C_1 and must be less than α times the circulating intensity of fiber B. So $I = \frac{I_0}{\alpha} = 100I_0$.

3. The intensity of light in fiber B is maximal when the light circulating in the fiber reaches the lower contact region in the same phase as the light from fiber A. Then the intensity going to fiber C is also maximal. Thus, fiber B must accommodate an integer of n wavelengths. From the graph we see that two successive resonances occur at wavelengths $\lambda_0 = 1660$ nm and $\lambda_1 = 1680$ nm. So $\lambda_1' = (n+1)\lambda_0' = l$, where l is the desired length and the second resonant wavelength in the fiber is $\lambda_1' = \lambda_0' \frac{\lambda_1}{\lambda_0}$. From this relation we find $n^{-1} = \frac{\lambda_1'}{\lambda_0'} - 1$ and

$$l = \frac{\lambda_0' \lambda_1}{\lambda_1 - \lambda_0} = \boxed{84\ \mu\text{m}}$$

Problem 17:



We find light source images in mirrors. The light incident around O can then be viewed as a superposition of the light emitted from the images. We first find $\sin\phi$.

$$x = 2a\cos\alpha \cdot \sin\alpha, y = 2a\cos\alpha \cdot \cos\alpha + a$$

$$\tan\phi = \frac{x}{y} = \frac{2a\cos\alpha \cdot \sin\alpha}{2a\cos\alpha \cdot \cos\alpha + a}, \sin\phi = \frac{1}{\sqrt{1 + (\cot\phi)^2}}$$

$$\sin\phi = \left[1 + \left(\frac{2a\cos\alpha \cdot \cos\alpha + a}{2a\cos\alpha \cdot \sin\alpha}\right)^2\right]^{-\frac{1}{2}} = \left[1 + \left(\frac{\cos\alpha + 1/(2\cos\alpha)}{\sin\alpha}\right)^2\right]^{-\frac{1}{2}}$$

$$= \frac{\sin\alpha}{\sqrt{\sin^2\alpha + (\cos\alpha + 1/(2\cos\alpha))^2}} = \frac{\sin\alpha}{\sqrt{2 + 1/(4\cos^2\alpha)}} = \frac{\sin 2\alpha}{\sqrt{8\cos^2\alpha + 1}}$$

Let M be the minimum interference at a point near O. If we were to move up from this point d, the light path from the lower image would be increased by $d\sin\phi$ and the light from the upper image would be shortened by $d\sin\phi$. So we get a path difference $2d\sin\phi$ compared to M. This is the minimum if $\lambda = 2d\sin\phi$. So we get the answer

$$\lambda = \frac{2d\sin 2\alpha}{\sqrt{8\cos^2\alpha + 1}}$$

Problem 18: (a) By energy conservation, the amplitudes of the output wave and input wave must be the same. The output fiber wave is formed by the sum of the wave in the fiber and the wave from the other fiber. According to the energy conservation, the amplitude of each component is $\sqrt{2}$ times smaller than the original when the wave enters only one fiber. Thus, while the amplitude of the incoming waves was A, the outgoing resultant wave is in an expressible form.

$$A = \sqrt{\left(\frac{A}{\sqrt{2}}\right)^2 \cdot 2 + 2\left(\frac{A}{\sqrt{2}}\right)\left(\frac{A}{\sqrt{2}}\right)\cos\phi}$$

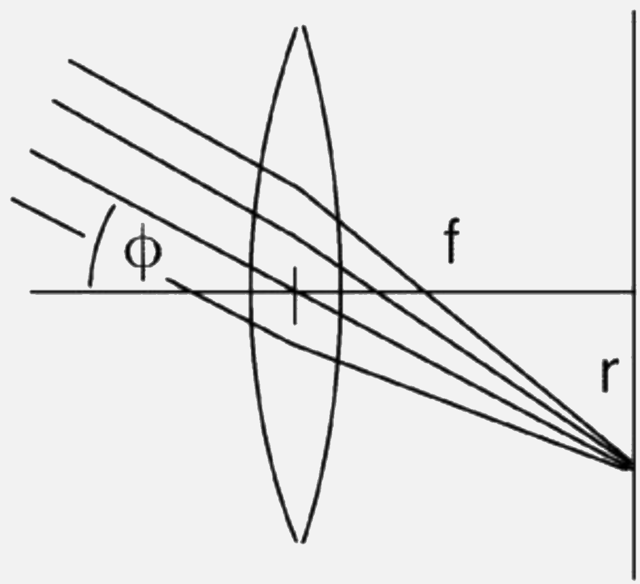
where ϕ is the phase shift. So $\cos(\phi/2) = 1/\sqrt{2}$ and consequently $\phi = \frac{\pi}{2}$

(b) Phase difference between the 2 fibers is π, the minima condition in fiber 1 is $\Delta l = n\lambda$, where n is an integer. Writing this as $n = \frac{\Delta l}{\lambda}$ we see that

$$\frac{\Delta l}{\lambda_{\text{min}}} \geq n \geq \frac{\Delta l}{\lambda_{\text{max}}}$$

thus $49.2 \geq n \geq 45.4$ and the values of n to be sought are $46, 47, 48$ and 49. The corresponding wavelengths are given by the formula $\lambda = \frac{n}{\Delta l}$; these are $612, 625, 638$ and 652 nm

Problem 19: (a) Since the lens is focused on infinity, we see that the groups of parallel arrays will focus onto the sensor.



If the first minimum of the interference pattern focuses a distance r away, then

$$\frac{d}{2}\sin\phi = \frac{\lambda}{2}$$

where d is the total width of the group of light rays. In small angle approximations, we see that

$$\tan\phi \approx \sin\phi \implies \sin\phi \approx \frac{r}{f}.$$

Going back to our original expression, we see that

$$\frac{d}{2}\frac{r}{f} = \frac{\lambda}{2} \implies d = \boxed{\frac{d\lambda}{r}}.$$

(b) There are several strange things about the picture on the paper:

- its colour is different—it is definitely not 404 nm violet, more like bluish white, longer wavelength;
- it is much brighter (could be explained by the better sensitivity of the camera to longer wavelengths, see above);
- it is 'smoother'—mainly in that speckles are not visible in it.

As long as there is light scattered off a surface, there should be speckles visible in the picture. The lack of these could be explained by an insufficient resolution, but that does not seem right: the speckle size S'measured for other images was much larger than the pixel size. Also, when light is scattered, its wavelength does not change (unlike what was observed). Therefore, the only possible answer is: it is not a photograph of the scattered laser light, but a photograph of some other light that is created within the paper by the laser light.

This can only be a fluorescent light created when a light of a suitable wavelength is absorbed by a molecule. Then, an electron is excited to a higher energy level, and later returns to a lower level while emitting a photon. This explains the observed change in wavelength. Paper often contains such fluorescent molecules in low concentrations, because by absorbing UV light and emitting visible light of a higher wavelength, they make the paper seem whiter to the human eye, which has low sensitivity to short wavelengths (digital cameras recognize colours in a similar way, to make the pictures more realistic). The time required for a single absorption–emission process is not constant, and the fluorescent molecules are often so complicated that there are many allowed transitions with similar energy difference, so there are various emission–absorption processes possible, and the emitted light wavelength is not constant. This means that the light generated at the bright spot is completely random—not coherent, not even monochromatic. Thus, every illuminated spot on the paper acts like an independent (non-coherent) point source, and therefore no speckles form.

The camera is much more sensitive to a blue light than to a violet one. Therefore, even if some of the light from the bright spot is a scattered laser light which gives rise to weak speckles, these are not bright enough to be detected by the sensor. In the wall, however, fluorescence is negligible, so speckles which arise from random reflection from the wall can be seen.

This solution is taken from the official 2012 Physics Cup Solution as it is much better at explaining things than us.