1 Isotropic visco-acoustic variable density second order self-adjoint system

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2 Introduction

This note shows the derivation of *time update equations* and the linearization for the isotropic variable density self-adjoint system. We implement attenuation with a monochromatic approximation to Maxwell bodies, and use this attenuation model to implement zero outgoing absorbing boundary conditions on the exterior of the modeling domain.

The time update equations are used to advance solutions in time, expressing the pressure wavefield at time $p_{(t+\Delta)}$ as a function of $p_{(t-\Delta)}$ and $p_{(t)}$.

2.1 Symbols

∂_t	$\dfrac{\partial}{\partial t}$
∇p	Laplacian: $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right) p$
Δ	Temporal sampling rate
ω	reference frequency for attenuation
Q	attenuation at frequency ω
P	Pressure wavefields
$S(x, y, z, t)$	Pressure source term
b	buoyancy $= 1/\rho$ (reciprocal density)
$\{\ V_p\ \}$	Material parameters

2.2 Modeling system

Equation 1 shows the modeling system with absorbing boundaries implemented using amplitude only (dissipation only, no dispersion) Q.

We apply the time derivative in the term $\frac{\omega}{Q}\partial_t p$ using a backward one-sided numerical difference. We tested both forward one-sided and centered difference alternatives and found them to be less stable.

$$\frac{1}{v^2}\left(\partial_t^2 p + \frac{\omega}{Q}\partial_t p\right) = \nabla p + S \tag{1}$$

3 Time update equations

3.1 Time update numerical difference formulas, first and second order

$$\partial_t p = \frac{1}{\Delta}\left[p_{(t)} - p_{(t-\Delta)}\right] \tag{2}$$

$$\begin{aligned} \partial_t^2 p &= \frac{1}{\Delta^2}\left[p_{(t+\Delta)} - 2p_{(t)} + p_{(t-\Delta)}\right] \\ p_{(t+\Delta)} &= \Delta^2 \partial_t^2 p + 2p_{(t)} - p_{(t-\Delta)} \end{aligned} \tag{3}$$

3.2 Rearrange equation 1 for $\partial_t^2 p$

$$\partial_t^2 p = v^2\left(\nabla p + S\right) - \frac{\omega}{Q}\partial_t p \tag{4}$$

3.3 Apply equations 2 and 3, and rearrange

$$p_{(t+\Delta)} = \Delta^2 v^2\left(\nabla p + S\right) - \Delta\frac{\omega}{Q}\left[p_{(t)} - p_{(t-\Delta)}\right] + 2p_{(t)} - p_{(t-\Delta)} \tag{5}$$

4 Linearization and Born modeling equation

4.1 Nonlinear modeling equation

$$\frac{1}{v^2}\left(\partial_t^2 p + \frac{\omega}{Q}\partial_t p\right) = \nabla p + S \tag{6}$$

4.2 Taylor expand $\frac{1}{v^2} \to \left(\frac{1}{v_0^2} - \frac{2}{v_0^3}\delta v\right)$ and replace $p \to (p_0 + \delta p)$

$$\left(\frac{1}{v_0^2} - \frac{2}{v_0^3}\delta v\right)\left(\partial_t^2 p_0 + \partial_t^2 \delta p + \frac{\omega}{Q}\partial_t p_0 + \frac{\omega}{Q}\partial_t \delta p\right) = \nabla p_0 + \nabla \delta p + S \tag{7}$$

4.3 Expand

$$\frac{1}{v_0^2}\partial_t^2 p_0 + \frac{1}{v_0^2}\partial_t^2 \delta p + \frac{1}{v_0^2}\frac{\omega}{Q}\partial_t p_0 + \frac{1}{v_0^2}\frac{\omega}{Q}\partial_t \delta p - \frac{2}{v_0^3}\delta v \partial_t^2 p_0 - \frac{2}{v_0^3}\delta v \partial_t^2 \delta p - \frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t p_0 - \frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t \delta p = \nabla p_0 + \nabla \delta p + S \tag{8}$$

4.4 Cancel reference terms

$$\cancel{\frac{1}{v_0^2}\partial_t^2 p_0} + \frac{1}{v_0^2}\partial_t^2 \delta p + \cancel{\frac{1}{v_0^2}\frac{\omega}{Q}\partial_t p_0} + \frac{1}{v_0^2}\frac{\omega}{Q}\partial_t \delta p - \frac{2}{v_0^3}\delta v \partial_t^2 p_0 - \frac{2}{v_0^3}\delta v \partial_t^2 \delta p - \frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t p_0 - \frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t \delta p = \cancel{\nabla p_0} + \nabla \delta p + \cancel{S} \tag{9}$$

4.5 Zero terms second order in perturbations $\delta p, \delta v$

$$\frac{1}{v_0^2}\partial_t^2 \delta p + \frac{1}{v_0^2}\frac{\omega}{Q}\partial_t \delta p - \frac{2}{v_0^3}\delta v \partial_t^2 p_0 - \cancel{\frac{2}{v_0^3}\delta v \partial_t^2 \delta p} - \frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t p_0 - \cancel{\frac{2}{v_0^3}\delta v \frac{\omega}{Q}\partial_t \delta p} = \nabla \delta p \tag{10}$$

4.6 Rearrange for the Born modeling equation

$$\begin{aligned}\frac{1}{v_0^2}\left(\partial_t^2 \delta p + \frac{\omega}{Q}\partial_t \delta p\right) &= \nabla \delta p + \frac{2}{v_0^3}\delta v \left(\partial_t^2 p_0 + \frac{\omega}{Q}\partial_t p_0\right) \\ &= \nabla \delta p + \frac{2}{v_0^3}\delta v \left(v_0^2 \nabla p_0 + v_0^2 S\right)\end{aligned} \tag{11}$$

Note: it is procedurally simplest to serialize the quantity $\left(v_0^2 \nabla p_0 + v_0^2 S\right)$ when we perform nonlinear forward modeling, and use that field as the Born source for the linearized forward and adjoint operators.