Mean-field interactions in evolutionary spatial games

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We introduce a mean-field term to an evolutionary spatial game model. Namely, we consider the game of Nowak and May, based on the Prisoner's dilemma, and augment the game rules by a mean-field term. This way, an agent operates based on both local information from its neighbors and non-local information via the mean-field coupling. We simulate the model and construct the steady state phase diagram, which shows significant new features due to the mean-field term. **TODO: rewrite**

Introduction— The mean-field approximation, initially devised in 1907 in the context of (ferro)magnetic properties of materials [1], is since playing a truly central role in a variety of branches of physics, ranging from superconductivity [2] to ferromagnetism of disordered metals [3]; from ultracold quantum gases [4] to spin glasses [5], to name just a few. The main idea—that a behavior of a collection of a macroscopic number of strongly interacting components can be reduced to a single-particle problem in a self-consistent external potential—proves fruitful for multiple research fields outside of traditional physics: for instance describing spatially extended networks [6], droplet traffic [7], traffic flow [8], and social dilemmas [9].

In this Letter, we demonstrate how mean-field ideas can be applied to a system which does not have an explicit representation in the language of statistical mechanics. Instead it is a dynamical system with a steady state, i.e., a long-lived state with a well-defined "thermodynamic limit" of the infinite system size. Introduction of the mean-field interaction leads to a nontrivial diagram of steady states and opens new directions for possible application of structured multi-agent systems.

Studying dynamics of ensembles of agents is a common theme in biology, economics, and social sciences. Collective behavior emerges through repeated pairwise interactions between individuals, where each agent acts to maximize its fitness. The evolutionary game theory approach is to model elementary interactions as game-theoretic contests between agents having varying strategies [10]. For well-mixed populations—in physics language this corresponds to a mean-field approximation—the macroscopic description proceeds via the master equation for the time evolution of populations of strategies, the so-called replicator equation [11, 12].

One limitation of the replicator equation approach is that it does not allow for *structured* populations, where the interaction range is restricted to a local neighborhood. Such populations are known to exhibit long-time behavior beyond standard replicator dynamics [13]. Various local arrangements have been considered: following the pioneering work [14], noisy dynamics was studied on regular, decorated and frustrated two-dimensional lattices [13, 15, 16] and coevolving random networks [17]. Deterministic imitation dynamics have been investigated on the regular square [18, 19] and triangular lattices [20], on a simple cubic lattice 3D [21], and diluted 2D lattices [19, 22].

In this Letter, we construct an evolutionary game model which features both kinds of effects: purely local interactions with nearest neighbors, *and* a self-consistent, mean-field-type coupling to the order parameter. We start with the classic game of Nowak and May (NM) [14, 26], where the order parameter can be chosen as the mean density of cooperators $\langle f_c \rangle$ in the steady-state regime of evolution. The mean-field term is the coupling to the instantaneous density of cooperators $f(t)$, which fluctuates due to the game dynamics and has a well defined mean value in the steady-state regime. The addition of the MF term to the decision-making rule is quite realistic and can be interpreted as the influence of the "averaged strategy of society" or a mass-media influence which both reflects and drives opinions in the society. In the limit of vanishing mean-field (MF) term, the phase diagram reproduces known results.

We note that our approach is very different from what is known as mean-field games in mathematical economics [23]: the latter is a class of stochastic differential games (which, in turn, can be mapped onto the nonlinear Schroedinger equation [24]). In our work, we only consider discrete games based on the Prisoner's dilemma.

The Prisoner's dilemma.— Consider two agents, α and β, which interact via the rules of the Prisoner's dilemma game. An agent is characterized by a *strategy*, which takes one of two values: cooperate, $\mathcal{C}$, or defect, $\mathcal{D}$. In an "interaction", agents receive payoffs which depend on their strategies [25]. Encoding strategies of an agent by length-two vectors, $\vec{s}$, such that $\mathcal{C} = (1,0)^T$ and $\mathcal{D} = (0,1)^T$, the payoff of an agent α is $P_\alpha = \vec{s}_\alpha^{\,T} \widehat{H} \vec{s}_\beta$, and the payoff of the agent β is $P_\beta = \vec{s}_\beta^{\,T} \widehat{H} \vec{s}_\alpha$, where $\widehat{H}$ is the payoff matrix, and $\vec{s}_\alpha$ and $\vec{s}_\beta$ are the states of agents α and β, respectively. Following Ref. [14], we take

$$\widehat{H} = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} . \tag{1}$$

Essentially, Eq. (1) means that in a $\mathcal{C}$-$\mathcal{C}$ interaction, both agents receive a unit payoff (we set it to unity to fix the payoff scale) and in a $\mathcal{D}$-$\mathcal{C}$ interaction, the $\mathcal{D}$ receives a payoff of b. All other payoffs we set to zero, so that the game is governed by a single parameter, b.

The rules of the Nowak-May spatial game [14, 26]. — Consider a population of L^2 players arranged at the vertices of a regular $L \times L$ square lattice. The game is played at discrete time steps, $t = 0, 1, 2, \ldots, M$. The transition of the system from time t to the next time step $t+1$ consists of the update of strategies of all players in parallel (we call this a game round). At time step t, an agent located at the lattice site $\mathbf{x}$ "interacts" with its neighbors and receives a total payoff,

$$P_{\mathbf{x}}(t) = \sum_{\mathbf{y} \in \text{n.n.}} \vec{s}_{\mathbf{x}}^{\,T} \widehat{H} \vec{s}_{\mathbf{y}} \;, \tag{2}$$

where $\vec{s}_{\mathbf{x}} \equiv \vec{s}_{\mathbf{x}}(t)$ is the strategy of an agent $\mathbf{x}$ at a time step t, the summation runs over the eight neighbors of the agent $\mathbf{x}$ (similar to the chess king moves)[32]. After all payoffs are calculated, each player changes its state, $\vec{s}_{\mathbf{x}}(t) \to \vec{s}_{\mathbf{x}}(t+1)$, to accept the strategy of their neighbor with the largest payoff,

$$\max\{P_{\mathbf{x}}(t), \max_{\mathbf{y} \in \text{n.n.}} \{P_{\mathbf{y}}(t)\}\} \;. \tag{3}$$

This concludes the game round, and the game repeats with the new strategies of players.

Note that the game is fully deterministic given the initial state of all strategies at $t = 0$. The strategy of a player $\mathbf{x}$ at the next time step, $\vec{s}_{\mathbf{x}}(\text{t}+1)$, depends on the current state of itself, $\vec{s}_{\mathbf{x}}(t)$, its eight neighbors, and their neighbors—i.e. on states of 25 players in total. In the cellular automata language, the spatial evolutionary game of Nowak-May (NM game) is a cellular automaton with the single-player's transition matrix containing 2^{25} rules.

The steady state of the spatial evolutionary game.— The game starts with some initial distribution of strategies at $t = 0$ and proceeds repeatedly. After some transition time the system reaches a steady state, with the nature of the steady state strongly dependent on the value of the game parameter b [14]. Following Ref. [14], we characterize the instantaneous state of the game by the density of cooperators, $f_c(t)$, and the steady state is characterized by the mean density of cooperators $\langle f_c \rangle$:

$$\langle f_c \rangle \;=\; \lim_{T \to \infty} \frac{1}{T - \tau} \sum_{t=\tau}^{T} f_c(t) \;, \tag{4}$$

$$f_c(t) \;=\; \frac{1}{L^2} \sum_{\mathbf{x}} \delta\left(\vec{s}_{\mathbf{x}}(t) = \mathcal{C}\right) \;. \tag{5}$$

The summation in Eq. (5) runs over the states of all agents at time t, and $\delta\left(\vec{s}_{\mathbf{x}}(t) = \mathcal{C}\right)$ is an indicator variable taking the value of 1 if $\vec{s}_{\mathbf{x}}(t) = C$ and zero otherwise. In other words, Eq. (5) is nothing but the number of cooperators divided by the total number of agents. In Eq. (4), T is the total number of game rounds, and τ is the burn-in time such that $|\langle f_c \rangle - f_c(t)| \ll \langle f_c \rangle$ for $t > \tau$.

The mean-field modification.— In this Letter, we include the mean-field term as follows: instead of Eq. (2) we consider the payoffs of the form

$$P_{\mathbf{x}}(t) = \sum_{\mathbf{y} \in \text{n.n.}} \vec{s}_{\mathbf{x}}^{\,T} \widehat{H} \vec{s}_{\mathbf{y}} + \lambda\, \vec{s}_{\mathbf{x}}^{\,T} \widehat{H}_{\text{mf}} \vec{s}_{\mathbf{x}} \;, \tag{6}$$

where

$$\widehat{H}_{\text{mf}} = \begin{pmatrix} f_c(t) & 0 \\ 0 & b\, f_c(t) \end{pmatrix} \;. \tag{7}$$

In other words, the interaction of a player with its neighbors is identical to the the NM game, Eq. (2), and additionally we introduce a coupling to the instantaneous density of cooperators, $f_c(t)$. This way, transition rules become non-local in space due to the dependence on $f_c(t)$, the density of cooperators at the current time step. We stress that here $f_c(t)$ is the instantaneous density so that the game is memory-less.

In Eqs. (6)-(7), λ is the relative strength of the MF coupling. The limit $\lambda \to 0$ is simply the NM game, Eq. (2). In this work only consider $\lambda = 1$, so that b is the only parameter that governs the game dynamics.

The transition from time step t to $t+1$ is identical to the NM game: After all payoffs are calculated, each agent changes its strategy according to Eq. (3), and the game repeats with the new strategies of players.

Dynamical regimes.— The discrete structure of the payoffs in the NM game (1)-(2) leads to a very specific dependence of the game dynamics on the payoff parameter b. Comparing the payoffs of $\mathcal{C}$ and $\mathcal{D}$ in the neighborhood of an agent, one finds for the NW game [18, 26] that the dynamics of the game changes at the values of $b = m/n$ with $1 \leqslant m, n \leqslant 8$. Here m and n are integers and are related to the numbers of $\mathcal{C}$ in a local neighborhood. The steady-state density of cooperators, $\langle f_c \rangle$ takes discontinuities at these special values of b, as shown in Fig. 1(a).

For the mean-field game (6)-(7) (MF game), a similar comparison of the payoffs of the neighboring agents gives $(m + f_c(t)) \gtrless b(n + f_c(t))$, which self-consistently depends on the density of cooperators at time t. Since in the steady state we expect $f_c(t) \approx \langle f_c \rangle$, we conjecture that the steady state density is related to the payoff parameter via

$$b = \frac{m + \langle f_c \rangle}{n + \langle f_c \rangle} \;, \tag{8}$$

where m and n are integers and $1 \leqslant m, n \leqslant 8$. Note that Eq. (8) cannot be directly interpreted as giving the boundaries of dynamical regimes. To clarify the status of Eq. (8) we compare its predictions to direct numerical simulations.

Numerical simulations.— We simulate the dynamics of the game on square lattice with linear size $L = 200$. We use periodic boundary conditions to minimize finite-size effects. Additional simulations with system sizes up to $L = 800$ demonstrate no significant finite size effects. Therefore we believe our simulations correspond to the thermodynamic limit of the model. For all values of b, we simulate 40 runs with random initial distributions of $\mathcal{C}$ and $\mathcal{D}$ using the initial density of cooperator states f_0. Each configuration converges to a particular steady state, and we compute averages with respect to both time fluctuations in the steady state and realizations of initial conditions. The steady-state behavior is largely insensitive to the precise value of f_0, and we fix $f_0 = 0.9$ in most simulation runs [33] , which is the value used in Refs [18, 26]. The number of game rounds is typically a few thousand for the burn-in time towards the steady state, τ, cf Eq. (8), which is larger than the typical relaxation time for the initial state. We further collect statistics for a few thousand further game rounds, T.

Figure 1(a) shows the dependence of the steady state mean density of cooperators, $\langle f_c \rangle$, on the payoff parameter b for both NM and MF games. The values of $\langle f_c \rangle$ are essentially identical outside of the interval $3/2 < b < 5/3$. In this interval, however, the NM and MF games are strikingly different. For the MF game, the steady state density, $\langle f_c \rangle$ varies continuously with b. This is to be contrasted with the NM game, where $\langle f_c \rangle$ only changes via discontinuous jumps at $b = 3/2$, $8/5$ and $5/3$. The values of $\langle f_c \rangle$ lies on the hyperbolas given by Expr. 8 with $(m, n) = (8, 5)$ and $(5, 3)$. The remarkable feature is the switch between them at the value $\langle f_c \rangle \approx 0.3$ which corresponds to the value $\langle f_c \rangle$ in the chaotic regime of NW game. The flat switch between two regime is reminiscent of the first order phase transitions, driven by the competition between two stable states. In our case it is the switch between two competing chaotic regimes driven by the mean-field coupling.

Spatial chaos.— A salient feature of the NM game is the appearance of the chaotic regime: for $8/5 < b < 5/3$ all agents change their strategy with typical time scales of several game rounds, while maintaining the mean density constant, $\langle f_c \rangle \approx 0.3$ [26]. (For the NM game with self-interactions, the chaotic regime occurs for $9/5 < b < 2$ with a slightly larger density, $\langle f_c \rangle \approx 12 \ln 2 - 8 \approx 0.318$ [14]).

The standard way of probing the chaotic behavior is via the so-called persistence [27–29] $P(t_1, t_2)$, which, in the present context, is essentially the fraction of players who never change their strategy between time steps t_1 and t_2. Note that we are primarily interested in the *asymptotic steady state persistence*, $P(\tau, t_2 \to \infty)$. To this end, we compute $P(\tau, T)$ with τ and T having the same meaning as in Eq. (4). We stress that $P(\tau, \infty)$ is qualitatively different from $P(0, \infty)$. The latter is sensitive to the overlap between the steady state and the initial distribution of strategies at $t = 0$.

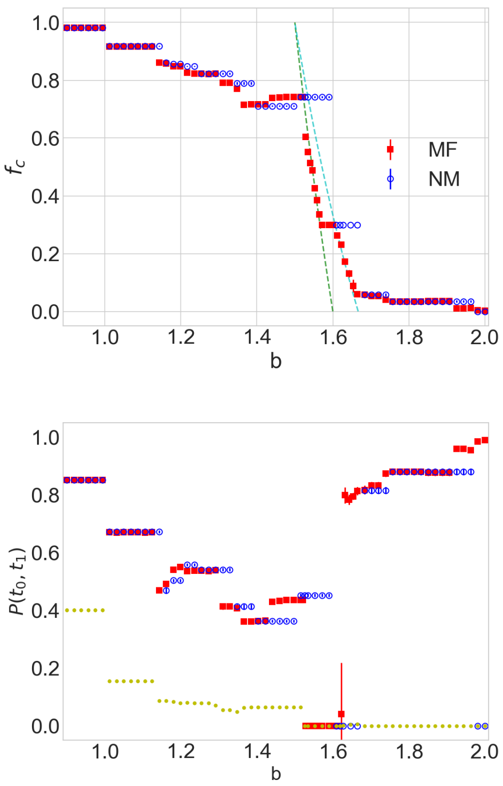


FIG. 1: (a) Steady-state mean density of cooperators $\langle f_c \rangle$ for the NM game (open blue circles) and MF game (red squares) as a function of the payoff parameter b. Errorbars are shown at all points and are typically less than symbol size. We also show hyperbolas, Eq. (8) for $m, n = (8, 5)$ (green dashed line) and $m, n = (5, 3)$ (cyan dash-dotted line). In these simulations, $f_0 = 0.9$, $L = 200$, $\tau = 10^4$, $T = 1.6 \times 10^4$. **TODO: check: 1.6 or 2.6, cf Eq (4), ANSW: Yes, it is 1.6**
(b) Steady-state persistence for the NM game (open blue circles) and the MF game (red squares). We also show $P(0, T)$ for the MF game (YELLOW circles). See text for discussion. **TODO: a) and b) place at the right middle box (between 1.8 and 2.0 and 0.4 and 0.6, symbols at a) - blue triangles, centering, plot widths, increase symbol size (c, P(t0,t1), b, and 0.0, 1.0, etc, remove lines in b)**

Fig. 1(b) compares the asymptotic persistence for the NM and MF games. Two kinds of regimes are clearly seen for both games: those with $P(\tau, \infty) \neq 0$ — where the steady state features finite clusters (or "cliques") of agents who cooperatively maintain a fixed strategy, and chaotic regimes where $P(\tau, \infty) = 0$ — where everyone changes their strategy at least once.

For the NM game, the chaotic regime occurs for $8/5 < b < 5/3$ as expected; the MF game displays chaotic steady states for $1.53 \lesssim b \lesssim 1.6$, which is exactly the range where the mean density of cooperators, $\langle f_c \rangle$, follows Eq. (8). **TODO: clarify 1.6 vs 1.63 etc: P=0 vs snapshots.**

Note that the chaotic regime for the MF game is slightly shifted towards smaller values of b as compared to the NM game. The most drastic change happens for $3/2 < b < 8/5$: the MF coupling drives a stationary steady state into spatial chaos.

We also show in Fig. 1(b) the behavior of $P(0,T)$ for the MF game. For $b < 1$, about 40% of agents keep their initial strategy. Upon increasing b, $P(0,\infty)$ jumps to below $\sim$ 20% at $b = 1$ and then further decreases before dropping off to zero at the onset of the chaotic regime. Note that $P(0,\infty)$ keeps being essentially zero for $b > 1.63$ where $P(\tau,T)$ is non-zero. This means that while the finite fraction of agents stabilizes, the steady state of an agent is not related to the initial state at $t = 0$ which is forgotten over first $< \tau$ time steps.

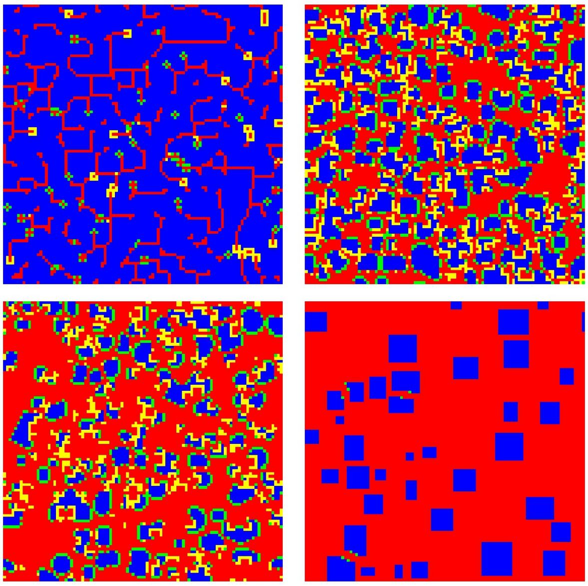


FIG. 2: Typical snapshots of the steady-state configurations of players for the MF game. Here $L = 100$, and values of $b = 1.2$ (top left), $b = 1.55$ (top right), $b = 1.59$ (bottom left) and $b = 1.63$ (bottom right). Blue (light grey) color corresponds to the cooperator state $\mathcal{C}$, red (dark grey) color is the defector state $\mathcal{D}$, green color is a $\mathcal{C}$ which was a $\mathcal{D}$ at the previous time step, and yellow is a $\mathcal{D}$ which was a $\mathcal{C}$.

Game field configurations.— Fig. 2 shows representative snapshots of steady states of the game field [34] of the MF game depending on the payoff parameter b. For $b < 1.53$ typical configurations feature web-like structures of $\mathcal{D}$ players. These webs are random, i.e. precise positions of the branches vary depending on an initial state of the field.

Upon reaching the steady state, however, these webs remain mostly static, with possible "blinking" players at the interfaces. The widths of the branches of defectors increases with increasing the payoff parameter b, and at around $b \approx 1.53$—which is the onset of the chaotic regime— the $\mathcal{D}$ webs melt into collections of smaller clusters of $\mathcal{C}$ and $\mathcal{D}$, which are no longer static and grow, shrink and collide instead.

Upon further increase of the payoff parameter, for $1.6 < b < 1.62$ **TODO: 1.6 vs 1.62: P=0 vs snapshots** the field is dominated by square-shaped clusters of $\mathcal{C}$. Isolated clusters grow in all directions, and disintegrate upon colliding with neighboring clusters. For $b > 1.63$ the fields feature static collections of small square-shaped clusters of $\mathcal{C}$ embedded into the $\mathcal{D}$ background.

It is instructive to examine the distribution of the sizes of clusters of $\mathcal{C}$ and $\mathcal{D}$ in different steady state regimes. Figure 3 shows the exponential decay in the distribution of the cluster sizes in chaotic regimes, $b = 1.55$ and 1.59, while it is slower than exponential for $b = 1.2$, the regime with the long memory of initial states (see Fig. 1(b)).

Asymptotically, the fractal dimension of the clusters as well as of the interfaces between clusters of cooperators $\mathcal{C}$ and defectors $\mathcal{D}$ tends to the dimension of the plane, $d = 2$, which also coincides with the scaling dimension of masses of clusters [18]. These interfaces form a special type of random fractals, quite different from those at the second-order or first order phase transitions [30].

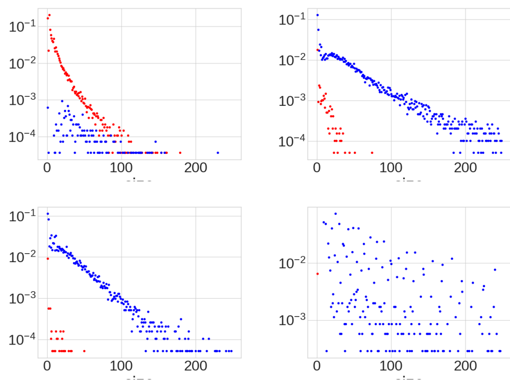
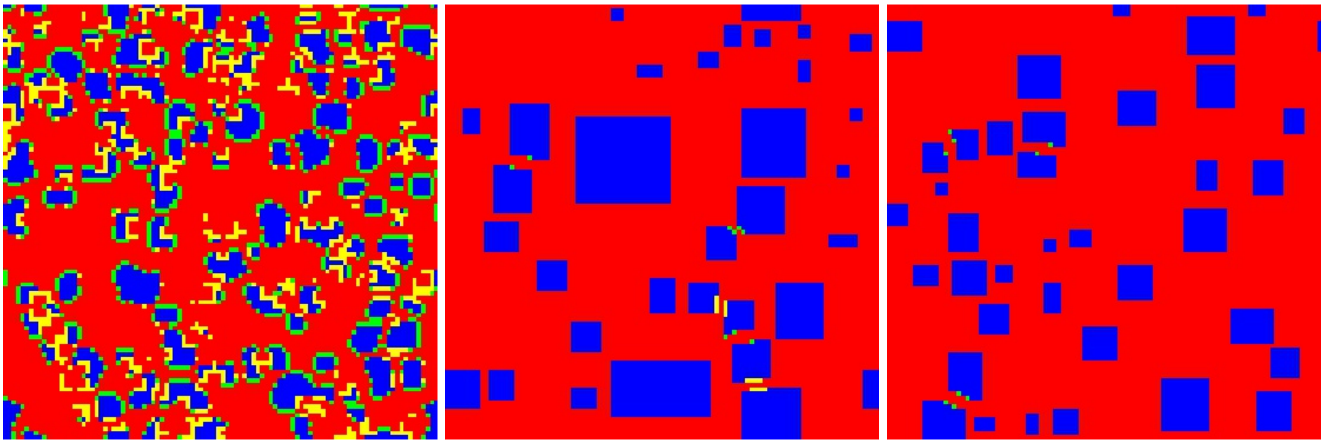


FIG. 3: Distribution of the cluster sizes (i.e. the number of agents in a cluster) of $\mathcal{D}$ (red) and $\mathcal{C}$ (blue) in the steady states for $L = 200$, and values of $b = 1.2$ (top left), $b = 1.55$ (top right), $b = 1.59$ (bottom left) and $b = 1.63$ (bottom right). We only show clusters of up to 250 agents.

Discussion.— We propose a way of introducing self-consistent mean-field-like couplings into evolutionary games with spatially structured populations. We perform direct numerical simulations of a simplest yet non-trivial model and find variety of steady states: chaotic



(a) $b = 1.6$ (b) $b = 1.62$ (c) $b = 1.63$

FIG. 4: Typical snapshots of the steady-state configurations of players for the MF game. Here $L = 100$, and values of $b = 1.6$, $b = 1.62$, $b = 1.63$ (bottom left) and $b = 1.63$ (bottom right). Blue (light grey) color corresponds to the cooperator state $\mathcal{C}$, red (dark grey) color is the defector state $\mathcal{D}$, green color is a $\mathcal{C}$ which was a $\mathcal{D}$ at the previous time step, and yellow is a $\mathcal{D}$ which was a $\mathcal{C}$.

states and self-organized stationary states where a finite fraction of agents form stable clusters (or "cliques"). These states are not critical, because cluster sizes are distributed exponentially.

We find that the mean-field coupling can induce qualitative changes for chaotic states and stationary states which are parametrically close to chaotic.

Introduction of the mean-field type couplings opens several intriguing avenues for possible future work. It would be interesting to clarify the effect of the strength of the MF coupling, $\lambda \neq 1$. Non-uniform, site-dependent couplings (e.g. $\lambda_{\mathbf{x}}$ drawn from some random distribution) would reflect natural variations between individuals. The role of the local connectivity of the population needs to clarified: this includes both higher-dimensional regular lattices and disordered and hierarchical random lattices and graphs. One may introduce an external field. It would be interesting to see the interplay between the mean-field couplings and noisy decision rules (e.g. where the decision rules contain pseudo-temperature [13, 15]).

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[32] Refs. [14, 18] also considered a modification of Eq. (2) to include a self-interaction term, $\vec{s}_{\mathbf{x}}^T \widehat{H} \vec{s}_{\mathbf{x}}$. This modification does not lead to any qualitative differences. We therefore follow Ref. [26] and do not include self-interaction.

[33] At lower values of f_0 there are initial states where all $\mathcal{C}$ are isolated at $t = 0$ and thus immediately disappear at $t = 1$, resulting in $\langle f_c \rangle = 0$. When these initial states are discarded, the steady state observables are independent on f_0.

[34] Here we use $L = 100$ for clarity only: qualitative features of snapshots are independent of L.