Empirical Comparison of Option Pricing Models

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Pricing Models

European Options:

- **Black Scholes** Model (GMB)
- Merton's **Jump Diffusion** (GMB with Poisson)

American Options:

- **Cox-Ross-Rubinstein** Model (Binomial Tree)
- **Longstaff-Schwartz** Model (Least Squares Monte Carlo)

Black Scholes

$$S(t_{i+1}) = S(t_i) \exp\left(\left(\mu - \frac{\sigma}{2}\right)\Delta t + \sigma\sqrt{\Delta t} Z_{i+1}\right)$$

Jump Diffusion

$$S(t_{i+1}) = S(t_i) \exp\left(\left(\mu - \frac{\sigma}{2}\right)\Delta t + \sigma\sqrt{\Delta t} Z_{i+1} + aN_{i+1} + b\sqrt{N_{i+1}} Z'_{i+1}\right)$$

$$\text{with } N_i \sim \mathrm{Pois}(\lambda \Delta t)$$

American Options

Allow for **early exercise**

Value at each time step: maximum between early exercise value and continuation value.

For an **American call** option:

$$O_t^i = \max\left(\underbrace{(S_t^i - K)^+}_{\text{early exercise}},\ \underbrace{e^{-(T-t)r}(E[S_T|S_t^i] - K)^+}_{\text{continuation value}}\right)$$

Binomial Trees 1: Simulate Stock Value

Stock price can:

- Move up to uS_0 with probability q
- Move down to dS_0 with probability $(1-q)$

$$u = e^{\sigma\sqrt{\Delta t}} \qquad d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \qquad q = \frac{e^{r\Delta t} - d}{u - d}$$

Fill the best outcome nodes with $S_{i,0} = uS_{i-1,0}$ and the others with $S_{i,j} = dS_{i-1,j-1}$

	0	1	2	3	4
0	1.000	0.000	0.000	0.000	0.000
1	1.221	0.819	0.000	0.000	0.000
2	1.492	1.000	0.670	0.000	0.000
3	1.822	1.221	0.819	0.549	0.000
4	2.226	1.492	1.000	0.670	0.449

Binomial Trees 2: Compute Option Price

Start from end of tree and move upwards:

At maturity, compute value as $\max(S_{T,j} - K, 0)$

For each previous node compute

$$O_{i,j} = \max\left(e^{-r\Delta t}(qO_{(i+1),j} + (1-q)O_{(i+1),(j+1)}),\ S_{i,j} - K\right)$$

	0	1	2	3	4
0	0.302	0.000	0.0	0.0	0.0
1	0.427	0.104	0.0	0.0	0.0
2	0.598	0.162	0.0	0.0	0.0
3	0.827	0.252	0.0	0.0	0.0
4	1.126	0.392	0.0	0.0	0.0

Why do we need the Longstaff-Schwartz algorithm?

- How can we evaluate the **continuation value** $E[S_T|S_t^i]$?

  Nested Monte Carlo simulations → **unfeasible** for large numbers

- *Binomial Tree:* **discretization** error if used with long time steps.

  Will **underestimate** the number of **early exercise opportunities** as it only provides two outcomes for the value of the underlying.

  Time complexity: $O(2^n)$

LSMC explanation: an example

We take as example an **American call** option with 1 year maturity, exercisable at times 1,2,3:

$$S_0 = 1,\ K = 1.1,\ r = 0.1,\ \sigma = 0.2$$

	t = 0	t = 1	t = 2	t = 3
0	1.0	1.15	0.92	1.05
1	1.0	1.20	1.19	1.33
2	1.0	0.96	0.97	1.04
3	1.0	0.96	1.19	1.46
4	1.0	1.04	1.02	0.99
5	1.0	1.07	1.02	1.08
6	1.0	1.13	1.18	1.36
7	1.0	1.08	1.18	1.19

Step 1

Setup **cash flow matrix**:
Determine expected payoff at **maturity**.
Since continuation value is zero = payoff of a vanilla European option

	0	1	2	3
0	0.0	0.0	0.0	0.00
1	0.0	0.0	0.0	0.23
2	0.0	0.0	0.0	0.00
3	0.0	0.0	0.0	0.36
4	0.0	0.0	0.0	0.00
5	0.0	0.0	0.0	0.00
6	0.0	0.0	0.0	0.26
7	0.0	0.0	0.0	0.09

Step 2

- One time step back: consider the paths were the option is **in the money** at $t = T - 1$.
- Discount the future **cash flow** of holding the option: $y_{t=2,i} = e^{-r}\pi_{t=3,i}$
- Get value of underlying at time T-1

	Disc val	S_2
0	0.23	1.19
1	0.35	1.19
2	0.25	1.18
3	0.09	1.18

Step 3

- Regress $y_{t=2}$ on a set of basis functions of $S_{t=2}$ to obtain the **continuation value**

$$\hat{C}_{t,i} = \sum_{j=0}^{n} a_{t,j} B_j(S_{t,i})$$

The parameters a_t are obtained minimizing

$$\frac{1}{I} \sum_{i=0}^{I} \left(y_{t,i} - \hat{C}_{t,i} \right)^2$$

```
X = np.column_stack([np.ones(M), S[:,i], S[:,i]**2, S[:,i]**3,S[:,i]**4,S[:,i]**5])
beta = np.linalg.lstsq(cond_x, Y, rcond=None)[0]
continue_val = np.dot(X, beta)
```

Step 4

If $S_{t,i} > \hat{C}_{t,i}$, fill cash flow matrix with resulting cash flow from this path.

Repeat until $t = 0$.

	Continue val	P_2
0	0.227	0.086
1	0.348	0.087
2	0.251	0.075
3	0.090	0.076

Data Collection

Goal: build database with market price of options and information about underlying

Stock data: `yfinance` Option data: `yahooquery`

```python
def BS(S0, K, T, sigma, r, type):
```

Option Chain:

symbol	expiration	optionType	contractSymbol	strike	currency	lastPrice	change	percentChange	volume	openInterest	bid	ask	contractSize	lastTradeDate	impliedVolatility
AAPL	2023-11-24	calls	AAPL231124C00050000	50.0	USD	141.35	0.0	0.0	6.0	0	0.0	0.0	REGULAR	2023-11-20 20:54:50	0.000010
		calls	AAPL231124C00075000	75.0	USD	100.21	0.0	0.0	1.0	0	0.0	0.0	REGULAR	2023-11-03 18:07:22	0.000010
		calls	AAPL231124C00090000	90.0	USD	81.35	0.0	0.0	1.0	0	0.0	0.0	REGULAR	2023-11-01 14:00:00	0.000010
		calls	AAPL231124C00095000	95.0	USD	77.10	0.0	0.0	0.0	0	0.0	0.0	REGULAR	2023-10-25 14:10:07	0.000010
		calls	AAPL231124C00100000	100.0	USD	85.83	0.0	0.0	7.0	0	0.0	0.0	REGULAR	2023-11-10 19:30:52	0.000010
	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
	2026-01-16	puts	AAPL260116P00250000	250.0	USD	75.00	0.0	0.0	0.0	0	69.3	73.5	REGULAR	2023-09-13 18:50:54	0.286048

Modifying Option Chain

Add price of underlying, maturity, variablility, mean returns.

```
import Data
Data.GetData('2018,11,22', '2022,11,22', 252, False)
```

	symbol	optionType	expiration	strike	lastPrice	lastTradeDate	inTheMoney	maturity	S0	sigma	returns	method
0	AAPL	puts	2024-06-21	280.0	143.10	2022-11-09	True	590	134.120331	0.346374	0.297289	A
1	AMZN	calls	2024-01-19	1040.0	1499.75	2022-06-03	False	595	122.349998	0.343390	0.138612	A
2	AMZN	calls	2024-01-19	1800.0	882.51	2022-06-03	False	595	122.349998	0.343390	0.138612	A
3	AMZN	calls	2024-01-19	2000.0	757.85	2022-06-03	False	595	122.349998	0.343390	0.138612	A
4	AMZN	calls	2024-01-19	2150.0	653.86	2022-06-03	False	595	122.349998	0.343390	0.138612	A
...	...	...	...	...	...	...	...	...	...	...	...	...
1448	^SPX	puts	2026-12-18	5900.0	1579.20	2022-07-25	True	1607	3966.840088	0.230712	0.112161	E
1449	^SPX	puts	2026-12-18	7000.0	2527.00	2022-10-13	True	1527	3669.909912	0.231177	0.085586	E
1450	^SPX	puts	2026-12-18	7200.0	2605.90	2022-09-29	True	1541	3640.469971	0.230415	0.084374	E
1451	^SPX	puts	2026-12-18	8800.0	4088.31	2022-06-23	True	1639	3795.729980	0.231362	0.102437	E
1452	^SPX	puts	2026-12-18	9000.0	4261.30	2022-06-23	True	1639	3795.729980	0.231362	0.102437	E

Jump Diffusion Calibration

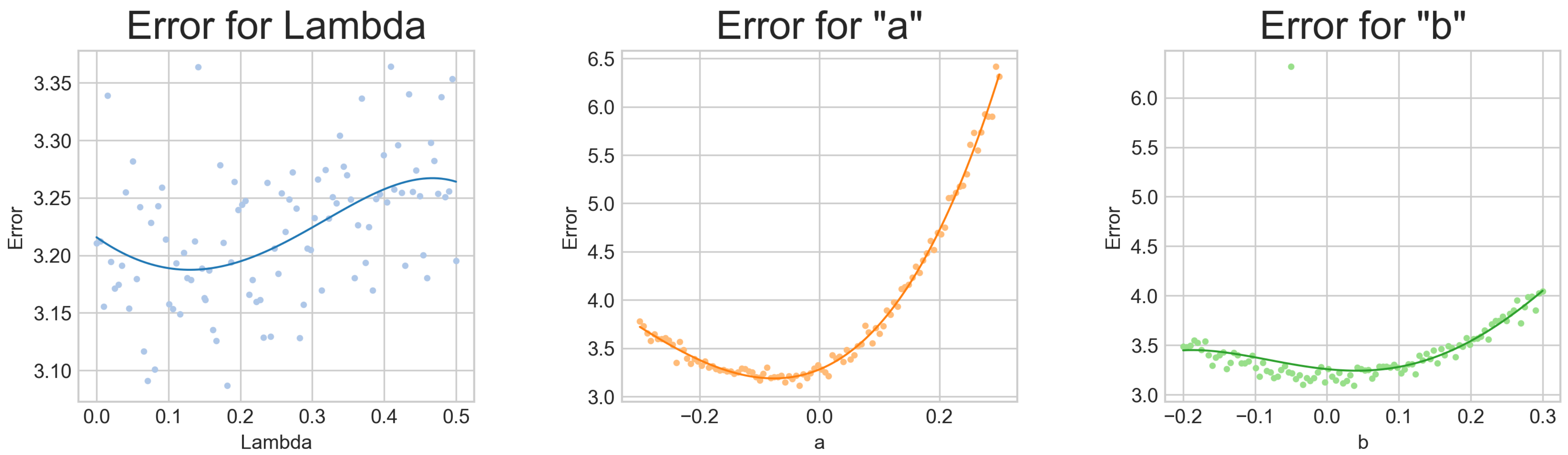
Parameters:

- $\lambda \rightarrow$ rate of Poisson process
- a and $b \rightarrow$ size of the jump

Test different values for each parameter holding the other two constant

Compute mean squared error of estimated prices

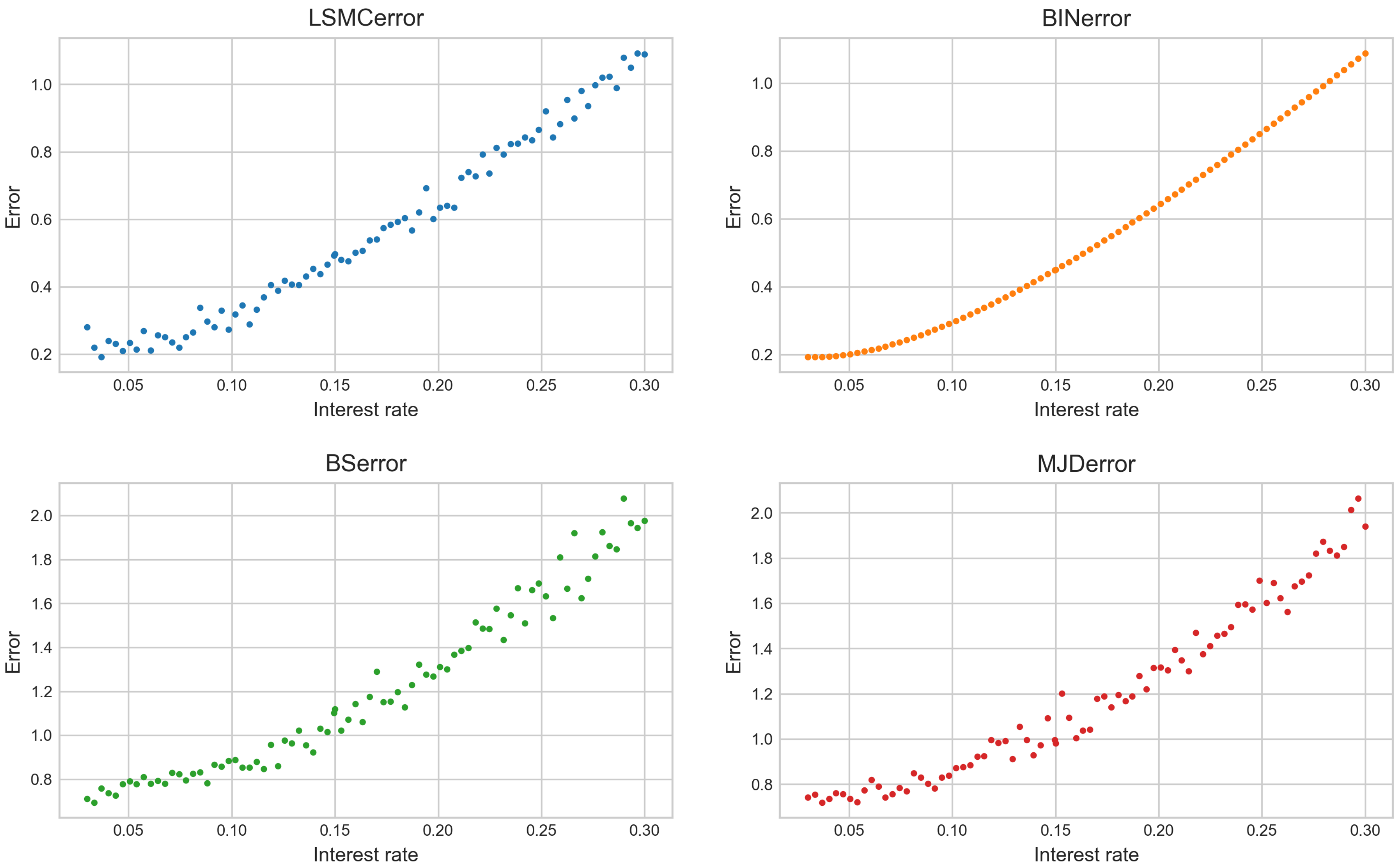
```
df = df.sample(n)
lamb_values = np.linspace(0,0.4, iterations)
for i in range(iterations):
  lamb = lamb_values[i]
  errors.append(compute_errors(lamb, a, b))
```



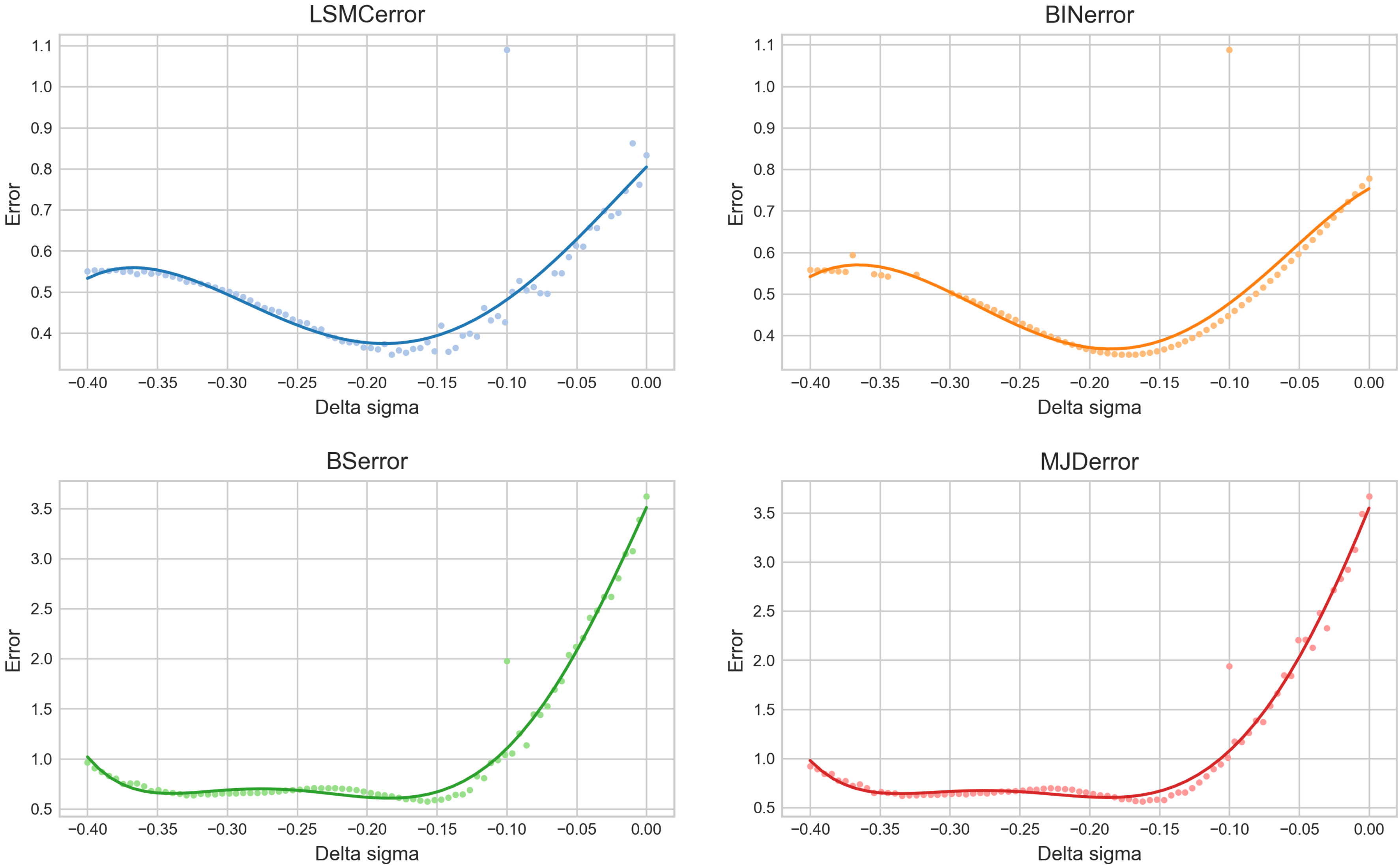
$$\lambda = 0.15, a = -0.1, b = 0.05$$

Interest rate and Standard Deviation calibration

Error given interest rate



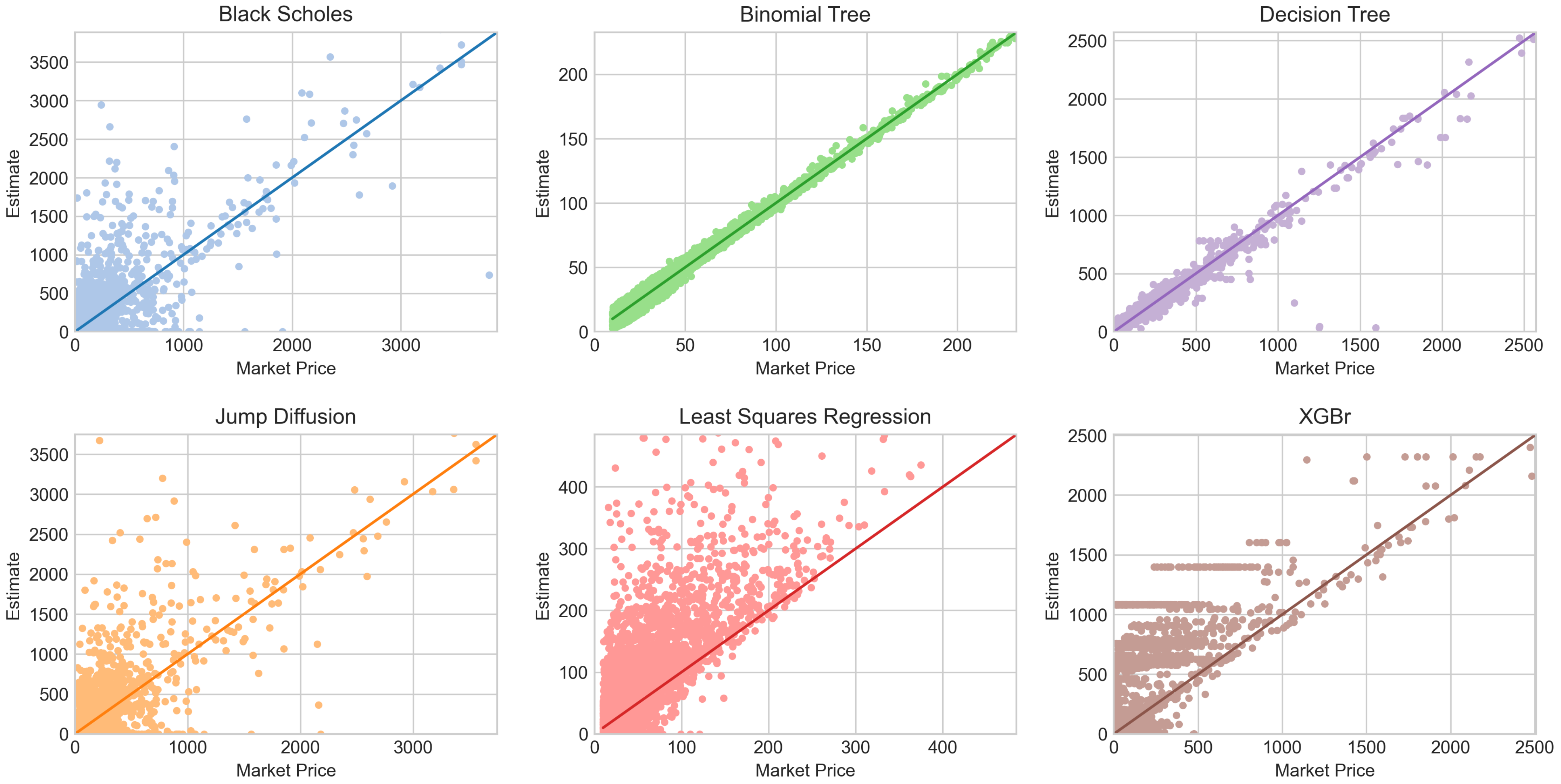
Error given volatility



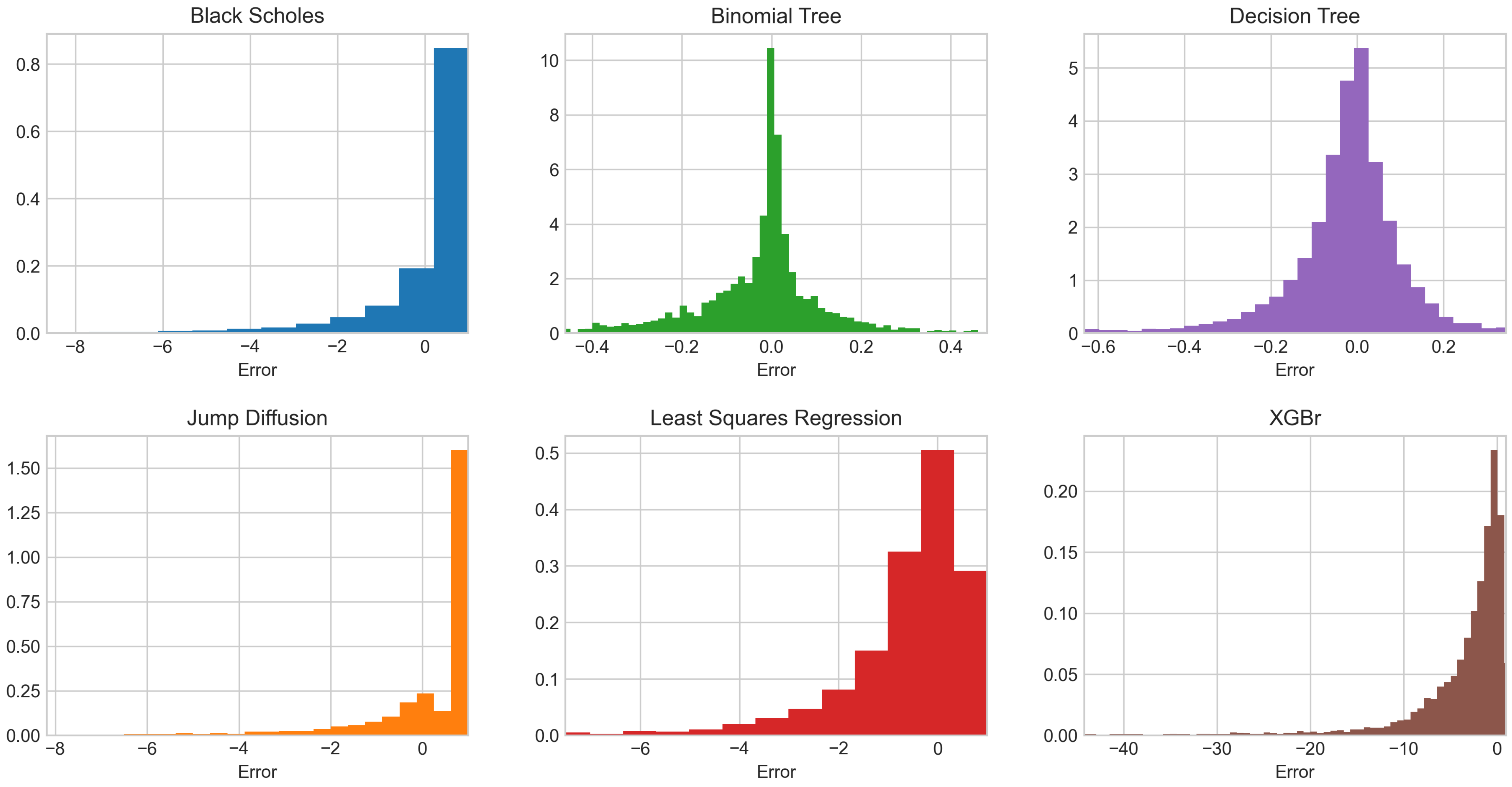
Results

Algorithm runtime

Option prices: Predicted against Market



Distribution of errors



	Average	MSE	Std Dev
BS	0.0392	6.7836	2.6042
MJD	0.0378	4.9287	2.2197
LSMC	-0.6832	4.8038	2.0825
BIN	-0.0196	0.0242	0.1545
DTR	-0.0307	0.0334	0.1802
XGBr	-4.1627	82.5343	8.0750