Investigation into Numerical Integration : Oscillators

School of Physics and Astronomy
University of Manchester

Amir El-hamdy

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1 Quartic Oscillator

The Quartic Oscillator equation:

$$\ddot{x} = -\omega^2(x + \lambda x^3) \tag{1}$$

The Quartic Oscillator is a well known one-dimensional problem in quantum mechanics.

$$U(X) = \frac{1}{2}k_4 x^4 \tag{2}$$

This potential can be thought of as bead constrained to slide vertically along a frictionless wire at small amplitudes around its equilibrium point.
Taking this idea I can construct the problem as follows in Lagrangian Dynamics :

$$L = T - V$$
$$T = \frac{1}{2}m\dot{x}^2$$
$$V = \frac{1}{2}kx^2 + \frac{1}{4}\lambda k x^4$$
$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - \frac{1}{4}\lambda k x^4$$

Then applying Lagrange's equations

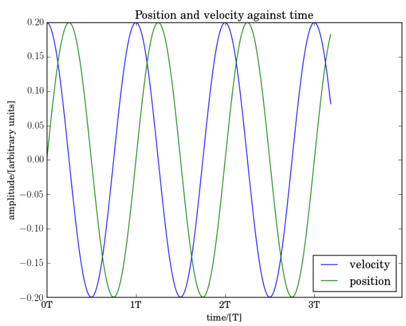
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{4}$$

we obtain Equation 1.

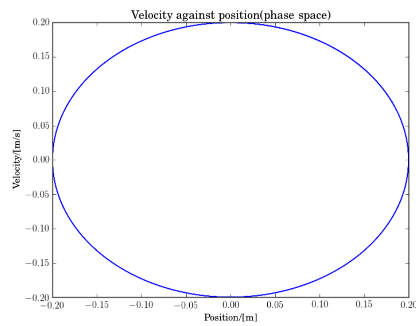
Therefore we can think of λ as being related to a kind of spring constant. Clearly if λ is zero, we will observe simple harmonic motion. However as λ is increased from zero we would expect the position to become more and more restricted as the restoring force is greater for a given displacement. At small magnitudes of λ we would expect SHM to dominate. For negative λ values we could imagine as some kind of force that weakens the restoration force more greatly at greater displacements. Eventually at very negative λ values we would expect the mass to break away and tend off to infinity. Similarly if the restoring force is quite weak, then we might expect the mass to break off at higher displacements as it is proportional to x^3.

1.1 $\lambda = 0$

Figure 1 shows the motion of the particle and the phase space for $\lambda = 0$.



(a) Position and Velocity against time.



(b) Phase space

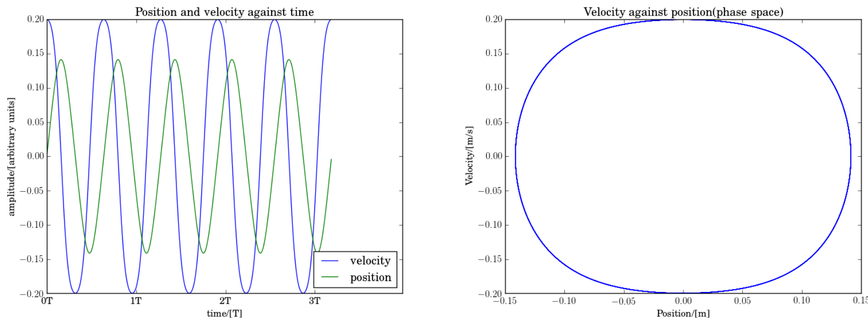
Figure 1: Motion of particle and Phase space.

We can clearly see that this corresponds to simple harmonic motion with the velocity leading the position by a quarter of a wavelength. We also observe the circular phase space.

1.2 $\lambda >> 1$

1.2.1 $\lambda = 100$

Figure 2 shows the motion and phase space of the quartic oscillator with $\lambda = 100$.



(a) Position and Velocity against time.

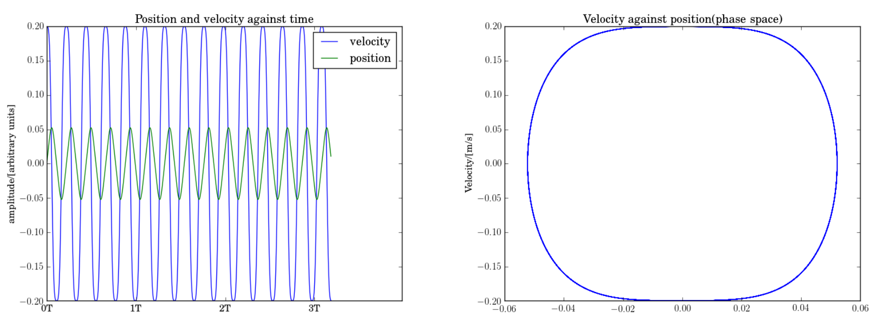
(b) Phase space

Figure 2: Motion of particle and Phase space.

Here we can clearly see the phase space has been reduced but the amplitude of the velocity is unaffected. We can also see that the frequency of oscillation is now slightly greater than that of the natural frequency. Both the velocity and position share the same frequency of oscillation. The velocity still leads the position by $\frac{\pi}{2}$.

1.2.2 $\lambda = 10000$

Figure 3 shows the motion and phase space of the quartic oscillator with $\lambda = 10000$.



(a) Position and Velocity against time.

(b) Phase space

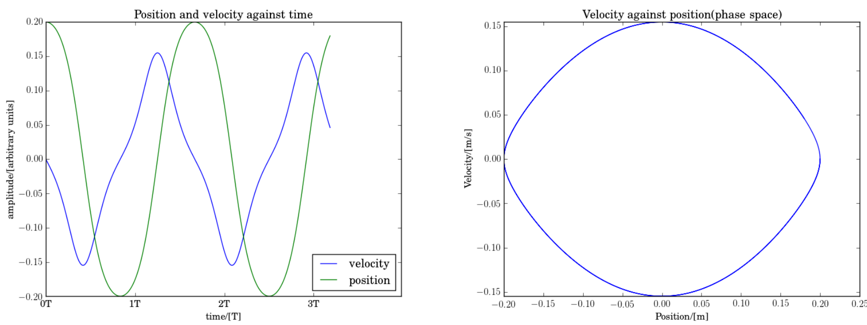
Figure 3: Motion of particle and Phase space.

Same as in the previous section but now the features are much more pronounced. We can see that the frequency of oscillation is much higher than the natural frequency of oscillation. The phase space of the particle is much more restricted. Once again the amplitude of the velocity is unaffected. Both the velocity and position share the same frequency of oscillation. The velocity still leads the position by $\frac{\pi}{2}$.

1.3 λ << -1

1.3.1 $\lambda = -20$

Figure 4 shows the motion and phase space of the quartic oscillator with $\lambda = -20$.



(a) Position and Velocity against time.

(b) Phase space

Figure 4: Motion of particle and Phase space.

Here we observe some interesting features. Firstly we can see the form of the velocity has been distorted from a sinusoid and correspondingly the phase space has also been distorted. The amplitude of the velocity has decreased but the amplitude of the position remains the same. We also observe the frequency of oscillation is smaller than the natural frequency.

2 Coupled Oscillator

Coupled Oscillator equations:

$$\ddot{x_1} = -\omega_1^2 x_1 - \lambda x_2, \tag{5a}$$

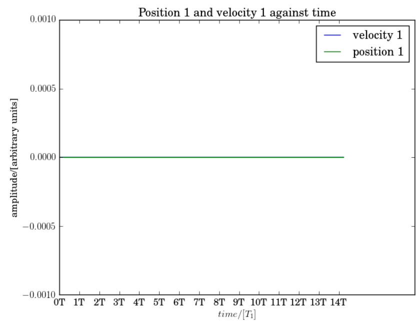
$$\ddot{x_2} = -\omega_2^2 x_2 - \lambda x_1, \tag{5b}$$

2.1 $\omega_1 = \omega_2$

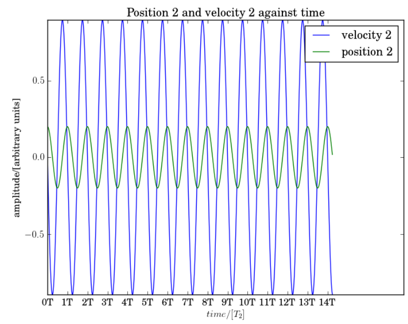
In order to observe the effect of the coupling parameter λ both particles will begin at rest, with particle 1 at equilibrium position and particle 2 released from some initial displacement.

2.1.1 $\lambda = 0$

We can clearly see from the equations that in the case $\lambda = 0$ we have two independent oscillators. This is shown in Figure 5



(a) Motion of particle 1

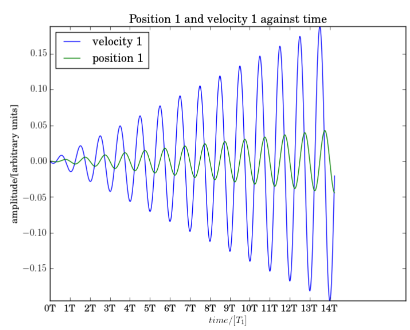


(b) Motion of particle 2

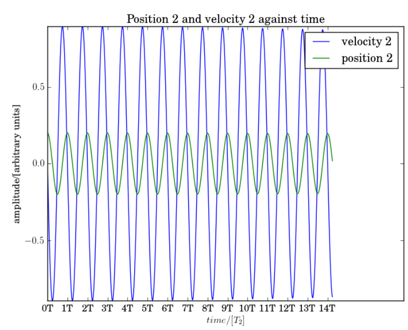
Figure 5: Motion of particle 1 and 2.

2.1.2 $\lambda = 0.1$

We now introduce the parameter λ. Since one particle is initially at rest in equilibrium position and the other released from displacement, we expect to see an oscillation of energy between the two masses. This is indeed what we observe in Figure 6.



(a) Motion of particle 1

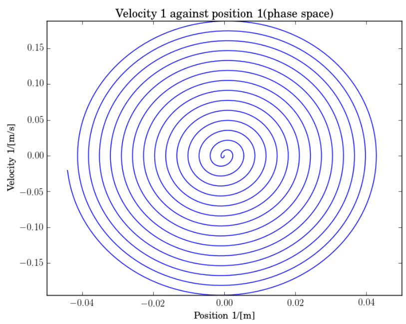


(b) Motion of particle 2

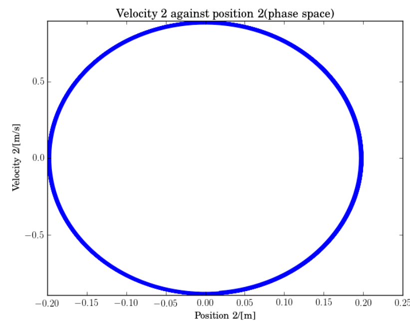
Figure 6: Motion of particle 1 and 2.

We can see that particle 1 is beginning to oscillate from rest due to the transfer of energy through the coupling. However this oscillation is very slight and we see almost no difference in the motion of particle 2.

?? shows the effect this has on the phase space of the two particles.



(a) Phase space of particle 1

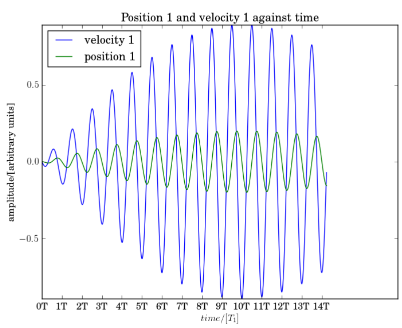


(b) Phase space of particle 2

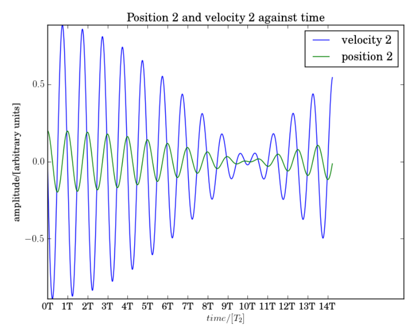
Figure 7: Phase space of particle 1 and 2.

2.1.3 $\lambda = 1$

We can now see the effect much more pronounced in Figure 8.



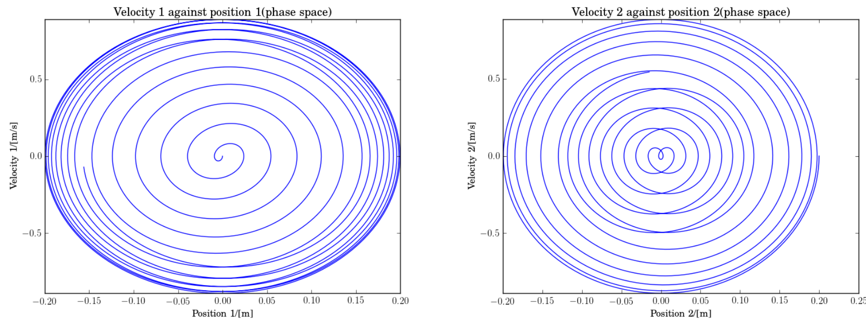
(a) Motion of particle 1



(b) Motion of particle 2

Figure 8: Motion of particle 1 and 2.

?? shows the effect this has on the phase space of the two particles.



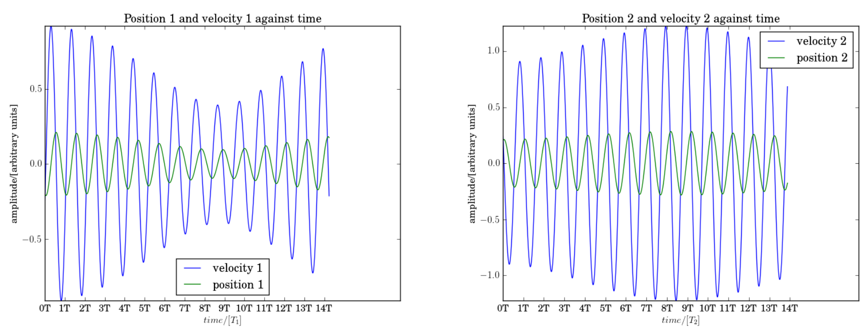
(a) Phase space of particle 1 (b) Phase space of particle 2

Figure 9: Phase space of particle 1 and 2.

2.2 $\omega_1 \neq \omega_2$

In order to see the effect of a changing frequency, both masses are set to oscillate in exact antiphase with $\lambda = 1$.

2.2.1 $\omega_1 \approx \omega_2$

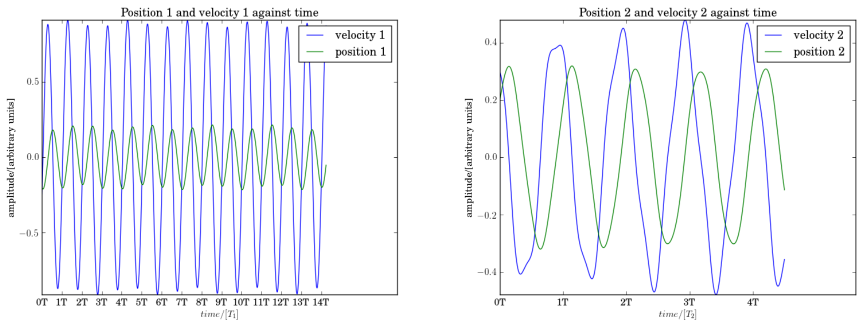


(a) Motion of particle 1 (b) Motion of particle 2

Figure 10: Motion of particle 1 and 2.

In Figure 10 we see an effect that looks very much like the beating effect of a superposition of two waves with similar frequency.

2.2.2 $\omega_1 >> \omega_2$

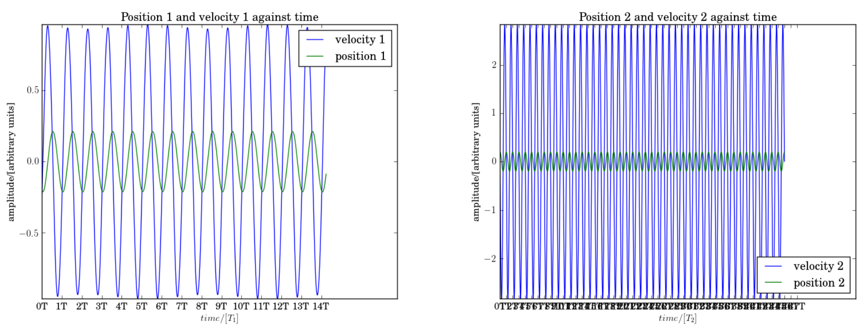


(a) Motion of particle 1 (b) Motion of particle 2

Figure 11: Motion of particle 1 and 2.

In Figure 11 we observe a transient effect where particle 2 has distorted waveform, but then settles down into a regular waveform.

2.2.3 $\omega_1 << \omega_2$



(a) Motion of particle 1 (b) Motion of particle 2

Figure 12: Motion of particle 1 and 2.

In Figure 12 we observe a very slight oscillation in the amplitude of the velocity of particle 1. This effect is not pronounced most likely due to the coupling parameter, λ, being too small to transfer any significant amount of energy in this high energy system.

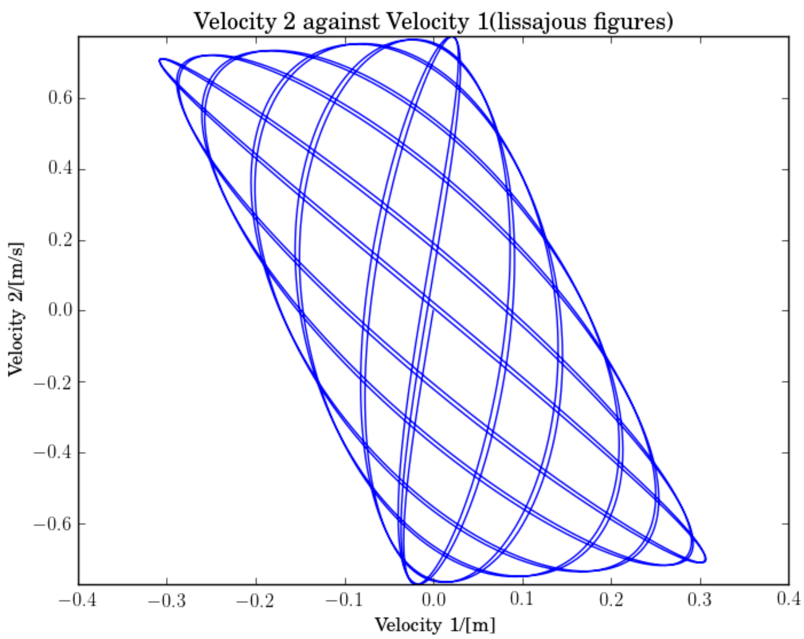
3 Lissajous figures

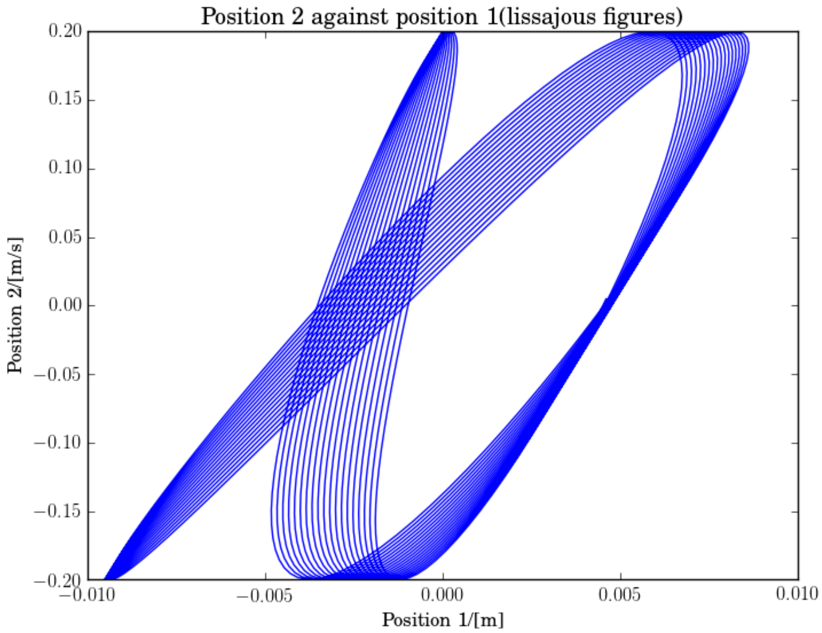
As an aside, it is nice to note that when the position/velocity of particle 1 is plotted against the position/velocity of particle 2 (provided they are oscillating at different frequencies) lissajous figures are produced. Lissajous figures are describe complex harmonic motion and are of the parametric form:

$$y = Bsin(bt) \tag{6}$$

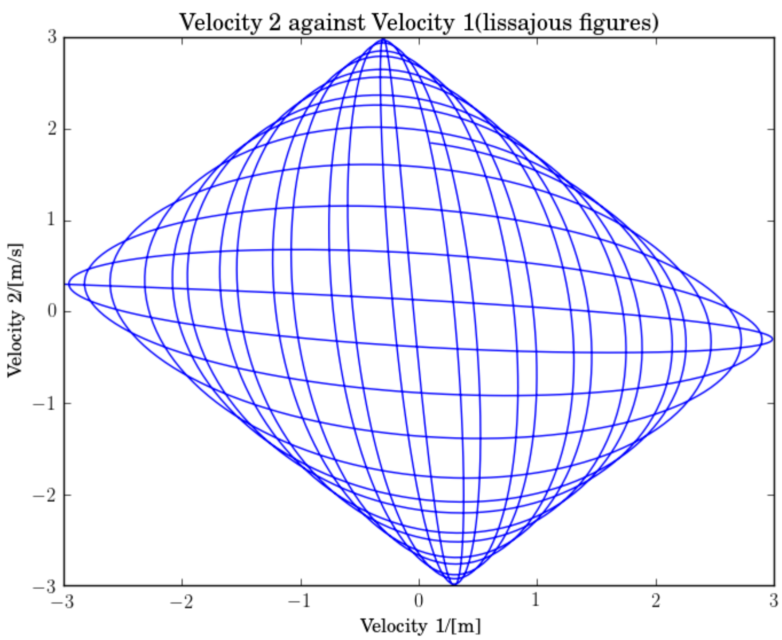
$$x = Asin(at + \delta) \tag{7}$$

A few that have been produced are shown below.

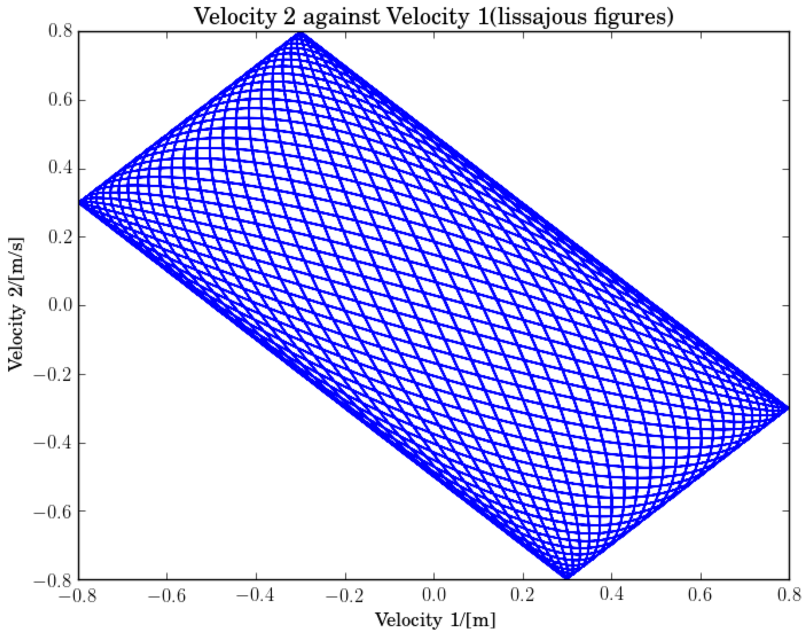


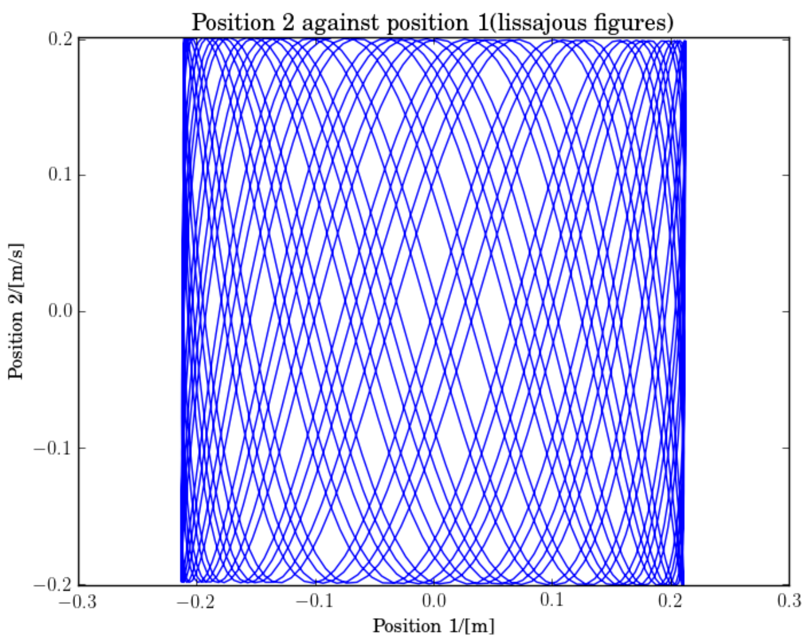


Position 2 against position 1(lissajous figures)
Position 2/[m/s]
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
−0.010
−0.005
0.000
0.005
0.010
Position 1/[m]



Velocity 2 against Velocity 1(lissajous figures)
Velocity 2/[m/s]
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
Velocity 1/[m]


Velocity 2 against Velocity 1(lissajous figures)
Velocity 2/[m/s]
Velocity 1/[m]
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8


Position 2 against position 1(lissajous figures)
Position 2/[m/s]
Position 1/[m]
0.2
0.1
0.0
−0.1
−0.2
−0.3
0.3