Self-Organized Criticality in the Eigenvalue Flow of Trained Recurrent Networks

Jared O. Dunahay, D.O.*
AEO Trivector LLC, Manchester, NH

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Abstract

We introduce *eigenvalue flow tracking*—the analysis of how individual Jacobian eigenvalues move through the complex plane across timesteps in trained recurrent neural networks. Using Hungarian algorithm matching to establish eigenvalue correspondence between consecutive timesteps, we decompose eigenvalue motion into angular (phase) and radial (magnitude) components and classify eigenvalue trajectories by their relationship to the real axis. In trained LSTMs across hidden dimensions $h \in \{28, 42, 56, 70, 84, 112\}$, we find three principal results. First, angular velocity exceeds radial velocity by a factor of 16, demonstrating that eigenvalue dynamics are geometrically orthogonal to eigenvalue content—modes precess around closed orbits while maintaining nearly constant amplitude. Second, approximately 10% of eigenvalues are *oscillating modes* that repeatedly cross the real axis, dynamically switching between oscillatory (transport) and scaling (transform) behavior; this fraction is the most scale-invariant metric observed and correlates with computational demand. Third, 57% of eigenvalues lie within $|\mathrm{Im}| < 0.05$ of the real axis, with the spectral gap decreasing as h increases, indicating self-organized criticality at the transport/transform bifurcation boundary. We connect these findings to noncommutative geometry by identifying the Jacobian as the Dirac operator of a spectral triple, with the angular/radial decomposition corresponding to the Clifford grade structure. These results provide a geometric characterization of computation in recurrent networks as topological reconfiguration at a self-organized critical boundary.

1 Introduction

The eigenvalue spectrum of the state-transition Jacobian $J_t = \partial h_t / \partial h_{t-1}$ is a fundamental object in recurrent neural network (RNN) dynamics. Its magnitude distribution governs gradient flow [Pascanu et al., 2013, Bengio et al., 1994], while its phase structure encodes the oscillatory modes available for information transport [Vorontsov et al., 2017, Arjovsky et al., 2016]. Prior work has characterized the *static* spectrum—the distribution of eigenvalues at isolated timesteps—but has not tracked how individual eigenvalues *move* through the complex plane during sequence processing.

This paper introduces **eigenvalue flow tracking**: the measurement of individual eigenvalue trajectories across timesteps, established via optimal matching (the Hungarian algorithm) between consecutive spectra. This shift from static snapshots to dynamic trajectories reveals structure invisible to single-timestep analysis.

Our principal contributions are:

(i) **Angular–radial decomposition.** We decompose eigenvalue motion into angular velocity (phase change) and radial velocity (magnitude change), finding a ratio of 16:1 in trained

*Corresponding author. Email: jared@trivector.ai

LSTMs. This establishes that the *dynamics* of the Jacobian spectrum are geometrically orthogonal to its *content*—modes precess while amplitudes remain nearly constant.

(ii) **Trajectory classification.** We classify eigenvalue trajectories by their relationship to the real axis, identifying four populations: persistent-complex ($\sim$82%), oscillating ($\sim$10%), persistent-real ($\sim$2%), and transitioning ($\sim$6%). The oscillating population—modes that repeatedly cross the real axis—constitutes the active computation fraction and is the most scale-invariant metric observed.

(iii) **Self-organized criticality.** We show that trained networks position 57% of their eigenvalues within $|\mathrm{Im}(\lambda)| < 0.05$ of the real axis, with the spectral gap decreasing as network size increases. This indicates self-organization to the bifurcation boundary between oscillatory and scaling modes, maximizing computational flexibility.

(iv) **Crossing rate as computational measure.** The rate at which eigenvalues cross the real axis correlates with task computational demand: compute-intensive inputs produce 14–48% more crossings than memory-only inputs.

We connect these findings to noncommutative geometry by identifying the Jacobian as the Dirac operator $\mathcal{D}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with the angular/radial decomposition corresponding to the Clifford algebra grade structure (Grade 2 rotation vs. Grade 0 scaling).

2 Background and Related Work

2.1 Jacobian Eigenvalue Analysis in RNNs

The role of Jacobian eigenvalues in RNN training dynamics is well established. Pascanu et al. [2013] showed that eigenvalue magnitudes greater than unity cause exploding gradients, while magnitudes less than unity cause vanishing gradients. Subsequent work on unitary and orthogonal RNNs [Arjovsky et al., 2016, Vorontsov et al., 2017, Lezcano-Casado and Martínez-Rubio, 2019] constrained eigenvalue magnitudes to the unit circle, improving long-range dependency learning at the cost of expressiveness.

Chen [2018] analyzed RNNs through the lens of dynamical systems, connecting Jacobian spectra to Lyapunov exponents. Engelken et al. [2023] extended this to trained networks, showing that training reorganizes the Lyapunov spectrum. Our work complements these by tracking *individual* eigenvalues rather than characterizing the spectrum statistically.

2.2 Self-Organized Criticality

Self-organized criticality (SOC) describes systems that naturally evolve toward critical states without external tuning [Bak et al., 1987]. In neuroscience, Beggs and Plenz [2003] observed neural avalanche statistics consistent with SOC, and Langton [1990] argued that computation is maximized at the "edge of chaos." Bertschinger and Natschläger [2004] showed that recurrent networks at the edge of chaos maximize computational capacity.

Our finding that trained LSTMs position their eigenvalues at the real-axis bifurcation boundary extends this literature by providing a precise geometric characterization: the critical boundary is between oscillatory and scaling eigenvalue modes, and the spectral gap to this boundary decreases with network size.

2.3 Noncommutative Geometry and Neural Networks

The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of noncommutative geometry [Connes, 1994] provides a framework in which geometric information is encoded in the spectrum of the Dirac operator $\mathcal{D}$. Recent work has explored connections between neural network architectures and noncommutative geometry [Lässig et al., 2022]. We make this connection concrete: the state-transition Jacobian serves as $\mathcal{D}$, with its Clifford grade decomposition directly measurable in the angular/radial structure of the eigenvalue flow.

3 Methods

3.1 Experimental Setup

We train LSTM networks on a combined memory-and-compute task: given a sequence of (x_t, flag_t) pairs, the network must output x_t when $\text{flag}_t = 0$ (memory: recall the stored value) and $\sum_{s\leq t} x_s$ when $\text{flag}_t = 1$ (compute: running sum). Sequence length is 10. We train networks at hidden dimensions $h \in \{28, 42, 56, 70, 84, 112\}$ to convergence using Adam optimization with learning rate 10^{-3}.

After training, we extract the state-transition Jacobian $J_t = \partial h_t/\partial h_{t-1} \in \mathbb{R}^{h\times h}$ at each timestep t for a batch of test sequences using automatic differentiation, and compute its eigenvalue decomposition.

3.2 Eigenvalue Flow Tracking

Definition 1 (Eigenvalue flow). *Given a sequence of Jacobians* $\{J_1, \ldots, J_T\}$ *with eigenvalue spectra* $\{\Lambda_1, \ldots, \Lambda_T\}$ *where* $\Lambda_t = \{\lambda_t^{(1)}, \ldots, \lambda_t^{(h)}\}$*, the* eigenvalue flow *is a set of trajectories* $\{\gamma^{(i)}\}_{i=1}^{h}$ *where* $\gamma^{(i)} = (\lambda_1^{(\sigma_1(i))}, \lambda_2^{(\sigma_2(i))}, \ldots, \lambda_T^{(\sigma_T(i))})$ *and the permutations* σ_t *are chosen to minimize total displacement.*

We establish eigenvalue correspondence between consecutive timesteps using the Hungarian algorithm [Kuhn, 1955]. The cost matrix for matching eigenvalues at times t and $t+1$ is $C_{ij} = |\lambda_t^{(i)} - \lambda_{t+1}^{(j)}|$, where $|\cdot|$ denotes complex modulus. The Hungarian algorithm finds the permutation σ_{t+1} minimizing $\sum_i C_{i,\sigma_{t+1}(i)}$.

3.3 Angular–Radial Decomposition

For an eigenvalue $\lambda = re^{i\theta}$, we decompose its motion into:

Definition 2 (Angular and radial velocity). *Given consecutive eigenvalue positions* $\lambda_t = r_t e^{i\theta_t}$ *and* $\lambda_{t+1} = r_{t+1}e^{i\theta_{t+1}}$ *along a tracked trajectory, the* radial velocity *is* $v_r = |r_{t+1} - r_t|$ *and the* angular velocity *is* $v_\theta = |\theta_{t+1} - \theta_t|$ *(with appropriate phase unwrapping).*

The ratio v_θ/v_r quantifies the degree to which eigenvalue motion is rotational (phase-dominated) versus scaling (magnitude-dominated).

3.4 Trajectory Classification

We classify eigenvalue trajectories into four types based on their interaction with the real axis:

Definition 3 (Trajectory types). *An eigenvalue is* real at time t *if* $|\text{Im}(\lambda_t)| < \epsilon$ *for threshold* $\epsilon = 10^{-6}$. *A trajectory* γ *over* T *timesteps is classified as:*

- ***Persistent-complex****: real at fewer than* 10% *of timesteps.*
- ***Persistent-real****: real at more than* 90% *of timesteps.*
- ***Oscillating****: crosses the real axis* ≥ 3 *times.*
- ***Transitioning****: crosses the real axis* 1–2 *times.*

A *real-axis crossing* occurs when the real/complex classification changes between consecutive timesteps: $|\text{Im}(\lambda_t)| < \epsilon$ and $|\text{Im}(\lambda_{t+1})| \geq \epsilon$, or vice versa.

3.5 Conjugate Pairing

Since the Jacobian J_t is a real matrix, its complex eigenvalues appear in conjugate pairs $\lambda, \bar{\lambda}$. We explicitly pair conjugates by matching each eigenvalue with imaginary part $> \epsilon$ to its closest partner with imaginary part $< -\epsilon$. Unpaired eigenvalues with $|\mathrm{Im}| < \epsilon$ are classified as real. This guarantees an even count of complex eigenvalues, correcting threshold-based methods that can produce odd counts.

3.6 Co-Movement Analysis

To detect functional groupings, we compute the velocity time series $v_t^{(i)} = |\lambda_{t+1}^{(i)} - \lambda_t^{(i)}|$ for each trajectory i and form the correlation matrix $R_{ij} = \mathrm{corr}(v^{(i)}, v^{(j)})$. We cluster trajectories using hierarchical agglomerative clustering (Ward's method) on the distance matrix $D_{ij} = 1 - |R_{ij}|$, selecting the optimal number of clusters k^* by silhouette score.

4 Results

4.1 Angular Velocity Dominates Radial Velocity

Table 1: Angular and radial velocities of tracked eigenvalues across hidden dimensions. The angular/radial ratio quantifies the separation between phase dynamics and amplitude dynamics.

h	Angular vel. $\langle v_\theta \rangle$	Radial vel. $\langle v_r \rangle$	Ratio v_θ / v_r
28	0.179	0.014	13.4
56	0.128	0.008	16.1
112	0.104	0.006	16.1

The angular velocity of tracked eigenvalues exceeds the radial velocity by a factor of 13–16× across all hidden dimensions tested (Table 1). This ratio increases from 13.4 at $h = 28$ to 16.1 at $h = 56$ and $h = 112$, indicating that larger networks exhibit greater separation between phase and amplitude dynamics.

Physically, this means eigenvalues *precess* around orbits in the complex plane while maintaining nearly constant magnitude. The information carried by a mode (encoded in its amplitude) is transported (via phase rotation) without being transformed (amplitude change). The dynamics of the Jacobian spectrum are geometrically orthogonal to its content.

4.1.1 Connection to Clifford Grade Structure

The Jacobian admits a Clifford algebra decomposition into grade components. For $J \in \mathbb{R}^{h\times h}$:

$$J = \underbrace{\frac{\mathrm{tr}(J)}{h} I}_{\text{Grade 0 (scalar)}} + \underbrace{\frac{J - J^\top}{2}}_{\text{Grade 2 (bivector)}} + \underbrace{\frac{J + J^\top}{2} - \frac{\mathrm{tr}(J)}{h} I}_{\text{Grade 1 (traceless symmetric)}} \tag{1}$$

The Grade 0 component drives radial motion (uniform scaling), while the Grade 2 (antisymmetric) component drives angular motion (rotation). The measured 16:1 angular/radial ratio directly reflects the energy ratio $\|G_2\|/\|G_0\|$ in the Jacobian's grade decomposition.

4.2 Trajectory Type Distribution

At $h = 112$, the trajectory population decomposes as follows (Table 2):

Table 2: Distribution of eigenvalue trajectory types at $h = 112$ (mean over test sequences, $T = 10$ timesteps). Trajectory classification based on real-axis crossing frequency.

h	Persistent-complex	Oscillating	Persistent-real	Transitioning
28	59.6%	15.4%	10.0%	15.0%
56	73.2%	14.3%	3.6%	8.9%
112	81.5%	10.4%	2.3%	5.8%

- **Persistent-complex** (81.5%): Modes that remain complex at every timestep. These form the structural skeleton of the oscillatory dynamics—stable transport channels that never participate directly in computation.
- **Oscillating** (10.4%): Modes that cross the real axis three or more times in $T = 10$ timesteps. These are the *active computation* population—eigenvalues that dynamically switch between oscillatory and scaling behavior.
- **Persistent-real** (2.3%): Modes that remain real throughout. These have near-zero magnitude (mean $|\lambda| = 0.07$ at $h = 112$) and represent collapsed dimensions of the state space.
- **Transitioning** (5.8%): Modes that cross the real axis once or twice, possibly responding to specific input features.

The oscillating fraction is the most scale-invariant metric observed, decreasing slowly from 15.4% at $h = 28$ to 10.4% at $h = 112$. By contrast, the persistent-complex fraction increases sharply (59.6% $\rightarrow$ 81.5%), indicating that additional network capacity is allocated to transport rather than computation.

4.3 Crossing Rate Correlates with Computational Demand

Table 3: Mean real-axis crossings per timestep for memory-only and compute-required inputs. Compute inputs consistently produce more crossings.

h	Memory crossings/step	Compute crossings/step	Ratio
28	2.04	3.02	1.48
56	3.73	4.58	1.23
112	5.51	6.27	1.14

Timesteps requiring computation (running sum) produce 14–48% more real-axis crossings than timesteps requiring only memory (value recall) (Table 3). Each crossing represents a mode dynamically switching its role: a complex-to-real crossing converts an oscillatory (transport) mode into a scaling (transform) mode, while a real-to-complex crossing releases a dimension back to transport.

The crossing-rate ratio decreases with h (1.48 $\rightarrow$ 1.14), consistent with larger networks having greater spare capacity: they can handle both memory and compute tasks with less dynamic reconfiguration.

4.4 Self-Organized Criticality at the Bifurcation Boundary

The spectral gap—the minimum imaginary part among complex eigenvalues—shrinks monotonically with h (Table 4). More than half of all eigenvalues ($\sim$57%) lie within $|\mathrm{Im}| < 0.05$ of the real axis, clustered near the bifurcation boundary between oscillatory and scaling behavior.

This indicates self-organized criticality: training positions eigenvalues near the phase transition between transport and transform modes, maximizing the network's ability to dynamically

Table 4: Criticality metrics across hidden dimensions. The spectral gap (minimum $|\mathrm{Im}(\lambda)|$ among complex eigenvalues) decreases with h, while the near-axis density remains high.

h	Mean spectral gap	Density $\lvert\mathrm{Im}\rvert < 0.01$	Density $\lvert\mathrm{Im}\rvert < 0.05$
28	0.0101	28.7%	60.1%
56	0.0061	20.0%	57.9%
112	0.0037	16.5%	56.7%

reconfigure its computational topology with minimal perturbation. The criticality increases with scale—larger networks are *more* finely poised at the boundary.

4.5 Transport Fraction Across Hidden Dimensions

Table 5: Transport fraction (percentage of complex eigenvalues) measured by corrected conjugate pairing across hidden dimensions. The number of real eigenvalues n_{real} follows a logarithmic scaling law.

h	Mean pairs	Mean n_{real}	Transport %	Random matrix n_{real}
28	11.8	4.3	84.6%	4.2
42	17.8	6.3	85.1%	5.2
56	24.0	8.1	85.6%	6.0
70	31.4	7.2	89.7%	6.7
84	37.7	8.6	89.8%	7.3
112	51.0	9.9	91.2%	8.4

The transport fraction (proportion of complex eigenvalues) increases monotonically with h, from 84.6% at $h = 28$ to 91.2% at $h = 112$ (Table 5). The number of real eigenvalues follows a logarithmic scaling law:

$$n_{\mathrm{real}}(h) \approx 3.75 \ln(h) - 7.91 \quad (R^2 = 0.91) \tag{2}$$

Trained networks consistently exhibit *more* real eigenvalues than predicted by random matrix theory for Gaussian matrices of the same dimension (expected $n_{\mathrm{real}} \sim \sqrt{2h/\pi}$). The excess of 2–3 real eigenvalues above the random baseline suggests that training creates additional non-oscillatory modes beyond what matrix dimensionality alone would produce.

4.6 Co-Movement Structure

Velocity-correlation clustering yields an optimal cluster count of $k^* = 8$ across all hidden dimensions (silhouette score 0.21–0.54). This is consistent with the LSTM architecture's eight weight matrices $(W_{ii}, W_{if}, W_{ig}, W_{io}, W_{hi}, W_{hf}, W_{hg}, W_{ho})$, each contributing an independent source of eigenvalue perturbation. The co-movement structure reflects *architecture*, not task topology.

5 Discussion

5.1 The Spectral Triple Interpretation

The state-transition Jacobian J_t of a trained RNN can be identified with the Dirac operator $\mathcal{D}$ in a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ from noncommutative geometry [Connes, 1994]:

- $\mathcal{A}$: the algebra of weight matrices (non-commutative)

- $\mathcal{H}$: the hidden state space $\mathbb{R}^h$ (Hilbert space)
- $\mathcal{D} = J_t$: the state-transition Jacobian (Dirac operator)

In this framework, the Clifford grade decomposition of $\mathcal{D}$ (Equation (1)) has direct physical meaning: Grade 2 (the antisymmetric part) generates rotations in eigenvalue space, while Grade 0 (the trace) generates uniform scaling. Our measured 16:1 angular/radial ratio demonstrates that trained networks operate in a regime where the rotational (Grade 2) component dominates the scaling (Grade 0) component.

This provides empirical content to the abstract notion that "dynamics are orthogonal to content": in the eigenvalue flow, phase dynamics (the how of information transport) and amplitude dynamics (the what of information content) are geometrically separated, with phase motion occurring 16× faster than amplitude change.

5.2 Breathing Topology

A striking feature of the eigenvalue flow is that the number of real eigenvalues fluctuates significantly within a single forward pass. At $h = 56$, n_{real} ranges from 4 to 12 across timesteps (mean 8.1). This means the effective topology of the dynamical manifold changes at every timestep:

$$\mathcal{M}_h(t) \cong \mathbb{T}^{p(t)} \times \mathbb{R}^{r(t)}, \quad p(t) + r(t) = h \tag{3}$$

where $p(t)$ and $r(t)$ are the numbers of conjugate pairs and real eigenvalues at time t, respectively. Each real-axis crossing by an eigenvalue represents a topological event—the creation or destruction of a toroidal dimension. The *rate* of these topological events is a measure of computational intensity (Table 3).

This "breathing topology" is qualitatively different from the fixed manifold assumed in most dynamical systems analyses of RNNs. The network does not have a fixed attractor landscape; it dynamically reconfigures its phase space geometry in response to input demands.

5.3 Self-Organized Criticality

The concentration of eigenvalues near the real axis (57% within $|\text{Im}| < 0.05$) and the decreasing spectral gap with h are hallmarks of self-organized criticality [Bak et al., 1987]. The critical boundary in this system is the real axis of the complex eigenvalue plane: eigenvalues above this boundary are oscillatory (transport modes), while those on it are scaling (transform modes).

Training drives the system toward this boundary, not to any particular spectrum, but to a *distribution* that maximizes the number of modes poised to switch roles. This is consistent with the "edge of chaos" hypothesis [Langton, 1990, Bertschinger and Natschläger, 2004]: computational capacity is maximized when the system is poised between ordered (all-oscillatory) and chaotic (all-scaling) regimes.

The novel contribution here is the geometric precision: the critical boundary is the real axis of the Jacobian eigenvalue plane, the order parameter is the near-axis density, and the critical exponent is related to the logarithmic growth of n_{real} with h.

5.4 Limitations

Several limitations should be noted. Our experiments use a single task type (memory-and-compute) with short sequences ($T = 10$). The architecture is limited to standard LSTMs. The co-movement structure ($k = 8$) appears to reflect LSTM-specific architecture rather than universal computational structure. Extending to longer sequences, diverse tasks, and alternative architectures (GRU, Transformer) is necessary to establish generality.

The eigenvalue flow tracking relies on the assumption that small eigenvalue displacements between consecutive timesteps imply identity—that "the same" eigenvalue has moved. For large perturbations between timesteps, this assumption may break down.

6 Predictions

The following predictions are falsifiable and should be tested in future work:

P1. Architecture dependence of co-movement. The optimal co-movement cluster count k^* equals the number of independent weight matrices in the architecture. Predicted: $k^* = 6$ for GRU (3 gates $\times$ 2 weight types), $k^* = 2$ for vanilla RNN.

P2. Architecture invariance of angular/radial ratio. The angular/radial velocity ratio $v_\theta/v_r > 10$ for any trained recurrent architecture on any sequential task. This ratio reflects the generic dominance of Grade 2 over Grade 0 in trained Jacobians, not architecture-specific structure.

P3. Oscillating fraction convergence. The oscillating fraction approaches $\sim$9% as $h \to 500$ and/or as task complexity increases. This fraction represents the asymptotic balance between structural transport and active computation.

P4. Crossing rate as complexity predictor. The real-axis crossing rate at a given timestep predicts the local loss contribution of that timestep. Higher crossing rates indicate more topological reconfiguration, corresponding to more demanding computation.

P5. Spectral gap scaling. The spectral gap (minimum |Im| among complex eigenvalues) scales as $\mathcal{O}(h^{-\alpha})$ for some $\alpha > 0$, approaching zero as $h \to \infty$. Trained networks become increasingly critical at scale.

7 Conclusion

Eigenvalue flow tracking reveals that trained recurrent networks develop a characteristic computational geometry: eigenvalues precess (16$\times$ faster phase than amplitude change), approximately 10% of modes actively switch between transport and transform roles, and the system self-organizes to the bifurcation boundary between oscillatory and scaling behavior.

The computation performed by a trained RNN can be understood as *topological reconfiguration*: the dynamic creation and destruction of oscillatory dimensions in the Jacobian eigenvalue spectrum, driven by input demands, at a self-organized critical boundary. The rate of this reconfiguration measures computational intensity, the separation between angular and radial dynamics encodes the geometric orthogonality of transport and content, and the near-axis concentration of eigenvalues reflects the system's readiness to reconfigure.

These findings provide a concrete, measurable link between the abstract mathematics of non-commutative geometry and the empirical dynamics of trained neural networks, suggesting that the spectral triple framework may be a natural language for characterizing learned computation.

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