Linear Programming

Lecture 07A - The Revised Simplex Algorithm

John Bagterp Jørgensen

Department of Applied Mathematics and Computer Sicence
Technical University of Denmark

02612 Constrained Optimization

Learning Objectives

- Describe different formulations of linear programs
- Convert a linear program to its standard form
- Describe active-set algorithms for linear programs
- Describe and implement the revised simplex algorithm

Reading material: Nocedal & Wright: Chap. 13

Literature:

- V. Chvatal: Linear Programming, W.H. Freeman and Company, 1983
- J.L. Nazareth: Computer Solutions of Linear Programs, Oxford Science Publications, 1987
- R.J. Vanderbei: Linear Programming. Foundations and Extensions, Springer, 2008
- M.C. Ferris, O.L. Mangaserian, S.J. Wright: Linear Programming with Matlab, SIAM, 2007
- D. Bertsimas, J.N. Tsitsiklis: Introduction to Linear Optimization, Athena Scientific, 1997
- H.P. Williams: Model Building in Mathematical Programming, 3rd Edition, Wiley, 1993

HISTORY of Linear Programming

1. **1940's** George Dantzig at Stanford invents the Simplex Algorithm. The simplex algorithm has exponential complexity.
2. **1984** Karmarkar publishes the interior-point algorithm with polynomial rather than exponential complexity.
3. **1987-1991** Primal-dual interior-point algorithms developed. Initiated by Megido 1987.
4. **1992** Mehrotra describes the predictor-corrector method for primal-dual interior point algorithms.
5. **1996** Stephen J. Wright publishes "Primal Dual Interior-Point Algorithms". Interior-point algorithms for LPs has become a matured technology.

Main classes of algorithms

- Active-set algorithms (simplex algorithms)
- Interior-point algorithms

Outline

Linear Program

- Example of a linear program (LP)

$$
\begin{aligned}
\min_{x_1,x_2,x_3} \quad & F(x_1,x_2,x_3) = -3x_1 + 2x_2 - 4x_3 + 5 && \text{(1a)}\\
s.t. \quad & x_1 + x_2 + x_3 = 1 && \text{(1b)}\\
& x_1 + x_2 \geq 0.7 && \text{(1c)}\\
& x_1 + x_3 \leq 0.8 && \text{(1d)}\\
& 0 \leq x_1 \leq 1 && \text{(1e)}\\
& 0 \leq x_2 \leq 1 && \text{(1f)}\\
& 0 \leq x_3 \leq 1 && \text{(1g)}
\end{aligned}
$$

Linear Program

- Affine objective function

$$F(x) = g'x + \gamma = \sum_{j=1}^{n} g_j x_j + \gamma \tag{2}$$

- Linear equality constraints

$$a_i' x = \sum_{j=1}^{n} a_{ij} x_j = b_i \tag{3}$$

- Linear inequality constraint

$$a_i' x = \sum_{j=1}^{n} a_{ij} x_j \geq b_i \tag{4}$$

$$a_i' x = \sum_{j=1}^{n} a_{ij} x_j \leq b_i \tag{5}$$

- Bound constraints: $l_i \leq x_i \leq u_i$

Linear Program

- Linear program

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & F(x) = g'x + \gamma && \text{(6a)}\\
s.t. \quad & a_i'x = b_i \quad i \in \mathcal{E} && \text{(6b)}\\
& a_i'x \geq b_i \quad i \in \mathcal{I}_L && \text{(6c)}\\
& a_i'x \leq b_i \quad i \in \mathcal{I}_U && \text{(6d)}\\
& l \leq x \leq u && \text{(6e)}
\end{aligned}$$

- Canonical form

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & F(x) = g'x && \text{(7a)}\\
s.t. \quad & Ax \geq b && \text{(7b)}\\
& x \geq 0 && \text{(7c)}
\end{aligned}$$

- Standard form

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & F(x) = g'x && \text{(8a)}\\
s.t. \quad & Ax = b && \text{(8b)}\\
& x \geq 0 && \text{(8c)}
\end{aligned}$$

CONVERSION TO STANDARD FORM LP

- LP in standard from

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{9a}$$

$$s.t. \quad Ax = b \tag{9b}$$

$$x \geq 0 \tag{9c}$$

- Greater than inequality: $a_i' x \geq b_i \quad \Rightarrow \quad a_i' x - s = b_i \quad s \geq 0$
- Less than inequality: $a_i' x \leq b_i \quad \Rightarrow \quad a_i' x + s = b_i \quad s \geq 0$
- Lower bound: $l_i \leq x_i \quad \Rightarrow \quad x_i - s = l_i \quad s \geq 0$
- Upper bound: $x_i \leq u_i \quad \Rightarrow \quad x_i + s = u_i \quad s \geq 0$
- Free variable: $x = s_1 - s_2$, $s_1 \geq 0$, $s_2 \geq 0$.

$$x - s_1 + s_2 = 0 \quad s_1 \geq 0, \, s_2 \geq 0$$

CONVERSION TO STANDARD FORM LP

- Example of a linear program (LP)

$$
\begin{aligned}
\min_{x_1,x_2,x_3} \quad & F(x_1,x_2,x_3) = -3x_1 + 2x_2 - 4x_3 + 5 && \text{(10a)}\\
s.t. \quad & x_1 + x_2 + x_3 = 1 && \text{(10b)}\\
& x_1 + x_2 \geq 0.7 && \text{(10c)}\\
& x_1 + x_3 \leq 0.8 && \text{(10d)}\\
& 0 \leq x_1 \leq 1 && \text{(10e)}\\
& 0 \leq x_2 \leq 1 && \text{(10f)}\\
& 0 \leq x_3 \leq 1 && \text{(10g)}
\end{aligned}
$$

CONVERSION TO STANDARD FORM LP

- Introduction of slack variables

$$
\begin{aligned}
&x_1 + x_2 + x_3 = 1 && \Rightarrow && x_1 + x_2 + x_3 = 1 && \text{(11a)}\\
&x_1 + x_2 \geq 0.7 && \Rightarrow && x_1 + x_2 - s_1 = 0.7 \quad s_1 \geq 0 && \text{(11b)}\\
&x_1 + x_3 \leq 0.8 && \Rightarrow && x_1 + x_3 + s_2 = 0.8 \quad s_2 \geq 0 && \text{(11c)}\\
&0 \leq x_1 \leq 1 && \Rightarrow && x_1 \geq 0 \quad x_1 + s_3 = 1 \quad s_3 \geq 0 && \text{(11d)}\\
&0 \leq x_2 \leq 1 && \Rightarrow && x_2 \geq 0 \quad x_2 + s_4 = 1 \quad s_4 \geq 0 && \text{(11e)}\\
&0 \leq x_3 \leq 1 && \Rightarrow && x_3 \geq 0 \quad x_3 + s_5 = 1 \quad s_5 \geq 0 && \text{(11f)}
\end{aligned}
$$

- Matrix-vector formulation
  $x = \begin{bmatrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & s_5 \end{bmatrix}'$.

$$
A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0.7 \\ 0.8 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (12)
$$

CONVERSION TO STANDARD FORM LP

Consider the general LP

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{13a}$$
$$s.t. \quad Ax = b \tag{13b}$$
$$A_l x \geq b_l \tag{13c}$$
$$A_u x \leq b_u \tag{13d}$$
$$l \leq x \leq u \tag{13e}$$

Introduce $x = x_u - x_l$ with $x_u \geq 0$ and $x_l \geq 0$ such that the general LP becomes

$$\min_{x_l, x_u, s_l, s_u, t_l, t_u} \quad F(x_l, x_s, s_l, s_u, t_l, t_u) = g'x_u - g'x_l \tag{14a}$$
$$s.t. \quad Ax_u - Ax_l = b \qquad x_u \geq 0, \quad x_l \geq 0 \tag{14b}$$
$$A_l x_u - A_l x_l - s_l = b_l \qquad s_l \geq 0 \tag{14c}$$
$$A_u x_u - A_u x_l + s_u = b_u \qquad s_u \geq 0 \tag{14d}$$
$$x_u - x_l - t_l = l \qquad t_l \geq 0 \tag{14e}$$
$$x_u - x_l + t_u = u \qquad t_u \geq 0 \tag{14f}$$

Define

$\bar{x} = [x_l; x_u; s_l; s_u; t_l; t_u]$, $\bar{g} = [-g; g; 0; 0; 0; 0]$, $\bar{b} = [b; b_l; b_u; l; u]$, and
$\bar{A} = [-A, A, 0, 0, 0, 0; -A_l, A_l, -I, 0, 0, 0; -A_u, A_u, 0, I, 0, 0; -I, I, 0, 0, -I, 0; -I, I, 0, 0, 0, I]$

such that the general LP can be expressed as the LP in standard form

$$\min_{\bar{x}} \quad \bar{g}'\bar{x} \tag{15a}$$
$$s.t. \quad \bar{A}\bar{x} = \bar{b} \tag{15b}$$
$$\bar{x} \geq 0 \tag{15c}$$

Duality

- Primal LP

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{16a}$$
$$s.t. \quad Ax \geq b \tag{16b}$$

- Lagrange function

$$L(x, \lambda) = g'x - \lambda'(Ax - b) = (g - A'\lambda)'x + b'\lambda \tag{17}$$

- First order necessary and sufficient optimality conditions

$$\nabla_x L(x, \lambda) = g - A'\lambda = 0 \tag{18a}$$
$$Ax \geq b \perp \lambda \geq 0 \tag{18b}$$

- Dual (Lagrange Dual)

$$\max_{x, \lambda} \quad L(x, \lambda) = (g - A'\lambda)'x + b'\lambda \tag{19a}$$
$$s.t. \quad \nabla_x L(x, \lambda) = g - A'\lambda = 0 \tag{19b}$$
$$\lambda \geq 0 \tag{19c}$$

Duality

- Dual (Lagrange Dual)

$$
\begin{aligned}
\max_{x,\lambda} \quad & L(x,\lambda) = (g - A'\lambda)'x + b'\lambda && (20a)\\
s.t. \quad & \nabla_x L(x,\lambda) = g - A'\lambda = 0 && (20b)\\
& \lambda \geq 0 && (20c)
\end{aligned}
$$

- The dual reformulated

$$
\begin{aligned}
\max_{\lambda} \quad & b'\lambda && (21a)\\
s.t. \quad & A'\lambda = g && (21b)\\
& \lambda \geq 0 && (21c)
\end{aligned}
$$

Dual of Standard LP

- Linear program in standard form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & F(x) = g'x && (22a)\\ s.t. \quad & Ax = b && (22b)\\ & x \geq 0 && (22c) \end{aligned}$$

- Lagrange function

$$L(x, \mu, \lambda) = g'x - \mu'(Ax - b) - \lambda' x \qquad (23)$$

- Dual program

$$\begin{aligned} \max_{x,\mu,\lambda} \quad & L(x, \mu, \lambda) = (g - A'\mu - \lambda)'x + b'\mu && (24a)\\ s.t. \quad & \nabla_x L(x, \mu, \lambda) = g - A'\mu - \lambda = 0 && (24b)\\ & \lambda \geq 0 && (24c) \end{aligned}$$

Dual of Standard LP

- Primal LP in standard form

$$\min_{x\in\mathbb{R}^n} \quad F(x) = g'x \tag{25a}$$
$$s.t. \quad Ax = b \tag{25b}$$
$$x \geq 0 \tag{25c}$$

- Dual program

$$\max_{x,\mu,\lambda} \quad L(x,\mu,\lambda) = (g - A'\mu - \lambda)'x + b'\mu \tag{26a}$$
$$s.t. \quad \nabla_x L(x,\mu,\lambda) = g - A'\mu - \lambda = 0 \tag{26b}$$
$$\lambda \geq 0 \tag{26c}$$

- Dual program reformulated

$$\max_{\mu,\lambda} \quad b'\mu \tag{27a}$$
$$s.t. \quad A'\mu + \lambda = g \tag{27b}$$
$$\lambda \geq 0 \tag{27c}$$

LINEAR PROGRAM IN STANDARD FORM.

Optimality Conditions.

- Linear program (LP) in standard form ($A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $g \in \mathbb{R}^n$)

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{28a}$$

$$s.t. \quad Ax = b \tag{28b}$$

$$x \geq 0 \tag{28c}$$

- Lagrange function

$$L(x, \mu, \lambda) = g'x - \mu'(Ax - b) - \lambda' x \tag{29}$$

- Necessary and sufficient optimality conditions

$$\nabla_x L(x, \mu, \lambda) = g - A'\mu - \lambda = 0 \tag{30a}$$

$$Ax = b \tag{30b}$$

$$x \geq 0 \perp \lambda \geq 0 \tag{30c}$$

LINEAR PROGRAM IN STANDARD FORM.

Optimality Conditions.

- Necessary and sufficient optimality conditions

$$\nabla_x L(x, \mu, \lambda) = g - A'\mu - \lambda = 0 \tag{31a}$$

$$Ax = b \tag{31b}$$

$$x \geq 0 \perp \lambda \geq 0 \tag{31c}$$

- Formulation in matrix form

$$\begin{bmatrix} 0 & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} g - \lambda \\ b \end{bmatrix} \tag{32a}$$

$$x \geq 0 \perp \lambda \geq 0 \tag{32b}$$

$$x \in \mathbb{R}^n \, \mu \in \mathbb{R}^m \, \lambda \in \mathbb{R}^n \, A \in \mathbb{R}^{m \times n}$$

LINEAR PROGRAM IN STANDARD FORM.

Optimality Conditions.

- Non-basic and basic sets

$$\mathcal{N} = \{i : x_i = 0\} \tag{33}$$
$$\mathcal{B} = \{1, 2, \dots, n\} \setminus \mathcal{N} \qquad x_i \geq 0 \, i \in \mathcal{B} \tag{34}$$

- Complementarity condition ($x \geq 0 \perp \lambda \geq 0$)

$$\lambda_i \geq 0 \qquad i \in \mathcal{N} \tag{35}$$
$$\lambda_i = 0 \qquad i \in \mathcal{B} \tag{36}$$

- Notation

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} \quad \lambda = \begin{bmatrix} \lambda_B \\ \lambda_N \end{bmatrix} \quad g = \begin{bmatrix} g_B \\ g_N \end{bmatrix} \tag{37}$$
$$A = \begin{bmatrix} B & N \end{bmatrix} \quad B \in \mathbb{R}^{m \times m} \text{ nonsingular} \tag{38}$$

LINEAR PROGRAM IN STANDARD FORM.

Optimality Conditions.

- Formulation in matrix form

$$\begin{bmatrix} 0 & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} g - \lambda \\ b \end{bmatrix} \tag{39a}$$

$$x \geq 0 \perp \lambda \geq 0 \tag{39b}$$

- Use $A = \begin{bmatrix} B & N \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & B' \\ 0 & 0 & N' \\ B & N & 0 \end{bmatrix} \begin{bmatrix} x_B \\ x_N \\ \mu \end{bmatrix} = \begin{bmatrix} g_B - \lambda_B \\ g_N - \lambda_N \\ b \end{bmatrix} \tag{40}$$

- $x_N = 0$, $\lambda_B = 0$, $x_B \geq 0$, $\lambda_N \geq 0$

$$B'\mu = g_B - \lambda_B = g_B \tag{41a}$$

$$\lambda_N = g_N - N'\mu \tag{41b}$$

$$Bx_B = b - Nx_N = b \tag{41c}$$

LINEAR PROGRAM IN STANDARD FORM.

Optimality Conditions.

- Select the active set, i.e. the non-basic set $\mathcal{N}$, and thereby the basic set $\mathcal{B}$. Then

$$x_N = 0 \tag{42a}$$

$$\lambda_B = 0 \tag{42b}$$

- Compute x_B, μ, and λ_N:

$$Bx_B = b \tag{42c}$$

$$B'\mu = g_B \tag{42d}$$

$$\lambda_N = g_N - N'\mu \tag{42e}$$

- Verify the complementarity conditions

$$x_B \geq 0 \tag{42f}$$

$$\lambda_N \geq 0 \tag{42g}$$

Revised Simplex Algorithm

Require: The sets $\mathcal{B}$ and $\mathcal{N}$ such that $x_B = B^{-1}b \geq 0$ and $x_N = 0$ in which $B = [a_i]_{i\in\mathcal{B}}$, $N = [a_i]_{i\in\mathcal{N}}$, $x_B = (x_i)_{i\in\mathcal{B}}$, $x_N = (x_i)_{i\in\mathcal{N}}$, and $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$.

while not STOP **do**

Set $B = [a_i]_{i\in\mathcal{B}}$ and $N = [a_i]_{i\in\mathcal{N}}$.

Solve for μ:

$$B'\mu = g_B, \quad g_B = (g_i)_{i\in\mathcal{B}} \tag{43}$$

Compute: $\lambda_N = g_N - N'\mu$ in which $g_N = (g_i)_{i\in\mathcal{N}}$ and $\lambda_N = (\lambda_i)_{i\in\mathcal{N}}$.

if $(\lambda_N)_i \geq 0 \forall i$ **then**

STOP = **true** (optimal solution found)

else

Select $s : (\lambda_N)_s < 0$. Let $i_s = \mathcal{N}(s)$ be the corresponding entry in $\mathcal{N}$.

Solve for h:

$$Bh = a_{i_s} \tag{44}$$

$\mathcal{J} = \left\{ \arg\min_{i:h_i>0} \frac{(x_B)_i}{h_i} \right\}$

if $\mathcal{J} = \emptyset$ **then**

STOP = **true** (unbounded problem, no solution)

else

Select $j \in \mathcal{J}$ and $\alpha = \frac{(x_B)_j}{h_j}$

$x_B \leftarrow x_B - \alpha h$, $((x_B)_j \leftarrow 0)$, $(x_N)_s \leftarrow \alpha$

$i_j \leftarrow \mathcal{B}(j)$, $\mathcal{B} \leftarrow (\mathcal{B} \setminus \{i_j\}) \cup \{i_s\}$, $\mathcal{N} \leftarrow (\mathcal{N} \setminus \{i_s\}) \cup \{i_j\}$

end if

end if

end while

$\lambda_B \leftarrow 0$ in which $\lambda_B = (\lambda_i)_{i\in\mathcal{B}}$

FINDING A FEASIBLE POINT

- LP in standard form

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{45a}$$
$$s.t. \quad Ax = b \tag{45b}$$
$$x \geq 0 \tag{45c}$$

- Constrained l_1- or l_∞-regression

$$\min_{x \in \mathbb{R}^n} \quad F(x) = \|Ax - b\|_1 \tag{46a}$$
$$s.t. \quad x \geq 0 \tag{46b}$$

$$\min_{x \in \mathbb{R}^n} \quad F(x) = \|Ax - b\|_\infty \tag{47a}$$
$$s.t. \quad x \geq 0 \tag{47b}$$

- Initial feasible point: $x = 0$.
- Feasible problem if: $F(x^*) = 0$

FINDING A FEASIBLE POINT

- l_∞-regression problem

$$\min_{x \in \mathbb{R}^n} \quad F(x) = \|Ax - b\|_\infty \tag{48a}$$

$$s.t. \quad x \geq 0 \tag{48b}$$

- The l_∞-regression problem as an LP

$$\min_{x \in \mathbb{R}^n, t \in \mathbb{R}} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}' \begin{bmatrix} x \\ t \end{bmatrix} \tag{49a}$$

$$s.t. \quad \begin{bmatrix} A & e \\ -A & e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \geq \begin{bmatrix} b \\ -b \end{bmatrix} \tag{49b}$$

$$\begin{bmatrix} x \\ t \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49c}$$

- Next: Convert this LP to standard form and solve using the simplex method. Easy to find a feasible point of this program.

FINDING A FEASIBLE POINT

- Introduce slack variables to have an LP in standard form

$$\min_{x\in\mathbb{R}^n, t\in\mathbb{R}, s_1\in\mathbb{R}^m, s_2\in\mathbb{R}^m} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix}' \begin{bmatrix}x\\t\\s_1\\s_2\end{bmatrix} \tag{50a}$$

$$s.t. \quad \begin{bmatrix}A & e & -I & 0\\-A & e & 0 & -I\end{bmatrix} \begin{bmatrix}x\\t\\s_1\\s_2\end{bmatrix} = \begin{bmatrix}b\\-b\end{bmatrix} \tag{50b}$$

$$\begin{bmatrix}x\\t\\s_1\\s_2\end{bmatrix} \geq \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{50c}$$

- Initial feasible point: $x = 0$, $t = \max |b_i|$, $s_1 = te - b$, $s_2 = te + b$

REVISED SIMPLEX PROCEDURE

1. Convert the stated LP problem to an LP in standard form:

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{51a}$$

$$s.t. \quad Ax = b \tag{51b}$$

$$x \geq 0 \tag{51c}$$

2. **Phase I simplex**: Find a feasible point or detect infeasibility by solution of a l_∞-regression problem

$$\min_{x \in \mathbb{R}^n} \quad F(x) = \|Ax - b\|_\infty \tag{52a}$$

$$s.t. \quad x \geq 0 \tag{52b}$$

   using the revised simplex algorithm.

3. **Phase II simplex**: Use the solution, x, of the l_∞-regression problem in phase I as an initial feasible solution for solution of (51) by the revised simplex algorithm.
4. Report the solution in terms of the original problem.

Revised Simplex Procedure

- LP in standard form

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{53a}$$
$$s.t. \quad Ax = b \tag{53b}$$
$$x \geq 0 \tag{53c}$$

- Phase 1: LP for finding a feasible solution [standard LP]

$$\min_{x,s,t} \quad F(x,s,t) = e's + e't \tag{54a}$$
$$s.t. \quad Ax + s - t = b \tag{54b}$$
$$x \geq 0,\ s \geq 0,\ t \geq 0 \tag{54c}$$

- Phase 2: If $s^* = 0$ and $t^* = 0$ for the phase 1 LP, set the initial point to $x = x^*$ and solve the standard LP

$$\min_{x \in \mathbb{R}^n} \quad F(x) = g'x \tag{55a}$$
$$s.t. \quad Ax = b \tag{55b}$$
$$x \geq 0 \tag{55c}$$

Presolving

The purpose of presolving is to reduce the size of the user-defined LP before passing it to the solver.

Andersen and Andersen (1995): Presolving in linear programming. Mathematical Programming, 71, pp. 221-245

$$\min_{x} \quad c'x \tag{56a}$$
$$s.t. \quad Ax = b \tag{56b}$$
$$l \leq x \leq u \tag{56c}$$

- Row singleton
- Free column singleton
- Zero rows and columns
- Forcing or dominated constraints