ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 3: Stochastic Models

3.3 Heath-Jarrow-Morton Framework

Interest Rate Models

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3.3 Heath-Jarrow-Morton Framework

- Heath–Jarrow–Morton (HJM) framework specifies forward rate dynamics directly
- Initial forward curve exogeneous
- Forward rate drift fully determined by volatility (HJM drift condition)
- HJM framework contains all interest rate models driven by Brownian motion

Ingredients

A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with objective probability measure $\mathbb{P}$

- d-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))^\top$
- d-dimensional market price of risk $\lambda(t)$ such that by Girsanov theorem

$$dW^*(t) = dW(t) + \lambda(t)\, dt$$

is a Brownian motion under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ with Radon-Nikodym density process

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathcal{E}\left(-\int_0^t \lambda(s)^\top dW(s)\right).$$

HJM Forward Rate Specification

The Heath–Jarrow–Morton (HJM) framework is very broad and contains all interest rate models driven by a finite number of Brownian motions.

The $\mathbb{Q}$-dynamics of the forward rates $f(t, T)$ is

$$df(t, T) = \alpha(t, T)\, dt + \sigma(t, T)\, dW^*(t)$$

with parameters

- $\alpha(t, T)$: drift process (is fully determined by $\sigma(t, T)$, see below!)
- $\sigma(t, T) = (\sigma_1(t, T), \ldots, \sigma_d(t, T))$: volatility process
- $f(0, T)$: exogeneous initial forward curve.

Towards Bond Price Dynamics

Bonds prices are explicit in terms of forward curve: $P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)du}$.

Formal differentiation under the integral and changing order of integration gives

$$
\begin{aligned}
d\left(-\int_t^T f(t,u)\,du\right) &= f(t,t)\,dt - \int_t^T df(t,u)\,du \\
&= \left(r(t) - \int_t^T \alpha(t,u)\,du\right)dt + v(t,T)\,dW^*(t)
\end{aligned}
$$

where we write

$$
v(t,T) = -\int_t^T \sigma(t,u)\,du.
$$

HJM Bond Price Dynamics

Itô formula applied to $P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)du}$ gives

$$\frac{dP(t,T)}{P(t,T)} = d\left(-\int_t^T f(t,u)\,du\right) + \frac{1}{2}\|v(t,T)\|^2\,dt$$
$$= \left(r(t) - \int_t^T \alpha(t,u)\,du + \frac{1}{2}\|v(t,T)\|^2\right)dt + v(t,T)\,dW^*(t).$$

Arbitrage Pricing Theorem implies $-\int_t^T \alpha(t,u)\,du + \frac{1}{2}\|v(t,T)\|^2 = 0$ for all T.

Differentiating in T gives the HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)^\top du.$$

HJM Theorem

Summarizing, the $\mathbb{Q}$-dynamics of $f(t,T)$ is fully determined by volatility $\sigma(t,T)$

$$df(t,T) = \underbrace{\sigma(t,T)\int_t^T \sigma(t,u)^\top du}_{\text{HJM drift}}\, dt + \sigma(t,T)\, dW^*(t).$$

The corresponding bond price dynamics is of the form

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + v(t,T)\,dW^*(t)$$

with bond return volatility given by

$$v(t,T) = -\int_t^T \sigma(t,u)\,du.$$

Example: Vasiček Short Rate Model

Which HJM model $\sigma(t, T)$ corresponds to the Vasiček short rate model?

Vasiček model $dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma\,dW^*(t)$ implies forward rate

$$f(t, T) = A'(T - t) + B'(T - t)r(t) = A'(T - t) + \mathrm{e}^{-\kappa(T-t)}r(t).$$

Integration by parts gives

$$df(t, T) = \underbrace{(\cdots)}_{\text{HJM drift!}}\,dt + \mathrm{e}^{-\kappa(T-t)}\sigma\,dW^*(t).$$

Hence $\sigma(t, T) = \mathrm{e}^{-\kappa(T-t)}\sigma$.