ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 3: Stochastic Models

3.2 Short Rate Models

Interest Rate Models

Damir Filipović

3.2 Short Rate Models

- Arbitrage Pricing Theorem implies a bond pricing formula in terms of short rates
- Earliest interest rate models: diffusion short rate models
  - Vasiček model
  - Cox–Ingersoll–Ross (CIR) model
- Fitting initial term structure: time-inhomogeneous models
  - Hull–White model

Ingredients

A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with objective probability measure $\mathbb{P}$

- Brownian motion $W(t)$
- Market price of risk $\lambda(t)$ such that by Girsanov theorem

$$dW^*(t) = dW(t) + \lambda(t)\,dt$$

  is a Brownian motion under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ with Radon-Nikodym density process

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathcal{E}\left(-\int_0^t \lambda(s)\,dW(s)\right)$$

Arbitrage Bond Pricing

Given adapted short rate process $r(t)$, Arbitrage Pricing Theorem implies that discounted bond price processes

$$\mathrm{e}^{-\int_0^t r(s)ds} P(t, T), \quad t \leq T,$$

are $\mathbb{Q}$-martingales, for all maturities T.

Using $P(T, T) = 1$, this leads us to the **bond pricing formula:**

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^T r(s)ds} \right].$$

Dynamics of $P(t, T)$ depend on model for $r(t)$ under $\mathbb{Q}$ (and $\lambda(t)$).

Vasiček Model

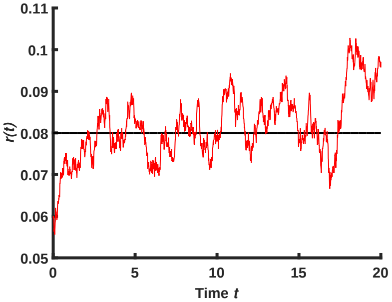
$\mathbb{Q}$-dynamics of the short rate is

$$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma\,dW^*(t)$$

for constant parameters

- θ: mean-reversion level
- κ: mean-reversion speed
- σ: volatility

$r(0) = 0.06,\ \theta = 0.08,\ \kappa = 0.86,\ \sigma = 0.01$



Vasiček Model is Gaussian

Variation of constants shows the solution to the Vasiček Model is

$$r(t) = \mathrm{e}^{-\kappa t} r(0) + \int_0^t \mathrm{e}^{-\kappa(t-s)} \kappa\theta \, ds + \int_0^t \mathrm{e}^{-\kappa(t-s)} \sigma \, dW^*(s).$$

Consequently, $r(t)$ is a Gaussian process with mean and covariance functions

$$m(t) = \mathrm{e}^{-\kappa t} r(0) + \theta \left(1 - \mathrm{e}^{-\kappa t}\right)$$
$$c(t_1, t_2) = \frac{\sigma^2}{2\kappa} \left(1 - \mathrm{e}^{-2\kappa(t_1 \wedge t_2)}\right)$$

and normal limiting distribution with mean θ and variance $\frac{\sigma^2}{2\kappa}$, for $t \to \infty$.

Calculating Bond Prices in the Vasiček Model

Changing order of integration $\int_t^T \int_0^s \cdot du\, ds = \int_0^t \int_t^T \cdot ds\, du + \int_t^T \int_u^T \cdot ds\, du$ shows

$$\int_t^T r(s)\, ds = \int_t^T \mathrm{e}^{-\kappa(s-t)} ds\, \mathrm{e}^{-\kappa t} r(0) + \int_t^T \int_0^s \mathrm{e}^{-\kappa(s-u)} \kappa\theta\, du\, ds$$
$$+ \int_t^T \int_0^s \mathrm{e}^{-\kappa(s-u)} \sigma\, dW^*(u)\, ds$$
$$= B(T-t) r(t) + \int_t^T B(T-u)\kappa\theta\, du + \int_t^T B(T-u)\sigma\, dW^*(u)$$

where the function $B(t)$ is defined as

$$B(t) = \int_0^t \mathrm{e}^{-\kappa s} ds.$$

Calculating Bond Prices in the Vasiček Model

This implies that $\int_t^T r(s)\,ds$ conditional on $\mathcal{F}_t$ is normal distributed with

$$\text{mean} = B(T-t)r(t) + \int_0^{T-t} B(u)\kappa\theta\,du, \quad \text{variance} = \int_0^{T-t} B(u)^2\sigma^2\,du.$$

The moment generating function of a normal distribution is well known and gives

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds}\right] = \mathrm{e}^{-A(T-t)-B(T-t)r(t)}$$

where the function $A(t)$ is defined as

$$A(t) = \int_0^t \left(B(u)\kappa\theta - B(u)^2\frac{\sigma^2}{2}\right) du.$$

Affine Bond Prices in the Vasiček Model

Summarizing, the Vasiček short rate model $dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma\,dW^*(t)$ yields exponential affine bond prices in the prevailing short rate $r(t)$

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds}\right] = \mathrm{e}^{-A(T-t)-B(T-t)r(t)}$$

where the functions $A(t)$ and $B(t)$ solve the Riccati equations

$$\begin{aligned} A'(t) &= \kappa\theta B(t) - \tfrac{\sigma^2}{2}B(t)^2 \\ B'(t) &= -\kappa B(t) + 1 \end{aligned}$$

along with initial conditions $A(0) = 0$ and $B(0) = 0$.

Cox-Ingersoll-Ross (CIR) Model

$\mathbb{Q}$-dynamics of the short rate is

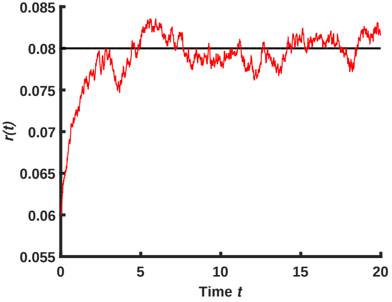
$$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dW^*(t)$$

for constant parameters

- θ: mean-reversion level
- κ: mean-reversion speed (s.t. $\kappa\theta \geq 0$)
- σ: volatility

Fact: there exists a unique nonnegative solution $r(t) \geq 0$ for all initial $r(0) \geq 0$.

$r(0) = 0.06,\ \theta = 0.08,\ \kappa = 0.86,\ \sigma = 0.01$



Affine Bond Prices in the CIR Model

Claim: The CIR short rate model $dr(t) = \kappa(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dW^*(t)$ yields exponential affine bond prices in the prevailing short rate $r(t)$

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds}\right] = \mathrm{e}^{-A(T-t)-B(T-t)r(t)}$$

where the functions $A(t)$ and $B(t)$ solve the Riccati equations

$$A'(t) = \kappa\theta B(t)$$

$$B'(t) = -\frac{\sigma^2}{2}B(t)^2 - \kappa B(t) + 1$$

along with initial conditions $A(0) = 0$ and $B(0) = 0$.

Closed-Form Expressions for $A(t)$ and $B(t)$

The solutions $A(t)$ and $B(t)$ to these Riccati equations are given in closed-form

$$A(t) = -\frac{2\kappa\theta}{\sigma^2} \log\left(\frac{2\gamma \mathrm{e}^{(\gamma+\kappa)t/2}}{(\gamma+\kappa)\left(\mathrm{e}^{\gamma t}-1\right)+2\gamma}\right)$$

$$B(t) = \frac{2\left(\mathrm{e}^{\gamma t}-1\right)}{(\gamma+\kappa)\left(\mathrm{e}^{\gamma t}-1\right)+2\gamma}$$

where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$. This can be verified by elementary calculus.

Proof of the Claim

To prove the claim

$$\mathbb{E}_t^{\mathbb{Q}}\Big[\underbrace{\mathrm{e}^{-\int_0^t r(s)ds}\mathrm{e}^{-\int_t^T r(s)ds}}_{=M(T)}\Big] = \underbrace{\mathrm{e}^{-\int_0^t r(s)ds}\mathrm{e}^{-A(T-t)-B(T-t)r(t)}}_{=M(t)}$$

show that $M(t)$ is a martingale with terminal value $M(T) = \mathrm{e}^{-\int_0^T r(s)ds}$. Indeed, integration by parts and Itô formula show that the drift of $\frac{dM(t)}{M(t)}$ equals

$$\underbrace{(A'(T-t) - \kappa\theta B(T-t))}_{=0} + \underbrace{\left(B'(T-t) + \frac{\sigma^2}{2}B(T-t)^2 + \kappa B(T-t) - 1\right)}_{=0} r(t).$$

Fitting the Initial Term Structure

Vasiček and CIR models imply parametric initial forward curve

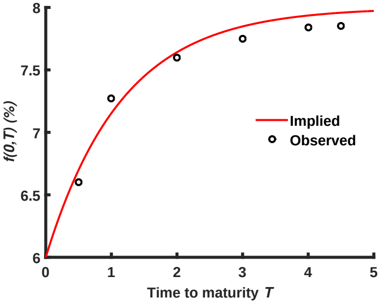
$$f(0, T) = A'(T) + B'(T)r(0)$$

with 4 parameters κ, θ, σ, and $r(0)$.

Problem: does not fit data in general

Solution: time-inhomogeneous models

CIR with $r(0) = 0.06$, $\theta = 0.08$, $\kappa = 0.86$, $\sigma = 0.01$



Time-Inhomogeneous Models

Build a time-inhomogeneous short rate model

$$r(t) = \phi(t) + \widetilde{r}(t)$$

fitting any given initial forward curve $f_0(t)$ using two ingredients:

- auxiliary Vasiček (or CIR) model

$$d\widetilde{r}(t) = -\kappa\widetilde{r}(t)\,dt + \sigma\,dW^*(t), \quad \widetilde{r}(0) = f_0(0)$$

  with closed-form bond prices $\widetilde{P}(t,T) = \mathrm{e}^{-\widetilde{A}(T-t)-\widetilde{B}(T-t)\widetilde{r}(t)}$,

- deterministic shift function $\phi(t)$ with $\phi(0) = 0$.

Calibrating the Shift Function

Bond prices are of the form

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T(\phi(s)+\widetilde{r}(s))ds}\right] = \mathrm{e}^{-\int_t^T\phi(s)ds}\,\widetilde{P}(t,T).$$

The model implied initial forward curve equals

$$f(0,t) = \phi(t) + \widetilde{A}'(t) + \widetilde{B}'(t)\widetilde{r}(0).$$

We obtain a perfect fit of the given initial forward curve, $f(0,t) = f_0(t)$, for

$$\phi(t) = f_0(t) - \left(\widetilde{A}'(t) + \widetilde{B}'(t)\widetilde{r}(0)\right).$$

Hull-White Model

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Calculate dynamics of $r(t) = \phi(t) + \widetilde{r}(t)$

$$dr(t) = \phi'(t)\,dt + d\widetilde{r}(t) = (\phi'(t) - \kappa\widetilde{r}(t))\,dt + \sigma\,dW^*(t).$$

Plugging in $\widetilde{r}(t) = r(t) - \phi(t)$, this can be written as

$$dr(t) = \kappa\,(\theta(t) - r(t))\,dt + \sigma\,dW^*(t)$$

with time-inhomogeneous mean-reversion level $\theta(t) = \phi(t) + \frac{\phi'(t)}{\kappa}$.

This extension of the Vasiček model is called Hull–White model.