ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 2: Estimating the Term Structure

2.3 Smoothing Methods

Interest Rate Models

Damir Filipović

2.3 Smoothing Methods

Exact methods:

- Discount curve is sensitive to small changes in cash flow matrix.
- Discount factors of similar maturity can be very different.
- Leads to ragged forward curves.

Smoothing methods: estimate a smooth forward curve from market rates at the cost of not exactly matching data.

Abstract Formulation

Set spot date $t_0 = 0$ for simplicity.

Data: N yields $y_i = y(0, T_i)$ with maturities $0 < T_1 < \cdots < T_N$.

Aim: find smooth forward curve $f(T) = f(0, T)$ matching the yields optimally

$$\int_0^{T_i} f(u)\, du = T_i y_i + \epsilon_i$$

for pricing errors ϵ_i subject to being minimized.

Key Criteria for Smoothing Methods

Smoothing methods mainly used by central banks.

- Smoothness: supply a market expectation for monetary policy purposes rather than precise pricing of all bonds in the market.
- Flexibility: sufficiently flexible to capture movements in the underlying term structure.
- Stability: small changes in data at one maturity do not have disproportionate effect on forward rates at other maturities.

Estimation Methods Used by Several Central Banks

Bank for International Settlements 2005: Nelson–Siegel (NS), Svensson (S), weighted prices (wp).

Central bank	Method	Minimized error
Belgium	S or NS	wp
Canada	Exponential spline	wp
Finland	NS	wp
France	S or NS	wp
Germany	S	yields
Italy	NS	wp
Japan	Smoothing spline	prices
Norway	S	yields
Spain	S	wp
Sweden	Smoothing spline or S	yields
Switzerland	S	yields
UK	Smoothing spline	yields
USA	Smoothing spline	bills: wp, bonds: prices

Nelson-Siegel and Svensson Curves

Parametric families of forward curves.

- Nelson–Siegel:
$$f(T) = \beta_0 + \beta_1 \mathrm{e}^{-aT} + \beta_2 a T \mathrm{e}^{-aT}$$
for parameters $\beta_0, \beta_1, \beta_2$, and a.
- Svensson:
$$f(T) = \beta_0 + \beta_1 \mathrm{e}^{-a_1 T} + \beta_2 a_1 T \mathrm{e}^{-a_1 T} + \beta_3 a_2 T \mathrm{e}^{-a_2 T}$$
for parameters $\beta_0, \beta_1, \beta_2, \beta_3$, and a_1, a_2.

Nelson-Siegel Yield Curves

Nelson–Siegel yield curves are

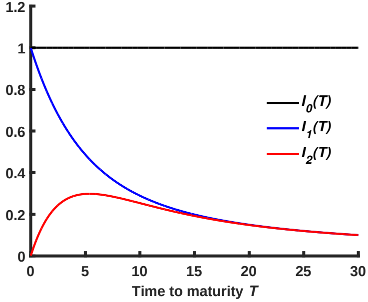
$$
\begin{aligned}
y(T) &= \frac{1}{T}\int_0^T f(u)\,du \\
&= \beta_0 l_0(T) + \beta_1 l_1(T) + \beta_2 l_2(T)
\end{aligned}
$$

with basis functions

- $l_0(T) = 1$ (level)
- $l_1(T) = \frac{1-\mathrm{e}^{-aT}}{aT}$ (slope)
- $l_2(T) = \frac{1-\mathrm{e}^{-aT}}{aT} - \mathrm{e}^{-aT}$ (curvature).

Issues: flexibility, stability

Figure with $a = 0.3$:



Smoothing Splines: Hilbert Space Approach

Find forward curve in Hilbert space H consisting of absolutely continuous functions on $[0, T_*]$ with scalar product

$$\langle g, h \rangle_H = g(0)h(0) + \int_0^{T_*} g'(u)h'(u)\, du.$$

Solve optimization problem

$$\min_{f \in H} \int_0^{T_*} (f'(u))^2\, du + \alpha \sum_{i=1}^{N} \left(T_i y_i - \int_0^{T_i} f(u)\, du \right)^2 \qquad \text{(P)}$$

where $\alpha > 0$ tunes trade-off between smoothness and correctness of fit.

Lorimier's Theorem

The unique solution f of (P) is a quadratic spline characterized by

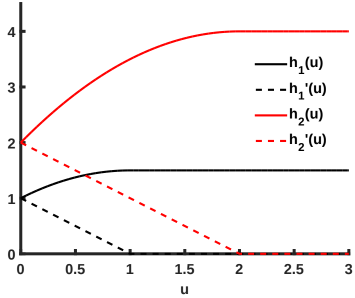
$$f(u) = \beta_0 + \sum_{i=1}^{N} \beta_i h_i(u)$$

where $h_i \in C^1[0, T_*]$ is a quadratic basis spline with

$$h_i'(u) = (T_i - u)^+, \quad h_i(0) = T_i$$

and $\beta_0, \ldots, \beta_N$ solve a linear system.

Quadratic basis splines h_1 and h_2 together with first derivatives for $T_1 = 1$ and $T_2 = 2$:



Lorimier's Theorem Continued

Lorimier's theorem continued: $\beta_0, \ldots, \beta_N$ solve the linear system of equations

$$\sum_{i=1}^{N} \beta_i T_i = 0,$$

$$\alpha \left(y_i T_i - \beta_0 T_i - \sum_{j=1}^{N} \beta_j \langle h_i, h_j \rangle_H \right) = \beta_i, \quad i = 1, \ldots, N.$$

Choice of Parameter α

Smoothing splines satisfy key criteria: smoothness, flexibility, stability.

Parameter α tunes trade-off between smoothness and correctness of fit:

- $\alpha \to 0$: maximal smoothness, constant forward curve $f(T) = \beta_0$
- $\alpha \to \infty$: perfect fit subject to minimal $\int_0^{T_*} (f'(u))^2 \, du$

Choice of α is critical.

Example: Swiss government bond yields in August 2011