ÉCOLE POLYTECHNIQUE
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Chapter 4: Interest Rate Derivatives

4.4 Calibration Example

Interest Rate Models

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4.4 Calibration Example

- Price caps in Gaussian HJM model
- Calibrate two-factor Gaussian HJM model to market data consisting of
  - swap rates
  - ATM cap quotes
- Minimize vega weighted squared cap price errors

Cap Pricing

Consider

- spot date $t_0 = 0$
- reset/settlement dates $0 < T_0 < T_1 < \cdots < T_n$ with $T_0 = T_i - T_{i-1} \equiv \delta$
- cap rate κ

Recall the time-0 price of the ith caplet is

$$Cpl(T_{i-1}, T_i) = (1 + \delta\kappa)\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^{T_{i-1}} r(s)ds} \underbrace{\left(\frac{1}{1+\delta\kappa} - P(T_{i-1}, T_i) \right)^+}_{\text{payoff of put option on } T_i\text{-bond}} \right]$$

Caplet Price Formula in Gaussian HJM Model

Recall the bond put option price formula in Gaussian HJM models is

$$Cpl(T_{i-1}, T_i) = (1+\delta\kappa)\left(\frac{1}{1+\delta\kappa}P(0, T_{i-1})\Phi(-d_2(i)) - P(0, T_i)\Phi(-d_1(i))\right)$$

with standard normal cumulative distribution function Φ and

$$d_{1,2}(i) = \frac{\log\left(\frac{P(0,T_i)}{P(0,T_{i-1})}(1+\delta\kappa)\right) \pm \frac{1}{2}\int_0^{T_{i-1}} \|v(s, T_{i-1}) - v(s, T_i)\|^2 ds}{\sqrt{\int_0^{T_{i-1}} \|v(s, T_{i-1}) - v(s, T_i)\|^2 ds}}$$

where $v(t, T)$ is the T-bond return volatility.

Two-Factor Gaussian HJM Model

Consider the two-factor Gaussian HJM model with volatility specification

$$\sigma(t, T) = \left(\mathrm{e}^{-\beta_1(T-t)} v_1,\ \mathrm{e}^{-\beta_2(T-t)} v_2\right)$$

for real parameters $v_1, v_2, \beta_1, \beta_2$ to be calibrated to cap market data.

We obtain

$$\int_0^{T_{i-1}} \|v(s, T_{i-1}) - v(s, T_i)\|^2 ds = \sum_{k=1}^{2} \frac{v_k^2}{\beta_k^2} \left(\mathrm{e}^{-\beta_k T_{i-1}} - \mathrm{e}^{-\beta_k T_i}\right)^2 \frac{\mathrm{e}^{2\beta_k T_{i-1}} - 1}{2\beta_k}.$$

Calibration Problem

Goal: calibrate the model to swap and cap data at spot date $t_0 = 0$.

Given data:

- spot swap rates (with annual fixed leg payments)
- ATM cap quotes (caps with semi-annual cash flows)

Procedure:

- Estimate discount curve from swap data
- Calculate forward swap rates with semi-annual fixed payments (ATM cap rates)
- Minimize weighted squared cap price errors for calibrating the volatility parameters $v_1, v_2, \beta_1, \beta_2$

Swap Data

Spot swap rates with annual fixed leg.

First quote is the 6M simple spot rate.

Maturity	Swap rate (%)
6m	0.3430
1y	0.4420
2y	0.6260
3y	0.8630
4y	1.1191
5y	1.3650
6y	1.5750
7y	1.7574
8y	1.9184
9y	2.0630
10y	2.1905
15y	2.5990
20y	2.7135
30y	2.7135

Estimated Discount Curve

Use exact method based on pseudoinverse to estimate weighted increments

$$\Delta = \left(\frac{P(0,T_0)-1}{\sqrt{\delta}}, \frac{P(0,T_1)-P(0,T_0)}{\sqrt{\delta}}, \dots, \frac{P(0,T_N)-P(0,T_{N-1})}{\sqrt{\delta}}\right)^\top$$

with $T_k = (k+1)\delta$ and $\delta = \frac{1}{2}$, for $k = 1, \dots, 59$ ($T_{59} = 30$ years).

Estimated $P(0, T_k)$:

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
T_{10i}	0.9983	0.9216	0.7865	0.6591	0.5642	0.4918
T_{1+10i}	0.9956	0.9093	0.7729	0.6489	0.5565	0.4855
T_{2+10i}	0.9916	0.8963	0.7595	0.6390	0.5489	0.4793
T_{3+10i}	0.9876	0.8833	0.7462	0.6291	0.5414	0.4731
T_{4+10i}	0.9810	0.8697	0.7331	0.6193	0.5340	0.4671
T_{5+10i}	0.9745	0.8562	0.7200	0.6096	0.5267	0.4610
T_{6+10i}	0.9653	0.8422	0.7072	0.6001	0.5195	0.4551
T_{7+10i}	0.9562	0.8283	0.6943	0.5905	0.5122	0.4493
T_{8+10i}	0.9450	0.8142	0.6818	0.5812	0.5052	0.4435
T_{9+10i}	0.9338	0.8001	0.6692	0.5719	0.4982	0.4378

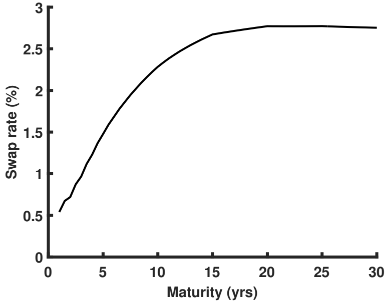
Estimated Forward Swap Curve

Six-month forward swap rates

- first reset date $T_0 = \frac{1}{2}$
- semi-annual fixed payments ($\delta = \frac{1}{2}$)

derived from estimated discount curve:

$$R_{swap}(T_0, T_n) = \frac{P(0, T_0) - P(0, T_n)}{\delta \sum_{i=1}^{n} P(0, T_i)}$$



Cap Data

ATM cap prices in terms of

- nominal prices
- Black implied volatilities
- Normal implied volatilities

and corresponding ATM strike rates (6M forward swap rates).

Caps have semi-annual cash flows.

Maturity	Price	Black IV (%)	Normal IV (bps)	ATM strike (%)
1y	0.0012	170.52	86.81	0.54
2y	0.0046	113.62	76.58	0.72
3y	0.0092	76.52	70.92	0.97
4y	0.0148	54.54	67.17	1.23
5y	0.0210	41.36	63.86	1.48
6y	0.0278	34.58	62.10	1.68
7y	0.0349	30.46	60.79	1.86
8y	0.0417	27.10	58.67	2.02
9y	0.0490	25.02	57.49	2.16
10y	0.0565	23.67	56.86	2.28
15y	0.0904	19.87	53.46	2.67
20y	0.1196	19.38	54.80	2.77
30y	0.1686	19.31	56.79	2.75

Calibration Problem

Calibration: find parameters $\theta = (v_1, v_2, \beta_1, \beta_2)$ such that model cap prices $\hat{C}_n^\theta$ are "as close as possible" to market cap prices C_n, for $n = 1 \dots 13$.

Minimize mean squared error of implied volatilities rather than nominal prices.

Reason: Black and normal implied volatilities

- standardize option prices
- are more comparable across maturities and strikes.

Problem: computing implied volatilities involves numerical inversion of the Black or Bachelier formula at each minimization step, which is computationally costly.

Solution: Vega Weights

Use vega $= \frac{\partial C_n}{\partial \sigma}$, the derivative of Black or Bachelier cap price w.r.t. volatility.

First order Taylor expansion gives

$$\hat{C}_n^\theta \approx C_n + \frac{\partial C_n}{\partial \sigma} \left(\hat{\sigma}_n^\theta - \sigma_n \right)$$

where $\hat{\sigma}_n^\theta$ is the model implied volatility and σ_n is the market implied volatility.

We obtain vega weighted squared cap price errors

$$\left(\hat{\sigma}_n^\theta - \sigma_n \right)^2 \approx \frac{1}{\left(\frac{\partial C_n}{\partial \sigma} \right)^2} \left(\hat{C}_n^\theta - C_n \right)^2 .$$

Black Cap Vega

Recall Black price of ith caplet is, writing $F_i = F(0, T_{i-1}, T_i)$,

$$Cpl(T_{i-1}, T_i) = \delta P(0, T_i)(F_i \Phi(d_{1,i}) - \kappa \Phi(d_{2,i})),$$

where

$$d_{1,i} = \frac{\log(F_i/\kappa) + \sigma^2 T_{i-1}}{\sigma \sqrt{T_{i-1}}} \quad \text{and} \quad d_{2,i} = d_{1,i} - \sigma \sqrt{T_{i-1}}.$$

Elemantary calculus, using $F_i \phi(d_{1,i}) = \kappa \phi(d_{2,i})$, gives Black caplet vega:

$$\frac{\partial Cpl(T_{i-1}, T_i)}{\partial \sigma} = \delta P(0, T_i) F_i \sqrt{T_{i-1}} \phi(d_{1,i}).$$

Black cap vega is sum of Black caplet vegas $\frac{\partial C_n}{\partial \sigma} = \sum_{i=1}^{n} \frac{\partial Cpl(T_{i-1}, T_i)}{\partial \sigma}$.

Bachelier Cap Vega

Recall Bachelier price of ith caplet is, writing $F_i = F(0, T_{i-1}, T_i)$,

$$Cpl(T_{i-1}, T_i) = \delta P(0, T_i)\sigma\sqrt{T_{i-1}}(D_i\Phi(D_i) + \phi(D_i)),$$

where

$$D_i = \frac{F_i - \kappa}{\sigma\sqrt{T_{i-1}}}.$$

Elemantary calculus, using $\frac{\partial D_i}{\partial \sigma} = -\frac{1}{\sigma}D_i$ and $\phi'(D_i) = -D_i\phi(D_i)$, gives Bachelier caplet vega:

$$\frac{\partial Cpl(T_{i-1}, T_i)}{\partial \sigma} = \delta P(0, T_i)\sqrt{T_{i-1}}\phi(D_i).$$

Bachelier cap vega is sum of Bachelier caplet vegas $\frac{\partial C_n}{\partial \sigma} = \sum_{i=1}^{n} \frac{\partial Cpl(T_{i-1}, T_i)}{\partial \sigma}$.

Weighted Least Squares Calibration

Using Bachelier vegas $\frac{\partial C_n}{\partial \sigma}$, the calibration problem boils down to the weighted least squares problem

$$\min_{\theta} \sum_{n=1}^{13} \frac{1}{\left(\frac{\partial C_n}{\partial \sigma}\right)^2} \left(\hat{C}_n^{\theta} - C_n\right)^2 .$$

Using standard optimization libraries, such as Matlab built in fminsearch, we find

$$\hat{v}_1 = 0.0149, \quad \hat{v}_2 = 0.0056, \quad \hat{\beta}_1 = 1.7381, \quad \hat{\beta}_2 = 0.0127.$$

We obtain similar results using Black vegas.

Fitted Normal and Black Implied Volatility Curves