ÉCOLE POLYTECHNIQUE
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Chapter 2: Estimating the Term Structure

2.2 Exact Methods

Interest Rate Models

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2.2 Exact Methods

- Bootstrapping is an example of an exact method
- Formalize the exact method
- Provide an alternative exact method based on pseudoinverse

General Formulation

Data: n market prices $p = (p_1, \dots, p_n)^\top$ at spot date t_0 with N cash flow dates $t_0 < t_1 < \cdots < t_N$ and corresponding $n \times N$-cash flow matrix $C = (c_{ij})$:

$$p_i = \sum_{j=1}^{N} P(t_0, t_j) c_{ij}.$$

Aim: find discount curve $d = (P(t_0, t_1), \dots, P(t_0, t_N))^\top$ exactly matching the market prices

$$C\, d = p.$$

Next step: bring data from bond and interest rate markets into the above format.

Bond Markets

Example: UK government bond (gilt) market on 4 September 1996

	Coupon (%)	Next coupon	Maturity date	Dirty price (p_i)
Bond 1	10	15/11/96	15/11/96	103.82
Bond 2	9.75	19/01/97	19/01/98	106.04
Bond 3	12.25	26/09/96	26/03/99	118.44
Bond 4	9	03/03/97	03/03/00	106.28
Bond 5	7	06/11/96	06/11/01	101.15
Bond 6	9.75	27/02/97	27/08/02	111.06
Bond 7	8.5	07/12/96	07/12/05	106.24
Bond 8	7.75	08/03/97	08/09/06	98.49
Bond 9	9	13/10/96	13/10/08	110.87

Cash Flow Dates

UK government bond market on 4/9/96 (spot date):

- Coupon bonds with semiannual coupons
- Day-count convention actual/365

Number of

- market prices $n = 9$
- cash flow dates $N = 1 + 3 + 6 + 7 + 11 + 12 + 19 + 20 + 25 = 104$

Cash flow dates $t_1 = 26/09/96$ (bond 3), $t_2 = 13/10/96$ (bond 9), $t_3 = 06/11/97$ (bond 5),...

Sparse Cash Flow Matrix

None of the bonds have cash flows at the same date. Cash flow matrix is sparse:

$$C = \begin{pmatrix} 0 & 0 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & 0 & \dots \\ 6.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.125 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 & 0 & \dots \\ 0 & 0 & 3.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4.25 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.875 & 0 & \dots \\ 0 & 4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

Money and FRA Markets: Formalization

LIBOR $L(t_0, T)$ with maturity T:

- Price $p = 1$
- Cash flow $c = 1 + \delta(t_0, T)L(t_0, T)$ at T

Forward rate $F(t_0, T_1, T_2)$ for $[T_1, T_2]$:

- Price $p = 0$
- Cash flows $c_1 = -1$ at T_1 and $c_2 = 1 + \delta(T_1, T_2)F(t_0, T_1, T_2)$ at T_2

Swap Markets: Formalization

Swap with swap rate K and dates $t_0 \leq T_0 < \cdots < T_n$ where $T_i - T_{i-1} \equiv \delta$.

Since

$$0 = -P(t_0, T_0) + \delta K \sum_{j=1}^{n-1} P(t_0, T_j) + (1 + \delta K) P(t_0, T_n),$$

we can choose

- if $T_0 = t_0$: $p = 1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$,
- if $T_0 > t_0$: $p = 0$, $c_0 = -1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$.

Formalization Summary

At spot date t_0

- LIBOR and swaps have notional price 1
- FRAs and forward starting swaps have notional price 0

Money and Swap Markets: Example

US market: LIBOR, futures, swaps

Spot date t_0 is 1 October 2012.

Day-count convention is actual/360.

Number of

- market prices $n = 3 + 5 + 9 = 17$
- cash flow dates $N = 3 + 6 + 30 = 39$

Cash flow dates $t_1 = 02/10/2012$, $t_2 = 05/11/2012$, $t_3 = 19/12/2012$ (first futures), ...

Source	Quote	Maturity
LIBOR	0.15	02/10/2012
	0.21	05/11/2012
	0.36	03/01/2013
Futures	99.68	20/03/2013
	99.67	19/06/2013
	99.65	18/09/2013
	99.64	18/12/2013
	99.62	19/03/2014
Swap	0.36	03/10/2014
	0.43	05/10/2015
	0.56	03/10/2016
	0.75	03/10/2017
	1.17	03/10/2019
	1.68	03/10/2022
	2.19	04/10/2027
	2.40	04/10/2032
	2.58	03/10/2042

Money and Swap Markets: Sparse Cash Flow Matrix

First 12 columns of the 17×39 cash flow matrix C:

$$C = \left(\begin{array}{ccccccccccccc}
c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & c_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & c_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & -1 & 0 & c_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -1 & c_{56} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -1 & c_{67} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & c_{79} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{8,10} & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{98} & 0 & 0 & c_{9,11} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{10,8} & 0 & 0 & c_{10,11} & c_{10,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11,8} & 0 & 0 & c_{11,11} & c_{11,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12,8} & 0 & 0 & c_{12,11} & c_{12,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{13,8} & 0 & 0 & c_{13,11} & c_{13,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14,8} & 0 & 0 & c_{14,11} & c_{14,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{15,8} & 0 & 0 & c_{15,11} & c_{15,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16,8} & 0 & 0 & c_{16,11} & c_{16,12} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{17,8} & 0 & 0 & c_{17,11} & c_{17,12} & \cdots
\end{array}\right)$$

Problem and Solutions

Typically $n \ll N$:

- The linear system $C\,d = p$ is under-determined.
- There exists many discount curve solutions d.
- Which one to choose?

Solution: Bootstrap (previous section)

- Synthetically create $N - n$ new market instruments, e.g. by linear interpolation of swap rates, such that $N \times N$ cash flow matrix C becomes invertible.
- Unique discount curve is given by $d = C^{-1}\,p$.

Alternative solution: Pseudoinverse of weighted increments (next)

Pseudoinverse: Weighted Increments

Idea: instead of estimating discount function $d = (P(t_0, t_1), \ldots, P(t_0, t_N))^\top$ estimate weighted increments vector

$$\Delta = \left(\frac{P(t_0,t_1)-1}{\sqrt{\delta(t_1,t_0)}}, \frac{P(t_0,t_2)-P(t_0,t_1)}{\sqrt{\delta(t_2,t_1)}}, \ldots, \frac{P(t_0,t_N)-P(t_0,t_{N-1})}{\sqrt{\delta(t_N,t_{N-1})}} \right)^\top = W \left(M\, d - (1, 0, \ldots, 0)^\top \right)$$

with $N \times N$ matrices $W = \operatorname{diag}\left(\frac{1}{\sqrt{\delta(t_1,t_0)}}, \ldots, \frac{1}{\sqrt{\delta(t_N,t_{N-1})}} \right)$ and $M = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & & \vdots \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$.

Pseudoinverse: Matching Equation

In consequence, Δ satisfies $d = M^{-1}\left(W^{-1}\Delta + (1,0,\dots,0)^\top\right)$ where

$$M^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 0 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Hence the discount curve matching problem $C\,d = p$ is equivalent to

$$CM^{-1}W^{-1}\Delta = p - CM^{-1}(1,0,\dots,0)^\top. \qquad (*)$$

Pseudoinverse Solution

Assume the $n \times N$-matrix $A = CM^{-1}W^{-1}$ has full rank then the pseudoinverse

$$\Delta^* = A^\top \left(AA^\top\right)^{-1} \left(p - CM^{-1}(1, 0, \dots, 0)^\top\right)$$

is the solution Δ of (*) with minimal Euclidian norm

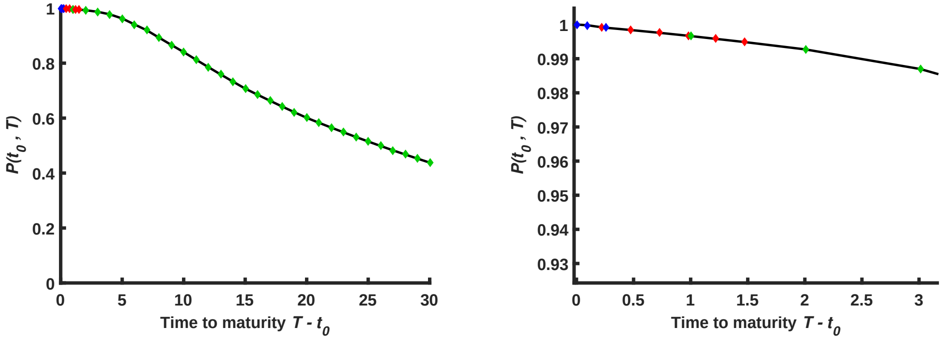
$$\|\Delta\|^2 = \sum_{j=1}^{N} \left| \frac{P(t_0, t_j) - P(t_0, t_{j-1})}{\sqrt{\delta(t_j, t_{j-1})}} \right|^2 \approx \int_{t_0}^{t_N} \left|\partial_T P(t_0, T)\right|^2 dT.$$

We thus obtain the smoothest possible matching discount curve.

Pseudoinverse Example: Discount Curve

US market on 1 October 2012: we find discount curve $P(t_0, t_i)$ for 40 points

$$t_i = t_0, \textcolor{blue}{S_1}, \textcolor{blue}{S_2}, \textcolor{red}{T_1}, \textcolor{blue}{S_3}, \textcolor{red}{T_2}, \textcolor{red}{T_3}, \textcolor{red}{T_4}, \textcolor{green}{U_1}, \textcolor{red}{T_5}, \textcolor{red}{T_6}, \textcolor{green}{U_2}, \ldots, \textcolor{green}{U_{30}}$$



Pseudoinverse Example: Yield and Forward Curves

Irregular implied forward curve but smoother than bootstrapping solution: