ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 1: Interest Rates and Related Contracts

1.2 Forward and Futures Rates

Interest Rate Models

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1.2 Forward and Futures Rates

- Forward rate agreements
- Forward rates
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- Futures rates

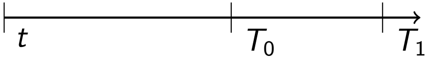
Forward Rate Agreements

A forward rate agreement (FRA) at time t is specified by

- a future period $[T_0, T_1]$ with length denoted by $\delta = T_1 - T_0$
- a fixed rate K
- a notional N

At T_1, the holder of the FRA contract

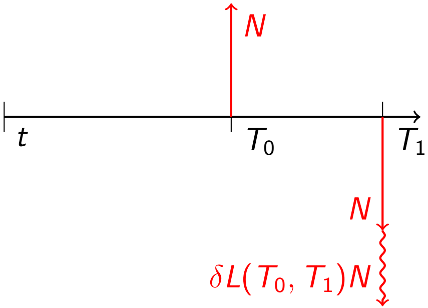
- pays fixed δKN
- receives floating $\delta L(T_0, T_1)N$



FRAs Lock In Rates

This FRA allows to lock in a fixed rate at time t over the future period $[T_0, T_1]$.

The FRA pays the holder the difference between the rate $L(T_0, T_1)$ prevailing at T_0 and the fixed rate K locked in at t on a loan of N over the future period $[T_0, T_1]$ (but it does not guarantee that the holder gets the loan at T_0).



Payoff of the FRA

The payoff of the FRA at T_1 is

$$N(\delta L(T_0, T_1) - \delta K) = N\left(\frac{1}{P(T_0, T_1)} - 1 - \delta K\right).$$

This can be rewritten as $N\left(I_1 - I_2\right)$ where

$$I_1 = \frac{1}{P(T_0, T_1)}, \quad I_2 = 1 + \delta K.$$

We value these components I_1, I_2 using discount bonds.

Value of the FRA

The time-T_0 value of $I_1 = \frac{1}{P(T_0,T_1)}$ at T_1 is

$$P(T_0, T_1)\frac{1}{P(T_0, T_1)} = 1$$

and hence its time-t value is $P(t, T_0)$.

The time-t value of $I_2 = 1 + \delta K$ at T_1 is

$$P(t, T_1) + P(t, T_1)\delta K.$$

The total time-t value of the FRA therefore is

$$V_{FRA}(t, T_0, T_1) = N\left(P(t, T_0) - P(t, T_1) - P(t, T_1)\delta K\right).$$

Forward Rate

The rate K which renders this value zero is the simple forward rate

$$F(t, T_0, T_1) = \frac{1}{\delta}\left(\frac{P(t, T_0)}{P(t, T_1)} - 1\right)$$

prevailing at time t.

For $t = T_0$ it equals the simple spot rate $F(T_0, T_0, T_1) = L(T_0, T_1)$.

The time-t value of the FRA can be expressed in terms of the forward rate as

$$V_{FRA}(t, T_0, T_1) = NP(t, T_1)\delta\left(F(t, T_0, T_1) - K\right).$$

Instantaneous Forward Rate

For infinitesimally small lending period, $T_1 \to T_0$, we obtain the instantaneous forward rate with maturity $T = T_0$,

$$f(t, T) = \lim_{T_1 \to T_0 = T} F(t, T_0, T_1) = -\frac{\partial \log P(t, T)}{\partial T}.$$

Because $P(T, T) = 1$, this is equivalent to

$$P(t, T) = \mathrm{e}^{-\int_t^T f(t,u)\, du}.$$

Forward Curve

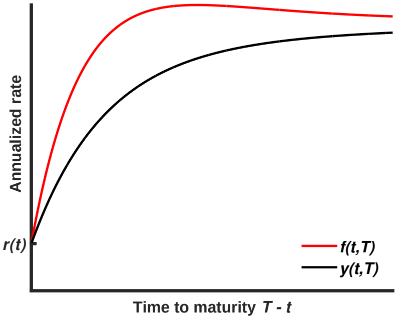
For varying maturities T we obtain the term structure of forward rates prevailing at t, also called the forward curve,

$$T \mapsto f(t, T).$$

Forward rates are related to yields by

$$y(t, T) = \frac{1}{T - t} \int_t^T f(t, u)\, du,$$

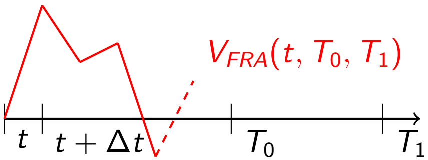
and to short rates by $f(t, t) = r(t)$.



Interest Rate Futures

Similar to a FRA, an interest rate futures contract allows to manage the exposure to the interest rate $L(T_0, T_1)$ prevailing over a future period $[T_0, T_1]$.

In contrast to FRAs, interest rate futures are daily marked to market (resettled).



Marking to Market

Marking to market works as follows:

- At $t \leq T_0$: the futures price is quoted as

$$P_{futures}(t, T_0, T_1) = 100 \times (1 - R_{futures}(t, T_0, T_1))$$

  where $R_{futures}(t, T_0, T_1)$ is the futures rate prevailing at t

- At $t + \Delta t$: cash flow to holder of futures contract

$$P_{futures}(t + \Delta t, T_0, T_1) - P_{futures}(t, T_0, T_1)$$

Futures Rates

The futures rate is chosen such that

- At $t = T_0$ (delivery): $R_{futures}(T_0, T_0, T_1) = L(T_0, T_1)$
- At $t < T_0$: the present value of cash flow

$$P_{futures}(t + \Delta t, T_0, T_1) - P_{futures}(t, T_0, T_1)$$

from holding the futures contract is zero.

Futures Rates vs. Forward Rates

Futures rates $R_{futures}(t, T_0, T_1)$ are different from forward rates $F(t, T_0, T_1)$.

The difference

$$R_{futures}(t, T_0, T_1) - F(t, T_0, T_1)$$

is called convexity adjustment. It is typically positive.

Market Example: Eurodollar Futures

The Eurodollar futures contract is tied to the LIBOR.

Introduced by the Chicago Mercantile Exchange (CME) in 1981.

Designed to protect owner from fluctuations in 3-month LIBOR $L(T_0, T_0 + 1/4)$.

Settlement (delivery) months are March, June, September and December.