Chapter 3: Stochastic Models

3.1 Stochastic Calculus

Interest Rate Models

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3.1 Stochastic Calculus

- Crash course in stochastic calculus
- Brownian motion
- Stochastic integral
- Itô formula
- Girsanov theorem
- Arbitrage pricing theorem

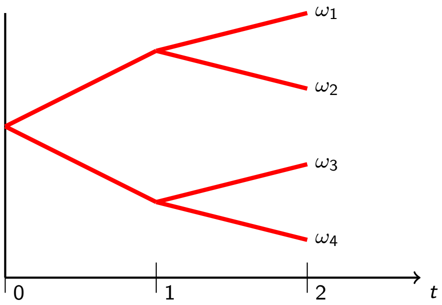
Stochastic Basis

Random variables are modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is the set of samples ω
- $\mathcal{F}$ is the σ-algebra of measurable (observable) events $A \subset \Omega$
- $\mathbb{P}$ is a probability measure assigning probabilities to events $A \mapsto \mathbb{P}[A]$

Filtration $\mathcal{F}_t$: information available by t

Example: $\Omega = \{\omega_1, \ldots, \omega_4\}$, $\mathbb{P}[\omega_i] = \frac{1}{4}$,
$\mathcal{F} = \mathcal{F}_2 = \sigma(\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\})$,
$\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\})$



Stochastic Process

A stochastic process $X(t) = X(\omega, t)$ is

- adapted if

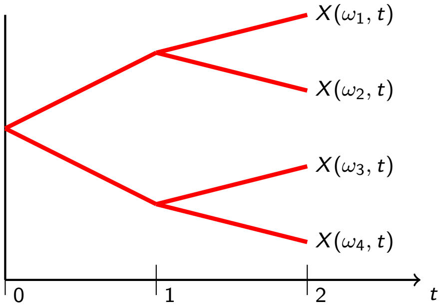
  $$\omega \mapsto X(\omega, t)$$

  is $\mathcal{F}_t$-measurable, for all t.

- a martingale (fair game) if adapted and $\mathcal{F}_t$-conditional expectation

  $$\mathbb{E}_t[X(T)] = X(t) \quad \text{for all } t \le T.$$

Example: $X(\omega, t)$ asset price at t, adapted: $X(\omega_1, 1) = X(\omega_2, 1)$, $X(\omega_3, 1) = X(\omega_4, 1)$



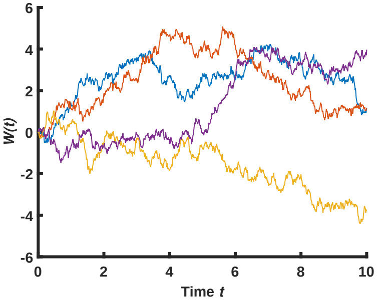
Brownian Motion

$W(t)$ is a standard Brownian motion if

- $W(t)$ is continuous and adapted,
- $W(0) = 0$,
- $W(t_2) - W(t_1)$ independent of $\mathcal{F}_{t_1}$,
- $W(t_2) - W(t_1)$ normal with mean 0 and variance $t_2 - t_1$, for all $t_1 < t_2$.

Fact: $W(t)$ is a martingale.

Four sample paths of the Brownian motion:



Gaussian Processes

A stochastic process $Z(t)$ is called Gaussian process if

$$(Z(t_1), Z(t_2), \ldots, Z(t_n))$$

is multivariate normal (Gaussian) distributed, for any $0 \leq t_1 < t_2 < \cdots < t_n$.

Fact: The distribution of a Gaussian process $Z(t)$ is determined by its

- mean function $m(t) = \mathbb{E}[Z(t)]$ and
- covariance function $c(t_1, t_2) = \mathrm{Cov}[Z(t_1), Z(t_2)]$.

Stochastic Integral

For adapted process $\sigma(t)$ with $\int_0^\infty \sigma(s)^2\,ds < \infty$ can define stochastic integral

$$\int_0^t \sigma(s)\,dW(s) = \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(t_{i-1})\,(W(t_i) - W(t_{i-1}))$$

for partitions $t_i = \frac{i}{n} t$.

Fact: $\int_0^t \sigma(s)\,dW(s)$ is a continuous martingale, if $\mathbb{E}\left[\int_0^\infty \sigma(s)^2\,ds\right] < \infty$.

Itô Processes

An Itô process $X(t)$ is adapted and of the form

$$X(t) = \underbrace{X(0)}_{\text{initial value}} + \underbrace{\int_0^t \mu(s)\,ds}_{\text{drift}} + \underbrace{\int_0^t \sigma(s)\,dW(s)}_{\text{martingale (noise)}}$$

In differential notation:

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)$$

Fact: if $X(0)$, $\mu(t)$, and $\sigma(t)$ are deterministic functions then $X(t)$ is a Gaussian process with mean and covariance functions given by

$$m(t) = X(0) + \int_0^t \mu(s)\,ds, \quad c(t_1, t_2) = \int_0^{t_1 \wedge t_2} \sigma(s)^2\,ds.$$

Itô Formula

For a C^2-function $f(x)$ and an Itô process

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t),$$

$f(X(t))$ is again an Itô process with decomposition

$$df(X(t)) = \frac{\partial f(X(t))}{\partial x}\,dX(t) + \frac{1}{2}\frac{\partial^2 f(X(t))}{\partial x^2}\sigma(t)^2\,dt.$$

Integration By Parts

For two Itô process

$$dX_i(t) = \mu_i(t)\,dt + \sigma_i(t)\,dW(t), \quad i = 1, 2,$$

their product $X_1(t)X_2(t)$ is again an Itô process with decomposition

$$d(X_1(t)X_2(t)) = X_1(t)\,dX_2(t) + X_2(t)\,dX_1(t) + \sigma_1(t)\sigma_2(t)\,dt.$$

Stochastic Exponential

For an Itô process

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)$$

the exponential $Y(t) = \mathrm{e}^{X(t)}$ satisfies, using Itô formula,

$$dY(t) = Y(t)\,dX(t) + \tfrac{1}{2}Y(t)\sigma(t)^2\,dt.$$

Define the stochastic exponential

$$\mathcal{E}(X(t)) = \exp\left(X(t) - \tfrac{1}{2}\int_0^t \sigma(s)^2\,ds\right).$$

Fact: $\mathcal{E}(X(t))$ satisfies the stochastic differential equation

$$d\mathcal{E}(X(t)) = \mathcal{E}(X(t))\,dX(t).$$

Bayes' Rule

Let $\mathbb{Q} \sim \mathbb{P}$ be equivalent probability measure with Radon–Nikodym density $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

Bayes' rule relates conditional expectations under $\mathbb{P}$ and $\mathbb{Q}$.

For any bounded random variable X we have:

$$\mathbb{E}_t^{\mathbb{Q}}[X] = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X\right]}{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]}.$$

Girsanov Theorem

For an adapted process $\lambda(t)$, satisfying technical conditions, the Itô process

$$dW^*(t) = dW(t) + \lambda(t)\, dt$$

is a Brownian motion under the equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with Radon–Nikodym density process

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathcal{E}\left(-\int_0^t \lambda(s)\, dW(s)\right).$$

Application: Arbitrage Pricing Theorem

Any traded asset with positive price process $S(t)$ must have return of the form

$$\frac{dS(t)}{S(t)} = (r(t) + \sigma(t)\lambda(t))\, dt + \sigma(t)\, dW(t)$$

where $r(t)$ is risk-free short rate, $\sigma(t)$ is volatility, $\lambda(t)$ is market price of risk, such that

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma(t)\, dW^*(t)$$

and the discounted price process

$$\mathrm{e}^{-\int_0^t r(s)ds} S(t) = \mathcal{E}\left(\int_0^t \sigma(s)\, dW^*(s)\right)$$

is a martingale under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$.