ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 4: Interest Rate Derivatives

4.2 Caps and Floors

Interest Rate Models

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4.2 Caps and Floors

- Caps protect against high rates
- Floors protect against low rates
- Cap-Floor parity
- Pricing under forward measures
- Black's price formula
- Bachelier's price formula
- Cap/Floor quotes in terms of implied volatilities

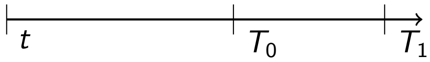
Caplets

A caplet with reset date T_0 and settlement date $T_1 = T_0 + \delta$ pays the holder the difference between simple spot rate $L(T_0, T_1)$ and strike rate κ.

The cash flow at T_1 is

$$\delta(L(T_0, T_1) - \kappa)^+.$$

Protects against rising interest rates.

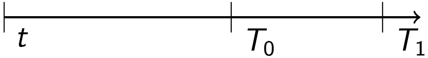


Price of Caplet

The time-T_0 value of the caplet is

$$
\begin{aligned}
&P(T_0, T_1)\delta(L(T_0, T_1) - \kappa)^+ \\
&\qquad = (1 + \delta\kappa)\left(\tfrac{1}{1+\delta\kappa} - P(T_0, T_1)\right)^+
\end{aligned}
$$

$(1 + \delta\kappa) \times$ put option on T_1-bond with expiry date T_0 and strike price $\frac{1}{1+\delta\kappa}$.



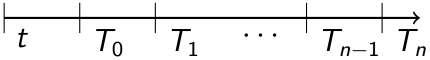
Time-t price of the caplet therefore is

$$
Cpl(t, T_0, T_1) = (1 + \delta\kappa) \times p_{\text{put}}.
$$

Caps

A cap is a strip of caplets, specified by

- reset/settlement dates $T_0 < \cdots < T_n$ (T_0=first reset date, T_n=maturity)
- a cap rate κ
- for simplicity assume: $T_i - T_{i-1} \equiv \delta$



The cap price at $t \le T_0$ is

$$Cp(t) = \sum_{i=1}^{n} Cpl(t, T_{i-1}, T_i)$$

with price of ith caplet $Cpl(t, T_{i-1}, T_i)$.

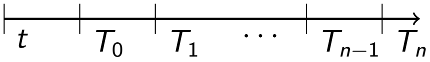
Floors

A floor

- is the converse to a cap,
- protects against low interest rates,

is a strip of floorlets with T_i-cash flows

$$\delta(\kappa - L(T_{i-1}, T_i))^+.$$



The floor price at $t \leq T_0$ is

$$Fl(t) = \sum_{i=1}^{n} Fll(t, T_{i-1}, T_i)$$

with price of ith floorlet $Fll(t, T_{i-1}, T_i)$.

Caps, Floors and Swaps

The following parity relation holds:

$$Cp(t) - Fl(t) = V_p(t)$$

where $V_p(t)$ is the time-t value of a payer swap with fixed rate κ, notional one, and the same tenor structure $T_0 < \cdots < T_n$ as the cap and floor.

The cap (floor) is said to be

- at-the-money (ATM) if $\kappa = R_{swap}(t) = \frac{P(t,T_0)-P(t,T_n)}{\delta \sum_{i=1}^n P(t,T_i)}$
- in-the-money (ITM) if $\kappa < R_{swap}(t)$ (floor: $\kappa > R_{swap}(t)$)
- out-of-the-money (OTM) if $\kappa > R_{swap}(t)$ (floor: $\kappa < R_{swap}(t)$)

Black's Formula: Underlying Assumptions

Black's formula assumes that $L(T_{i-1}, T_i) = F(T_{i-1}, T_{i-1}, T_i)$ is log-normal with

$$dF(t, T_{i-1}, T_i) = F(t, T_{i-1}, T_i)\sigma\, dW^{T_i}(t)$$

with constant $\sigma > 0$ and Brownian motion $W^{T_i}(t)$ under the T_i-forward measure.

Time-t prices of ith caplet and floorlet are

$$Cpl(t, T_{i-1}, T_i) = P(t, T_i)\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\delta(L(T_{i-1}, T_i) - \kappa)^+\right]$$

$$Fll(t, T_{i-1}, T_i) = P(t, T_i)\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\delta(\kappa - L(T_{i-1}, T_i))^+\right]$$

Black's Formula for Caplets and Floorlets

Black's formula for the ith caplet and floorlet price is

$$Cpl(t, T_{i-1}, T_i) = \delta P(t, T_i)\left(F(t, T_{i-1}, T_i)\Phi(d_1) - \kappa\Phi(d_2)\right)$$

$$Fll(t, T_{i-1}, T_i) = \delta P(t, T_i)\left(\kappa\Phi(-d_2) - F(t, T_{i-1}, T_i)\Phi(-d_1)\right)$$

where Φ is the standard normal cumulative distribution function and

$$d_{1,2} = \frac{\log\left[\frac{F(t;T_{i-1},T_i)}{\kappa}\right] \pm \frac{1}{2}\sigma^2(T_{i-1} - t)}{\sigma\sqrt{T_{i-1} - t}}.$$

σ: Black (or relative) volatility (same for all caplets/floorlets of a cap/floor).

Bachelier's Formula: Underlying Assumptions

Bachelier's formula assumes that $L(T_{i-1}, T_i) = F(T_{i-1}, T_{i-1}, T_i)$ is normal with

$$dF(t, T_{i-1}, T_i) = \sigma \, dW^{T_i}(t)$$

with constant $\sigma > 0$ and Brownian motion $W^{T_i}(t)$ under the T_i-forward measure.

Time-t prices of ith caplet and floorlet are

$$Cpl(t, T_{i-1}, T_i) = P(t, T_i)\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\delta(L(T_{i-1}, T_i) - \kappa)^+\right]$$

$$Fll(t, T_{i-1}, T_i) = P(t, T_i)\mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\delta(\kappa - L(T_{i-1}, T_i))^+\right]$$

Bachelier's Formula for Caplets and Floorlets

Bachelier's formula for the ith caplet and floorlet price is

$$Cpl(t, T_{i-1}, T_i) = \delta P(t, T_i)\sigma\sqrt{T_{i-1} - t}\,(D\Phi(D) + \phi(D))$$

$$Fll(t, T_{i-1}, T_i) = \delta P(t, T_i)\sigma\sqrt{T_{i-1} - t}\,(-D\Phi(-D) + \phi(-D))$$

where Φ is the standard normal cumulative distribution function, $\phi = \Phi'$, and

$$D = \frac{F(t, T_{i-1}, T_i) - \kappa}{\sigma\sqrt{T_{i-1} - t}}.$$

σ: normal (basis point, absolute) volatility (same for all caplets/floorlets of a cap/floor).

Cap and Floor Quotes

Cap/floor prices are quoted in terms of their Black or normal implied volatilities.

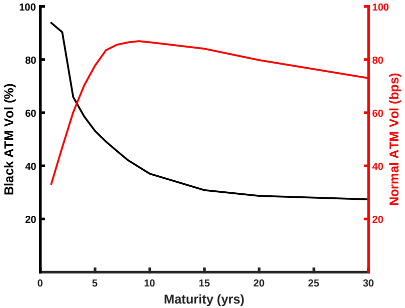
Typically: $t = 0$, $T_0 = \delta = T_i - T_{i-1}$ with

- $\delta =$ three months (US market)
- $\delta =$ half a year (euro market)

Example of Cap Quotes

EUR ATM Cap Quotes, 30 August 2013:

Maturity (yrs)	Cap ATM Price (%)	Black ATM Vol (%)	Normal ATM Vol (bps)
1	0.08%	93.81%	33.23
2	0.35%	90.31%	46.94
3	0.84%	65.94%	60.15
4	1.61%	58.65%	70.25
5	2.54%	53.14%	77.76
6	3.60%	49.12%	83.54
7	4.60%	45.59%	85.60
8	5.62%	42.17%	86.46
9	6.65%	39.59%	86.94
10	7.67%	37.04%	86.52
15	12.38%	30.86%	84.08
20	16.15%	28.70%	79.81
30	22.35%	27.39%	73.01



It is a challenge for any interest rate model to match the given volatility curve.