ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 1: Interest Rates and Related Contracts

1.4 Duration and Convexity

Interest Rate Models

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1.4 Duration and Convexity

- Duration
- Convexity
- Bond portfolio risk management

Sensitivity of Bond Prices

We consider a fixed coupon bond with

- cash flow dates $0 < T_1 < \cdots < T_n$
- cash flows c_i at T_i (for simplicity c_n contains the principal).

Its price at $t = 0$ is

$$p = \sum_{i=1}^{n} \mathrm{e}^{-y_i T_i} c_i$$

where we write $y_i = y(0, T_i)$.

How does the price change under parallel shift of the yield curve, $y_i \mapsto y_i + s$?

Duration

The duration of the bond is defined as relative first-order sensitivity with respect to a parallel shift of the yield curve:

$$D = -\frac{1}{p}\frac{d}{ds}\left(\sum_{i=1}^{n} c_i \mathrm{e}^{-(y_i+s)T_i}\right)\Big|_{s=0} = \frac{\sum_{i=1}^{n} T_i c_i \mathrm{e}^{-y_i T_i}}{p}.$$

Duration is a weighted average of the coupon dates $T_1, \ldots, T_n$.

Obtain first-order approximation for relative bond price change $\frac{\Delta p}{p}$ with respect to a parallel shift Δy of the yield curve:

$$\frac{\Delta p}{p} \approx -D\Delta y$$

Duration: Example 1

Assume a flat yield curve $y(0, T) \equiv 3\%$.

Price of a $10Y$ 6% bond with annual coupon and principal 100 is

$$p_{10} = \sum_{i=1}^{10} 6 \times \mathrm{e}^{-0.03 \times i} + 100 \times \mathrm{e}^{-0.03 \times 10} = 125.14$$

Its duration is

$$D_{10} = \frac{\sum_{i=1}^{10} i \times 6 \times \mathrm{e}^{-0.03 \times i} + 10 \times 100 \times \mathrm{e}^{-0.03 \times 10}}{p} = 8.06$$

Duration: Example 2

Assume a flat yield curve $y(0, T) \equiv 3\%$.

Price of a $5Y$ 3% bond with annual coupon and principal 100 is

$$p_5 = \sum_{i=1}^{5} 3 \times \mathrm{e}^{-0.03 \times i} + 100 \times \mathrm{e}^{-0.03 \times 5} = 99.79$$

Its duration is

$$D_5 = \frac{\sum_{i=1}^{5} i \times 3 \times \mathrm{e}^{-0.03 \times i} + 5 \times 100 \times \mathrm{e}^{-0.03 \times 5}}{p_5} = 4.72$$

Duration Hedging

Aim: immunize the value of a bond portfolio with respect to small parallel shifts of the yield curve!

- $\Pi(s)$ value a portfolio to be hedged as function of yield shift s
- $H(s)$ value of the hedging instrument as function of s

Find q such that

$$\frac{d}{ds}\left(\Pi(s) + q\,H(s)\right)\big|_{s=0} = 0.$$

The solution is given by

$$q = \frac{-D_\Pi \times \Pi(0)}{D_H \times H(0)}$$

where $D_\Pi =$ duration of Π, $D_H =$ duration of H.

Duration Hedging: Examples 1 and 2 revisited

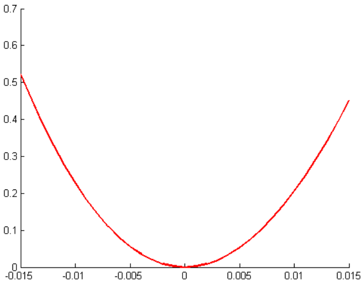
Immunize a long position in one $10Y$ 6% bond by holding q $5Y$ 3% bonds.

The solution is given by

$$q = \frac{-D_{10} \times p_{10}}{D_5 \times p_5} = -2.14.$$

This means that one should short 2.14 units of the $5Y$ bond in order to hedge one long unit of the $10Y$ bond.

Hedging Performance



Value change $(\Pi(s) - 2.14\, H(s)) - (\Pi(0) - 2.14\, H(0))$ of immunized portfolio as function of yield shift s.

Convexity

The convexity of the bond is defined as relative second-order sensitivity with respect to a parallel shift of the yield curve:

$$C = \frac{1}{p}\frac{d^2}{ds^2}\left(\sum_{i=1}^{n} c_i \mathrm{e}^{-(y_i+s)T_i}\right)\Big|_{s=0} = \frac{1}{p}\sum_{i=1}^{n} c_i \mathrm{e}^{-y_i T_i}(T_i)^2.$$

Obtain second-order approximation for relative bond price change $\frac{\Delta p}{p}$ with respect to a parallel shift Δy of the yield curve:

$$\frac{\Delta p}{p} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

Convexity Hedging

Aim: immunize the value of a bond portfolio with respect to small and not so small parallel shifts of the yield curve!

- $\Pi(s)$ value a portfolio to be hedged as function of yield shift s
- $H_1(s)$ value of 1st hedging instrument as function of s
- $H_2(s)$ value of 2nd hedging instrument as function of s

Find q_1 and q_2 such that

$$\frac{d}{ds}\left(\Pi(s) + q_1\, H_1(s) + q_2\, H_2(s)\right)|_{s=0} = 0$$
$$\frac{d^2}{ds^2}\left(\Pi(s) + q_1\, H_1(s) + q_2\, H_2(s)\right)|_{s=0} = 0$$

Convexity Hedging: Solution

The above system is equivalent to

$$\begin{aligned} -q_1\, D_{H_1}\, H_1(0) - q_2\, D_{H_2}\, H_2(0) &= D_\Pi\, \Pi(0) \\ q_1\, C_{H_1}\, H_1(0) + q_2\, C_{H_2}\, H_2(0) &= -C_\Pi\, \Pi(0) \end{aligned}$$

The solution is

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -D_{H_1}\, H_1(0) & -D_{H_2}\, H_2(0) \\ C_{H_1}\, H_1(0) & C_{H_2}\, H_2(0) \end{pmatrix}^{-1} \begin{pmatrix} D_\Pi\, \Pi(0) \\ -C_\Pi\, \Pi(0) \end{pmatrix}$$

where $D_\Pi(C_\Pi) =$ duration(convexity) of Π, $D_{H_i}(C_{H_i}) =$ duration(convexity) of H_i.

Convexity Hedging: Example

Portfolio: price $\Pi = 32,863.5$
Duration 6.76
Convexity 85.329

1st hedging instrument: price $H_1 = 97.962$
Duration: 8.813
Convexity: 99.081

2nd hedging instrument: price $H_2 = 108.039$
Duration: 2.704
Convexity: 10.168

Convexity Hedging: Example Solution

The hedging portfolio is given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -8.813 \times 97.962 & -2.704 \times 108.039 \\ 99.081 \times 97.962 & 10.168 \times 108.039 \end{pmatrix}^{-1} \begin{pmatrix} 6.76 \times 32,863.5 \\ -85.329 \times 32,863.5 \end{pmatrix}$$
$$= \begin{pmatrix} -304.8 \\ 140.3 \end{pmatrix}$$

In order to hedge Π against parallel shifts of the yield curve you should buy 140.3 units of H_2 and sell 304.8 units of H_1 short.