ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Chapter 4: Interest Rate Derivatives

4.1 Interest Rate Futures and Convexity Adjustment

Interest Rate Models

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4.1 Interest Rate Futures and Convexity Adjustment

- Recap interest rate futures
- Derive futures rate formula
- Calculate convexity adjustment for Gaussian HJM models
- Example: Vasiček model

Recap: Interest Rate Futures

Similar to a FRA, an interest rate futures contract allows to manage the exposure to the simple spot rate $L(T_0, T_1)$ prevailing over a future period $[T_0, T_1]$ with length $\delta = T_1 - T_0$.

In contrast to FRAs, interest rate futures are daily marked to market (resettled).

Recap: Marking to Market

Marking to market works as follows:

- At $t \leq T_0$: the futures price is quoted as

$$P_{futures}(t, T_0, T_1) = 100 \times (1 - R_{futures}(t, T_0, T_1))$$

  where $R_{futures}(t, T_0, T_1)$ is the futures rate prevailing at t

- At $t + \Delta t$: cash flow to holder of futures contract

$$\Delta P_{futures}(t + \Delta t) := P_{futures}(t + \Delta t, T_0, T_1) - P_{futures}(t, T_0, T_1)$$

Futures Rates

The futures rate is chosen such that

- At $t = T_0$ (delivery): $R_{futures}(T_0, T_0, T_1) = L(T_0, T_1)$
- At $t < T_0$: the present value of cash flow $\Delta P_{futures}(t + \Delta t)$ from holding the futures contract is zero, for small Δt:

$$0 = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^{t+\Delta t} r(s) ds} \Delta P_{futures}(t + \Delta t) \right] \approx \mathrm{e}^{-r(t)\Delta t} \, \mathbb{E}_t^{\mathbb{Q}} \left[\Delta P_{futures}(t + \Delta t) \right]$$

Consequence: futures price process $P_{futures}(t, T_0, T_1)$ is a $\mathbb{Q}$-martingale.

Futures Rates Formula

Consequence: futures rate process $R_{futures}(t, T_0, T_1)$ is a $\mathbb{Q}$-martingale,

$$R_{futures}(t, T_0, T_1) = \mathbb{E}_t^{\mathbb{Q}}\left[L(T_0, T_1)\right].$$

Recap: forward rate process $F(t, T_0, T_1)$ is a $\mathbb{Q}^{T_1}$-martingale,

$$F(t, T_0, T_1) = \mathbb{E}_t^{\mathbb{Q}^{T_1}}\left[L(T_0, T_1)\right].$$

The difference

$$\gamma(t, T_0, T_1) = R_{futures}(t, T_0, T_1) - F(t, T_0, T_1)$$

is called convexity adjustment. It is a model dependent value.

Convexity Adjustment in Gaussian HJM Models

Consider a Gaussian HJM model with deterministic forward rate volatility $\sigma(t, T)$.

Denote T-bond volatility $v(t, T) = -\int_t^T \sigma(t, u)du$.

Some stochastic calculus shows

$$\frac{P(t,T_0)}{P(t,T_1)} = \underbrace{\frac{P(0,T_0)}{P(0,T_1)}\mathcal{E}\left(\int_0^t (v(s, T_0) - v(s, T_1))dW^*(s)\right)}_{\mathbb{Q}\text{-martingale}} \underbrace{\mathrm{e}^{\int_0^t v(s,T_1)(v(s,T_1)-v(s,T_0))^\top ds}}_{\text{deterministic}}.$$

Hence

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{P(T_0,T_1)}\right] = \frac{P(t,T_0)}{P(t,T_1)}\mathrm{e}^{\int_t^{T_0} v(s,T_1)(v(s,T_1)-v(s,T_0))^\top ds}.$$

Convexity Adjustment in Gaussian HJM Models

The convexity adjustment is

$$\gamma(t, T_0, T_1) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{\delta}\left(\frac{1}{P(T_0,T_1)} - 1\right)\right] - \frac{1}{\delta}\left(\frac{P(t,T_0)}{P(t,T_1)} - 1\right).$$

Hence

$$\gamma(t, T_0, T_1) = \frac{1}{\delta}\frac{P(t,T_0)}{P(t,T_1)}\left(\mathrm{e}^{\int_t^{T_0}\left(\int_s^{T_1}\sigma(s,u)du\right)\left(\int_{T_0}^{T_1}\sigma(s,v)^\top dv\right)ds} - 1\right).$$

The convexity adjustment $\gamma(t, T_0, T_1) \geq 0$ if $\sigma(s, u)\sigma(s, v)^\top \geq 0$ for all s, u, v.

Convexity Adjustment in the Vasiček Model

Vasiček short rate model is Gaussian HJM model with $\sigma(t,T) = \mathrm{e}^{-\kappa(T-t)}\sigma$.

Parameters: $\kappa = 0.86$, $\theta = 0.08$, $\sigma = 0.01$, $r(0) = 0.06$, $T_0 - t = \frac{1}{2}$ and $\delta = \frac{1}{4}$: