Chapter 3: Stochastic Models

3.4 Forward Measures

Interest Rate Models
Damir Filipović

3.4 Forward Measures

- Bond price processes discounted by the savings account are martingales under the risk-neutral measure
- Bond price processes discounted by T-bond are martingales under the T-forward measure
- This “change of numeraire” technique proves most useful for option pricing
- Derive closed form bond option price formula for Gaussian HJM models

Setup

A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$

- Short rate process $r(t)$
- T-bonds with prices $P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^T r(s)\,ds} \right]$

such that T-bond price process discounted by the savings account

$$\mathrm{e}^{-\int_0^t r(s)\,ds} P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_0^T r(s)\,ds} \right], \quad t \leq T$$

is a $\mathbb{Q}$-martingale.

Forward Measure

Define T-forward measure $\mathbb{Q}^T \sim \mathbb{Q}$ with Radon-Nikodym density process

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right] = \mathrm{e}^{-\int_0^t r(s)ds}\frac{P(t,T)}{P(0,T)} \qquad \left(= \mathrm{e}^{-\int_0^T r(s)ds}\frac{1}{P(0,T)} \text{ for } t = T.\right)$$

Application in derivatives pricing: Consider a T-claim X. Its time-t price is, using Bayes' rule,

$$p(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds} X\right] = P(t,T)\frac{\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(s)ds}\frac{1}{P(0,T)}X\right]}{\mathrm{e}^{-\int_0^t r(s)ds}\frac{P(t,T)}{P(0,T)}} = P(t,T)\mathbb{E}_t^{\mathbb{Q}^T}[X].$$

Fundamental Property

Fact: for any T_0, T_1, the T_0-bond price process discounted by the T_1-bond

$$\frac{P(t,T_0)}{P(t,T_1)}, \quad t \leq T_0 \wedge T_1,$$

is a $\mathbb{Q}^{T_1}$-martingale. Indeed, Bayes' rule gives, for any $t < T \leq T_0 \wedge T_1$,

$$\mathbb{E}_t^{\mathbb{Q}^{T_1}}\left[\frac{P(T,T_0)}{P(T,T_1)}\right] = \frac{1}{P(t,T_1)}\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds} P(T, T_1)\frac{P(T,T_0)}{P(T,T_1)}\right] = \frac{P(t,T_0)}{P(t,T_1)}.$$

Consequence: the simple forward rate process $F(t, T_0, T_1)$ is a $\mathbb{Q}^{T_1}$-martingale

$$F(t, T_0, T_1) = \frac{1}{T_1 - T_0}\left(\frac{P(t,T_0)}{P(t,T_1)} - 1\right) = \mathbb{E}_t^{\mathbb{Q}^{T_1}}\left[L(T_0, T_1)\right].$$

Forward Measure in the HJM Framework

Rewrite T-forward rate dynamics

$$df(t,T) = \sigma(t,T)\underbrace{\left(dW^*(t) - v(t,T)^\top dt\right)}_{=dW^T(t)}$$

with bond return volatility $v(t,T) = -\int_t^T \sigma(t,u)du$ such that

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right] = \mathrm{e}^{-\int_0^t r(s)ds}\frac{P(t,T)}{P(0,T)} = \mathcal{E}\left(\int_0^t v(s,T)\,dW^*(s)\right).$$

Girsanov theorem implies that $W^T(t)$ is a Brownian motion under $\mathbb{Q}^T$ and the T-forward rate process $f(t,T)$ is a $\mathbb{Q}^T$-martingale, $f(t,T) = \mathbb{E}_t^{\mathbb{Q}^T}[r(T)]$.

Bond Option Pricing

European call option on T_1-bond with expiry date $T_0 < T_1$ and strike price K.

Price at $t = 0$ is

$$p_{\text{call}} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds}\,(P(T_0,T_1)-K)^+\right].$$

Decompose $(P(T_0,T_1)-K)^+ = P(T_0,T_1)1_{\{P(T_0,T_1)>K\}} - K1_{\{P(T_0,T_1)>K\}}$:

$$p_{\text{call}} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds} P(T_0,T_1)1_{\{P(T_0,T_1)>K\}}\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)ds} K1_{\{P(T_0,T_1)>K\}}\right]$$

$$= P(0,T_1)\mathbb{Q}^{T_1}\left[P(T_0,T_1)>K\right] - KP(0,T_0)\mathbb{Q}^{T_0}\left[P(T_0,T_1)>K\right].$$

Bond Option Pricing in HJM Framework

Observe that

$$\mathbb{Q}^{T_1}\left[P(T_0, T_1) > K\right] = \mathbb{Q}^{T_1}\left[\frac{P(T_0,T_0)}{P(T_0,T_1)} < \frac{1}{K}\right]$$

$$\mathbb{Q}^{T_0}\left[P(T_0, T_1) > K\right] = \mathbb{Q}^{T_0}\left[\frac{P(T_0,T_1)}{P(T_0,T_0)} > K\right]$$

and the **fact** (interchangeably for $T_0 \leftrightarrow T_1$):

$$\frac{P(t, T_0)}{P(t, T_1)} = \frac{P(0, T_0)}{P(0, T_1)}\,\mathcal{E}\left(\int_0^t (v(s, T_0) - v(s, T_1))\,dW^{T_1}(s)\right)$$

Bond Option Pricing in Gaussian HJM Models

Assume deterministic volatility $\sigma(t, T)$ then (interchangeably for $T_0 \leftrightarrow T_1$)

$$\log \frac{P(t,T_0)}{P(t,T_1)} = \log \frac{P(0,T_0)}{P(0,T_1)} + \int_0^t \|v(s, T_0) - v(s, T_1)\|^2 \, ds$$
$$+ \int_0^t (v(s, T_0) - v(s, T_1)) \, dW^{T_1}(s)$$

is normal under $\mathbb{Q}^{T_1}$ with mean

$$\log \frac{P(0,T_0)}{P(0,T_1)} + \int_0^t \|v(s, T_0) - v(s, T_1)\|^2 \, ds$$

and variance

$$\int_0^t \|v(s, T_0) - v(s, T_1)\|^2 \, ds.$$

Bond Option Pricing Formula for Gaussian HJM Models

In a Gaussian HJM model (deterministic $\sigma(t, T)$) the time-0 price of a European call option on a T_1-bond with expiry date T_0 and strike price K is

$$p_{\text{call}} = P(0, T_1)\Phi[d_1] - KP(0, T_0)\Phi[d_2]$$

where Φ is the standard normal cumulative distribution function and

$$d_{1,2} = \frac{\log\left[\frac{P(0,T_1)}{KP(0,T_0)}\right] \pm \frac{1}{2}\int_0^{T_0} \|v(s, T_0) - v(s, T_1)\|^2\, ds}{\sqrt{\int_0^{T_0} \|v(s, T_0) - v(s, T_1)\|^2\, ds}}.$$

Put-Call Parity

The put-call parity derives from the elementary identity

$$(P(T_0, T_1) - K)^+ - (K - P(T_0, T_1))^+ = P(T_0, T_1) - K$$

such that

$$p_{\text{call}} - p_{\text{put}} = P(0, T_1) - KP(0, T_0).$$

Consequence: In the above Gaussian HJM model the time-0 price of a European put option on a T_1-bond with expiry date T_0 and strike price K is

$$p_{\text{put}} = KP(0, T_0)\Phi[-d_2] - P(0, T_1)\Phi[-d_1].$$

Example: Vasiček Short Rate Model

Vasiček short rate model is Gaussian HJM model with $\sigma(t,T) = \mathrm{e}^{-\kappa(T-t)}\sigma$.

Hence $v(t,T_0) - v(t,T_1) = \frac{\sigma}{\kappa}\left(\mathrm{e}^{-\kappa T_0} - \mathrm{e}^{-\kappa T_1}\right)\mathrm{e}^{\kappa t}$ such that

$$\int_0^{T_0} \|v(s,T_0) - v(s,T_1)\|^2\, ds = \frac{\sigma^2}{\kappa^2}\left(\mathrm{e}^{-\kappa T_0} - \mathrm{e}^{-\kappa T_1}\right)^2 \frac{\mathrm{e}^{2\kappa T_0} - 1}{2\kappa}$$

and $P(0,T) = \mathrm{e}^{-A(T)-B(T)r(0)}$ yield closed form bond option price formulas.

Example: Vasiček Short Rate Model

Parameters: $\kappa = 0.86$, $\theta = 0.08$,
$\sigma = 0.01$, $r(0) = 0.06$.

Put options on T_i-bond with

- expiry date T_{i-1}
- ATM strike $K_i = P(0, T_i)/P(0, T_{i-1})$

for $T_i = (i + 1)/4$, $i = 0, \ldots, 39$.