Exercise (5 minutes)

Scenario:

- You have been asked to predict current house prices in the Pittsburgh area based on a labeled dataset with features like number of bedrooms, location, square footage, etc.

Instructions

- Get into groups of 3-5
- Brainstorm approaches you could use to solve the problem given the different sizes of training data.

N	**Method**
1 Million	
100,000	
10,000	Random Forest
1,000	
100	
50	
10	
0?	

Reasoning About Data: Applying Bayesian Data Analysis

Factory Day

23 February 2024

CPT Bobby Nelson

Why Bayesian?

- Intuitive
- Flexible
- Expressive

When Bayesian?

<u>Machine Learning</u>

- Big data
- Big model
- Few assumptions

Data does the work

<u>Bayesian Analysis</u>

- Small(er) data
- Strongly structured model
- Strong priors

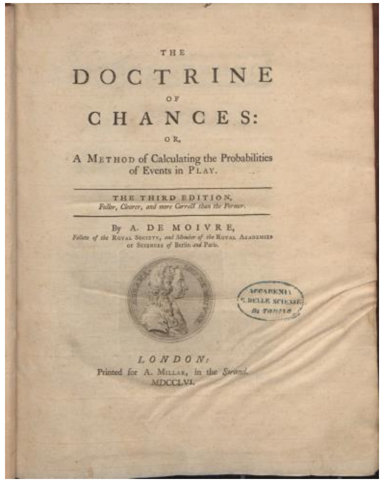
Model does the work

Agenda

- **Introduction**
  - Brief History
  - Bayes Theorem
- **Simple Problems**
- **Bean Bag Toss Experiment**
  - Binomial Process
  - Grid Approximation and MCMC
- **Predicting Ice Thickness**
  - Linear Mixed Models
  - Hierarchical Models

A Brief History of Bayesian Statistics

Abraham de Moivre

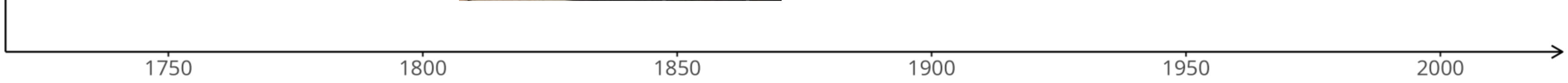


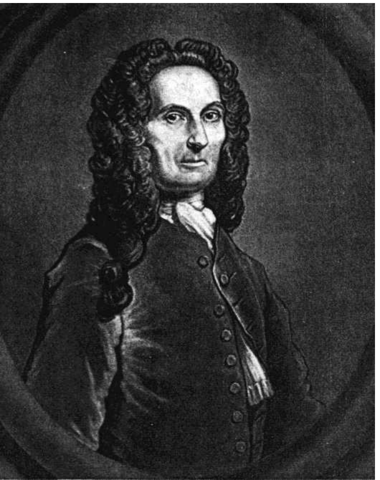
THE
DOCTRINE
OF
CHANCES:
OR,
A METHOD of Calculating the Probabilities of Events in PLAY.

THE THIRD EDITION,
Fuller, Clearer, and more Correct than the Former.

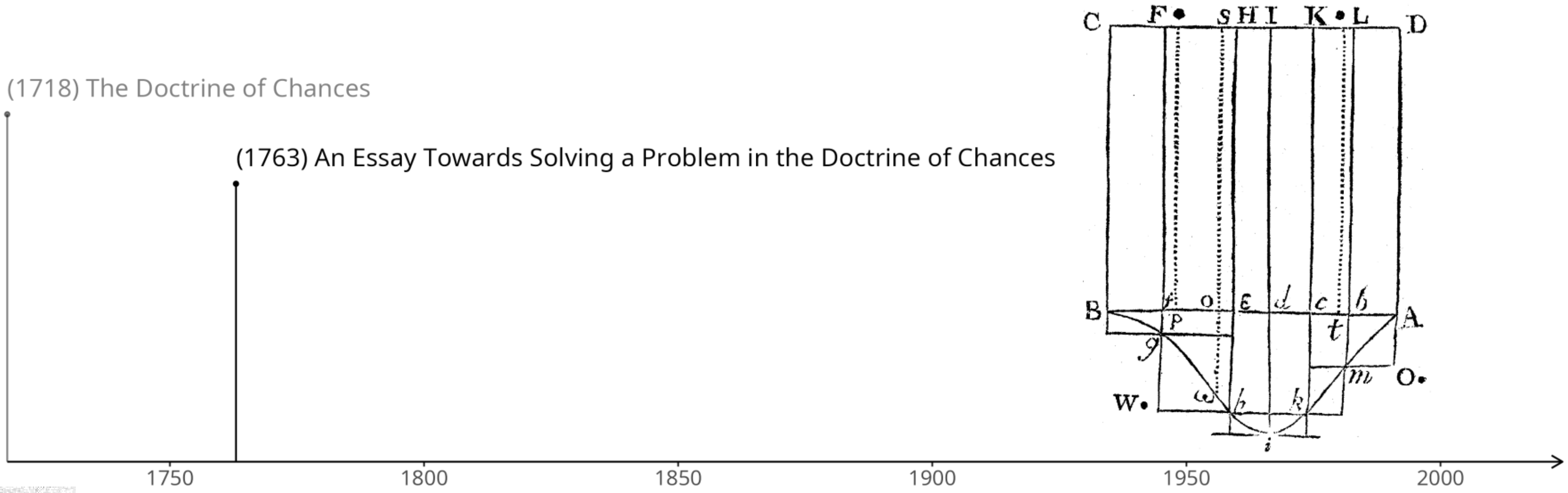
By A. DE MOIVRE,
Fellow of the ROYAL SOCIETY, *and Member of the* ROYAL ACADEMIES OF SCIENCES *of Berlin and Paris.*

LONDON:
Printed for A. MILLAR, in the *Strand.*
MDCCLVI.





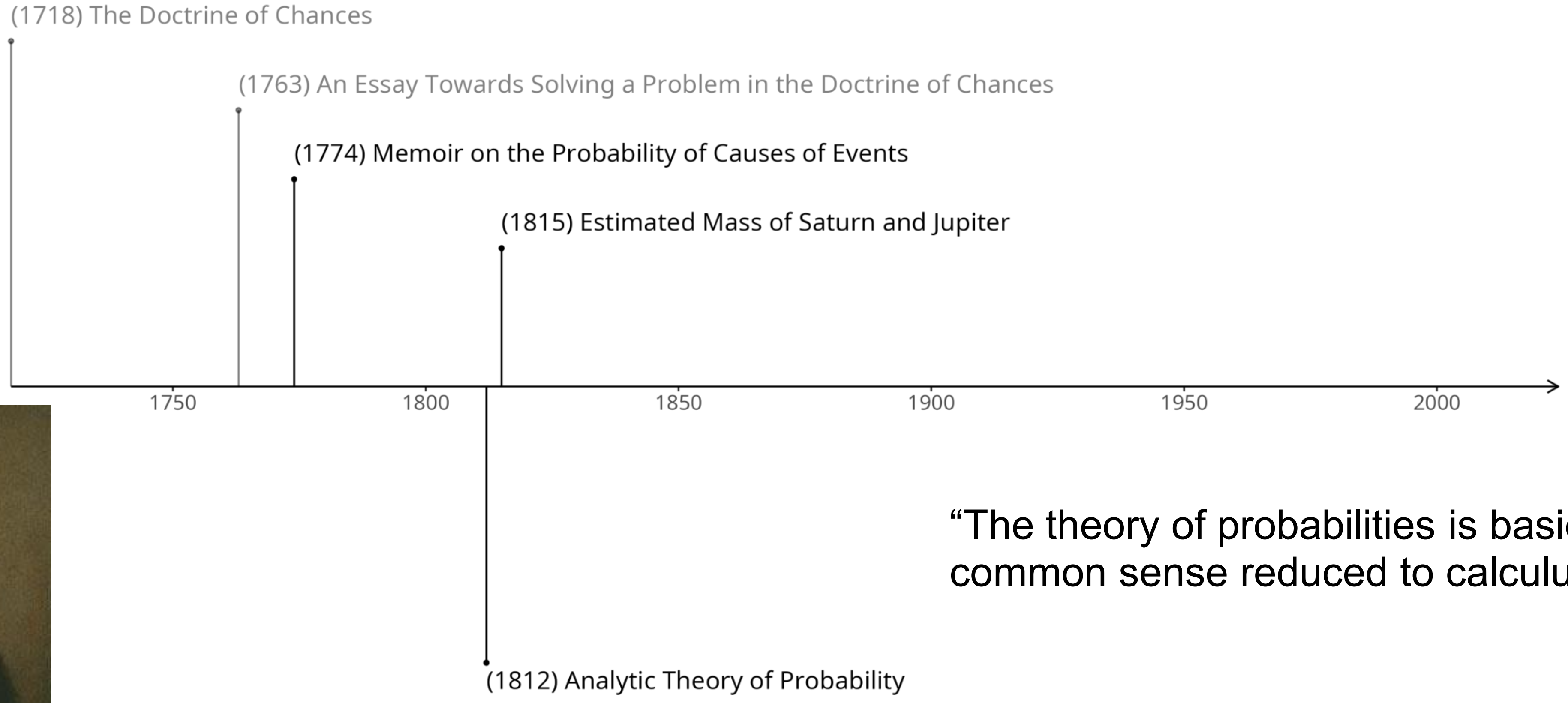
Reverend Thomas Bayes



PROBLEM.

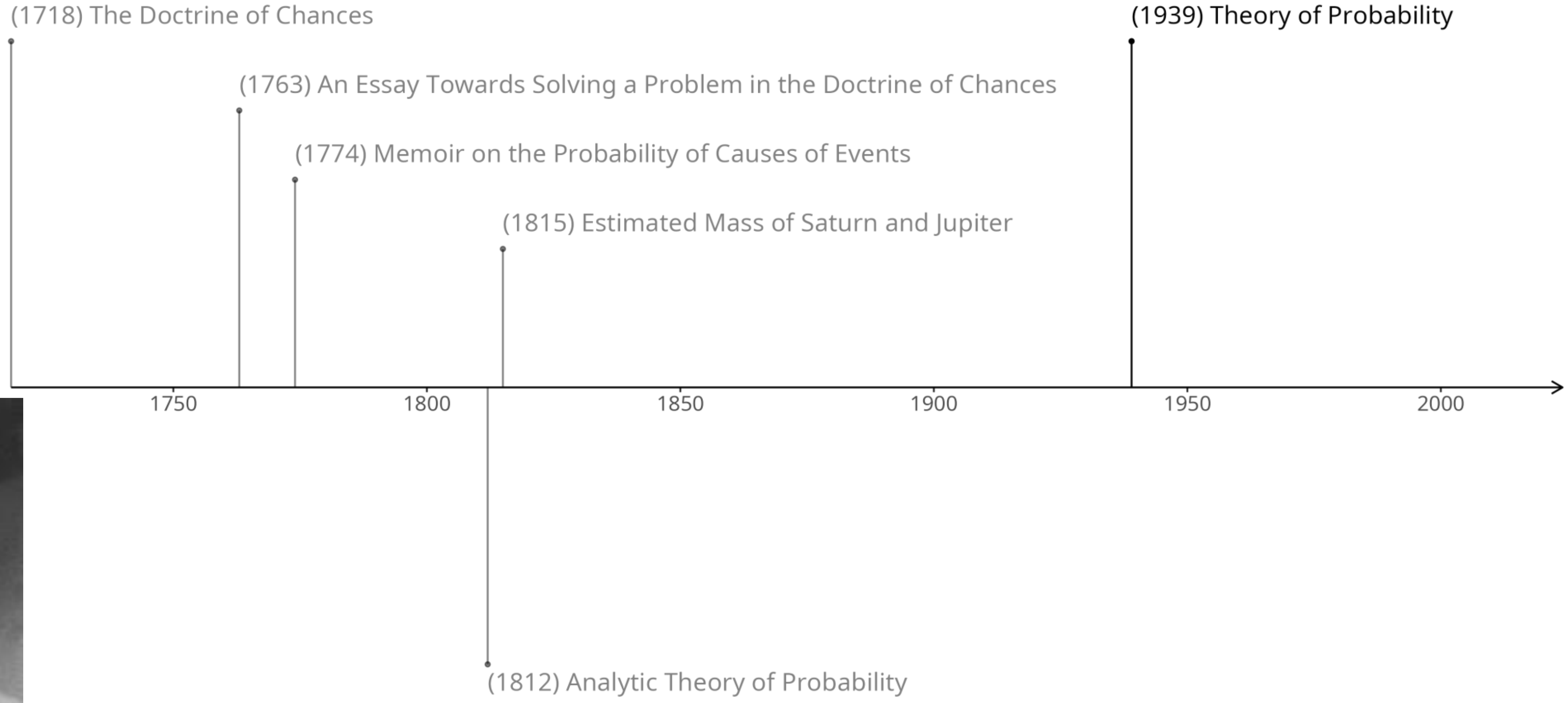
Given the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a ſingle trial lies ſomewhere between any two degrees of probability that can be named.

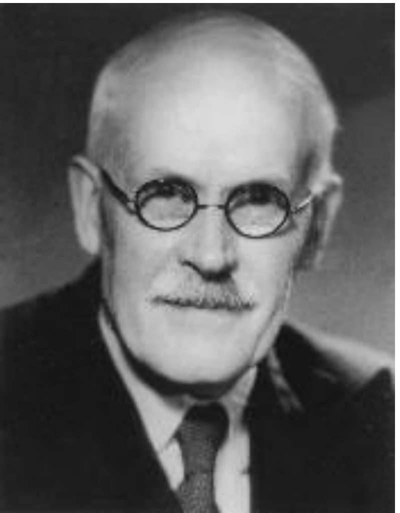
Pierre Simon Laplace



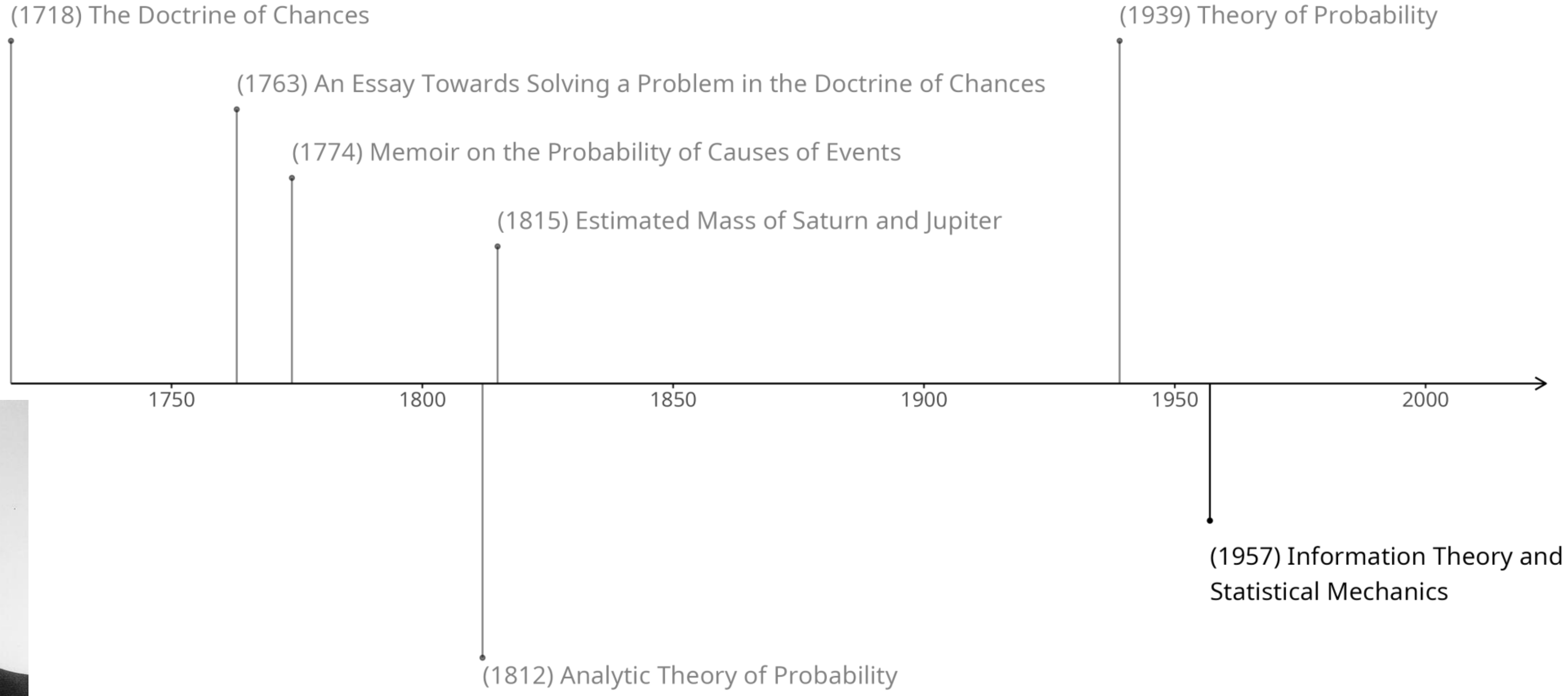
"The theory of probabilities is basically just common sense reduced to calculus…"

Harold Jeffreys





Edwin Jaynes



"Probability theory as extended logic"

Bayes' Theorem

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Bayes' Theorem

$$\boxed{P(H \mid E)} = \frac{P(H)P(E \mid H)}{P(E)}$$

Posterior Probability

Bayes' Theorem

Prior Probability

$$P(H \mid E) = \frac{\boxed{P(H)}\, P(E \mid H)}{P(E)}$$

Bayes' Theorem

Sampling Probability

(proportional to) Likelihood

$$P(H \mid E) = \frac{P(H)\boxed{P(E \mid H)}}{P(E)}$$

Bayes' Theorem

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{\boxed{P(E)}}$$

Model Evidence

Probability of the evidence averaged over the prior

Bayes’ Theorem

Sampling Probability

Prior Probability

(proportional to) Likelihood

$$\boxed{P(H \mid E)} = \frac{\boxed{P(H)}\,\boxed{P(E \mid H)}}{\boxed{P(E)}}$$

Posterior Probability

Model Evidence

Probability of the evidence averaged over the prior

Simple Inferential Probability

- MREs
- Boy or Girl Paradox
- Monte Hall

Picking MREs

You have two boxes of MREs with different proportions of veggie omelet and buffalo chicken given by the following table.

	Veggie Omelet	Buffalo Chicken
Box 1	4	8
Box 2	7	5

What is the probability that you get a veggie omelet at random?

Picking MREs

You have two boxes of MREs with different proportions of veggie omelet and buffalo chicken given by the following table.

What is the probability that you get a veggie omelet at random?

$$P(V) = P(V \mid Box\ 1)P(Box\ 1) + P(V \mid Box\ 2)P(Box\ 2)$$

$$= {}^{4}/_{12} \times {}^{1}/_{2} + {}^{7}/_{12} \times {}^{1}/_{2}$$

$$= {}^{11}/_{24}$$

Picking MREs

Your friend opens one of the boxes and pulls out a veggie omelet.

What is the probability that the opened box is Box 1?

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Box 1				
Box 2				

Picking MREs

Your friend opens one of the boxes and pulls out a veggie omelet.

What is the probability that the opened box is Box 1?

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Box 1	1/2	4/12	4/24	4/11
Box 2	1/2	7/12	7/24	7/11

Picking MREs

Your other friend pulls a buffalo chicken MRE out of the open box.

What is the probability that the opened box is Box 1?

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Box 1				
Box 2				

Picking MREs

Your other friend pulls a buffalo chicken MRE out of the open box.

What is the probability that the opened box is Box 1?

$$P(H \mid E) = \frac{P(H)P(E \mid H)}{P(E)}$$

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Box 1	4 / 11	8 / 11	32 / 121	32 / 67
Box 2	7 / 11	5 / 11	35 / 121	35 / 67

Picking MREs

After your two observations, what is the probability that you get a veggie omelet at random from the open box?

$$P(V) = P(V \mid Box\ 1)P(Box\ 1) + P(V \mid Box\ 2)P(Box\ 2)$$

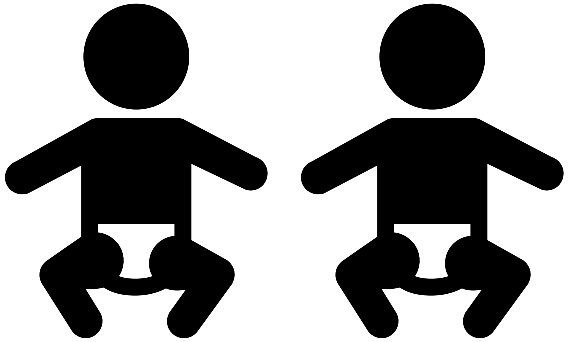
$$= {}^{3}/_{10} \times {}^{32}/_{67} + {}^{6}/_{10} \times {}^{35}/_{67}$$

$$= {}^{306}/_{670}$$

$$\approx 0.457$$

Boy or Girl?

Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?



Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

Given:

- 2 children
- Older child is a girl

Question:

- P(G, G)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B				
B, G				
G, B				
G, G				

Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

<u>Given:</u>

- 2 children
- Older child is a girl

<u>Question:</u>

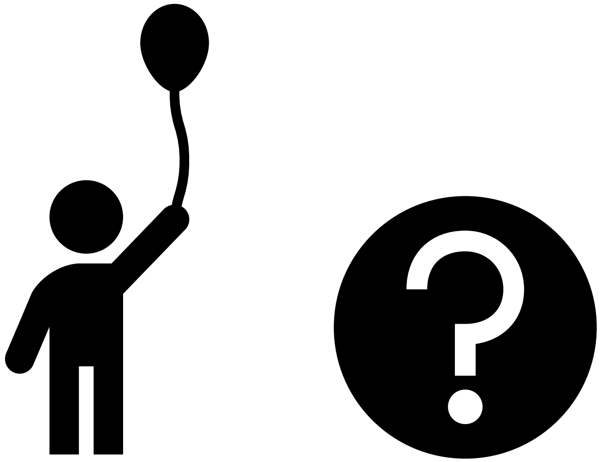
- P(G, G)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B	1 / 4	0	0	0
B, G	1 / 4	0	0	0
G, B	1 / 4	1	1 / 4	1 / 2
G, G	1 / 4	1	1 / 4	1 / 2

Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

Mr. Smith has two children. At least one of them is a boy.
What is the probability that both children are boys?



Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

Given:

- 2 children
- At least 1 is a boy

Question:

- P(B, B)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B				
B, G				
G, B				
G, G				

Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

Given:

- 2 children
- At least 1 is a boy

Question:

- P(B, B)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B	1 / 4	1	1 / 4	1 / 3
B, G	1 / 4	1	1 / 4	1 / 3
G, B	1 / 4	1	1 / 4	1 / 3
G, G	1 / 4	0	0	0

Martin Gardner (1961). The Second Scientific American Book of Mathematical Puzzles and Diversions. Simon & Schuster. ISBN 978-0-226-28253-4.

Boy or Girl?

Mr. Smith has two children. You see him walk to a parent-teacher conference with a boy. What is the probability that both children are boys?

Boy or Girl?

<u>Given:</u>

- 2 children
- You observe one child is a boy

<u>Question:</u>

- P(B, B)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B				
B, G				
G, B				
G, G				

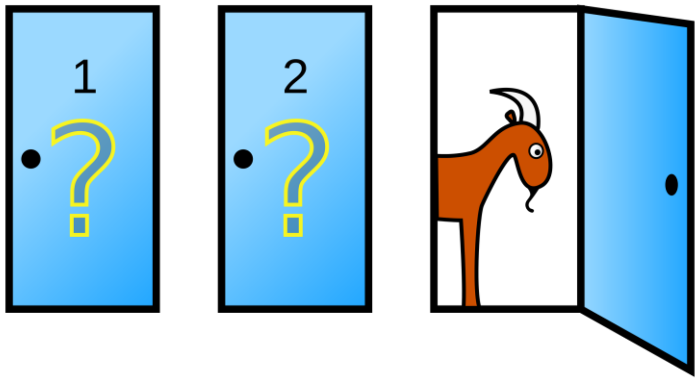
Boy or Girl?

Given:
- 2 children
- You observe one child is a boy

Question:
- P(B, B)

Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
B, B	1 / 4	1	1 / 4	1 / 2
B, G	1 / 4	1 / 2	1 / 8	1 / 4
G, B	1 / 4	1 / 2	1 / 8	1 / 4
G, G	1 / 4	0	0	0

Monte Hall



Given:

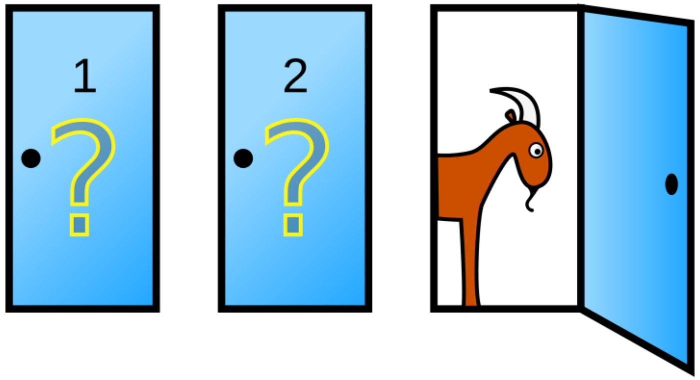
- You pick door 1
- Monte Hall opens door 3 showing a goat

Assumptions:

- Monte Hall always offers a switch
- Monte Hall always opens a door with a goat behind it

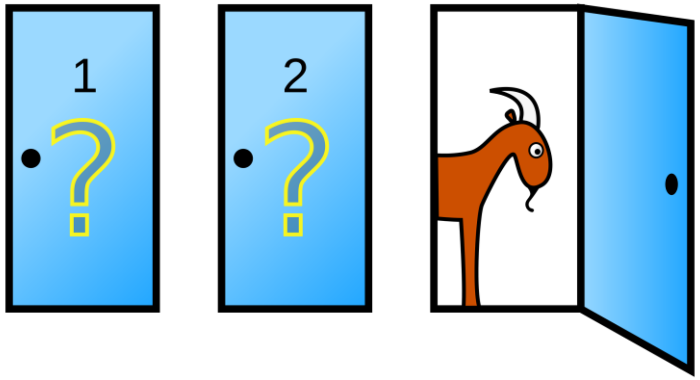
What are the probabilities that the car is behind each door?C

Monte Hall



Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Car in Door 1				
Car in Door 2				
Car in Door 3				

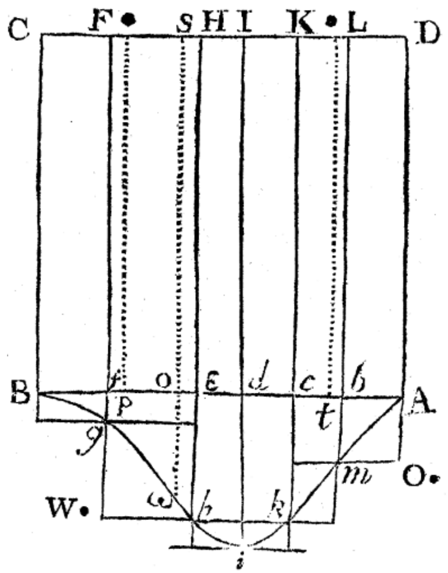
Monte Hall



Hypothesis (H)	Prior P(H)	Sampling Probability P(E | H)	Plausibility P(H)P(E | H)	Posterior Probability P(H | E)
Car in Door 1	1 / 3	1 / 2	1 / 6	1 / 3
Car in Door 2	1 / 3	1	1 / 3	2 / 3
Car in Door 3	1 / 3	0	0	0

Bayes' Table

First known application of Bayes' Theorem!



PROBLEM.

Given the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named.

Bayes’ Table

Process:

- Throw a ball on a table and mark its position
- Throw a second ball repeatedly and measure if it landed to the left or right of the mark
- Calculate the probability of the position of the first ball

Bayes’ Table – Experiment

Instructions:

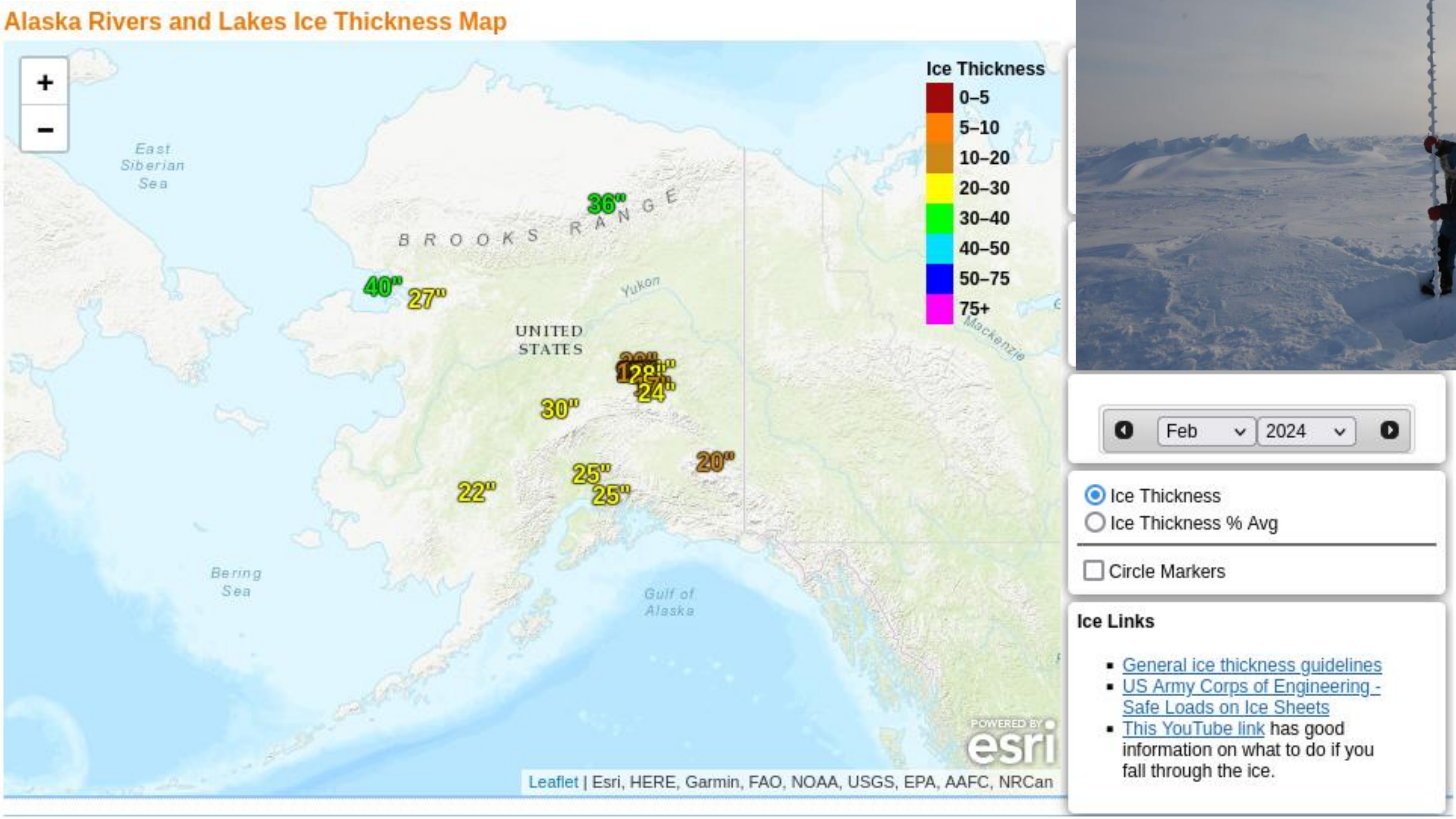
- On my command, throw a single bag
- Throw backwards with your non-dominant hand
- After throwing all four bags, collect them and throw from the other set of chairs

While waiting:

- Clone the factory day repo from AI2C GitHub
- Set up a python virtual environment based on the requirements file

Predicting Ice Thickness

Predicting Ice Thickness



https://www.weather.gov/aprfc/IceThickness

Predicting Ice Thickness

Stefan's Equation:

$$h_j^2 - h_k^2 = \alpha^2 \Delta U$$

DEPARTMENT OF THE ARMY
U.S. Army Corps of Engineers
Washington, DC 20314-1000

EM 1110-2-1612
Change 4

CECW-EE

Manual
No. 1110-2-1612

31 August 18

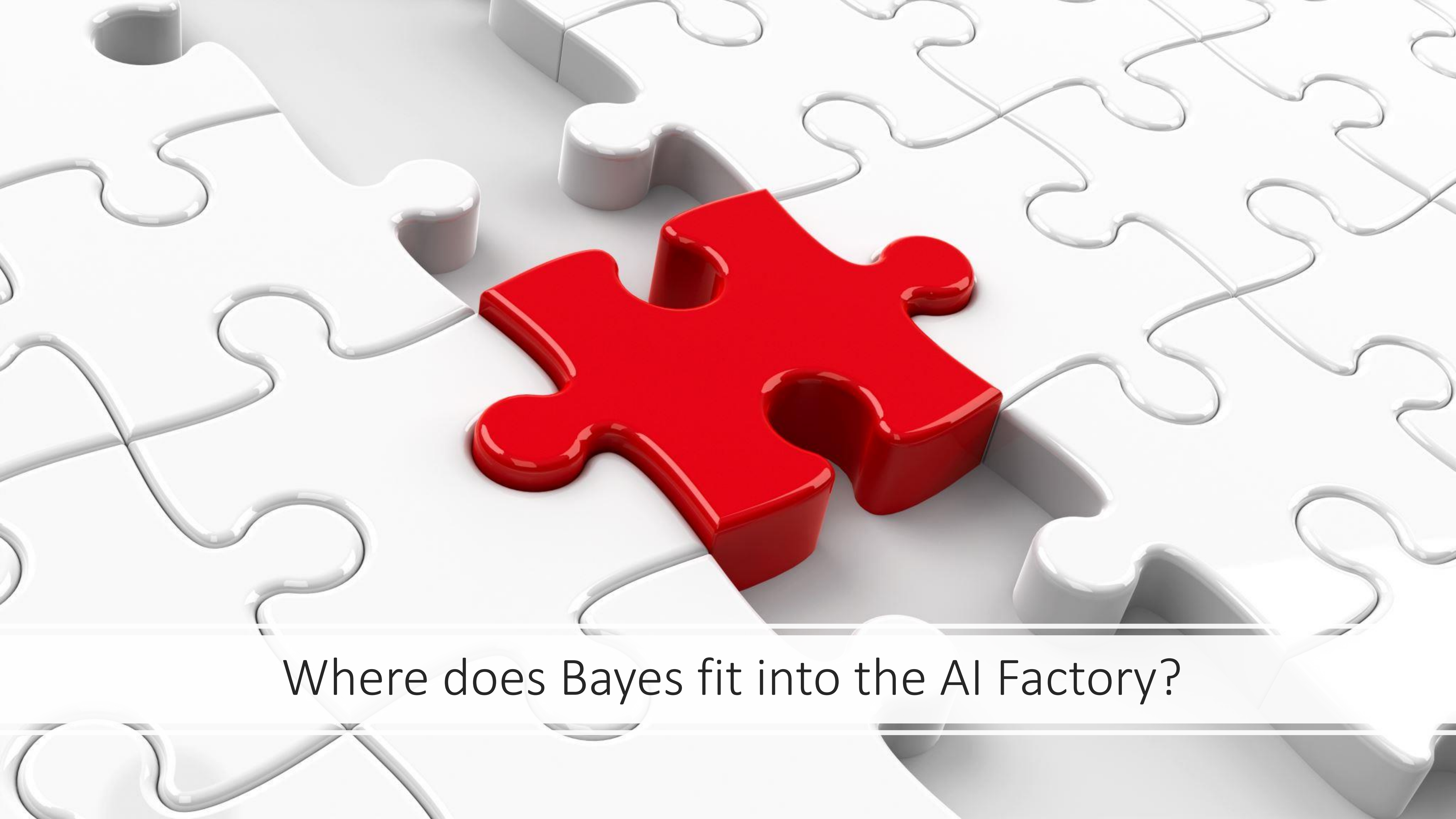
Engineering and Design
ICE ENGINEERING

Typical values for α are presented in Table 2-2. In this case the ice thickness is proportional to the square root of the accumulated freezing degree-days.

Table 2-2
Typical Values of α (after Michel 1971)

Ice Cover Condition	α *	α †
Windy lake w/no snow	2.7	0.80
Average lake with snow	1.7–2.4	0.50–0.70
Average river with snow	1.4–1.7	0.40–0.50
Sheltered small river	0.7–1.4	0.20–0.40

* AFDD calculated using degrees Celsius. The ice thickness is in centimeters.
† AFDD calculated using degrees Fahrenheit. The ice thickness is in inches.



Where does Bayes fit into the AI Factory?

Books to Read

- Clayton, Aubrey. *Bernoulli's Fallacy: Statistical Illogic and the Crisis of Modern Science*, New York Chichester, West Sussex: Columbia University Press, 2021. https://doi.org/10.7312/clay19994

- McElreath, R. (2020). *Statistical Rethinking: A Bayesian Course with Examples in R and STAN* (2nd ed.). Chapman and Hall/CRC. https://doi.org/10.1201/9780429029608

- Gelman, A., Carlin, J.B., Stern, H.S., Dunson, D.B., Vehtari, A., & Rubin, D.B. (2013). *Bayesian Data Analysis* (3rd ed.). Chapman and Hall/CRC. https://doi.org/10.1201/b16018