STEP Homework Questions

Here are three homework questions adapted from various STEP practice papers. Use the space provided (or extra paper) to write your solutions. Be sure to show all your working and reasoning.

1. (11-S1-Q1)

(i)

Show that the gradient of the curve

$$\frac{a}{x} + \frac{b}{y} = 1,$$

where $b \neq 0$, is

$$-\frac{a\,y^2}{b\,x^2}.$$

The point (p, q) lies on both the straight line

$$ax + by = 1$$

and the curve

$$\frac{a}{x} + \frac{b}{y} = 1,$$

where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.

Show further that either

$$(a-b)^2 = 1 \quad \text{or} \quad (a+b)^2 = 1.$$

(ii)

Show that if the straight line

$$ax + by = 1, \quad (ab \neq 0),$$

is a normal to the curve

$$\frac{a}{x} - \frac{b}{y} = 1,$$

then

$$a^2 - b^2 = \frac{1}{2}.$$

Space for your solution to Q1 (i) and (ii)

2. (15-S1-Q7)

Let

$$f(x) = 3ax^2 - 6x^3$$

and, for each real number a, let $M(a)$ be the greatest value of $f(x)$ in the interval $-\frac{1}{3} \leqslant x \leqslant 1$.

Task: Determine $M(a)$ for $a \geqslant 0$. Note that the formula for $M(a)$ is **different in different ranges of** a; you will need to identify **three** such ranges and give an expression for $M(a)$ in each.

Space for your solution to Q2

3. (08-Sa-Q4)

A curve is given by

$$x^2 + y^2 + 2a\,x\,y = 1,$$

where a is a constant satisfying $0 < a < 1$.

1. Show that the gradient of the curve at the point P with coordinates (x, y) is

$$-\frac{x + ay}{ax + y},$$

   provided $ax + y \neq 0$.
2. Show that θ, the acute angle between OP (the line from the origin to P) and the **normal** to the curve at P, satisfies

$$\tan\theta = a\left|y^2 - x^2\right|.$$

3. Show further that if $\frac{d\theta}{dx} = 0$ at P, then:

   (i)

$$a(x^2 + y^2) + 2xy = 0;$$

   (ii)

$$(1 + a)\left(x^2 + y^2 + 2xy\right) = 1;$$

   (iii)

$$\tan\theta = \frac{a}{\sqrt{1 - a^2}}.$$

Space for your solution to Q3