STEP Lesson 1: Calculus – Differentiation

Handout for Students

(Feel free to write your name here and take notes throughout.)

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Throughout this handout, **blank spaces** are provided for you to take notes and work out solutions.

Overview of STEP 2/3

- **STEP（Sixth Term Examination Paper）** is a mathematics examination, often used for university admissions in the UK.
- **STEP Mathematics 1** is no longer offered since June 2021.

- **STEP Mathematics 2 and 3** remain with the same structure in 2025.
- **Content Assumptions**:
  - STEP 2 assumes knowledge of A-level Mathematics (and what was in STEP 1).
  - STEP 3 assumes knowledge of A-level Mathematics and Further Mathematics (and what was in STEP 2).

Notes:

Exam Format & Preparation

- Based on **A Level Mathematics** content (some topics removed, some added).
- **STEP 2 & 3** each is a 3-hour paper, split into:
  - **Section A:** Pure Mathematics (8 questions)
  - **Section B:** Mechanics (2 questions)
  - **Section C:** Probability/Statistics (2 questions)
- Grading is based on the best 6 answers (each worth up to 20 marks).
- **Important**:
  - No formula booklet provided.
  - No calculators or bilingual dictionaries allowed.
- Questions may test familiar knowledge in unfamiliar ways, requiring insight and creativity.

Notes:

Limits

Theorem 1: Basic Limit Rules

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist, then:

1. **Sum/Difference Rule**

$$\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

2. **Product Rule**

$$\lim_{x \to c}[f(x)g(x)] = \left(\lim_{x \to c} f(x)\right) \cdot \left(\lim_{x \to c} g(x)\right)$$

3. **Quotient Rule**

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

(assuming the denominator limit $\neq 0$)

Notes:

Theorem 2: Composition Rule

If:

1. $\lim_{x \to c} f(x) = A$
2. $\lim_{y \to A} g(y) = B$
3. Range of f is in the domain of g

Then:

$$\lim_{x \to c} g(f(x)) = B.$$

Example:
$\lim_{x \to 0} \sin(\sin x)$
$= \sin\left(\lim_{x \to 0} \sin x\right)$
$= \sin(0) = 0.$

Theorem 3: Squeeze Theorem

If, on an interval, we have

$$f(x) \leq h(x) \leq g(x)$$

and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = A,$$

then

$$\lim_{x \to c} h(x) = A.$$

Key Application: $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Notes:

Important Limits & Applications

- $\displaystyle \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$
- $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1.$

These are crucial for defining e, finding the derivative of $\sin x$, small-angle approximations, and Taylor expansions.

Notes:

Differentiation

Differentiation measures how a function changes as its input changes.

- **Physical Interpretation**: Rate of change, slope of tangent, velocity, acceleration, etc.
- **Mathematical Definition** (First Principle):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Key Applications:

- Rate of change problems
- Optimization
- Motion (velocity, acceleration)
- Tangent lines

Notes:

First Principle of Differentiation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- **Interpretation**: Instantaneous rate of change or slope of the tangent line at a point.
- **Examples**:

- Velocity = derivative of position.
- Acceleration = derivative of velocity.

Notes:

Power Functions via First Principle

Example: $\frac{d}{dx}(x^2)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x. \end{aligned}$$

General result for $f(x) = x^n$:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Notes:

Exponential & Logarithmic Derivatives

1. $\frac{d}{dx}(e^x)$
   Using first principle:

$$\frac{d}{dx}(e^x) = e^x.$$

   Key limit used:
   $\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$
2. $\frac{d}{dx}(\ln x)$
   Using first principle:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

   Key limit used:
   $\lim_{h \to 0} \frac{\ln(1+h)}{h} = 1.$

Notes:

Homework: Trigonometric Derivatives

Problem 1: Prove from first principles that

$$\frac{d}{dx} \sin x = \cos x.$$

Hints:

1. $\sin(x+h) = \sin x \cos h + \cos x \sin h.$

2. Use the limits $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

Problem 2: Prove from first principles that

$$\frac{d}{dx} \cos x = -\sin x.$$

Hints:

1. $\cos(x+h) = \cos x \cos h - \sin x \sin h$.
2. Similar limit usage as above.

Space for your working/notes:

Real STEP Example: Osculating Circle (STEP 2010, Paper 2 Q1)

Question:

A curve C is given by $y = 1 - x + \tan x$. Let P be the point on this curve with x-coordinate $\frac{\pi}{4}$.
The **osculating circle** at P is the circle that:

1. Touches C at P.
2. Has the same rate of change of gradient as C at P.

Task: Find the centre and radius of the osculating circle at P.

Space for your working/notes:

(The detailed solution typically involves matching coordinates, first derivative, and second derivative with that of a general circle. You might also use the radius of curvature formula $R = \frac{[1+(y')^2]^{3/2}}{|y''|}$.)

Basic Derivatives (Reference)

1. **Power Functions**
   $\frac{d}{dx}x^a = ax^{a-1}$.
2. **Trigonometric Functions**

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\tan x = \sec^2 x.$$

3. **Inverse Trigonometric Functions**

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

4. **Exponential & Logarithm**

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}.$$

Differentiation Rules

Chain Rule

For $h(x) = g(f(x))$:

$$h'(x) = g'(f(x)) \cdot f'(x).$$

Example: $\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x$.

Product Rule

$$\frac{d}{dx}[u \cdot v] = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example: $\frac{d}{dx}[x \sin x] = \sin x + x\cos x$.

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}.$$

Example: $\frac{d}{dx}\left[\frac{x}{\sin x}\right] = \frac{\sin x \cdot 1 - x \cdot \cos x}{(\sin x)^2}$.

Notes:

Real STEP Example: Function Analysis (STEP 2000, Paper 1 Q7)

Question:
Let $f(x) = ax - \frac{x^3}{1+x^2}$, where a is a constant. Show that if $a \geq \frac{9}{8}$, then $f'(x) \geq 0$ for all x.

Outline:

1. Differentiate $f(x)$ using the quotient rule.
2. Show that the resulting expression is non-negative for $a \geq \frac{9}{8}$.

Space for your working/notes:

End of Handout

Remember: Practice questions from past STEP papers often involve applying these derivative rules in non-standard ways. Focus on mastering first principles, the chain/product/quotient rules, and practicing problems under exam conditions.

Good Luck!