Chapter 1: Transformations and rigid movements

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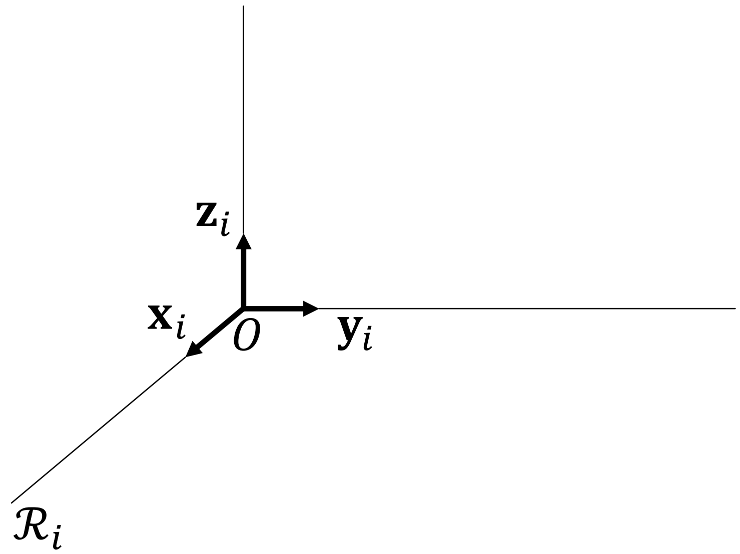
Notations and definitions

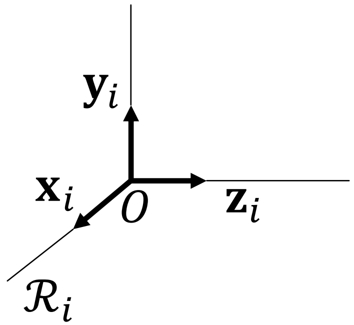
- Point : lowercase italic letter
- Vector : lowercase bold letter
- Null vector: $\mathbf{0}$
- Matrix: bold capital letter
- Norm of a vector : ‖ ‖
- Scalar product : $\mathbf{a} \cdot \mathbf{b}$
- Cross product: $\mathbf{a} \wedge \mathbf{b}$
- Pose = position + orientation

NOTATIONS AND DEFINITIONS

Let $\mathcal{R}_i = (O, \mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ be a direct orthonormal frame

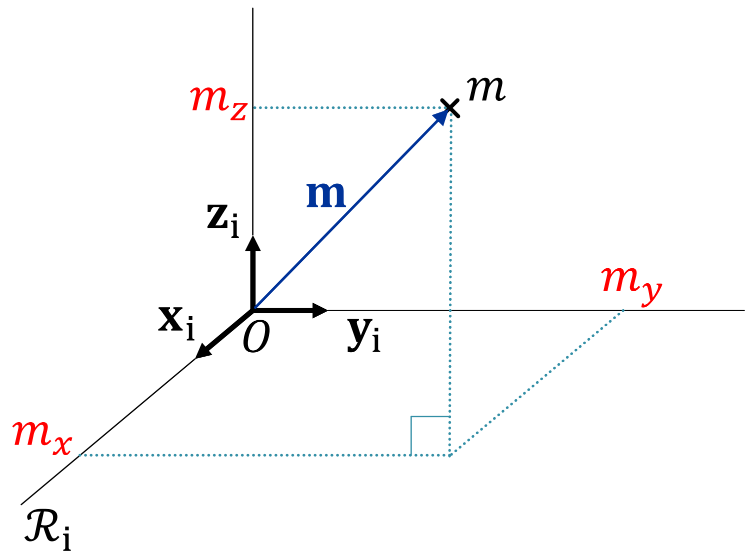


- Orthogonal frame: $\mathbf{x}_i \perp \mathbf{y}_i \perp \mathbf{z}_i$
- Normalized frame : $\|\mathbf{x}_i\| = \|\mathbf{y}_i\| = \|\mathbf{z}_i\| = 1$



Indirect frame

Point representation



- The coordinates of a point m are represented by the vector:

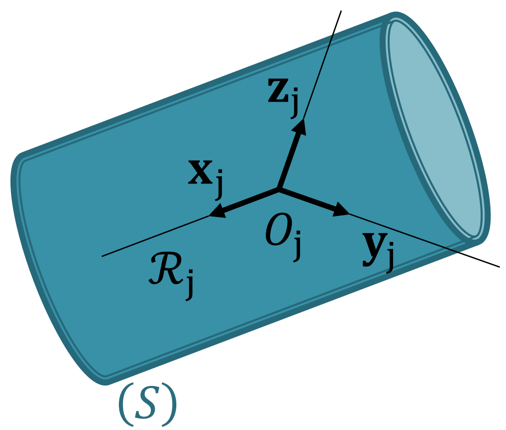
$$^{\mathrm{i}}\mathbf{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

i to say that the vector $\mathbf{m}$ is expressed in the frame $\mathcal{R}_\mathrm{i}$

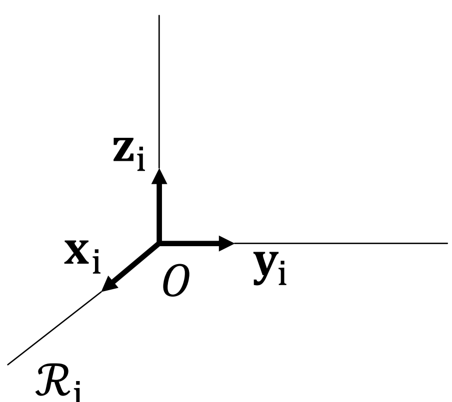
Or:

$$^{\mathrm{i}}\mathbf{m} = m_x \mathbf{x}_\mathrm{i} + m_y \mathbf{y}_\mathrm{i} + m_z \mathbf{z}_\mathrm{i}$$

SOLID POSE REPRESENTATION



$\mathcal{R}_\mathrm{j}$: frame attached the solid (S)



$\mathcal{R}_\mathrm{i}$: reference frame

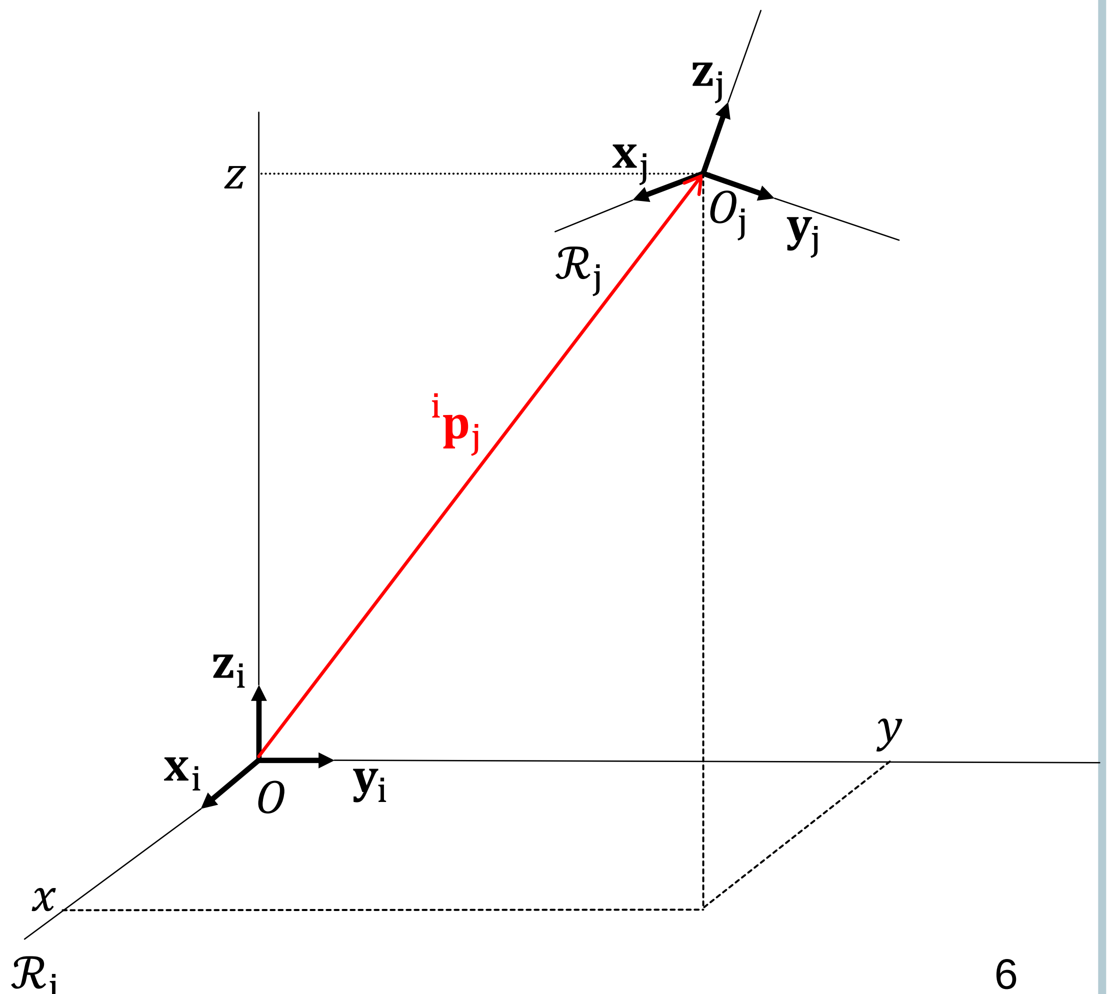
Q: What is the pose (position + orientation) of the solid with respect to the frame $\mathcal{R}_\mathrm{i}$?

Solid pose representation

Position of a solid (S)

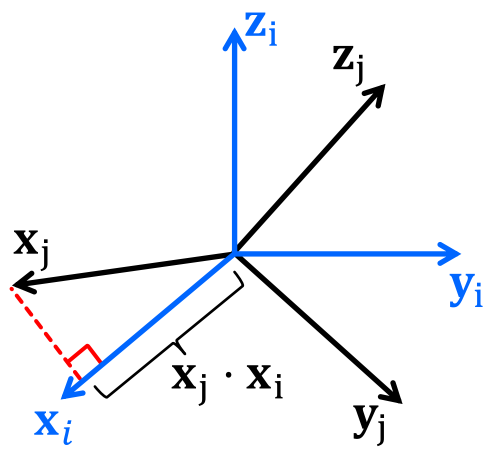
- The position $\mathcal{R}_\mathrm{j}$ with respect to $\mathcal{R}_\mathrm{i}$ is represented by the vector:

$$ {}^{\mathrm{i}}\mathbf{p}_\mathrm{j} = \overrightarrow{OO_\mathrm{j}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} $$



Solid pose representation

Orientation of the solid (S)



The vector of the base $\mathcal{R}_j$ can be expressed in the frame $\mathcal{R}_i$ as:

$$ {}^{i}\mathbf{x}_j = \begin{pmatrix} \mathbf{x}_j \cdot \mathbf{x}_i \\ \mathbf{x}_j \cdot \mathbf{y}_i \\ \mathbf{x}_j \cdot \mathbf{z}_i \end{pmatrix}; \qquad {}^{i}\mathbf{y}_j = \begin{pmatrix} \mathbf{y}_j \cdot \mathbf{x}_i \\ \mathbf{y}_j \cdot \mathbf{y}_i \\ \mathbf{y}_j \cdot \mathbf{z}_i \end{pmatrix}; \qquad {}^{i}\mathbf{z}_j = \begin{pmatrix} \mathbf{z}_j \cdot \mathbf{x}_i \\ \mathbf{z}_j \cdot \mathbf{y}_i \\ \mathbf{z}_j \cdot \mathbf{z}_i \end{pmatrix} $$

Solid pose representation

The orientation of the frame $\mathcal{R}_\mathrm{j}$ with respect to $\mathcal{R}_\mathrm{i}$ is represented by:

$$ {}^{\mathrm{i}}\mathbf{R}_\mathrm{j} = \left({}^{\mathrm{i}}\mathbf{x}_\mathrm{j} \quad {}^{\mathrm{i}}\mathbf{y}_\mathrm{j} \quad {}^{\mathrm{i}}\mathbf{z}_\mathrm{j} \right) $$

${}^{\mathrm{i}}\mathbf{R}_\mathrm{j}$: 3×3 rotation matrix, expressing the transition from the frame $\mathcal{R}_\mathrm{i}$ towards the frame $\mathcal{R}_j$

Properties:

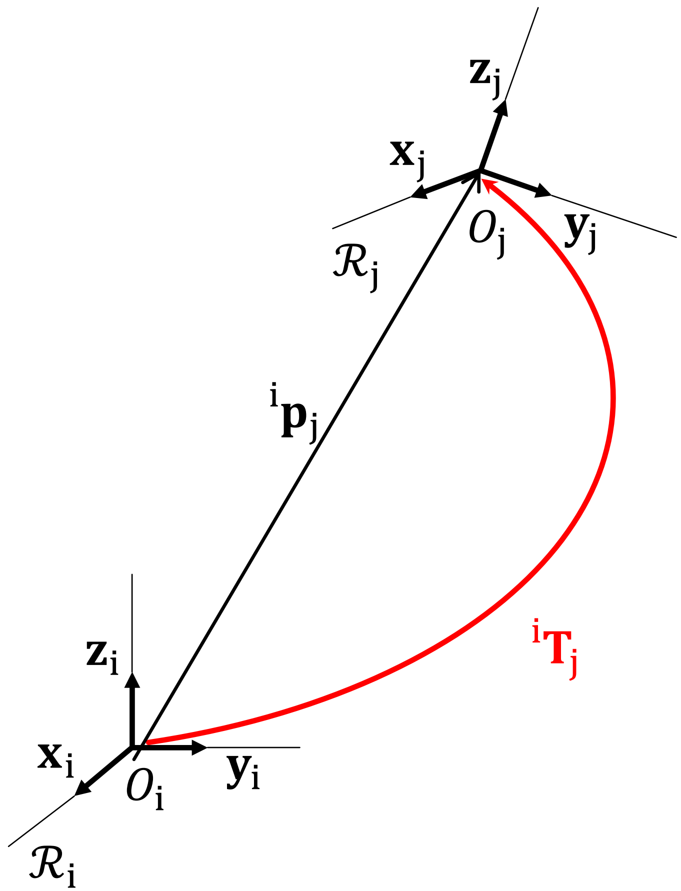
- $\| {}^{\mathrm{i}}\mathbf{x}_\mathrm{j} \| = \| {}^{\mathrm{i}}\mathbf{y}_\mathrm{j} \| = \| {}^{\mathrm{i}}\mathbf{z}_\mathrm{j} \| = 1$
- ${}^{\mathrm{i}}\mathbf{x}_\mathrm{j} \cdot {}^{\mathrm{i}}\mathbf{y}_\mathrm{j} = {}^{\mathrm{i}}\mathbf{x}_\mathrm{j} \cdot {}^{\mathrm{i}}\mathbf{z}_\mathrm{j} = {}^{\mathrm{i}}\mathbf{y}_\mathrm{j} \cdot {}^{\mathrm{i}}\mathbf{z}_\mathrm{j} = 0$

The pose of the solid (S), represented by the vector ${}^{\mathrm{i}}\mathbf{p}_\mathrm{j}$, and the matrix ${}^{\mathrm{i}}\mathbf{R}_\mathrm{j}$ are gathered inside a matrix ${}^{\mathrm{i}}\mathbf{T}_\mathrm{j}$, known as homogeneous transformation matrix

Homogeneous transformation matrix

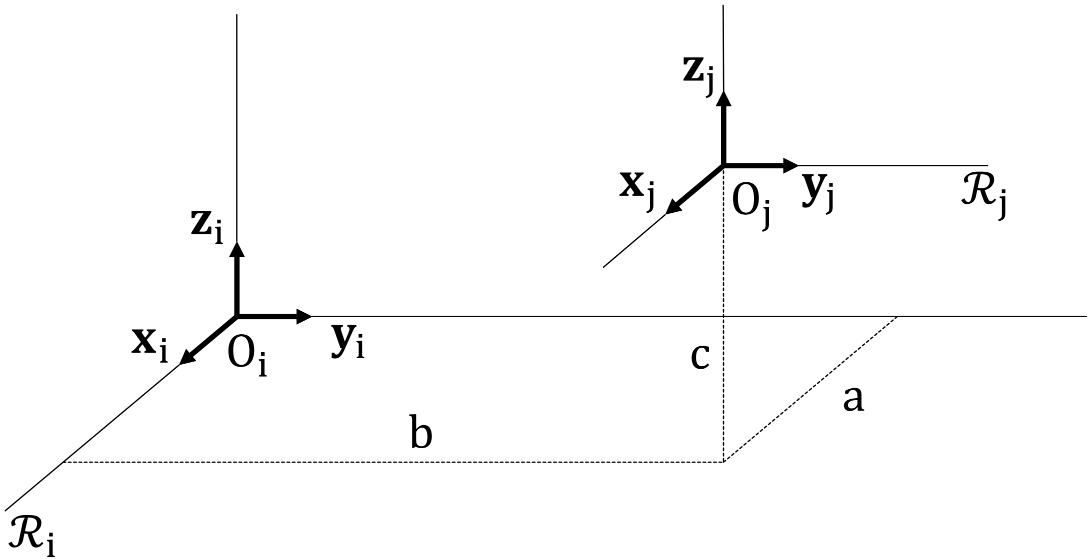
${}^{i}\mathbf{T}_{j}$: transition matrix from the frame $\mathcal{R}_i$ towards the frame $\mathcal{R}_j$

$${}^{i}\mathbf{T}_{j} = \begin{pmatrix} & {}^{i}\mathbf{R}_{j} & & {}^{i}\mathbf{p}_{j} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Homogeneous transformation matrix

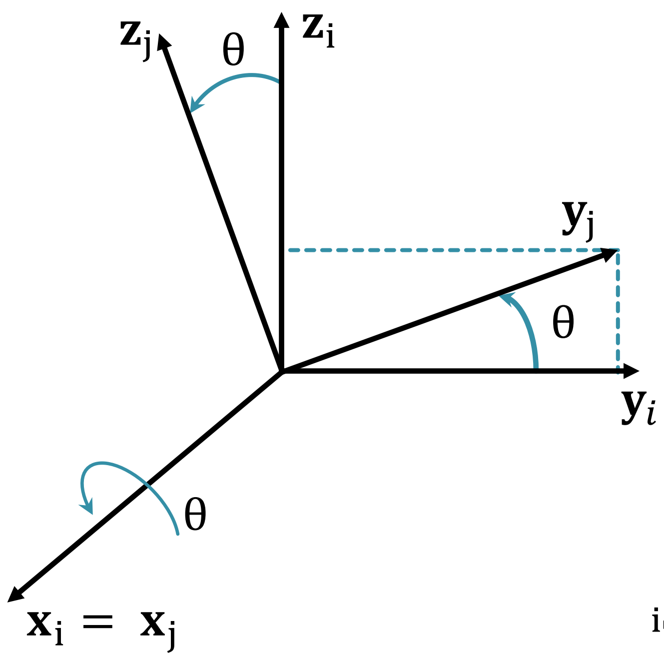
Pure translation



$$^{i}\mathbf{T}_{j} = \mathbf{Trans}(a, b, c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Homogeneous transformation matrix

Pure rotation (1)



$$ {}^{\mathrm{i}}\mathbf{x}_{\mathrm{j}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; $$

$$ {}^{\mathrm{i}}\mathbf{y}_{\mathrm{j}} = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}; $$

$$ {}^{\mathrm{i}}\mathbf{z}_{\mathrm{j}} = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} $$

$$ {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}} = \mathbf{Rot}(\mathbf{x}, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Homogeneous transformation matrix

Pure rotation (2)

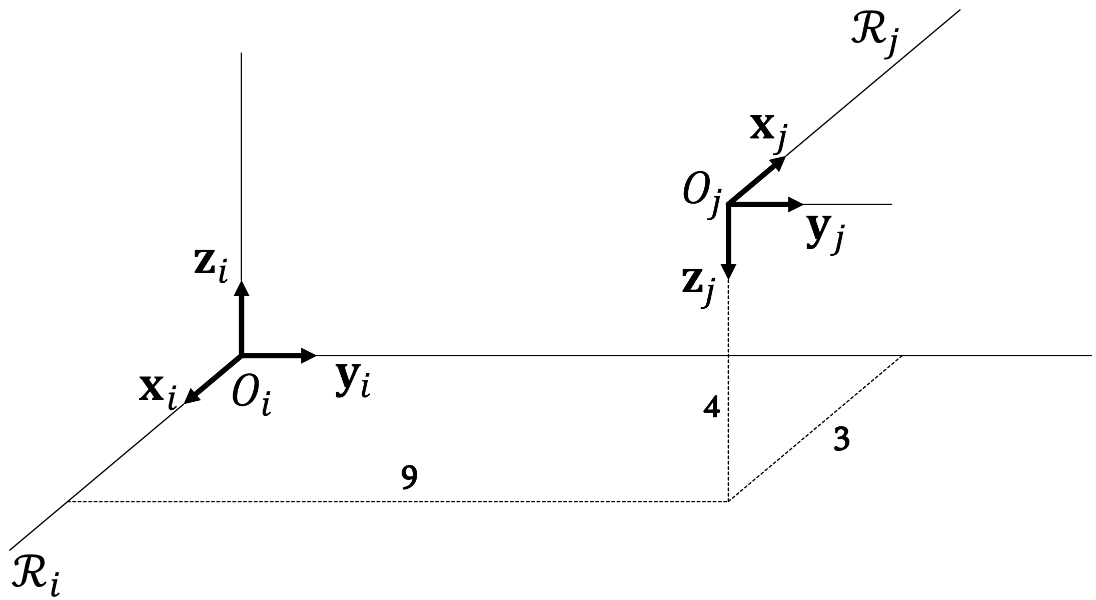
$$\mathbf{Rot}(\mathbf{x}, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Rot}(\mathbf{y}, \theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Rot}(\mathbf{z}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

HOMOGENEOUS TRANSFORMATION MATRIX

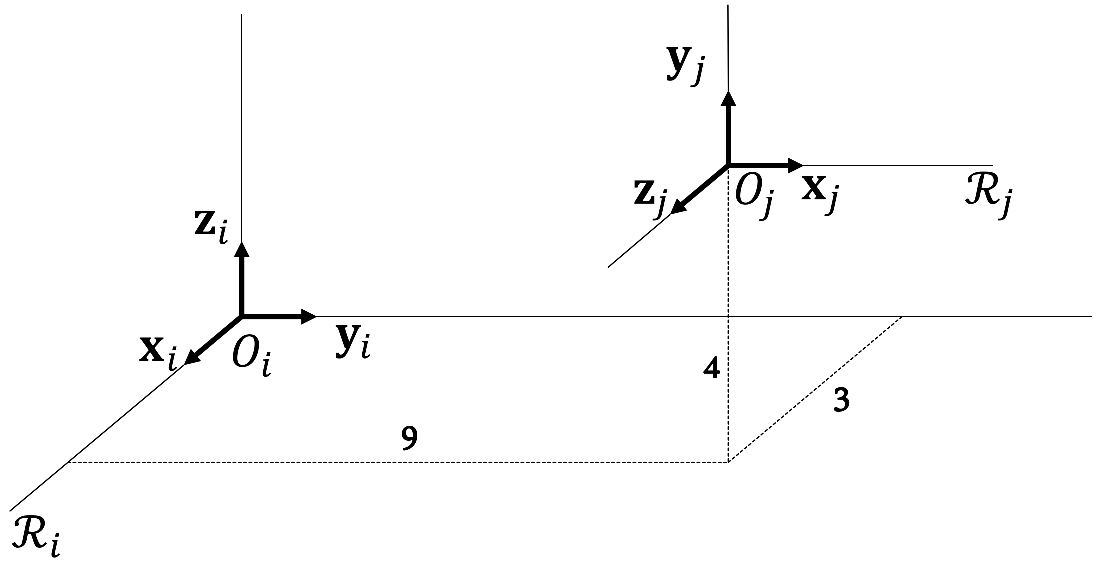
Example 1: compute the homogeneous transformation matrix ${}^{i}T_{j}$



Response:

HOMOGENEOUS TRANSFORMATION MATRIX

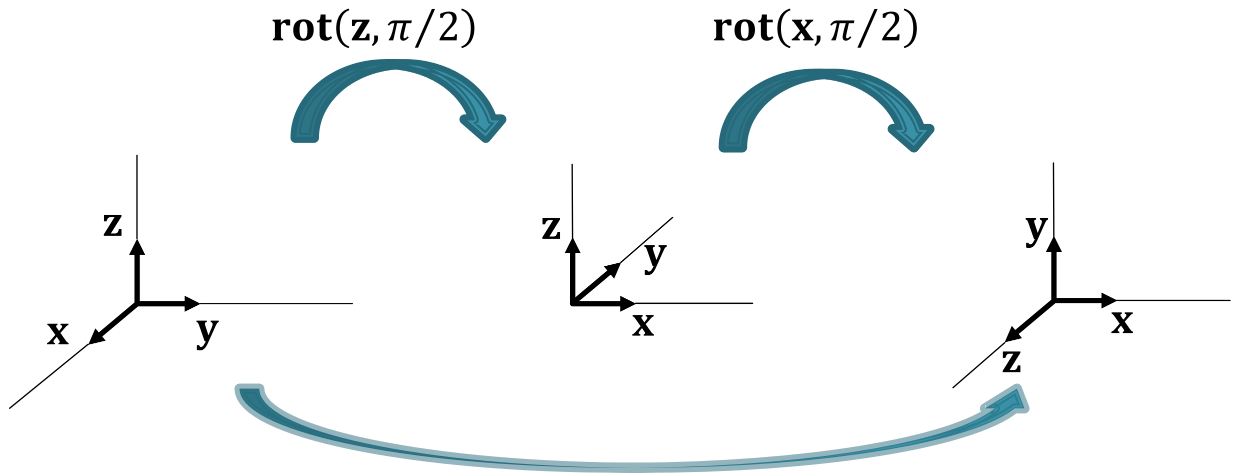
Example 2: compute the homogeneous transformation matrix ${}^{i}T_{j}$



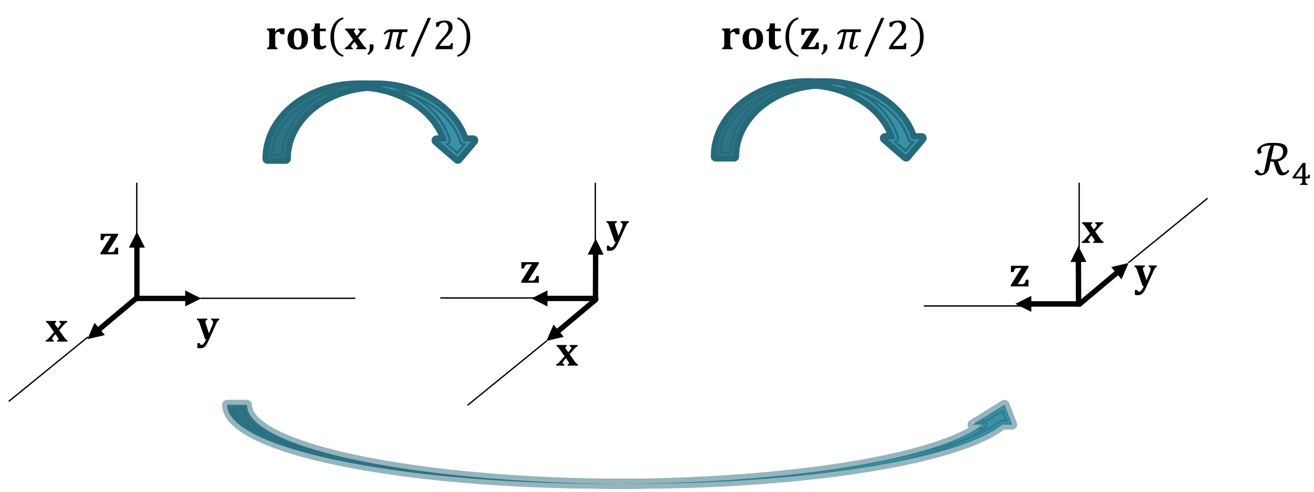
Response:

HOMOGENEOUS TRANSFORMATION MATRIX

Remarks:



$$\mathbf{R}_1 = \mathbf{rot}(\mathbf{z}, \pi/2) * \mathbf{rot}(\mathbf{x}, \pi/2)$$



$$\mathbf{R}_2 = \mathbf{rot}(\mathbf{x}, \pi/2) * \mathbf{rot}(\mathbf{z}, \pi/2) \neq \mathbf{R}_1$$

Homogeneous transformation matrix

Remarks:

Let $\mathbf{T}_1 = \begin{pmatrix} & \mathbf{R}_1 & & \mathbf{p}_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{T}_2 = \begin{pmatrix} & \mathbf{R}_2 & & \mathbf{p}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$$\mathbf{T}_1\mathbf{T}_2 = \begin{pmatrix} & \mathbf{R}_1\mathbf{R}_2 & & \mathbf{R}_1\mathbf{p}_2 + \mathbf{p}_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{T}_2\mathbf{T}_1 = \begin{pmatrix} & \mathbf{R}_2\mathbf{R}_1 & & \mathbf{R}_2\mathbf{p}_1 + \mathbf{p}_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$$

Homogeneous transformation matrix

Properties:

1) $\mathbf{R}^{-1} = \mathbf{R}^{T}$

2) $\mathbf{R}^{T}\mathbf{R} = \mathbf{I}_3$

3) $\det(\mathbf{R}) = 1$

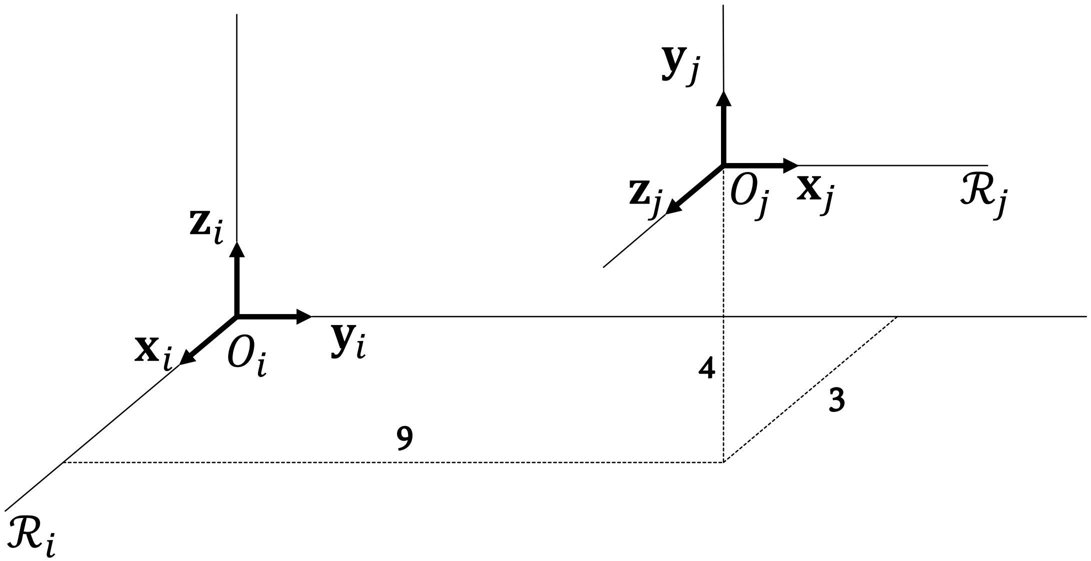
4) ${}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}}{}^{-1} = {}^{\mathrm{j}}\mathbf{T}_{\mathrm{i}}$

5) $\mathbf{T} = \begin{pmatrix} & \mathbf{R} & & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T}^{-1} = \begin{pmatrix} & \mathbf{R}^{T} & & -\mathbf{R}^{T}\mathbf{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

6) ${}^{0}\mathbf{T}_{k} = {}^{0}\mathbf{T}_{1}\,{}^{1}\mathbf{T}_{2}\,{}^{2}\mathbf{T}_{3} \ldots {}^{k-1}\mathbf{T}_{k}$

MATRICES DE TRANSFORMATION HOMOGÈNES

Example 3: from the example 2, compute ${}^{j}T_i$ using the properties 4 and 5



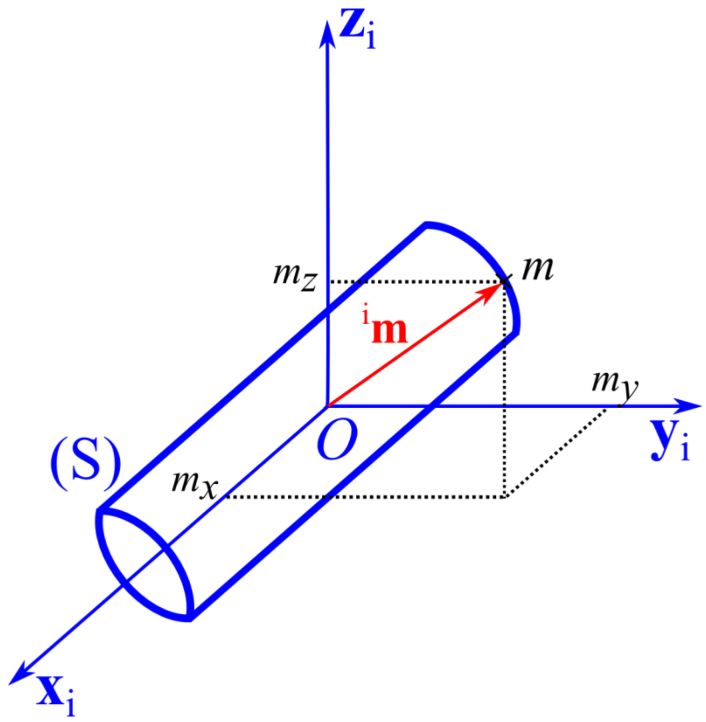
Response:

$$
{}^{j}T_i = \begin{pmatrix} 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Point belonging to a solid in pure rotation

- Let m be a point belonging to a solid (S). The coordinates of m, expressed in the frame $\mathcal{R}_{\mathrm{i}}(O, \mathbf{x}_{\mathrm{i}}, \mathbf{y}_{\mathrm{i}}, \mathbf{z}_{\mathrm{i}})$, are represented by a vector:

$$^{\mathrm{i}}\mathbf{m} = (m_x \quad m_y \quad m_z)^T$$



POINT BELONGING TO A SOLID IN PURE ROTATION

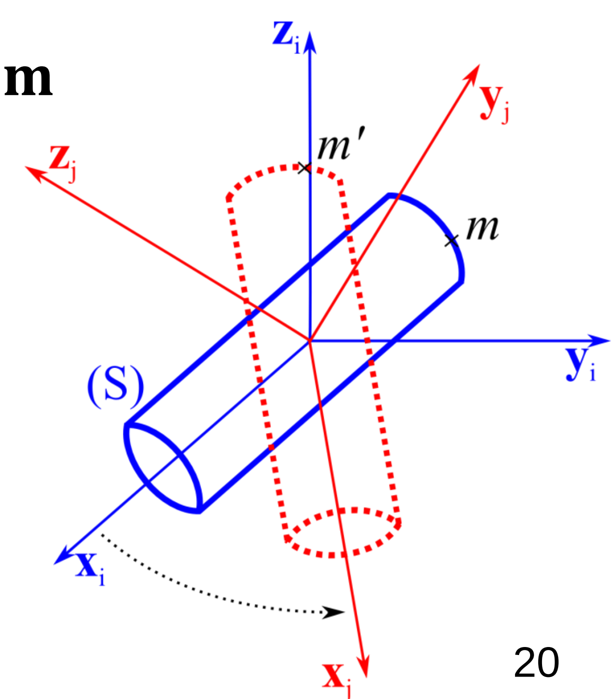
- After a pure rotation of the solid around the origin O, the coordinates of m', expressed in the frame $\mathcal{R}_\mathrm{j}(O, \mathbf{x}_\mathrm{j}, \mathbf{y}_\mathrm{j}, z_\mathrm{j})$, are:

$$^\mathrm{j}\mathbf{m}' = (m_x \quad m_y \quad m_z)^T = \ ^\mathrm{i}\mathbf{m}$$

- The coordinates of m', expressed un the frame $\mathcal{R}_\mathrm{i}(O, \mathbf{x}_\mathrm{i}, \mathbf{y}_\mathrm{i}, \mathbf{z}_\mathrm{i})$, are:

$$^\mathrm{i}\mathbf{m}' = \ ^\mathrm{i}\mathbf{R}_\mathrm{j} \ ^\mathrm{j}\mathbf{m}' = \ ^\mathrm{i}\mathbf{R}_\mathrm{j} \ ^\mathrm{i}\mathbf{m}$$

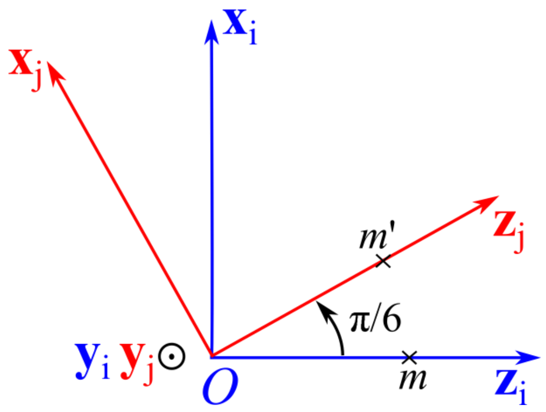
with $^\mathrm{i}\mathbf{R}_\mathrm{j}$ being the rotation matrix $\mathcal{R}_\mathrm{i} \longrightarrow \mathcal{R}_\mathrm{j}$



Point belonging to a solid in pure rotation

Example 1:

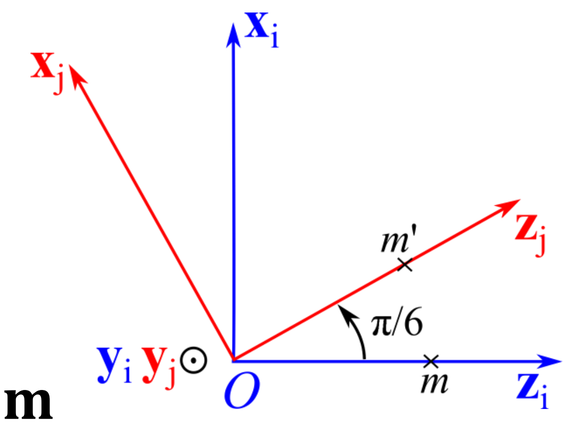
Let m be a point with the coordinates $(0 \quad 1 \quad \sqrt{3})^T$, expressed in the frame $\mathcal{R}_\mathrm{i}(O, \mathbf{x}_\mathrm{i}, \mathbf{y}_\mathrm{i}, \mathbf{z}_\mathrm{i})$. Compute the coordinates of this transformed point (now point m') after a rotation of the frame $\mathcal{R}_\mathrm{i}$ by $\pi/6$ around the axis $\mathbf{y}_\mathrm{i}$.



Response:

POINT BELONGING TO A SOLID IN PURE ROTATION

- Solution :



$$^{\mathrm{i}}\mathbf{m} = \begin{pmatrix} 0 \\ 1 \\ \sqrt{3} \end{pmatrix} = \ ^{\mathrm{j}}\mathbf{m}'$$

$$^{\mathrm{i}}\mathbf{m}' = \ ^{\mathrm{i}}\mathbf{R}_\mathrm{j}\ ^{\mathrm{j}}\mathbf{m}' = \ ^{\mathrm{i}}\mathbf{R}_\mathrm{j}\ ^{\mathrm{i}}\mathbf{m}$$

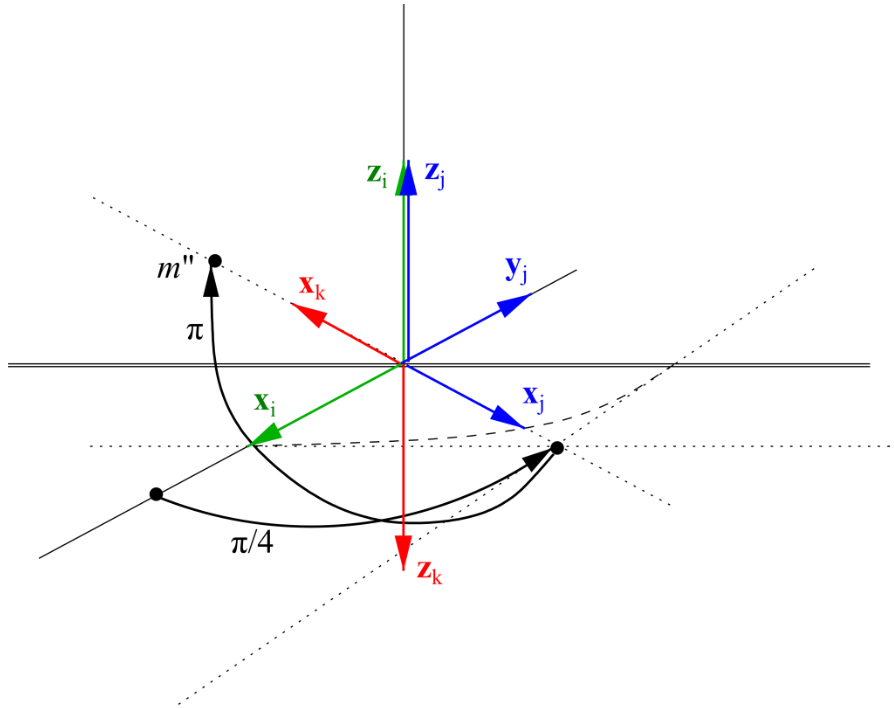
$$^{\mathrm{i}}\mathbf{R}_\mathrm{j} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$^{\mathrm{i}}\mathbf{m}' = \ ^{\mathrm{i}}\mathbf{R}_\mathrm{j}\ ^{\mathrm{j}}\mathbf{m}' = \ ^{\mathrm{i}}\mathbf{R}_\mathrm{j}\ ^{\mathrm{i}}\mathbf{m} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 1 \\ \frac{3}{2} \end{pmatrix}$$

POINT BELONGING TO A SOLID IN PURE ROTATION

Exemple 2:

Let $m"$ be a point of coordinates $(\sqrt{2} \quad 0 \quad 0)^T$, expressed in the frame $\mathcal{R}_\mathrm{k}(O, \mathbf{x}_\mathrm{k}, \mathbf{y}_\mathrm{k}, \mathbf{z}_\mathrm{k})$. Determine the coordinates of the same point in the frame $\mathcal{R}_\mathrm{i}$

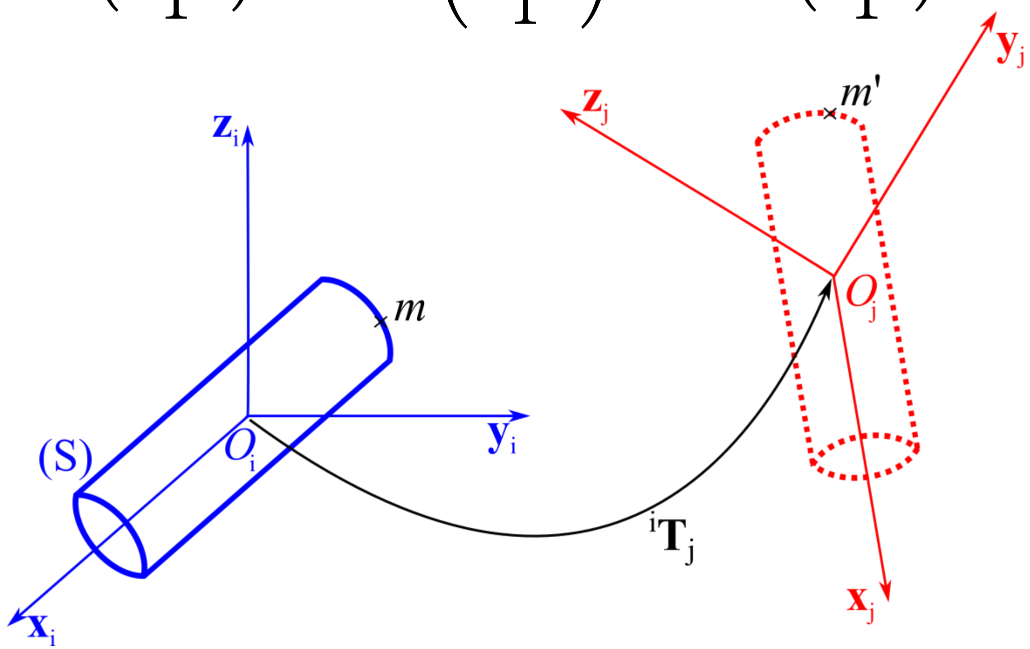


Réponse :

Point belonging to a solid in a free movement

- ${}^{i}\mathbf{m} = (m_x \quad m_y \quad m_z)^T$, coordinate of m expressed in $\mathcal{R}_{\mathrm{i}}$
- ${}^{\mathrm{j}}\mathbf{m}' = (m_x \quad m_y \quad m_z)^T$, coordinates of m' expressed in $\mathcal{R}_{\mathrm{j}}$
- The coordinates of m', expressed in the frame $\mathcal{R}_{\mathrm{i}}$ are:

$$\begin{pmatrix} {}^{\mathrm{i}}\mathbf{m}' \\ 1 \end{pmatrix} = {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}} \begin{pmatrix} {}^{\mathrm{j}}\mathbf{m}' \\ 1 \end{pmatrix} = {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}} \begin{pmatrix} {}^{\mathrm{i}}\mathbf{m} \\ 1 \end{pmatrix}$$

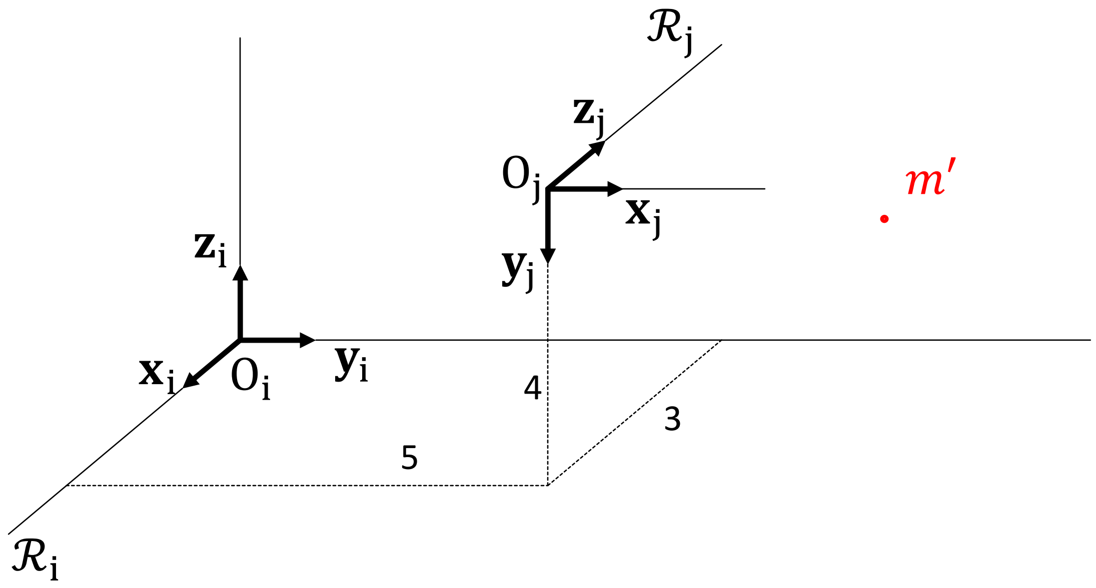


POINT BELONGING TO A SOLID IN A FREE MOVEMENT

Example 3:

Coordinates of m' in $\mathcal{R}_\mathrm{j}$: ${}^{\mathrm{j}}\mathbf{m}' = (\sqrt{3} \quad 4 \quad 5)^T$

Compute the coordinates of m' in $\mathcal{R}_\mathrm{i}$



Response:

Point belonging to a solid in a free movement

- Solution:

$$ {}^{\mathrm{j}}\mathbf{m}' = \begin{pmatrix} \sqrt{3} \\ 4 \\ 5 \end{pmatrix} $$

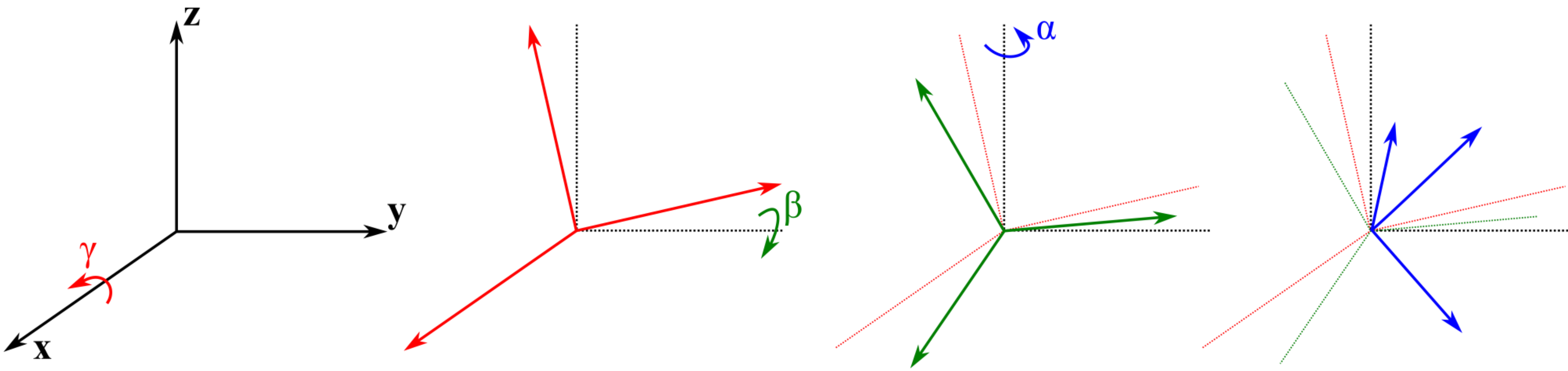
$$ {}^{\mathrm{i}}\mathbf{m}' = {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}}\ {}^{\mathrm{j}}\mathbf{m}' $$

$$ {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}} = \begin{pmatrix} 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{\mathrm{i}}\mathbf{T}_{\mathrm{j}}\ {}^{\mathrm{i}}\mathbf{m}' = \begin{pmatrix} 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ \sqrt{3}+5 \\ 0 \\ 1 \end{pmatrix} $$

Orientations representation

Knowing the Roll (γ) Pitch (β) Yaw (α) angles of a solid , the associated rotation matrix can be expressed as:



$$\mathbf{R} = \mathbf{rot}(\mathbf{z}, \alpha)\ \mathbf{rot}(\mathbf{y}, \beta)\ \mathbf{rot}(\mathbf{x}, \gamma)$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

Orientations representation

Inversely, to determine the Roll Pitch Yaw angles from a given rotation matrix $\mathbf{R} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$, one can proceed as follow:

- if $x_z \neq \pm 1$

$$\alpha = \mathrm{atan2}(x_y, x_x)$$
$$\beta = \mathrm{atan2}\left(-x_z, \sqrt{{x_x}^2 + {x_y}^2}\right)$$
$$\gamma = atan2(y_z, z_z)$$

- if $x_z = \pm 1$

$$\alpha - sign(\beta)\gamma = atan2(z_y, z_x)$$

α et γ are indeterminate

Orientations representation

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

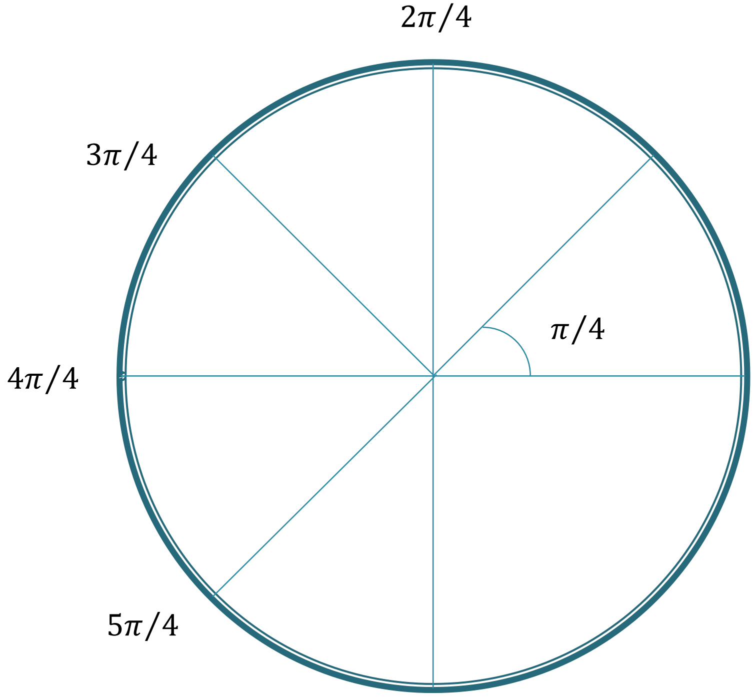
$$\Longrightarrow \tan\frac{\pi}{4} = 1$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\Longrightarrow \tan\frac{5\pi}{4} = 1$$

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$



$$\tan^{-1}(1) = \text{atan}(1) = ???$$

ORIENTATIONS REPRESENTATION

$$\mathrm{atan2}(\sin\theta, \cos\theta)$$

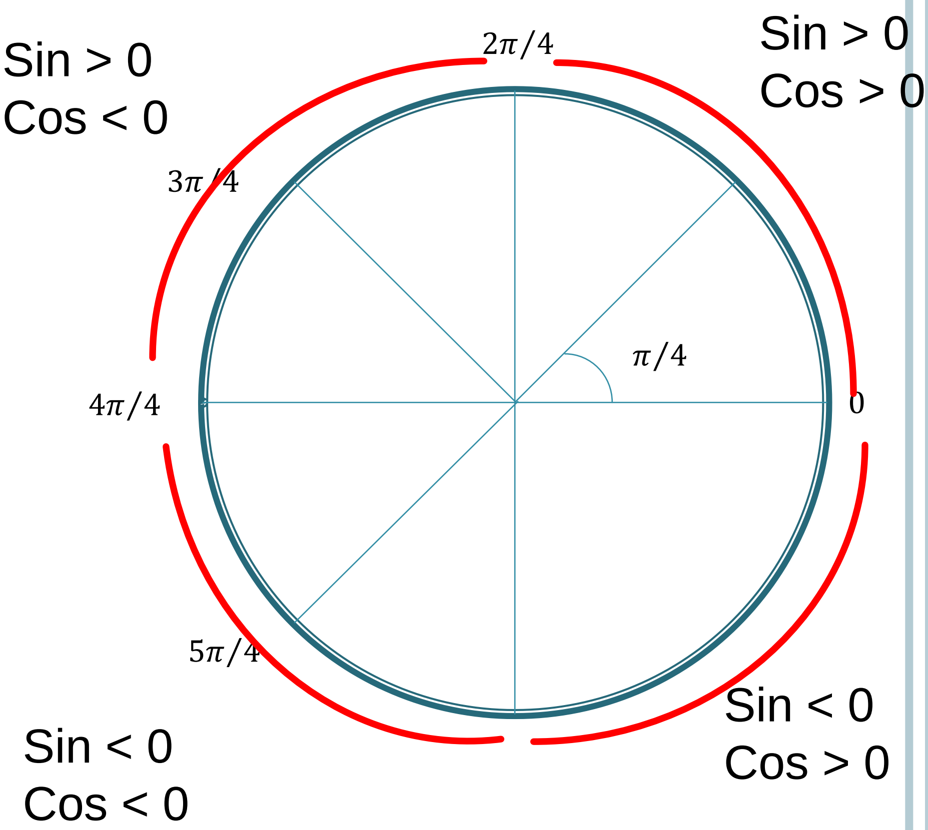
Example :

$$\mathrm{atan2}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\mathrm{atan2}\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{7\pi}{4}$$

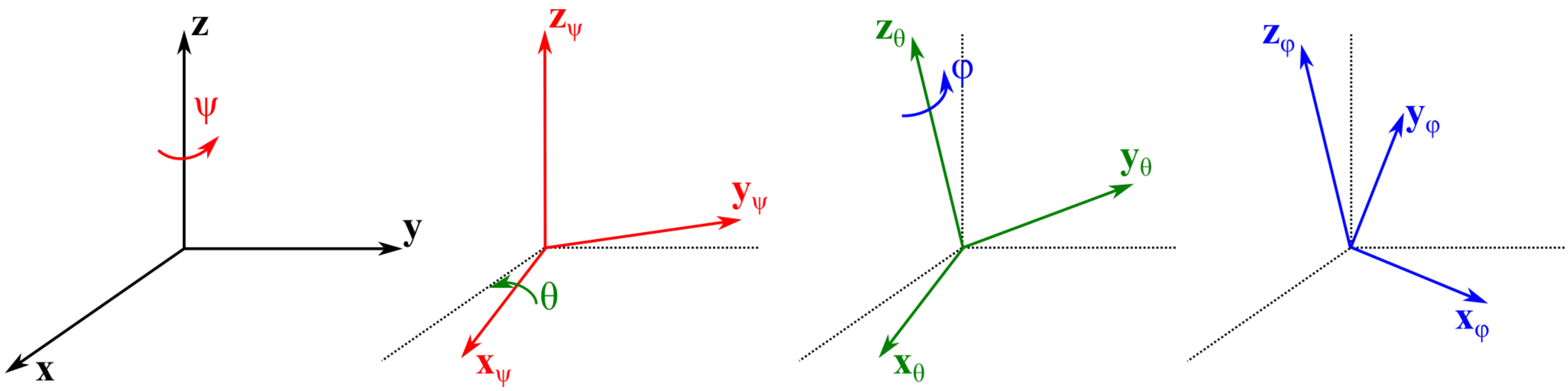
$$\mathrm{atan2}\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{5\pi}{4}$$

$$\mathrm{atan2}\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$



ORIENTATIONS REPRESENTATION

Knowing the Euler angles (ψ, θ, φ) angles of a solid , the associated rotation matrix can be expressed as:



$$\mathbf{R} = \mathbf{rot}(\mathbf{z}, \psi)\, \mathbf{rot}(\mathbf{x}_\psi, \theta)\, \mathbf{rot}(\mathbf{z}_\theta, \varphi)$$

$$= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi & \sin\psi\sin\theta \\ \sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & -\cos\psi\sin\theta \\ \sin\theta\sin\varphi & \sin\theta\cos\varphi & \cos\theta \end{pmatrix}$$

ORIENTATIONS REPRESENTATION

Inversely, to determine the Euler angles from a given rotation matrix

$\mathbf{R} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$, we can proceed as follow:

- if $z_z \neq \pm 1$

$$\psi = \mathrm{atan2}(z_x, -z_y)$$
$$\theta = \mathrm{acos}(z_z)$$
$$\varphi = \mathrm{atan2}(x_z, y_z)$$

- If $z_z = \pm 1$

$$\theta = \pi(1 - z_z)/2$$
$$\psi + z_z\varphi = atan2(y_x, x_x)$$

ψ et φ are indeterminate

ORIENTATIONS REPRESENTATION

Example :

Let the rotation matrix be $\mathbf{R} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$

Compute either the Euler angles and/or the Roll-Pitch-Yaw angles associated to this rotation matrix

$$\alpha = \text{atan2}\left(0, \frac{\sqrt{3}}{2}\right) \text{ impossible}$$

$$\beta = \text{atan2}\left(-1/2, \frac{\sqrt{3}}{2}\right) \text{ impossible}$$

$$\gamma = atan2\left(\frac{\sqrt{3}}{2}, 0\right) \text{ impossible}$$

Orientations representation

Example :

Let the rotation matrix be $\mathbf{R} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & -1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$

Compute either the Euler angles and/or the Roll-Pitch-Yaw angles associated to this rotation matrix

$$\psi = \mathrm{atan2}(0, 1) = 0$$
$$\theta = \mathrm{acos}(0) = \pm\frac{\pi}{2}$$
$$\varphi = \mathrm{atan2}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Orientations representation

Example :

Let the rotation matrix be $\mathbf{R} = \begin{pmatrix} 0.7071 & -0.3536 & -0.6124 \\ 0.7071 & 0.3536 & 0.6124 \\ 0 & -0.8660 & 0.5 \end{pmatrix}$

Compute either the Euler angles and/or the Roll-Pitch-Yaw angles associated to this rotation matrix

$$\alpha = \text{atan2}(0.7071, 0.7071) = \frac{\pi}{4}$$
$$\beta = \text{atan2}(0, 1) = 0$$
$$\gamma = atan2\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = -\frac{\pi}{3}$$

ORIENTATIONS REPRESENTATION

Example :

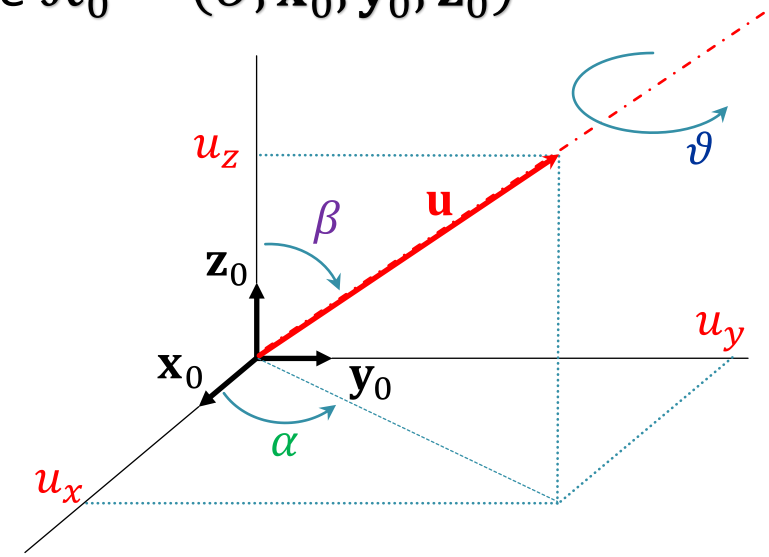
Let the rotation matrix be $\mathbf{R} = \begin{pmatrix} 0.3536 & -0.3536 & 0.8660 \\ 0.6124 & -0.6124 & -0.5 \\ 0.7071 & 0.7071 & 0 \end{pmatrix}$

Compute either the Euler angles and/or the Roll-Pitch-Yaw angles associated to this rotation matrix

$$\psi = \text{atan2}(0.8660, 0.5) = \frac{\pi}{3}$$
$$\theta = \text{acos}(0) = \frac{\pi}{2}$$
$$\varphi = \text{atan2}(0.7071, 0.7071) = \frac{\pi}{4}$$

Orientation representation: angle and axis

- Let $\mathbf{u} = [u_x \quad u_y \quad u_z]^T$ be a unit vector of a rotation axis with respect to the reference frame $\mathcal{R}_0 = (O, \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$



$$\mathbf{rot}(\mathbf{u}, \vartheta) = \mathbf{rot}(z, \alpha)\mathbf{rot}(y, \beta)\mathbf{rot}(z, \vartheta)\mathbf{rot}(y, -\beta)\mathbf{rot}(z, -\alpha)$$

$$\sin\alpha = \frac{u_y}{\sqrt{{u_x}^2 + {u_y}^2}}; \qquad \cos\alpha = \frac{u_x}{\sqrt{{u_x}^2 + {u_y}^2}};$$

$$\sin\beta = \sqrt{{u_x}^2 + {u_y}^2}; \qquad \cos\beta = u_z$$

Orientation representation: angle and axis

$$\mathbf{rot}(\mathbf{u},\vartheta)=\begin{pmatrix} {u_x}^2(1-C_\vartheta)+C_\vartheta & u_xu_y(1-C_\vartheta)-u_zS_\vartheta & u_xu_z(1-C_\vartheta)+u_yS_\vartheta \\ u_xu_y(1-C_\vartheta)+u_zS_\vartheta & {u_y}^2(1-C_\vartheta)+C_\vartheta & u_yr_z(1-C_\vartheta)-u_xS_\vartheta \\ u_xu_z(1-C_\vartheta)-u_yS_\vartheta & u_yu_z(1-C_\vartheta)+u_xS_\vartheta & {u_z}^2(1-C_\vartheta)+C_\vartheta \end{pmatrix}$$

The vector $\mathbf{u}$ is constrained by the following relationship:

$${u_x}^2+{u_y}^2+{u_z}^2=1$$

Orientation representation: angle and axis

To solve the inverse problem and determine the angle ϑ and the unit vector $\mathbf{u}$ from a given rotation matrix $\mathbf{R} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$, we can proceed as follow:

$$C\vartheta = \frac{1}{2}\left(x_x + y_y + z_z - 1\right)$$

$$S\vartheta = \frac{1}{2}\sqrt{\left(y_z - z_y\right)^2 + (z_x - x_z)^2 + \left(x_y - y_x\right)^2}$$

- The vector $\mathbf{u}$ and the angle ϑ are computed as:

$$\vartheta = atan2(S\vartheta, C\vartheta)$$

$$\mathbf{u} = \frac{1}{2\sin\vartheta}\begin{pmatrix} y_z - z_y \\ z_x - x_z \\ x_y - y_x \end{pmatrix}$$

Orientation representation: angle and axis

Important:

If $\sin\vartheta = 0$, the expression of $\mathbf{u}$ becomes meaningless. To solve this problem, it is necessary to refer to particular expressions for $\vartheta = 0$ and $\vartheta = \pi$ (not given in this course).

Orientation representation: unit Quaternion

- The drawbacks of the axis/angle representation can be overcome by a different four-parameter representation; namely the *unit quaternion. In other word, Euler parameters,* defined as $\mathfrak{Q} = \{\eta, \boldsymbol{\epsilon}\}$ where :

$$\eta = \cos\frac{\vartheta}{2}$$

$$\boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\mathbf{u}$$

η **is** called the scalar part of the quaternion and $\boldsymbol{\epsilon} = [\epsilon_{\boldsymbol{x}} \quad \epsilon_{\boldsymbol{y}} \quad \epsilon_{\boldsymbol{z}}]^{\boldsymbol{T}}$ is called the vector part of the quaternion. They are constrained by the condition:

$$\eta^2 + \epsilon_x{}^2 + \epsilon_y{}^2 + \epsilon_z{}^2 = 1$$

Hence the name of unit quaternion

Orientation representation: unit Quaternion

- The associated rotation matrix is given by:

$$\mathbf{rot}(\eta, \boldsymbol{\epsilon}) = \begin{pmatrix} 2(\eta^2 + {\epsilon_x}^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + {\epsilon_y}^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + {\epsilon_z}^2) - 1 \end{pmatrix}$$

Orientation representation: unit Quaternion

- To solve the inverse problem and compute the quaternion from a given rotation matrix $\mathbf{R} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$, one can proceed as follow:

$$\eta = \frac{1}{2}\sqrt{x_x + y_y + z_z + 1}$$

$$\boldsymbol{\epsilon} = \frac{1}{2}\begin{pmatrix} \mathrm{sign}(y_z - z_y)\sqrt{x_x - y_y - z_z + 1} \\ \mathrm{sign}(z_x - x_z)\sqrt{y_y - z_z - x_x + 1} \\ \mathrm{sign}(x_y - y_x)\sqrt{z_z - x_x - y_y + 1} \end{pmatrix}$$

End of chapter 1