Chapter 2: kinematic and differential kinematic models

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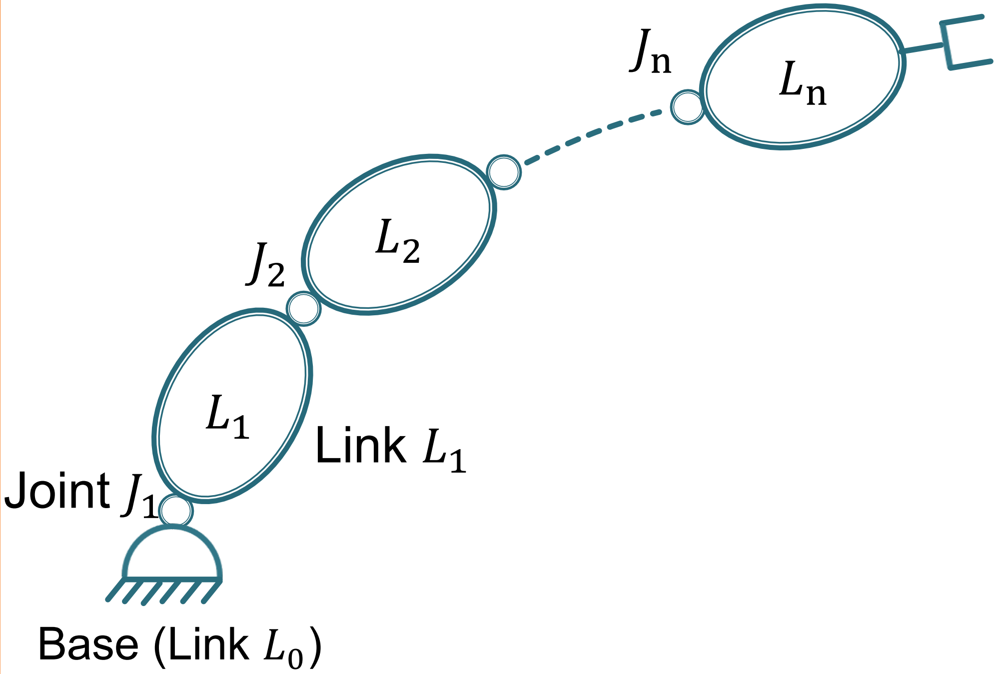
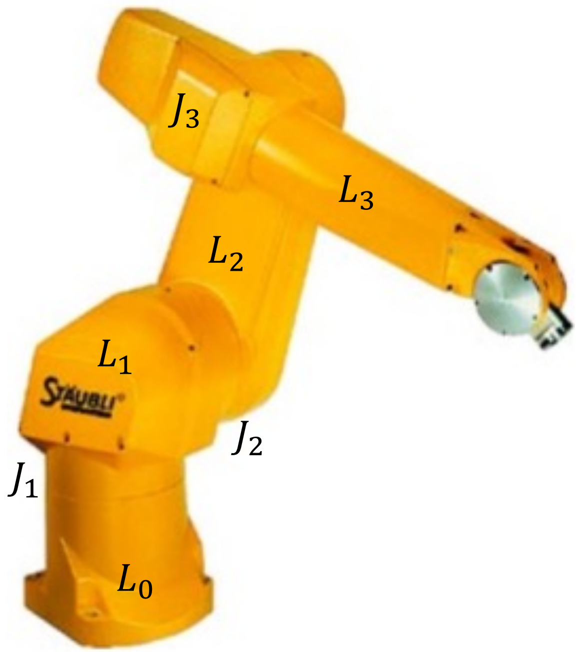
Notations and definition

- Pose = position + orientation
- n: number of joints
- m: number of DOF
- **DKM**: Direct Kinematic Model
- **IKM**: Inverse Kinematic Model
- **DDKM** : Direct Differential Kinematic Model
- **IDKM**: Inverse Differential Kinematic Model

DESCRIPTION OF ROBOTIC ARM

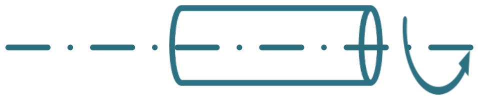




Kinematic chain of a simple open structure

DESCRIPTION OF ROBOTIC ARM

- Revolute joint (R)

Joint axis 

- Prismatic joint (P)

Joint axis 

- For a joint j, we can define the kinematic parameter:

$$\sigma_{\mathrm{j}} = \begin{cases} 0 & \text{for a revolute joint} \\ 1 & \text{for a prismatic joint} \end{cases}$$

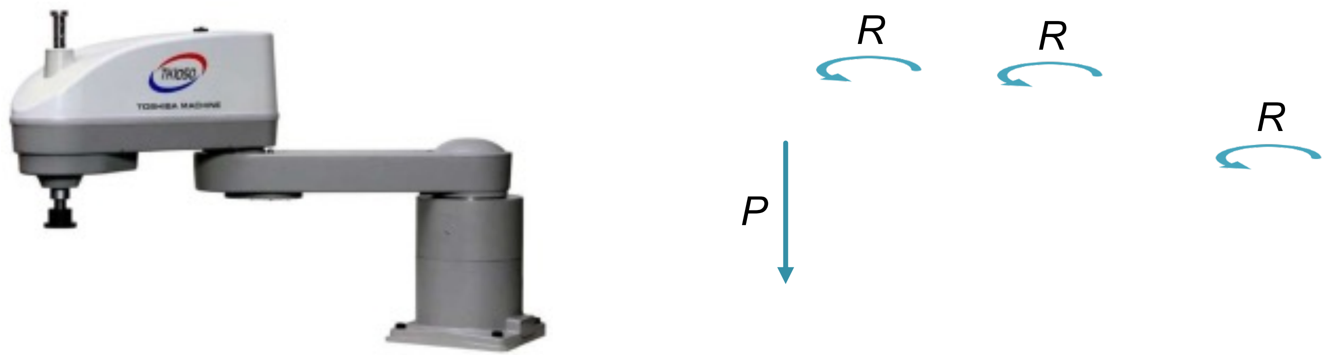
DESCRIPTION OF ROBOTIC ARM

- Anthropomorphic arm 6*R*

6*R*: 6 revolute joints

Stäubli robot RX-90

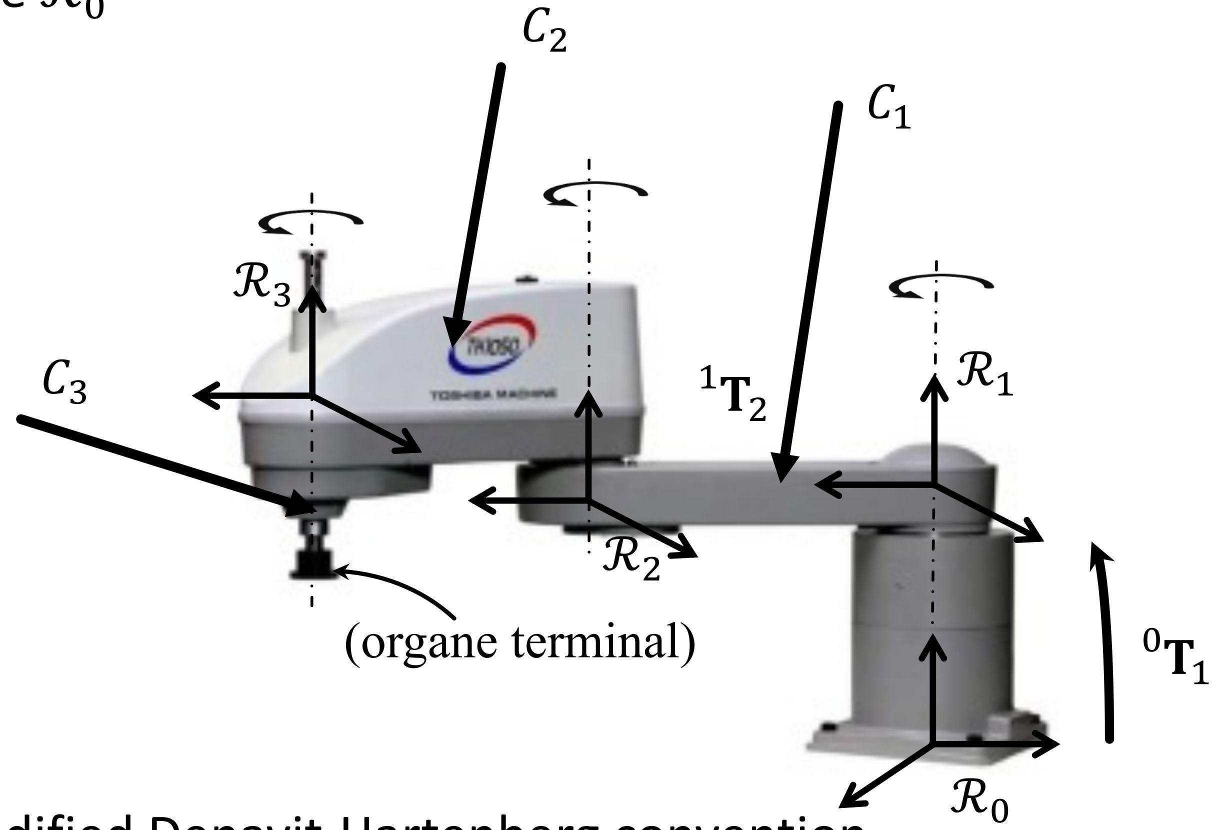
- SCARA manipulator *RRRP*



SCARA robot (Toshiba)

KINEMATIC MODELLING OF A ROBOTIC MANIPULATOR

Objective: establish a relationship between the frames $\mathcal{R}_i$ in order to determine the pose (position and orientation) of the end effector with respect to the reference frame $\mathcal{R}_0$

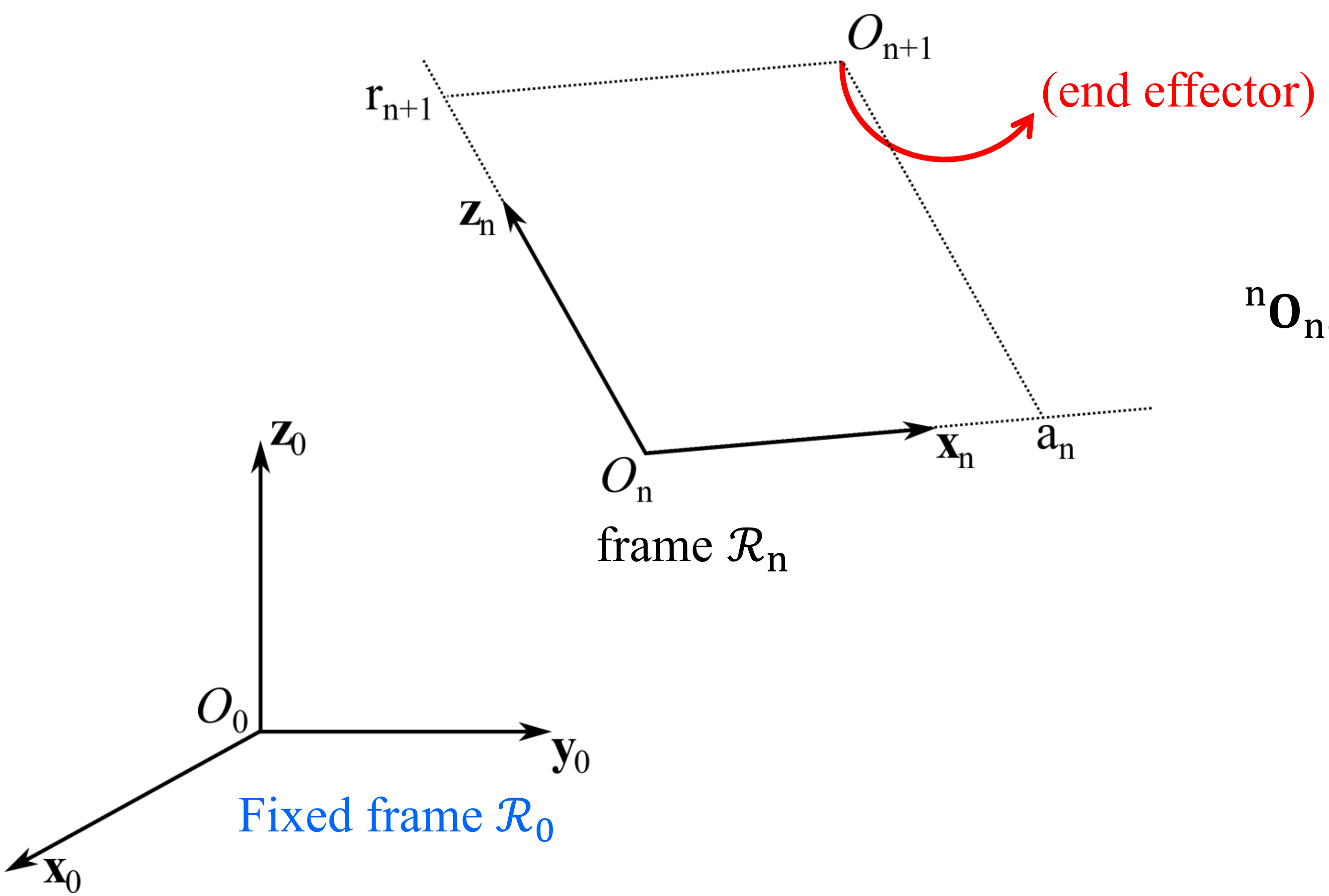


Approach: Modified Denavit-Hartenberg convention

Modified Denavit-Hartenberg convention

Placement of the frames $\mathcal{R}_0$ *et* $\mathcal{R}_n$:

- $\mathcal{R}_0$ is freely chosen
- O_{n+1} is placed at the end effector
- $\mathcal{R}_n$ is chosen so that $O_{n+1} \in (O_n, \mathbf{x}_n, \mathbf{z}_n)$



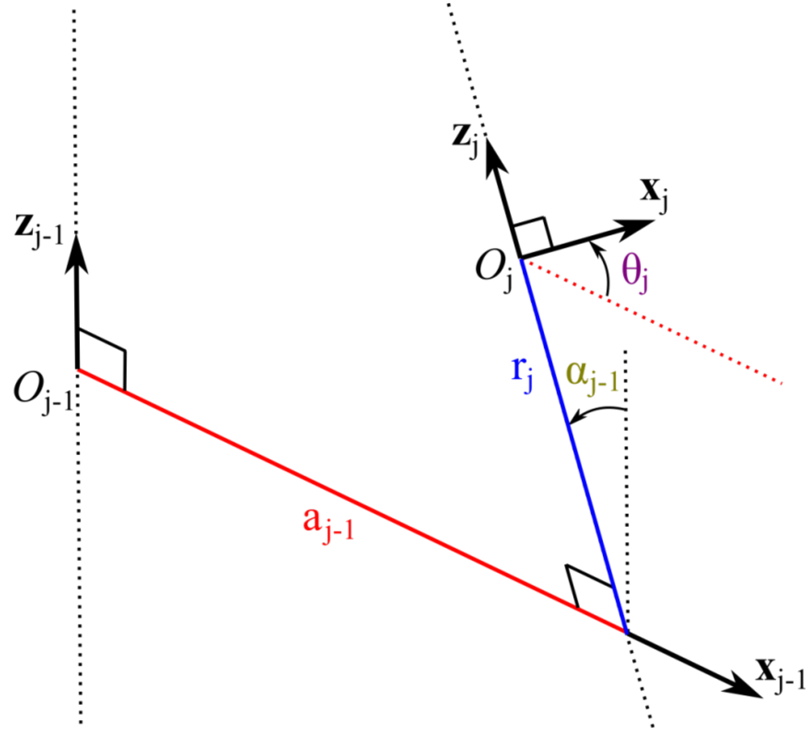
$$ {}^{n}\mathbf{O}_{n+1} = \begin{pmatrix} a_n \\ 0 \\ r_{n+1} \end{pmatrix} $$

Modified Denavit-Hartenberg convention

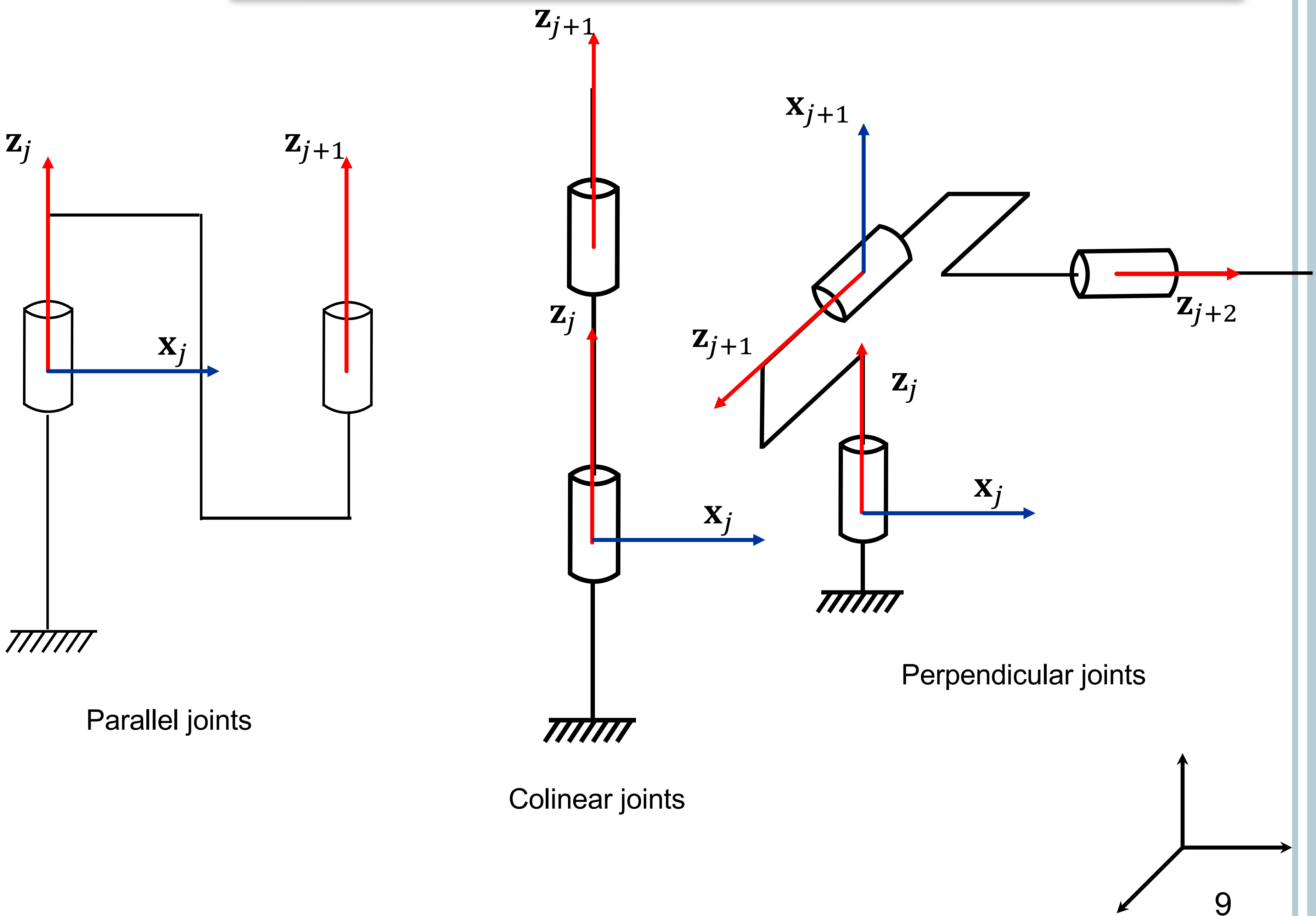
Placement of the frames $\mathcal{R}_1, \dots, \mathcal{R}_{n-1}$

The frame $\mathcal{R}_j$, fixed to the link L_j, is defined so that:

- $\mathbf{z}_j$ is chosen along the axis of the joint j
- Locate the origin O_j at the intersection of the common normal to the
- choose $\mathbf{x}_j$ along the common normal to axes $\mathbf{z}_j$et $\mathbf{z}_{j+1}$
- If the axes $\mathbf{z}_j$ and $\mathbf{z}_{j+1}$ are parallel or colinear ➔ the choice of $\mathbf{x}_j$ is not unique



MODIFIED DENAVIT-HARTENBERG CONVENTION



Modified Denavit-Hartenberg convention

DH parameters:

- α_{j-1} : angle between $\mathbf{z}_{j-1}$ and $\mathbf{z}_j$ (rotation round $\mathbf{x}_{j-1}$)
- a_{j-1} : distance between $\mathbf{z}_{j-1}$ and $\mathbf{z}_j$ along $\mathbf{x}_{j-1}$
- θ_j : angle between $\mathbf{x}_{j-1}$ and $\mathbf{x}_j$ (rotation around $\mathbf{z}_j$)
- r_j : distance between $\mathbf{x}_{j-1}$ and $\mathbf{x}_j$ along $\mathbf{z}_j$

Passage matrix $\mathcal{R}_{j-1} \longrightarrow \mathcal{R}_j$:

$$^{j-1}\mathbf{T}_j = \mathbf{Rot}(\mathbf{x}, \alpha_{j-1})\mathbf{Trans}(\mathbf{x}, a_{j-1})\mathbf{Rot}(\mathbf{z}, \theta_j)\mathbf{Trans}(\mathbf{z}, r_j)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{j-1} & -S\alpha_{j-1} & 0 \\ 0 & S\alpha_{j-1} & C\alpha_{j-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_{j-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C\theta_j & -S\theta_j & 0 & 0 \\ S\theta_j & C\theta_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r_j \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Modified Denavit-Hartenberg convention

$$^{j-1}\mathbf{T}_j = \begin{pmatrix} C\theta_j & -S\theta_j & 0 & a_{j-1} \\ C\alpha_{j-1}S\theta_j & C\alpha_{j-1}C\theta_j & -S\alpha_{j-1} & -r_j S\alpha_{j-1} \\ S\alpha_{j-1}S\theta_j & S\alpha_{j-1}C\theta_j & C\alpha_{j-1} & r_j C\alpha_{j-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The joint variable q_j of the joint j is defined by:

$$q_j = \sigma_j r_j + \bar{\sigma}_j \theta_j$$

  - Revolute joint j: σ_j=0 ➔ $\bar{\sigma}_j$=1 ➔ $q_j = \theta_j$
  - Prismatic joint j: σ_j=1 ➔ $\bar{\sigma}_j$=0 ➔ $q_j = r_j$

with:

$$\sigma_j = \begin{cases} 0 & \text{for a revolute joint} \\ 1 & \text{for a prismatic joint} \end{cases}$$

- $^{j-1}\mathbf{T}_j$ depends on the joint variable q_j:

$$^{j-1}\mathbf{T}_j = {}^{j-1}\mathbf{T}_j\left(q_j\right)$$

DIRECT KINEMATIC MODEL

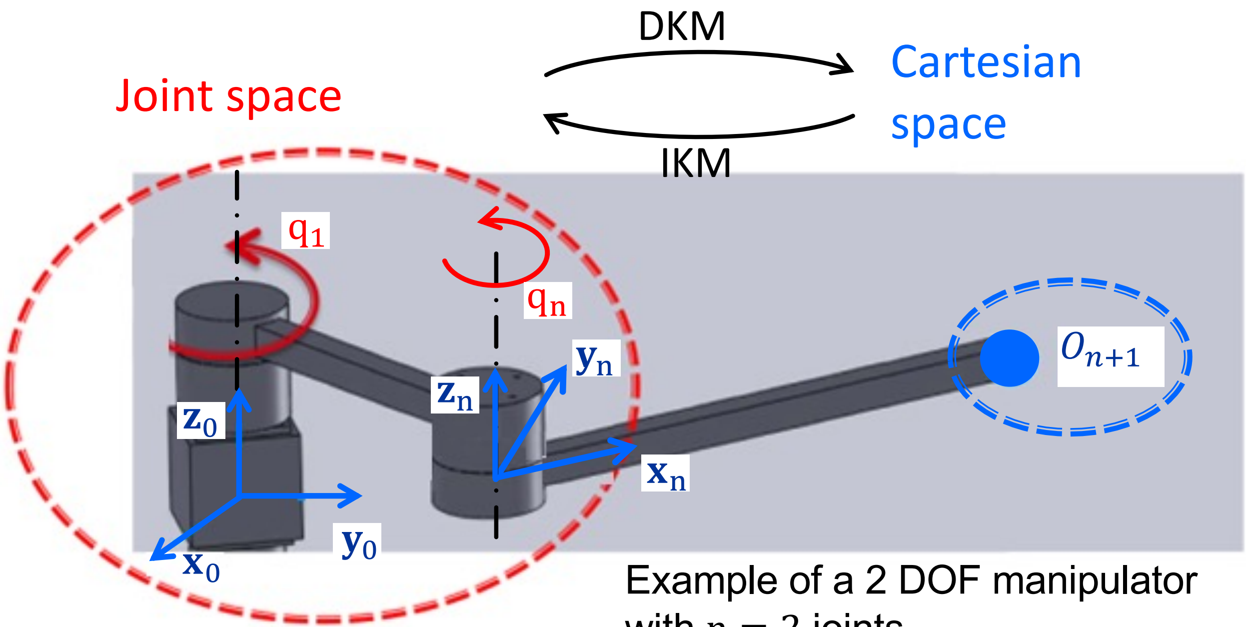
DKM: consists in computing, from the joint variables, the pose (position + orientation) of the end effector

Joint space:

Joints position Vector : $\mathbf{q} = [\mathrm{q}_1 \quad \mathrm{q}_2 \quad \cdot \quad \mathrm{q}_\mathrm{n}]^T$

Cartesian space:

$$\text{Pose of the end effector} \equiv \begin{cases} \text{position of } O_{\mathrm{n}+1} \text{ in the frame } \mathcal{R}_0(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \\ \text{orientation of the frame } \mathcal{R}_\mathrm{n} \text{ with respect to } \mathcal{R}_0 \end{cases}$$



Example of a 2 DOF manipulator with $\mathrm{n} = 2$ joints

Direct kinematic model

Methodology of computation (1)

- Compute the homogeneous matrices, from the $\boldsymbol{\mathcal{R}_0}$ to $\boldsymbol{\mathcal{R}_n}$:

$$^{0}\mathbf{T}_{n}(\mathbf{q}) = {}^{0}\mathbf{T}_{1}(q_1) \cdot {}^{1}\mathbf{T}_{2}(q_2) \cdot \ldots \cdot {}^{n-1}\mathbf{T}_{n}(q_n)$$

- Compute the position of the end effector in the reference frame $\mathcal{R}_0$:

$$\begin{pmatrix} {}^{0}\mathbf{O}_{n+1} \\ 1 \end{pmatrix} = {}^{0}\mathbf{T}_{n} \begin{pmatrix} {}^{n}\mathbf{O}_{n+1} \\ 1 \end{pmatrix}$$

- Compute the matrix $^{0}\mathbf{R}_{n}$, that corresponds to the orientation of the end effector:

$$^{0}\mathbf{R}_{n} = {}^{0}\mathbf{T}_{n}(1:3, 1:3)$$

DIRECT KINEMATIC MODEL

Methodology of computation (2)

- We denote by $\mathbb{x}$ the pose of the end effector:

$$\mathbb{x} = (x \quad y \quad z \quad \gamma \quad \beta \quad \alpha)^T$$

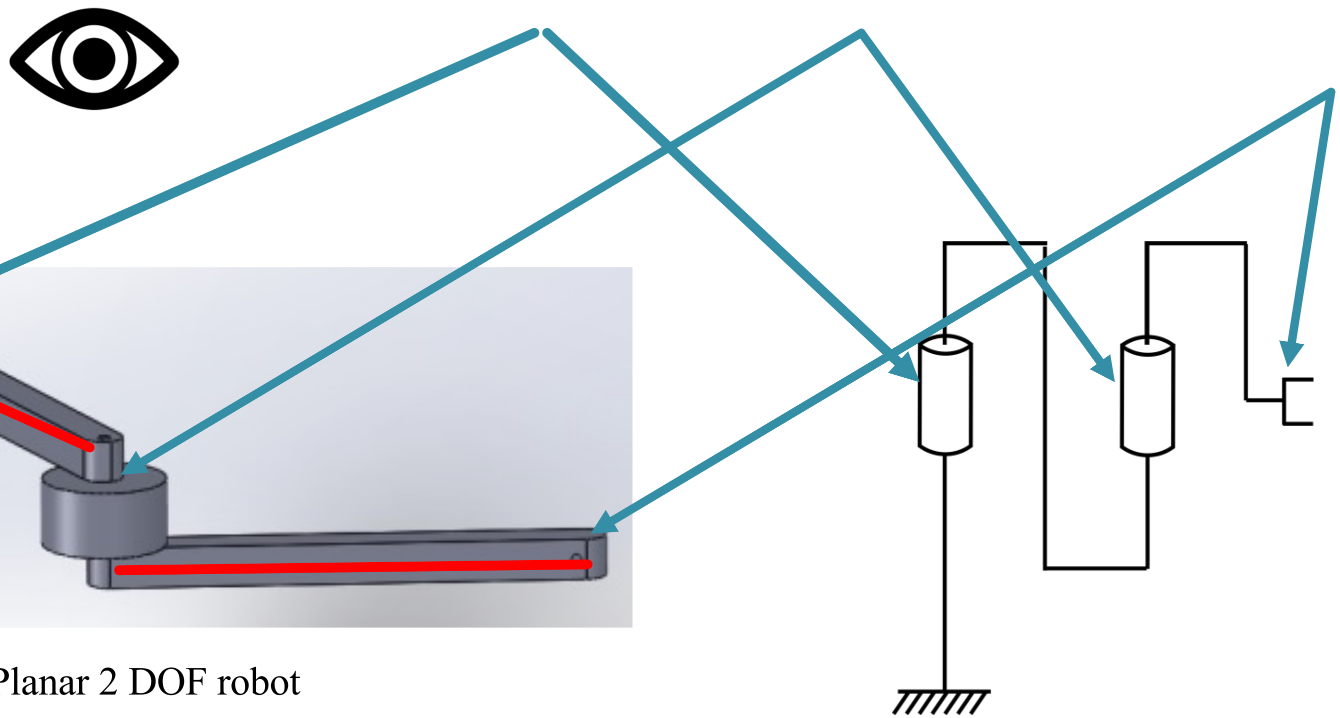
The first 3 components of $\mathbb{x}$ correspond to the position of the end effector:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^{0}\mathbf{O}_{\mathrm{n+1}}$$

γ, β et α are either Roll-Pich-Yaw angles, Euler angles, Euler parameters or the quaternion. They represent the orientation of the end effector. They are computed from the rotation matrix ${}^{0}\mathbf{R}_{\mathrm{n}}$ (see chapter 1)

DIRECT KINEMATIC MODEL

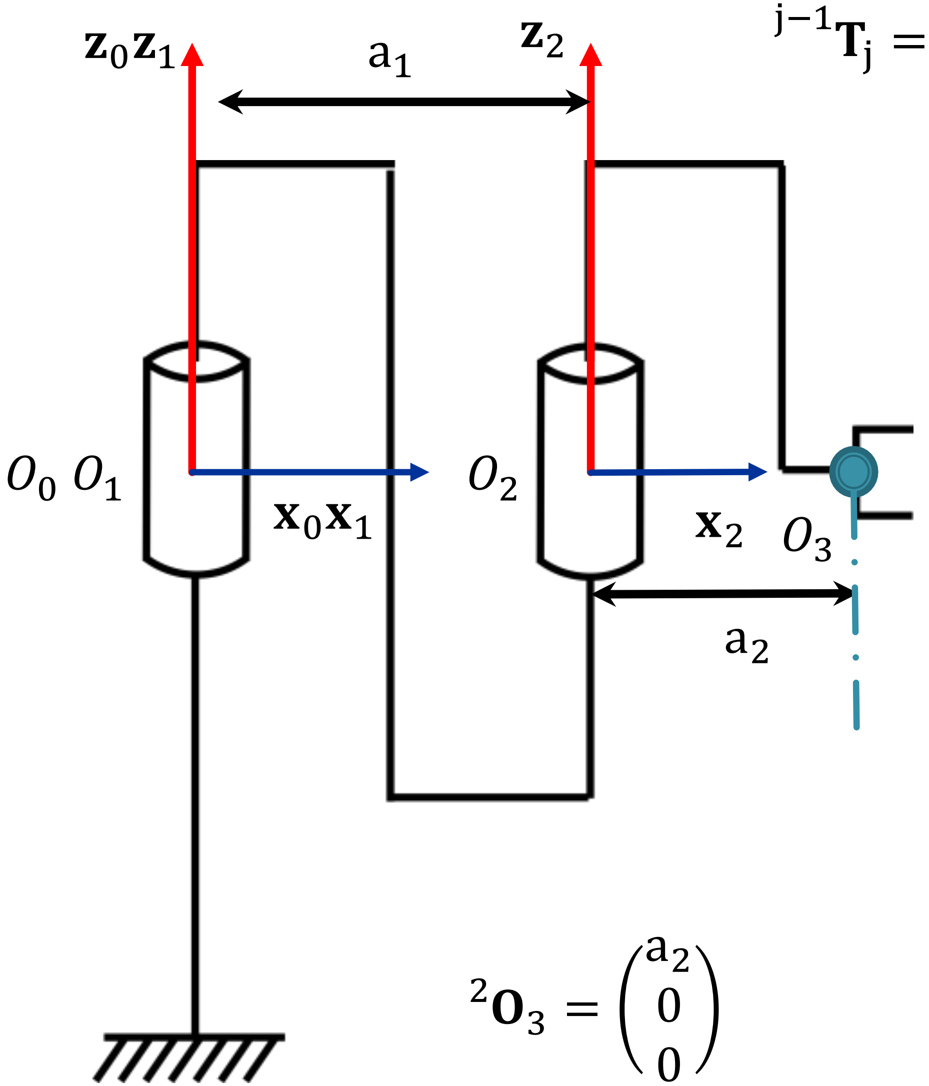
Example1 : compute the direct kinematic model



Planar 2 DOF robot

Equivalent kinematics

EXAMPLE 1



$$^{j-1}\mathbf{T}_j = \begin{pmatrix} C\theta_j & -S\theta_j & 0 & a_{j-1} \\ C\alpha_{j-1}S\theta_j & C\alpha_{j-1}C\theta_j & -S\alpha_{j-1} & -r_jS\alpha_{j-1} \\ S\alpha_{j-1}S\theta_j & S\alpha_{j-1}C\theta_j & C\alpha_{j-1} & r_jC\alpha_{j-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

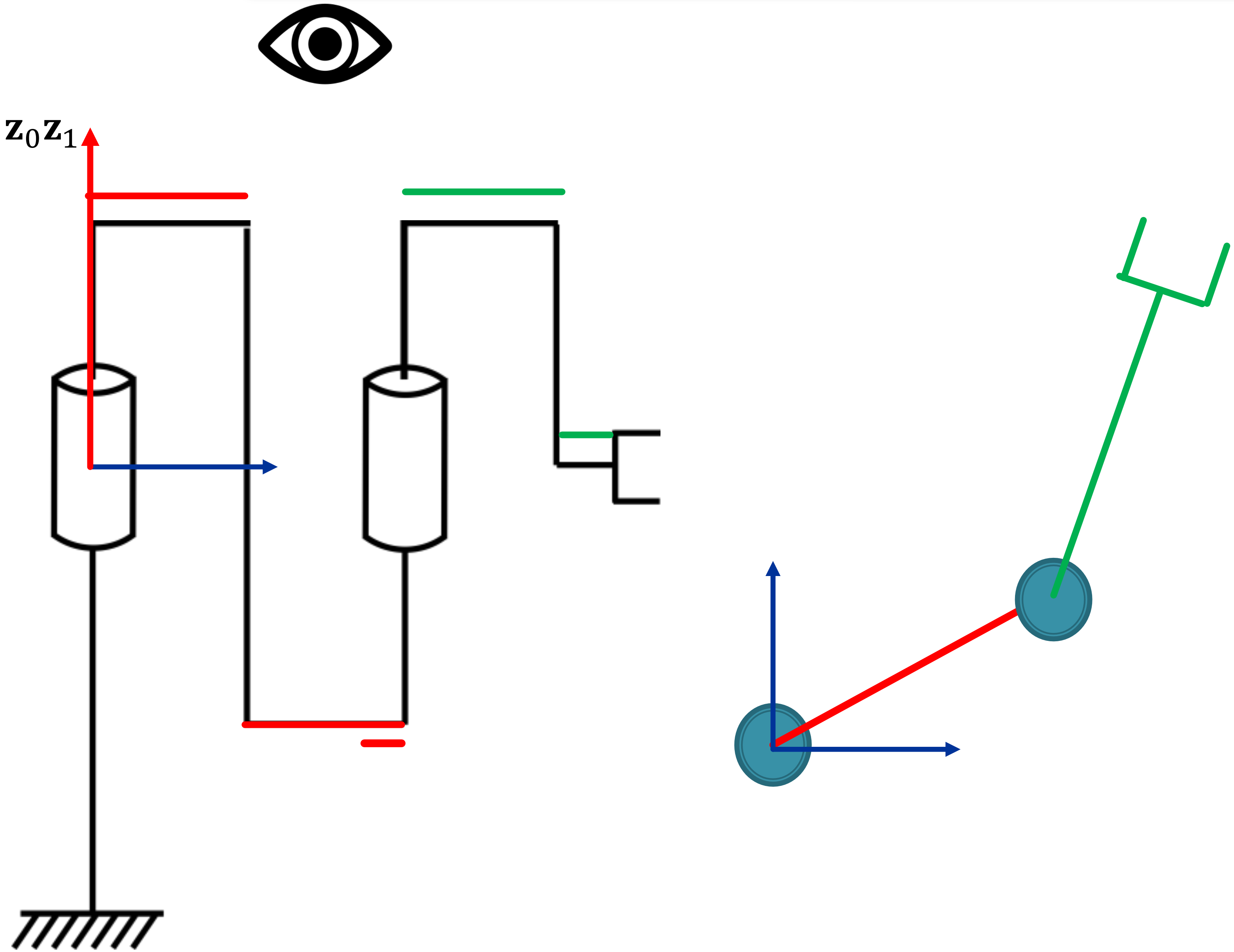
$$^{0}\mathbf{T}_1 = ?$$

$$^{1}\mathbf{T}_2 = ?$$

$$^{2}\mathbf{O}_3 = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}$$

j	σ_j	α_{j-1}	a_{j-1}	θ_j	r_j
1	0	0	0	θ_1	0
2	0	0	a_1	θ_2	0

EXAMPLE 1



Example 1 : direct kinematic model

$$
{}^{0}\mathbf{T}_1 = \begin{array}{cccc} & & {}^{0}\mathbf{z}_1 & {}^{0}\mathbf{p}_1 \end{array}
\begin{pmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{1}\mathbf{T}_2 = \begin{pmatrix} C\theta_2 & -S\theta_2 & 0 & a_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$q_1 = \theta_1$

$q_2 = \theta_2$

The vector of joint variables is:

$$
\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}
$$

j	σ_j	α_{j-1}	a_{j-1}	θ_j	r_j
1	0	0	0	θ_1	0
2	0	0	a_1	θ_2	0

EXAMPLE 1 : DIRECT KINEMATIC MODEL

- Question 6: determination of the DKM

$$ {}^{0}\mathbf{T}_2 = {}^{0}\mathbf{T}_1 \; {}^{1}\mathbf{T}_2 $$

$$ = \begin{pmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C\theta_2 & -S\theta_2 & 0 & a_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} C\theta_1 C\theta_2 - S\theta_1 S\theta_2 & -C\theta_1 S\theta_2 - S\theta_1 C\theta_2 & 0 & a_1 C\theta_1 \\ S\theta_1 C\theta_2 + C\theta_1 S\theta_2 & -S\theta_1 S\theta_2 + C\theta_1 C\theta_2 & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ \cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 $$
$$ \cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 $$
$$ \sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 $$
$$ \sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 $$

EXAMPLE 1 : DIRECT KINEMATIC MODEL

- Question 6: determination of the DKM:

$$\begin{array}{cc} & \begin{array}{cccc} & & {}^0\mathbf{z}_2 & {}^0\mathbf{p}_2 \end{array} \\ {}^0\mathbf{T}_2 = & \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1 C\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Computation of the end effector position in the reference frame $\mathcal{R}_0$:

$$\begin{pmatrix} {}^0\mathbf{O}_3 \\ 1 \end{pmatrix} = {}^0\mathbf{T}_2 \begin{pmatrix} {}^2\mathbf{O}_3 \\ 1 \end{pmatrix}$$

$${}^2\mathbf{O}_3 = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix}$$

EXAMPLE 1 : DIRECT KINEMATIC MODEL

$$\begin{pmatrix} {}^{0}\mathbf{O}_3 \\ 1 \end{pmatrix} = {}^{0}\mathbf{T}_2 \begin{pmatrix} {}^{2}\mathbf{O}_3 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1 C\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 0 \\ 1 \end{pmatrix}$$

The position of the end effector is:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^{0}\mathbf{O}_3 = \begin{pmatrix} a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 0 \end{pmatrix}$$

EXAMPLE 1 : DIRECT KINEMATIC MODEL

- Determination of the rotation matrix ${}^{0}\mathbf{R}_2$, which corresponds to the orientation of the end effector:

$$ {}^{0}\mathbf{R}_2 = {}^{0}\mathbf{T}_2(1:3,1:3) $$

$$ {}^{0}\mathbf{R}_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

- Roll-Pitch-Yaw angles : if $x_z = 0 \neq \pm 1$

$$ \alpha = \mathrm{atan2}(\sin(\theta_1+\theta_2), \cos(\theta_1+\theta_2)) = \theta_1 + \theta_2 $$
$$ \beta = \mathrm{atan2}(0,1) = 0 $$
$$ \gamma = atan2(0,1) = 0 $$

EXAMPLE 1 : DIRECT KINEMATIC MODEL

- $\mathbb{x}$ denote the pose of the end effector:

$$\mathbb{x} = \begin{pmatrix} x \\ y \\ z \\ \gamma \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathrm{a}_2 \cos(\theta_1 + \theta_2) + \mathrm{a}_1 \cos(\theta_1) \\ \mathrm{a}_2 \sin(\theta_1 + \theta_2) + \mathrm{a}_1 \sin(\theta_1) \\ 0 \\ 0 \\ 0 \\ \theta_1 + \theta_2 \end{pmatrix}$$

The number of joints:

$$n = 2$$

The number of operational degrees of freedom is:

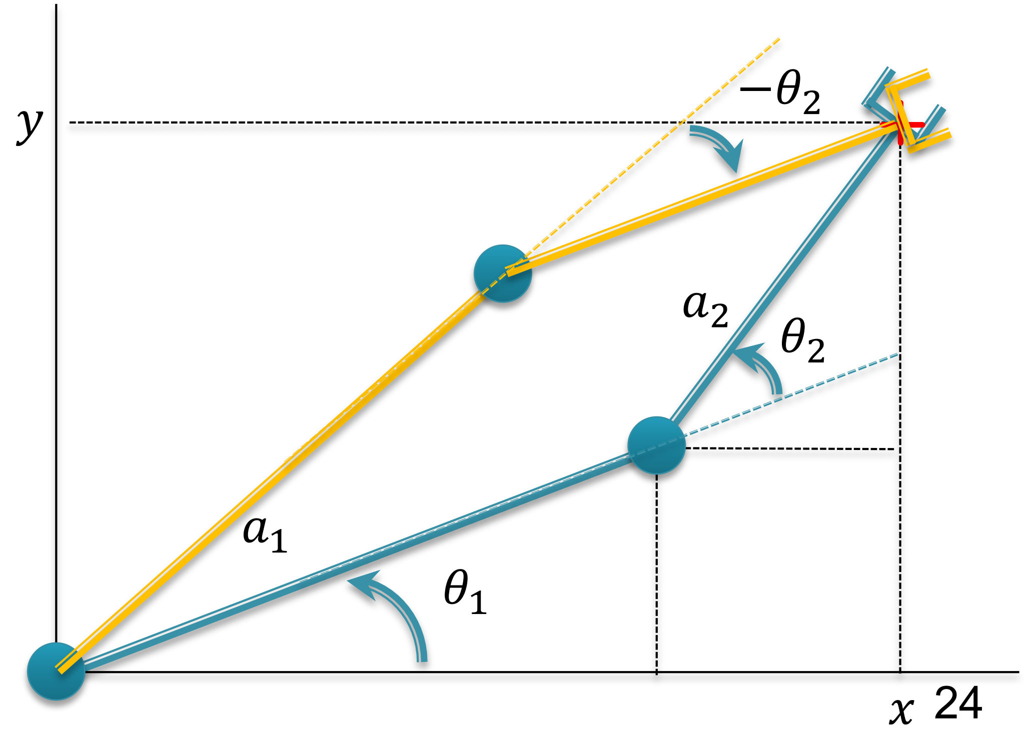
$$m = 2$$

EXAMPLE 1 : INVERSE KINEMATIC MODEL

$$\text{DKM :}\begin{cases}\begin{cases}x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2)\\ y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2)\end{cases}\\ \alpha = \theta_1+\theta_2\end{cases}$$

$$\theta_2 = \pm\text{acos}\left(\frac{x^2+y^2-{a_1}^2-{a_2}^2}{2a_1a_2}\right)$$

$$\theta_1 = atan2(S_1, C_1)$$



Example 1 : Inverse kinematic model

DKM: $\begin{cases} x = \mathrm{a}_2 \cos(\theta_1 + \theta_2) + \mathrm{a}_1 \cos(\theta_1) \\ y = \mathrm{a}_2 \sin(\theta_1 + \theta_2) + \mathrm{a}_1 \sin(\theta_1) \end{cases}$

IKM: the objective is to determinate the expressions of θ_1 and θ_2 ?

Here, we suppose to know x and y and we search to compute θ_1 and θ_2

$$(\cos(\theta_1))^2 + (\sin(\theta_1))^2 = 1$$
$$(A+B)^2 = A^2 + B^2 + 2 * A * B$$

$$x^2 = (a_1 C_1 + a_2 C_{12})^2$$
$$x^2 = (a_1 C_1)^2 + (a_2 C_{12})^2 + 2 a_1 a_2 C_1 C_{12}$$

$$C_1 C_{12} + S_1 S_{12} = cos\theta_1 \cos(\theta_1 + \theta_2) + \sin(\theta_1) \sin(\theta_1 + \theta_2)$$
$$= \cos\big(\theta_1 - (\theta_1 + \theta_2)\big) = \cos(-\theta_2) = \cos(\theta_2) = C_2$$
$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

EXAMPLE 1: DIRECT DIFFERENTIAL KINEMATIC MODEL

$$
{}^{0}\mathbf{J}_{2} = \begin{pmatrix} \sigma_1\, {}^{0}\mathbf{z}_1 + \bar{\sigma}_1\left({}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,2}\right) & \sigma_2\, {}^{0}\mathbf{z}_2 + \bar{\sigma}_2\left({}^{0}\mathbf{z}_2 \wedge {}^{0}\mathbf{p}_{2,2}\right) & \ldots . & \sigma_2\, {}^{0}\mathbf{z}_2 + \bar{\sigma}_2\left({}^{0}\mathbf{z}_2 \wedge {}^{0}\mathbf{p}_{2,2}\right) \\ \bar{\sigma}_1\, {}^{0}\mathbf{z}_1 & \bar{\sigma}_2\, {}^{0}\mathbf{z}_2 & \ldots & \bar{\sigma}_2\, {}^{0}\mathbf{z}_2 \end{pmatrix}
$$

$$
{}^{0}\mathbf{J}_{2} = \begin{pmatrix} {}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,2} & {}^{0}\mathbf{z}_2 \wedge {}^{0}\mathbf{p}_{2,2} \\ {}^{0}\mathbf{z}_1 & {}^{0}\mathbf{z}_2 \end{pmatrix}
$$

$$
{}^{0}\mathbf{p}_{\mathrm{j,n}} = {}^{0}\mathbf{p}_{\mathrm{n}} - {}^{0}\mathbf{p}_{\mathrm{j}}
$$

$$
{}^{0}\mathbf{p}_{1,2} = {}^{0}\mathbf{p}_{2} - {}^{0}\mathbf{p}_{1}
$$

$$
{}^{0}\mathbf{p}_{2,2} = {}^{0}\mathbf{p}_{2} - {}^{0}\mathbf{p}_{2} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

EXAMPLE 1: DIRECT DIFFERENTIAL KINEMATIC MODEL

- $${}^{0}\mathbf{J}_2 = \begin{pmatrix} {}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,2} & \mathbf{0} \\ {}^{0}\mathbf{z}_1 & {}^{0}\mathbf{z}_2 \end{pmatrix}$$

- $${}^{0}\mathbf{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad {}^{0}\mathbf{z}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
- $${}^{0}\mathbf{p}_{1,2} = {}^{0}\mathbf{p}_2 - {}^{0}\mathbf{p}_1$$
- $${}^{0}\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
- $${}^{0}\mathbf{p}_2 = \begin{pmatrix} \mathrm{a}_1 \mathrm{C}\theta_1 \\ \mathrm{a}_1 S\theta_1 \\ 0 \end{pmatrix}$$
- $${}^{0}\mathbf{p}_{1,2} = \begin{pmatrix} \mathrm{a}_1 \mathrm{C}\theta_1 \\ \mathrm{a}_1 S\theta_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{a}_1 \mathrm{C}\theta_1 \\ \mathrm{a}_1 S\theta_1 \\ 0 \end{pmatrix}$$

Example 1: Direct Differential Kinematic Model

- $$ {}^{0}\mathbf{J}_2 = \begin{pmatrix} {}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,2} & \mathbf{0} \\ {}^{0}\mathbf{z}_1 & {}^{0}\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} -\mathrm{a}_1 S\theta_1 & 0 \\ \mathrm{a}_1 C\theta_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} $$

- $$ {}^{0}\mathbf{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad {}^{0}\mathbf{p}_{1,2} = \begin{pmatrix} \mathrm{a}_1 C\theta_1 \\ \mathrm{a}_1 S\theta_1 \\ 0 \end{pmatrix} $$

Let's $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$

The cross product $\mathbf{a} \wedge \mathbf{b}$ is computed as:

$$ \mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} $$

EXAMPLE 1: DIRECT DIFFERENTIAL KINEMATIC MODEL

- $${}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,2} = \begin{pmatrix} -\mathrm{a}_1 S\theta_1 \\ \mathrm{a}_1 \mathrm{C}\theta_1 \\ 0 \end{pmatrix}$$

- 2nd step:

- $$\mathbf{D} = \begin{pmatrix} 0 & 0 & -\mathrm{a}_2\sin(\theta_1+\theta_2) \\ 0 & 0 & \mathrm{a}_2\cos(\theta_1+\theta_2) \\ \mathrm{a}_2\sin(\theta_1+\theta_2) & -\mathrm{a}_2\cos(\theta_1+\theta_2) & 0 \end{pmatrix}$$

- $${}^{0}\mathbf{R}_{\mathrm{n}} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$$

- $${}^{0}\mathbf{R}_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

EXAMPLE 1: DIRECT DIFFERENTIAL KINEMATIC MODEL

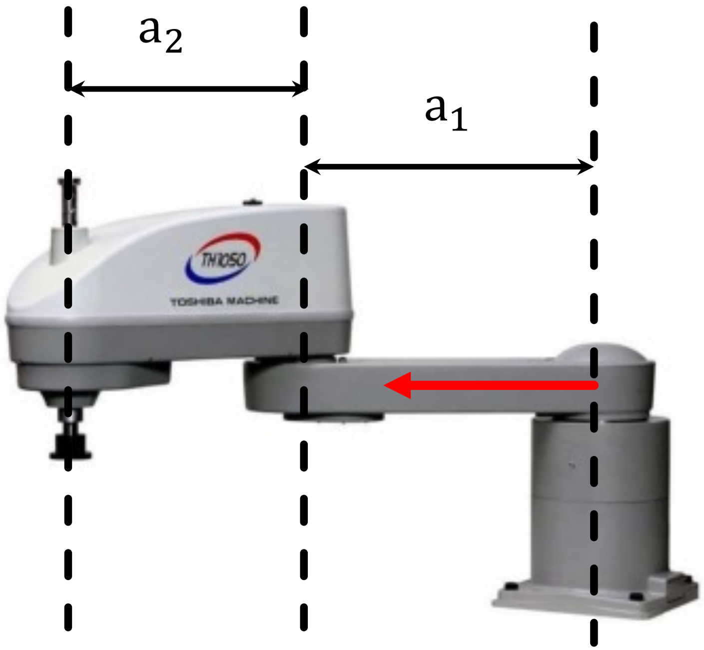
- $\mathbf{J} = \begin{pmatrix} \mathbf{I}_{3\times3} & \mathbf{D} \\ \mathbf{0}_{3\times3} & \mathbf{C} \end{pmatrix} \mathbf{J}_\mathrm{n}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -a_2S_{12} \\ 0 & 1 & 0 & 0 & 0 & a_2C_{12} \\ 0 & 0 & 1 & -a_2S_{12} & -a_2C_{12} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -a_1S_1 & 0 \\ a_1C_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

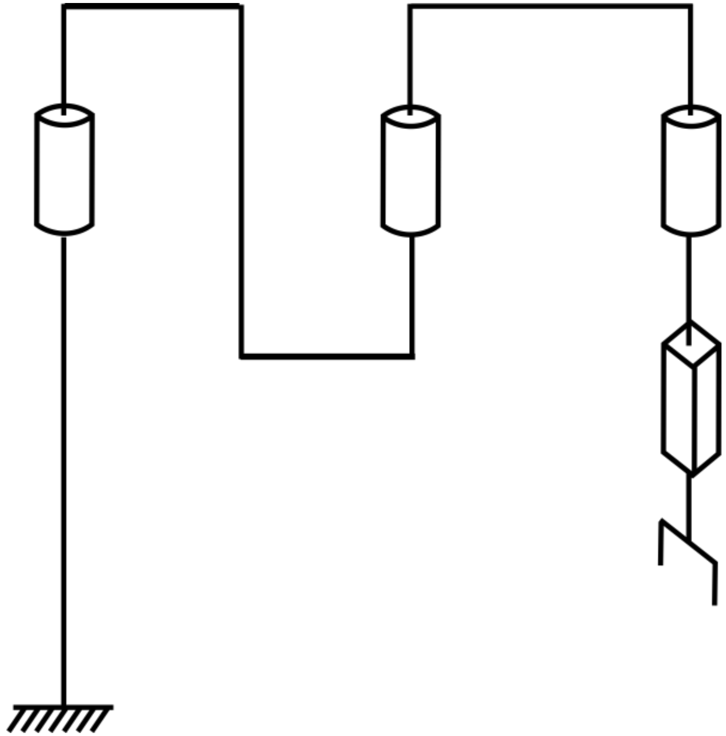
$$\mathbf{J} = \begin{pmatrix} -a_1S_1 - a_2S_{12} & -a_2S_{12} \\ a_1C_1 + a_2C_{12} & a_2C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

EXAMPLE 2

Example2: SCARA robot modelling



SCARA Robot (Toshiba)



SCARA architecture

EXEMPLE 2

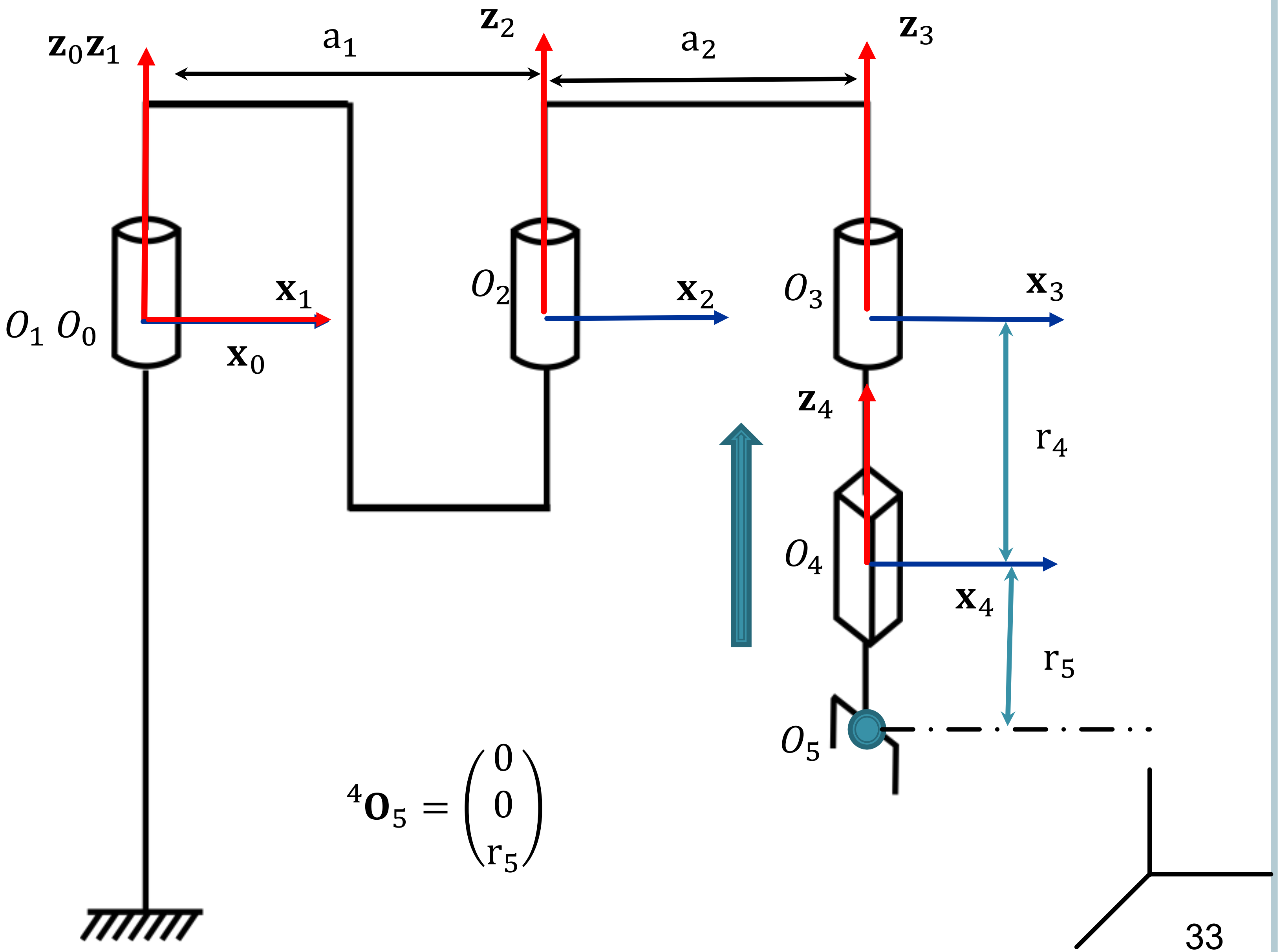
- n=4

1 . The vector of joint variables is:

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ r_4 \end{pmatrix}$$

2. The dimension of $\mathbb{x}$ is 6 if a Roll-Pitch-Yaw angle of Euler angles are used. Nevertheless, its dimension is equal to 7 if angle-axis or a quaternion representation is considered to define the orientation of the end effector.

EXAMPLE 2: DIRECT KINEMATIC MODEL



$$
{}^{4}\mathbf{O}_5 = \begin{pmatrix} 0 \\ 0 \\ r_5 \end{pmatrix}
$$

EXAMPLE 2: DIRECT KINEMATIC MODEL

4-

j	σ_j	α_{j-1}	a_{j-1}	θ_j	r_j
1	0	0	0	θ_1	0
2	0	0	a_1	θ_2	0
3	0	0	a_2	θ_3	0
4	1	0	0	0	r_4

5-

$$ {}^{0}\mathbf{T}_1 = \begin{pmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}\mathbf{T}_3 = \begin{pmatrix} C\theta_3 & -S\theta_3 & 0 & a_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{1}\mathbf{T}_2 = \begin{pmatrix} C\theta_2 & -S\theta_2 & 0 & a_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{3}\mathbf{T}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

EXAMPLE 2: DIRECT KINEMATIC MODEL

6- determination of the direct kinematic model:

- Computation of ${}^{0}\mathbf{T}_4$:

$$ {}^{0}\mathbf{T}_4(\mathbf{q}) = {}^{0}\mathbf{T}_1(q_1) \cdot {}^{1}\mathbf{T}_2(q_2) \cdot {}^{2}\mathbf{T}_3(q_3) \cdot {}^{3}\mathbf{T}_4(q_4) $$

$$ {}^{0}\mathbf{T}_2 = {}^{0}\mathbf{T}_1 * {}^{1}\mathbf{T}_2 = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1 C\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}\mathbf{T}_3 = {}^{0}\mathbf{T}_2 * {}^{2}\mathbf{T}_3 $$

$$ = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & a_1 C\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & a_1 S\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} C\theta_3 & -S\theta_3 & 0 & a_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

EXAMPLE 2: DIRECT KINEMATIC MODEL

$$
{}^{0}\mathbf{T}_3 = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{0}\mathbf{T}_4 = {}^{0}\mathbf{T}_3 * {}^{3}\mathbf{T}_4
$$

$$
= \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ 0 & 0 & 1 & r_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

EXAMPLE 2: DIRECT KINEMATIC MODEL

❖ Computation of the position of the end effector in the reference frame $\mathcal{R}_0$:

$$\begin{pmatrix} {}^{0}\mathbf{O}_5 \\ 1 \end{pmatrix} = {}^{0}\mathbf{T}_4 \begin{pmatrix} {}^{4}\mathbf{O}_5 \\ 1 \end{pmatrix}$$

The position of the end effector in the frame $\mathcal{R}_4$ is:

$${}^{4}\mathbf{O}_5 = \begin{pmatrix} 0 \\ 0 \\ \mathrm{r}_5 \end{pmatrix}$$

Its position in the reference frame $\mathcal{R}_0$ is:

$$\begin{pmatrix} {}^{0}\mathbf{O}_5 \\ 1 \end{pmatrix} = {}^{0}\mathbf{T}_4 \begin{pmatrix} {}^{4}\mathbf{O}_5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ 0 & 0 & 1 & \mathrm{r}_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \mathrm{r}_5 \\ 1 \end{pmatrix}$$

EXAMPLE 2: DIRECT KINEMATIC MODEL

$$= \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ 0 & 0 & 1 & r_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r_5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ r_5+r_4 \\ 1 \end{pmatrix}$$

The position of the end effector is:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_2\cos(\theta_1+\theta_2)+a_1\cos(\theta_1) \\ a_2\sin(\theta_1+\theta_2)+a_1\sin(\theta_1) \\ r_5+r_4 \end{pmatrix}$$

EXAMPLE 2: DIRECT KINEMATIC MODEL

- Determination of the rotation matrix ${}^{0}\mathbf{R}_4$, which corresponds to the orientation of the end effector:

$$ {}^{0}\mathbf{R}_4 = {}^{0}\mathbf{T}_4(1{:}3, 1{:}3) $$

$$ {}^{0}\mathbf{R}_4 = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

The Roll-Pitch-Yaw angles are:

if $x_z = 0 \neq \pm 1$

$$ \alpha = \mathrm{atan2}(\sin(\theta_1+\theta_2+\theta_3), \cos(\theta_1+\theta_2+\theta_3)) = \theta_1+\theta_2+\theta_3 $$
$$ \beta = \mathrm{atan2}(0, 1) = 0 $$
$$ \gamma = atan2(0, 1) = 0 $$

EXAMPLE 2: DIRECT KINEMATIC MODEL

- $\mathbb{x}$ represents the pose of the end effector:

$$\mathbb{x} = \begin{pmatrix} x \\ y \\ z \\ \gamma \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} a_2 \cos(\theta_1 + \theta_2) + a_1 \cos(\theta_1) \\ a_2 \sin(\theta_1 + \theta_2) + a_1 \sin(\theta_1) \\ r_5 + r_4 \\ 0 \\ 0 \\ \theta_1 + \theta_2 + \theta_3 \end{pmatrix}$$

$n = 4$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ r_4 \end{pmatrix}$$

$m = 4$

EXAMPLE 2 : INVERSE KINEMATIC MODEL

- Objective: determine the vector of joint variables $\mathbf{q}$ in function of the pose element of the vector pose $\mathbb{x}$

- From the direct kinematic model:

$$\begin{cases} x = \mathrm{a}_2\cos(\theta_1+\theta_2)+\mathrm{a}_1\cos(\theta_1) \\ y = \mathrm{a}_2\sin(\theta_1+\theta_2)+\mathrm{a}_1\sin(\theta_1) \\ z = \mathrm{r}_5+\mathrm{r}_4 \\ \alpha = \theta_1+\theta_2+\theta_3 \end{cases}$$

Example 2 : inverse kinematic model

$$\theta_2 = \pm \mathrm{acos}\left(\frac{x^2 + y^2 - {a_1}^2 - {a_2}^2}{2a_1 a_2}\right)$$

$$\theta_1 = atan2(S_1, C_1)$$

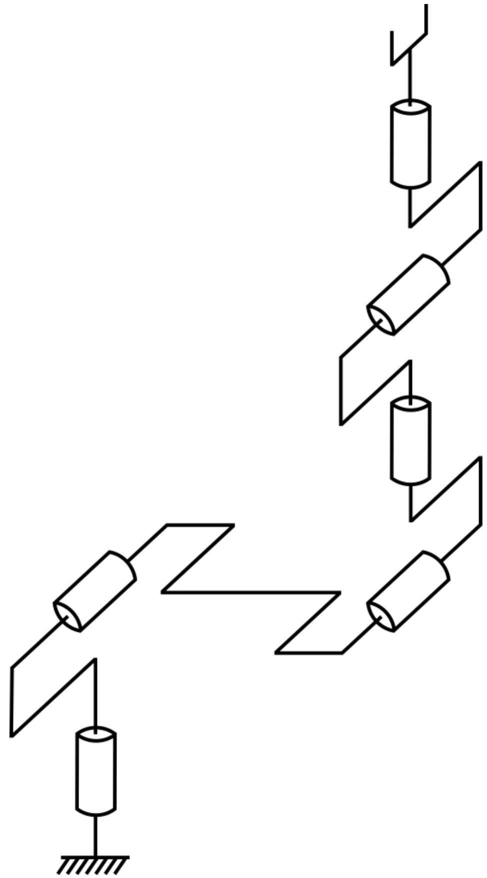
$$\theta_3 = \alpha - \theta_1 - \theta_2$$

$$r_4 = z - r_5$$

Example 3

Example3: modelling of an anthropomorphic 6-axis robot

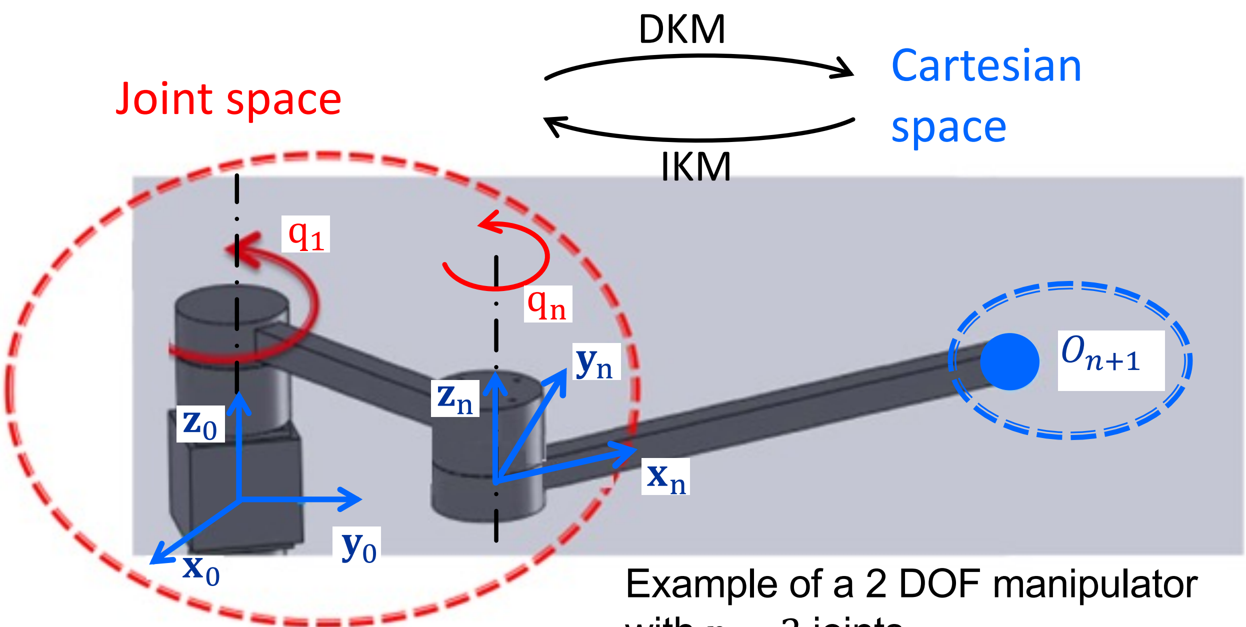
Robot Stäbli RX-90



Stäubli RX-90 architecture

INVERSE KINEMATIC MODEL

IKM: consists in computing the joint variables, from the pose $\mathbb{x}$ of the end effector**.** This model allows to pass from the operational space to the joint space.



Example of a 2 DOF manipulator with $n = 2$ joints

INVERSE KINEMATIC MODEL

Problematic: solve m system of non linear equations. This set of equations is usually complex to solve. It is therefore necessary to check before starting the resolution to:

- The existence of a solution
- the number of solution
- The resolution approach

Existence of a solution

- If $n < m$: no solutions
- If $n = m$: finite number of solutions
- if $n > m$: infinite number of solutions

Remark

An open-structure serial robot can have up to 16 solutions to m nonlinear system of equations, and thus 16 different expressions of the IKM.

In this course, we limit ourselves to the computation of the IKM of robot manipulators having at most 4 DOF.

DIFFERENTIAL KINEMATIC: DIRECT AND INVERSE

DDKM: allows to express the operational speed of the end effector as a function of the joint speeds by the relationship:

$$\mathbb{V} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

$\mathbf{J}(\mathbf{q})$: jacobian matrix of dimension $\mathrm{m}\times\mathrm{n}$

$\dot{\mathbf{q}} = [\dot{\mathrm{q}}_1 \quad \dot{\mathrm{q}}_2 \quad \cdot \quad \dot{\mathrm{q}}_\mathrm{n}]^T$: joints speed vector

$\mathbb{V} = \begin{pmatrix}\mathbf{V}\\ \boldsymbol{\omega}\end{pmatrix}$: operational speed vector

With $\quad \mathbf{V} = (\dot{x} \quad \dot{y} \quad \dot{z})^T$: end effector translation speed

$\boldsymbol{\omega} = (\omega_x \quad \omega_y \quad \omega_z)^T$: end effector rotational speed

DIFFERENTIAL KINEMATIC: DIRECT AND INVERSE

IDKM: allows to express the joints speed as a function of the end effector operational speed :

$$\dot{\mathbf{q}} = \mathbf{J}^{\#}(\mathbf{q})\mathbb{V}$$

if $\mathrm{n} = \mathrm{m}$: $\mathbf{J}^{\#} = \mathbf{J}^{-1}$

If not (non-square matrix): $\mathbf{J}^{\#} = \mathbf{J}^{+}$ (Pseudo inverse)

The IDKM can be obtained by reversing the DDKM. Implementation can be done analytically or numerically. Analytical methods are not discussed in this course.

Note: for the practical, we will be using the numerical approach

DIRECT DIFFERENTIAL KINEMATIC

Computation methodology

DKM derivation for simple structures

- Two-step procedure for complex structures:

Step 1: Compute the speed of the frame $\mathcal{R}_{\mathrm{n}}$:

$$\mathbb{V}_{\mathrm{n}} = \begin{pmatrix} \mathbf{V}_{\mathrm{n}} \\ \boldsymbol{\omega}_{\mathrm{n}} \end{pmatrix}$$

$\mathbf{V}_{\mathrm{n}}$: translation speed of O_{n} (origin of the frame $\mathcal{R}_{\mathrm{n}}$)

$\boldsymbol{\omega}_{\mathbf{n}}$: rotation speed of frame $\mathcal{R}_{\mathrm{n}}$

It should be noted that the speed of each joint of the robot contributes to the speed of the end effector. This contribution depends on the joint nature

DIRECT DIFFERENTIAL KINEMATIC

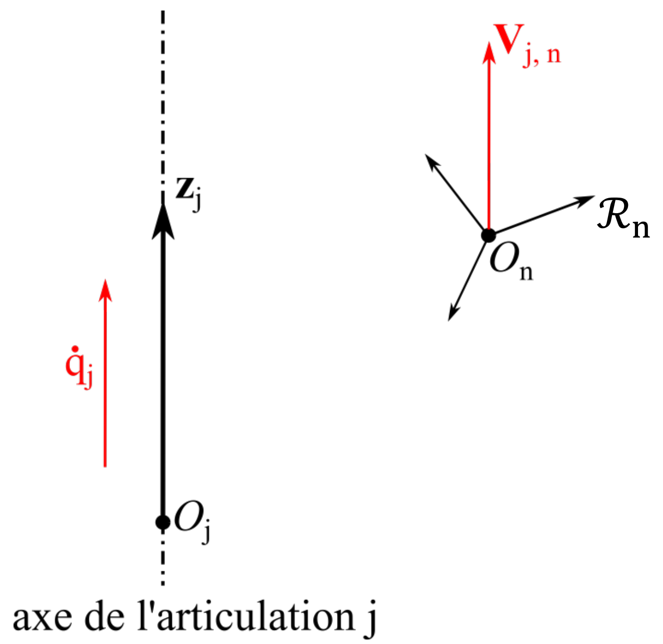
- If the joint j is prismatic : $\sigma_{\mathrm{j}} = 1$

Contribution of the joint j to the translational speed of $\mathcal{R}_{\mathrm{n}}$:

$$\mathbf{V}_{\mathrm{j,\,n}} = \dot{\mathrm{q}}_{\mathrm{j}}\ \mathbf{z}_{\mathrm{j}}$$

Contribution of the joint j to the rotational speed of $\mathcal{R}_{\mathrm{n}}$:

$$\boldsymbol{\omega}_{\mathrm{j,\,n}} = 0$$



DIRECT DIFFERENTIAL KINEMATIC

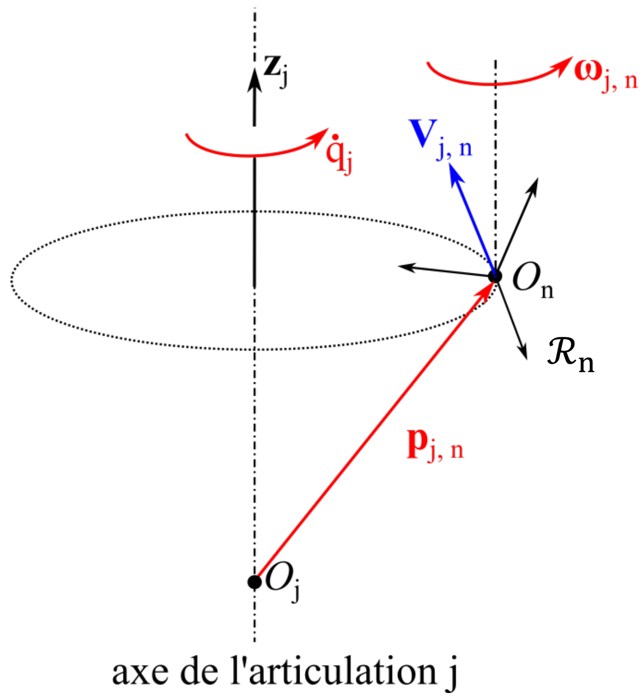
- If the joint j is revolute : $\sigma_j = 0$

Contribution of the j joint to the translational speed of $\mathcal{R}_n$:

$$\mathbf{V}_{j,n} = \dot{q}_j(\mathbf{z}_j \wedge \mathbf{p}_{j,n})$$

Contribution of the joint j to the rotational speed of $\mathcal{R}_n$:

$$\boldsymbol{\omega}_{j,n} = \dot{q}_j\, \mathbf{z}_j$$



DIRECT DIFFERENTIAL KINEMATIC

To resume:

- Contribution of the joint j to the translational speed of $\mathcal{R}_n$:

$$\mathbf{V}_{j,\,n} = \left(\sigma_j \mathbf{z}_j + \bar{\sigma}_j \left(\mathbf{z}_j \wedge \mathbf{p}_{j,n}\right)\right) \dot{q}_j$$

- Contribution of the joint j to the rotational speed of $\mathcal{R}_n$:

$$\boldsymbol{\omega}_{j,\,n} = \bar{\sigma}_j \dot{q}_j \, \mathbf{z}_j$$

- Translational speed of $\mathcal{R}_n$:

$$\mathbf{V}_n = \sum_{j=1}^{n} \left(\sigma_j \mathbf{z}_j + \bar{\sigma}_j \left(\mathbf{z}_j \wedge \mathbf{p}_{j,n}\right)\right) \dot{q}_j$$

- Rotational speed of $\mathcal{R}_n$:

$$\boldsymbol{\omega}_n = \sum_{j=1}^{n} \bar{\sigma}_j \dot{q}_j \, \mathbf{z}_j$$

DIRECT DIFFERENTIAL KINEMATIC

- Writing in vector form:
-

$$\begin{pmatrix} \mathbf{V}_n \\ \boldsymbol{\omega}_n \end{pmatrix} = \mathbf{J}_n \dot{\mathbf{q}}$$

With :

$$\mathbf{J}_n = \begin{pmatrix} \sigma_1 \mathbf{z}_1 + \bar{\sigma}_1 (\mathbf{z}_1 \wedge \mathbf{p}_{1,n}) & \sigma_2 \mathbf{z}_2 + \bar{\sigma}_2 (\mathbf{z}_2 \wedge \mathbf{p}_{2,n}) & . & \sigma_n \mathbf{z}_n + \bar{\sigma}_n (\mathbf{z}_n \wedge \mathbf{p}_{n,n}) \\ \bar{\sigma}_1 \mathbf{z}_1 & \bar{\sigma}_2 \mathbf{z}_2 & . & \bar{\sigma}_n \mathbf{z}_n \end{pmatrix}$$

The speeds $\mathbf{V}_n$ and $\boldsymbol{\omega}_n$ are often expressed in the reference frame $\mathcal{R}_0$ or in frame $\mathcal{R}_n$. The corresponding Jacobian matrices are denoted ${}^0\mathbf{J}_n$ et ${}^n\mathbf{J}_n$

DIRECT DIFFERENTIAL KINEMATIC

$$
{}^{0}\mathbf{J}_{\mathrm{n}} = \begin{pmatrix} \sigma_1\,{}^{0}\mathbf{z}_1 + \bar{\sigma}_1\left({}^{0}\mathbf{z}_1 \wedge {}^{0}\mathbf{p}_{1,\mathrm{n}}\right) & \sigma_2\,{}^{0}\mathbf{z}_2 + \bar{\sigma}_2\left({}^{0}\mathbf{z}_2 \wedge {}^{0}\mathbf{p}_{2,\mathrm{n}}\right) & . & \sigma_{\mathrm{n}}\,{}^{0}\mathbf{z}_{\mathrm{n}} + \bar{\sigma}_{\mathrm{n}}\left({}^{0}\mathbf{z}_{\mathrm{n}} \wedge {}^{0}\mathbf{p}_{\mathrm{n,n}}\right) \\ \bar{\sigma}_1\,{}^{0}\mathbf{z}_1 & \bar{\sigma}_2\,{}^{0}\mathbf{z}_2 & . & \bar{\sigma}_{\mathrm{n}}\,{}^{0}\mathbf{z}_{\mathrm{n}} \end{pmatrix}
$$

with:

$$
{}^{0}\mathbf{p}_{\mathrm{j,n}} = {}^{0}\mathbf{p}_{\mathrm{n}} - {}^{0}\mathbf{p}_{\mathrm{j}}
$$

Let be the vectors $\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$

The cross product $\mathbf{a} \wedge \mathbf{b}$ is computed as:

$$
\mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}
$$

Direct differential kinematic

Step 2: computation of the speed (translation + rotation) of the end effector:

$$\mathbb{V} = \begin{pmatrix} \mathbf{V} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{3\times3} & \mathbf{D} \\ \mathbf{0}_{3\times3} & \mathbf{C} \end{pmatrix} \mathbf{J}_{\mathrm{n}}\, \dot{\mathbf{q}}$$

$$= \mathbf{J}\, \dot{\mathbf{q}}$$

with :

$\mathbf{J} = \begin{pmatrix} \mathbf{I}_{3\times3} & \mathbf{D} \\ \mathbf{0}_{3\times3} & \mathbf{C} \end{pmatrix} \mathbf{J}_{\mathrm{n}}$ the Jacobian matrix of the robot

DIRECT DIFFERENTIAL KINEMATIC

$$\mathbf{D} = \begin{pmatrix} 0 & \mathrm{a_n} x_z + \mathrm{r_{n+1}} z_z & -\mathrm{a_n} x_y - \mathrm{r_{n+1}} z_y \\ -\mathrm{a_n} x_z - \mathrm{r_{n+1}} z_z & 0 & \mathrm{a_n} x_x + \mathrm{r_{n+1}} z_x \\ \mathrm{a_n} x_y + \mathrm{r_{n+1}} z_y & -\mathrm{a_n} x_x - \mathrm{r_{n+1}} z_x & 0 \end{pmatrix}$$

- The values x_x, x_y, x_z, z_x, z_y and z_z are determined from the rotation matrix ${}^{0}\mathbf{R}_{\mathrm{n}}$ (already computed in the DKM) :

$${}^{0}\mathbf{R}_{\mathrm{n}} = \begin{pmatrix} x_x & y_x & z_x \\ x_y & y_y & z_y \\ x_z & y_z & z_z \end{pmatrix}$$

Exercises:

Example 1: compute the Jacobian matrix of the 2 dof robot (using the two approaches)

Example 2: compute the Jacobian matrix of the SCARA robot

Fin du chapitre 2