Applicative, traversable, foldable

Advanced functional programming - Lecture 3

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Beyond the monad

So far, we have seen how *monads* define a common abstraction over many programming patterns.

This kind of abstraction occurs more often in Haskell's libraries.

In this lecture we will cover:

- applicative functors
- foldable
- traversable
- arrows

We'll motivate the need for applicative functors starting with *examples*.

Sequencing IO operations

```
sequenceIO :: [IO a] -> IO [a]
sequenceIO [] = return []
sequenceIO (c : cs) = do x <- c
                         xs <- sequenceIO cs
                         return (x : xs)
```

There is nothing ‘wrong’ with this code – but using do notation may seem like overkill. The variable x isn’t used in the rest of the computation!

We would like to ‘apply’ one monadic computation to another.

Using ap

The ap function defined as follows:

```
ap : Monad m => m (a -> b) -> m a -> m b
ap mf mx = do f <- mf
              x <- mx
              return (f x)
```

Using ap we can write:

```
sequenceIO :: [IO a] -> IO [a]
sequenceIO [] = return []
sequenceIO (c:cs) =
  return (:) `ap` c `ap` sequenceIO cs
```

This even works for *any monad*, not just the `IO` monad.

Evaluating expressions

Another example:

```
data Expr v = Var v | Val Int
  | Add (Expr v) (Expr v)

type Env v = Map v Int

eval : Expr v -> Env v -> Int
eval (Var v) env   = lookup v env
eval (Val i) env   = i
eval (Add l r) env = (eval l env) + (eval r env)
```

Once again, we are passing around an environment that is only really used in the `Var` branch.

An applicative alternative

```
const : a -> (env -> a)
const x = \env -> a

s : (env -> a -> b) -> (env -> a) -> (env -> b)
s ef es env = (ef env) (es env)

eval : Expr v -> Env v -> Int
eval (Var v) = lookup v
eval (Val i) = const i
eval (Add l r) env =
  const (+) `s` (eval l) `s` (eval r)
```

The s combinator lets us 'apply' one computation expecting an environment to another.

Transposing matrices

```
transpose :: [[a]] -> [[a]]
transpose [] = repeat []
transpose (xs : xss) =
  zipWith (:) xs (transpose xss)
```

Can we play the same trick and find a combinator that will ‘apply’ a list of functions to a list of arguments?

```
zapp : [a -> b] -> [a] -> [b]
zapp (f : fs) (x : xs) = (f x) : (zapp fs xs)

transpose (xs : xss) =
  repeat (:) ‘zapp‘ xs ‘zapp‘ transpose xss
```

What is the pattern?

What do these functions have in common?

```
ap : IO (a -> b) -> IO a -> IO b
s : (env -> a -> b) -> (env -> a) -> (env -> b)
zapp : [a -> b] -> [a] -> [b]
```

Applicative (applicative functors)

```
class (Functor f) => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

Note that `Functor` is a superclass of `Applicative`.

We can also define `map` in terms of the applicative operations:

```
(<$>)  ::  Functor f => (a -> b) -> f a -> f b
```

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We can also define `map` in terms of the applicative operations:

```
(<$>)  ::  Functor f => (a -> b) -> f a -> f b

(<$>) f fx = pure f <*> fx
```

Relating Applicative functors and Monads

- Every monad can be given an applicative functor interface.

```
instance Monad m => Applicative m where
  pure :: a -> m a
  pure = return
  mf <*> mx = do f <- mf; x <- mx; return (f x)
```

- But this may not always be the ‘right’ choice. For example, we have seen the ‘zippy’ applicative instance for lists; using the instance arising from the list monad gives very different behaviour!
- But not every applicative functor is a monad…

Monads vs. applicative functors - I

```
(<*>) :: (Applicative f) =>
  f (a -> b) -> f a -> f b
(>>=) :: (Monad m) =>
  m a -> (a -> m b) -> m b
```

- The arguments to <*> are (typically) first-order structures (that may contain higher-order data).
- Monadic bind is inherently higher order.
- With monads, subsequent actions can depend on the results of effects: depending on the character the user enters, respond differently.

Monads vs applicative functors - II

- There are more `Applicative` functors than there are monads; but monads are more powerful!
- If you *have* an `Applicative` functor, that's good; having a monad is better.
- If you *need* a monad, that's good; only needing an `Applicative` functor is better.
- With applicative functors, the structure is statically determined (and can be analyzed or optimized). Consider the following example:

```
miffy : Monad m => m Bool -> m a -> m a -> ma
```

Composing monads

- Given two monads `m1` and `m2`, is `m1 . m2` a monad?

Composing monads

- Given two monads `m1` and `m2`, is `m1 . m2` a monad?

```
data Compose m1 m2 a = Compose (m1 (m2 a))
instance (Monad m1, Monad m2) =>
  Monad (Compose m1 m2) where
    return :: a -> m1 (m2 a)
    (>>=) :: m1(m2 a) -> (a -> m1(m2 b)) -> m1(m2 b)
```

- Unfortunately, there is no guarantee that such an instance can be defined.
- As a result, there has been a great deal of work on *monad transformers*, that allow complex monads to be assembled from smaller pieces.
- For applicative functors however…

Composing applicative functors

For any pair of applicative functors `f` and `g`:

```
data Compose f g a = Compose (f (g a))

instance (Applicative f, Applicative g) =>
  Applicative (Compose f g) where
    pure :: a -> f (g a)
    pure x = ...
    (<*>) :: f (g (a -> b)) -> (f (g a)) -> f (g b)
    fgf <*> fgx = ...
```

We can define the desired `pure` and <*> operations!

This is a *guarantee of compositionality*.

Composing applicative functors

For any pair of applicative functors `f` and `g`:

```
data Compose f g a = Compose (f (g a))

instance (Applicative f, Applicative g) =>
  Applicative (Compose f g) where
    pure :: a -> f (g a)
    pure x = pure (pure x)
    (<*>) :: f (g (a -> b)) -> (f (g a)) -> f (g b)
    fgf <*> fgx = (pure <*>) <*> fgf <*> fgx
```

We can define the desired `pure` and <*> operations!

This is a *guarantee of compositionality*.

Imprecise but catchy slogans

Monads are programmable semi-colons!

Applicatives are programmable function application!

Applicative functor laws

- identity

```
pure id <*> u = u
```

- composition

```
pure (.) <*> u <*> v <*> w = u <*> (v <*> w)
```

- homomorphism

```
pure f <*> pure x = pure (f x)
```

- interchange

```
u <*> pure x = pure ( f -> f x) <*> u
```

Spot the pattern

```
sequenceIO :: [IO a] -> IO [a]
sequenceIO [] = pure []
sequenceIO (c:cs) =
  pure (:) <*> c <*> sequenceIO cs

transpose :: [[a]] -> [[a]]
transpose [] = pure []
transpose (xs : xss) =
  pure (:) <*> xs <*> transpose xss
```

Both these functions take a *list* of applicative actions as argument.

They traverse this list, performing the actions one by one, collecting the results in a list.

Traversing lists

We can define a new function to capture this pattern:

```
traverse :: Applicative f => [f a] -> f [a]
traverse [] = pure []
traverse (x:xs) = pure (:) <*> x <*> traverse xs
```

Clearly we can traverse lists in this fashion – but what other data types support such an operation?

Traversable

The `Traversable` class captures those types that can be traversed in this fashion:

```
class Traversable t where
  traverse :: Applicative f => (a -> f b) ->
                               t a -> f (t b)
  sequenceA :: Applicative f => t (f a) -> f (t a)
  sequenceA = traverse id
```

It requires a slightly more general `traverse` than the one we have seen so far.

Traversable: example

```
data Expr v = Var v | Val Int
            | Add (Expr v) (Expr v)

instance Traversable Expr where
  traverse :: Applicative f => (a -> f b) ->
              Expr a -> f (Expr b)
  traverse f (Var v) = pure Var <*> f v
  traverse f (Val x) = pure (Val x)
  traverse f (Add l r) =
    pure Add <*> traverse f l <*> traverse f r
```

In general, `traverse` is just like `fmap` – only in applicative style.

fmap via traverse

```haskell
newtype Id a = Id {getId :: a}

myFmap :: Traversable f => (a -> b) -> f a -> f b
myFmap f = getId . traverse (Id . f)
```

Introducing `Foldable`

In the Haskell libraries, `Traversable` is defined slightly differently.

```
class (Functor t, Foldable t) =>
        Traversable t where
```

What is the `Foldable` class?

Folding a list

The `foldr` function on lists captures a common pattern – think of it as a functional for-loop.

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr f y []     = y
foldr f y (x:xs) = f x (foldr f y xs)
```

We can use it to define all kinds of simple list traversals:

```
sum = foldr (+) 0
max = foldr max 0 xs
(++) = \ys -> foldr (:) ys xs
concat = foldr (++) []
map = \f -> foldr (xs -> f x : xs) []
```

Folding: beyond lists

There are many other data types that support some form of `fold` operator.

```
data Tree a = Leaf a | Node (Tree a) (Tree a)
            | Empty

foldTree :: (a -> b -> b) -> b -> Tree a -> b
foldTree f y Empty       = y
foldTree f y (Leaf x)    = f x y
foldTree f y (Node l r) =
  foldTree f (foldTree f y r) l
```

Note that *generic programming* gives a slightly more precise account.

Foldable

```
class Foldable f where
  foldr :: (a -> b -> b) -> b -> f a -> b
  foldMap :: Monoid m => (a -> m) -> t a -> m
```

Sometimes it can be easier to define the `foldMap` function – but what is a `Monoid`?

Intermezzo: monoids

```
class Monoid a where
  mempty :: a
  mappend :: a -> a -> a
```

Here `mempty` should be the unit of the associative operator `mappend`.

Examples?

Monoids everywhere

- `Bool` using `&&` and `True`;
- `Bool` using `or` and `False`;
- `Int` using `+` and `0`;
- `Int` using `*` and `1`;
- `Int` using `max` and `minBound`;
- `List a` using `++` and `[]`;
- Imperative programs using `;` and `skip`;
- `a -> a` using `.` and `id`;
- …

Monoids pop up everywhere!

Defining `foldMap`

Instead of defining `fold`, sometimes it can be easier to define `foldMap`:

```
foldMap :: Monoid m => (a -> m) -> Tree a -> m
foldMap f Empty = mempty
foldMap f (Leaf x) = f x
foldMap f (Node l r) =
  foldMap f l `mappend` foldMap f r
```

You need to apply `f` all `a` values in the tree and combine subtrees using `mappend`.

Why?

What is the point of all this abstraction?

We all agree (I hope!) that `foldr` is useful for lists.

```
sum = foldr (+) 0
max = foldr max 0 xs
(++) ys = foldr (:) ys xs
concat = foldr (++) []
map f = foldr (xs -> f x : xs) []
```

Why?

What is the point of all this abstraction?

We all agree (I hope!) that `foldr` is useful for lists.

```
sum :: Foldable f => f Int -> Int
sum = foldr (+) 0
max :: Foldable f => f Int -> Int
max = foldr max 0
flatten :: Foldable f => f a -> [a]
flatten = foldMap (-> [x]) ...
```

… but we can now give definitions that work for *any* foldable structure.

Generalizing any

As a slightly less trivial example, consider the **any** function:

```
any :: (a -> Bool) -> [a] -> Bool
any p [] = False
any p (x:xs) = p x || any p xs
```

How can we generalize this to work on *any* traversable type?

Mighty Booleans

Let's start by finding the right monoidal structure.

Instead of defining an instance for `Bool`, introducing a new type can help clarify the monoidal structure we are using.

```
newtype Might = Might {might : Bool}

instance Monoid Might where
  mempty = Might False
  (Might b) `mappend` (Might b') = Might (b || b')
```

Generic any

```
any :: Foldable f => (a -> Bool) -> f a -> Bool
any p = might . foldMap (Might . p)
```

Many other functions can be generalized similarly.

The Foldable-Traversible Proposal

As of GHC 7.10, many Prelude functions have been generalized to work over *any* traversable structure – and not just lists.

Suppose we have a data type for binary trees, with the obvious traversable/foldable instances:

```
data Tree a where
  Leaf :: a -> Tree a
  Node :: Tree a -> Tree a -> Tree a
```

We can use the prelude functions we are used to over this data structure too.

Example folds over trees

```
> let t = (Node (Leaf 1) (Node (Leaf 2) (Leaf 3))
> any isEven t
True
> length t
3
> elem 3 t
True
```

We no longer need to define specialized functions for trees.

Exercise

What is the foldable instance for `Maybe`?

What about the foldable instance for pairs?

Drawbacks

But there are also quite surprising examples:

```
minimum (1,1000)

length (lookup 4 [(2,“Hello”), (4,“World”), (5,“!”)]

null (lookup 3 [])
```

Drawbacks

But there are also quite surprising examples:

```
minimum (1,1000)

length (lookup 4 [(2,“Hello”), (4,“World”), (5,“!”)]

null (lookup 3 [])
```

Sometimes code may type check, where you would have liked to see a type error.

Introducing arrows

If applicative functors generalize the notion of application, can we find a similar abstraction over functions and function composition?

Yes! There is more than a decade of work investigating functional programming using *Arrows*.

Arrows

```
class Arrow a where
  arr :: (b -> c) -> a b c
  (>>>) :: a b c -> a c d -> a b d
  first :: a b c -> a (b,d) (c,d)
```

- Just like applicative functors and monads, arrows have several associated laws.
- Many programs using arrows require additional operations – similar to classes such as `MonadPlus`.
- GHC supports special syntax for programming with Arrows, similar to the `do` notation for monads.

Historical context

- Monads were originally studied in the context of program language semantics.
- Only later, was their importance for *structuring* programs discovered (and subsequently, modelling IO in Haskell)
- Arrows (Hughes 2000) were proposed as a weaker alternative to monads, but they have not been widely adopted.
- More recently, applicative functors have gained a lot of traction in the Haskell community (McBride and Paterson 2008), generalising the interface by Duponcheel and Swierstra (1996).
- Applicative functors, together with the associated Traversable and Foldable classes, are now part of the Haskell standard.

Why care?

Functional programmers are adicted to abstraction: as soon as they spot a pattern, they typically want to abstract over it.

The type classes we have seen today, such as monads, applicative functors, foldable, and traversable, all capture some common pattern.

Using these patterns can save you some boilerplate code.

But *understanding* these patterns can help guide your design.

Is my type a monad? Or is it just applicative? Can I find a Traversable instance? Why not?

Further reading

- *Applicative programming with effects*, McBride and Paterson
- *Monoids: Theme and Variations*, Brent Yorgey
- *Programming with arrows*, John Hughes
- *Idioms are oblivious, arrows are meticulous, monads are promiscuous*, Lindley, Wadler and Yallop
- and much much more!