



**Universiteit Utrecht**

[Faculty of Science  
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# Advanced Functional Programming

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### 3. Correctness and Testing



- ▶ Gain confidence in the correctness of your program.
- ▶ Show that common cases work correctly.
- ▶ Show that corner cases work correctly.
- ▶ Testing cannot prove the absence of bugs.



- ▶ When is a program correct?



- ▶ When is a program correct?
- ▶ What is a specification?
- ▶ How to establish a relation between the specification and the implementation?
- ▶ What about bugs in the specification?



- ▶ Equational reasoning with Haskell programs
- ▶ QuickCheck, an automated testing library/tool for Haskell



- ▶ Understand how to prove simple properties using equational reasoning.
- ▶ Understand how to define QuickCheck properties and how to use QuickCheck.
- ▶ Understand how QuickCheck works and how to make QuickCheck usable for your own larger programs.



## 3.1 Equational reasoning





- ▶ “Equals can be substituted for equals”
- ▶ In other words: if an expression has a value in a context, we can replace it with any other expression that has the same value in the context without affecting the meaning of the program.



SML is (like most languages) **not** referentially transparent:

```
let val x = ref 0
    fun f n = (x := !x + n; !x)
in f 1 + f 2
end
```

The expression evaluates to 4.



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```
let val x = ref 0
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end
```

The expression evaluates to 4. The value of `f 1` is 1. But

```
let val x = ref 0
    fun f n = (x := !x + n; !x)
in 1 + f 2
end
```

evaluates to 3.



Also

```
let val x = ref 0
    fun f n = (x := !x + n; !x)
in f 1 + f 1
```

cannot be replaced by

```
let val x = ref 0
    fun f n = (x := !x + n; !x)
    val r = f 1
in r + r
```



- ▶ Haskell is referentially transparent.
- ▶ The SML example breaks down because Haskell has no untracked side-effects.

**do**

```
x ← newIORef 0
let f n = do modifyIORef x (+n); readIORef x
r ← f 1
s ← f 2
return (r + s)
```

The type of  $f$  is  $\text{Int} \rightarrow \text{IO Int}$ , not  $\text{Int} \rightarrow \text{Int}$  as in SML.



- ▶ Because of referential transparency, the definitions of functions give us rules for reasoning about Haskell programs.
- ▶ Properties regarding datatypes can be proved using induction:

| **data**  $[a] = [] \mid a : [a]$

To prove  $\forall (xs :: [a]). P \ xs$ , we prove

- ▶  $P \ []$
- ▶  $\forall (x :: a) \ (xs :: [a]). P \ xs \rightarrow P \ (x : xs)$



```
length :: [a] → Int
length []      = 0
length (x : xs) = 1 + length xs

isort :: Ord a ⇒ [a] → [a]
isort []       = []
isort (x : xs) = insert x (isort xs)
```

```
insert :: Ord a ⇒ a → [a] → [a]
insert x []      = [x]
insert x (y : ys)
    | x ≤ y      = x : y : ys
    | otherwise  = y : insert x ys
```



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$$\forall (xs :: [a]). \text{length } (\text{isort } xs) \equiv \text{length } xs$$




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## Lemma

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Proof by induction on the list.

Case []:

$$\begin{aligned} & \text{length} (\text{insert } x []) \\ \equiv & \quad \{ \text{Definition of insert} \} \\ & \text{length } [x] \\ \equiv & \quad \{ \text{Definition of length} \} \\ & 1 + \text{length } [] \end{aligned}$$



## Lemma

$$\forall (x :: a) (ys :: [a]). \text{length} (\text{insert } x \text{ } ys) \equiv 1 + \text{length } ys$$

Case  $y : ys$ , case  $x \leq y$ :

$$\begin{aligned} & \text{length} (\text{insert } x (y : ys)) \\ \equiv & \quad \{ \text{Definition of insert} \} \\ & \text{length } (x : y : ys) \\ \equiv & \quad \{ \text{Definition of length} \} \\ & 1 + \text{length } (y : ys) \end{aligned}$$



## Lemma

$$\forall (x :: a) (ys :: [a]). \text{length} (\text{insert } x \text{ } ys) \equiv 1 + \text{length } ys$$

Case  $y : ys$ , case  $x > y$ :

$$\begin{aligned} & \text{length} (\text{insert } x (y : ys)) \\ \equiv & \quad \{ \text{Definition of insert} \} \\ & \text{length } (y : \text{insert } x \text{ } ys) \\ \equiv & \quad \{ \text{Definition of length} \} \\ & 1 + \text{length} (\text{insert } x \text{ } ys) \\ \equiv & \quad \{ \text{Induction hypothesis} \} \\ & 1 + (1 + \text{length } ys) \\ \equiv & \quad \{ \text{Definition of length} \} \\ & 1 + \text{length } (y : ys) \end{aligned}$$



## Theorem

$$\forall(xs :: [a]). \text{length} (\text{isort } xs) \equiv \text{length } xs$$

Proof by induction on the list.

Case []:

$$\begin{aligned} & \text{length} (\text{isort } []) \\ \equiv & \quad \{ \text{Definition of isort} \} \\ & \text{length } [] \end{aligned}$$



## Theorem

$$\forall (xs :: [a]). \text{length (isort xs)} \equiv \text{length xs}$$

Case  $x : xs$ :

$$\begin{aligned} & \text{length (isort (x : xs))} \\ \equiv & \quad \{ \text{Definition of isort} \} \\ & \text{length (insert x (isort xs))} \\ \equiv & \quad \{ \text{Lemma} \} \\ & 1 + \text{length (isort xs)} \\ \equiv & \quad \{ \text{Induction hypothesis} \} \\ & 1 + \text{length xs} \\ \equiv & \quad \{ \text{Definition of length} \} \\ & \text{length (x : xs)} \end{aligned}$$



- ▶ Equational reasoning can be an elegant way to prove properties of a program.
- ▶ Equational reasoning can be used to establish a relation between an “obviously correct” Haskell program (a specification) and an efficient Haskell program.
- ▶ Equational reasoning is usually quite lengthy.
- ▶ Careful with special cases (laziness):
  - ▶ undefined values;
  - ▶ infinite values
- ▶ It is infeasible to prove properties about every Haskell program using equational reasoning.



- ▶ Type systems.
- ▶ Proof assistants.





## 3.2 QuickCheck



- ▶ QuickCheck is a Haskell library developed by Koen Claessen and John Hughes.
- ▶ An embedded domain-specific language (EDSL) for defining properties.
- ▶ Automatic datatype-driven generation of random test data.
- ▶ Extensible by the user.
- ▶ Shrinks failing test cases.



To use QuickCheck in your program:

```
| import Test.QuickCheck
```

The simplest interface is to use

```
| quickCheck :: Testable prop => prop -> IO ()
```

```
| class Testable prop where  
  property :: prop -> Property
```

```
| instance Testable Bool
```

```
| instance (Arbitrary a, Show a, Testable prop) =>  
  Testable (a -> prop)
```



- ▶ Classes declare predicates on types.

```
class Testable prop where  
  property :: prop → Property
```

Here, any type can either be Testable or not.

- ▶ If a predicate holds for a type, this implies that the class methods are supported by the type.  
For any type prop such that Testable prop, there is a method  $\text{property} :: \text{prop} \rightarrow \text{Property}$ .  
Outside of a class declaration, Haskell denotes this type as

```
property :: Testable prop ⇒ prop → Property
```



- ▶ Instances declare which types belong to a predicate.

**instance** Testable Bool

**instance** (Arbitrary a, Show a, Testable prop)  $\Rightarrow$   
Testable (a  $\rightarrow$  prop)

Booleans are in Testable.

Functions  $a \rightarrow \text{prop}$  are in Testable if prop is Testable and a is in Arbitrary and in Show.

- ▶ Instance declarations have to provide implementations of the class methods (in this case, of property), as a proof that the predicate does indeed hold for the type.
- ▶ Other functions that use class methods inherit the class constraints:

**quickCheck** :: Testable prop  $\Rightarrow$  prop  $\rightarrow$  IO ()



**instance** Testable Bool

```
sortAscending :: Bool
sortAscending = sort [2, 1] == [1, 2]

sortDescending :: Bool
sortDescending = sort [2, 1] == [2, 1]
```

Running QuickCheck:

```
Main> quickCheck sortAscending
+++ OK, passed 100 tests.

Main> quickCheck sortDescending
*** Failed! Falsifiable (after 1 test):
```



- ▶ Nullary properties are static properties.
- ▶ QuickCheck can be used for unit testing.
- ▶ By default, QuickCheck tests 100 times (which is wasteful for static properties, but configurable).



**instance** (Arbitrary a, Show a, Testable prop)  $\Rightarrow$   
Testable (a  $\rightarrow$  prop)

sortPreservesLength :: ([Int]  $\rightarrow$  [Int])  $\rightarrow$  [Int]  $\rightarrow$  Bool  
sortPreservesLength isort xs = length (isort xs) == length xs

Main> quickCheck (sortPreservesLength isort)  
+++ OK, passed 100 tests.

Read parameterized properties as universally quantified.  
QuickCheck automatically generates lists of integers.





```
import Data.Set  
setSort = toList o fromList
```



```
import Data.Set  
setSort = toList o fromList
```

```
Main> quickCheck (sortPreservesLength setSort)  
*** Failed! Falsifiable (after 6 tests and 2 shrinks):  
[1, 1]
```



```
import Data.Set  
setSort = toList o fromList
```

```
Main> quickCheck (sortPreservesLength setSort)  
*** Failed! Falsifiable (after 6 tests and 2 shrinks):  
[1, 1]
```

- ▶ The function `setSort` eliminates duplicate elements, therefore a list with duplicate elements causes the test to fail.
- ▶ `QuickCheck` shows evidence of the failure, and tries to present minimal test cases that fail (shrinking).



## Property 1

A sorted list should be ordered:

```
sortOrders :: [Int] → Bool
sortOrders xs = ordered (sort xs)

ordered :: Ord a ⇒ [a] → Bool
ordered []      = True
ordered [x]     = True
ordered (x : y : ys) = x ≤ y ∧ ordered (y : ys)
```



## Property 2

A sorted list should have the same elements as the original list:

```
sortPreservesElements :: [Int] → Bool
sortPreservesElements xs = sameElements xs (sort xs)

sameElements :: Eq a ⇒ [a] → [a] → Bool
sameElements xs ys = null (xs \\ ys) ∧ null (ys \\ xs)
```



| `collect :: (Testable prop, Show a)  $\Rightarrow$  a  $\rightarrow$  prop  $\rightarrow$  Property`

The function `collect` gathers statistics about test cases. This information is displayed when a test passes:



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The function `collect` gathers statistics about test cases. This information is displayed when a test passes:

```
Main> let p = sortPreservesLength isort
Main> quickCheck ( $\lambda$ xs  $\rightarrow$  collect (null xs) (p xs))
+++ OK, passed 100 tests:
92% False
 8% True
```



```
Main> quickCheck ( $\lambda$ xs  $\rightarrow$  collect (length xs 'div' 10) (p xs))  
+++ OK, passed 100 tests:  
31% 0  
24% 1  
16% 2  
9% 4  
9% 3  
4% 8  
4% 6  
2% 5  
1% 7
```





In the extreme case, we can show the actual data that is tested:

```
Main> quickCheck ( $\lambda$ xs  $\rightarrow$  collect xs (p xs))  
+++ OK, passed 100 tests:  
6% []  
1% [9, 4, -6, 7]  
1% [9, -1, 0, -22, 25, 32, 32, 0, 9, ...  
...
```

## Question

Why is it important to have access to the test data?



The function `insert` preserves an ordered list:

```
implies :: Bool → Bool → Bool  
implies x y = ¬ x ∨ y
```

Problematic:

```
insertPreservesOrdered :: Int → [Int] → Bool  
insertPreservesOrdered x xs =  
    ordered xs 'implies' ordered (insert x xs)
```



```
Main> quickCheck insertPreservesOrdered  
+++ OK, passed 100 tests.
```



```
Main> quickCheck insertPreservesOrdered  
+++ OK, passed 100 tests.
```

But:

```
Main> let iPO = insertPreservesOrdered  
Main> quickCheck ( $\lambda x\ xs \rightarrow \text{collect (ordered xs) (iPO x xs)}$ )  
+++ OK, passed 100 tests.  
88% False  
12% True
```

Only 12 lists have really been tested!



The solution is to use the QuickCheck implication operator:

```
( $\impl$ ) :: (Testable prop)  $\impl$  Bool  $\rightarrow$  prop  $\rightarrow$  Property
```

```
instance Testable Property
```

The type Property allows to encode not only True or False, but also to reject the test case.

```
iPO :: Int  $\rightarrow$  [Int]  $\rightarrow$  Property
```

```
iPO x xs = ordered xs  $\impl$  ordered (insert x xs)
```

Now we get:

```
Main> quickCheck ( $\lambda$ x xs  $\rightarrow$  collect (ordered xs) (iPO x xs))
```

```
*** Gave up! Passed only 43 tests (100% True).
```



```
quickCheckWith ::
```

```
  (Testable prop) ⇒ Int    -- maximum number tests  
                    → Int    -- maximum number attempts  
                    → Int    -- maximum size  
                    → prop  
                    → IO Bool
```

```
quickCheck p =
```

```
  do
```

```
    quickCheckWith 100 500 100 p  
  return ()
```

- ▶ Increasing the number of attempts might work.
- ▶ Better solution: use a custom generator (discussed next).



- ▶ Generators belong to an abstract data type `Gen`. Think of `Gen` as a restricted version of `IO`. The only effect available to us is access to random numbers.
- ▶ We can define our own generators using another domain-specific language. We can define default generators for new datatypes by defining instances of class `Arbitrary`:

```
class Arbitrary a where
```

```
  arbitrary :: Gen a
```

```
  shrink :: a → [a]
```



choose :: Random a  $\Rightarrow$  (a, a)  $\rightarrow$  Gen a  
oneof :: [Gen a]  $\rightarrow$  Gen a  
frequency :: [(Int, Gen a)]  $\rightarrow$  Gen a  
elements :: [a]  $\rightarrow$  Gen a  
sized :: (Int  $\rightarrow$  Gen a)  $\rightarrow$  Gen a





**instance** Arbitrary Bool **where**

arbitrary = choose (False, True)

**instance** (Arbitrary a, Arbitrary b)  $\Rightarrow$  Arbitrary (a, b) **where**

arbitrary = **do**

$x \leftarrow$  arbitrary

$y \leftarrow$  arbitrary

    return (x, y)

**data** Dir = North | East | South | West

**instance** Arbitrary Dir **where**

arbitrary = elements [North, East, South, West]



- ▶ A simple possibility:

```
instance Arbitrary Int where  
    arbitrary = choose (-20, 20)
```

- ▶ Better:

```
instance Arbitrary Int where  
    arbitrary = sized ( $\lambda n \rightarrow$  choose  $(-n, n)$ )
```

- ▶ QuickCheck automatically increases the size gradually, up to the configured maximum value.



A bad approach to generating more complex values is a frequency table:

**data** Tree a = Leaf a | Node (Tree a) (Tree a)

**instance** Arbitrary a  $\Rightarrow$  Arbitrary (Tree a) **where**  
arbitrary =  
frequency [(1, liftM Leaf arbitrary),  
(2, liftM2 Node arbitrary arbitrary)]

Here:

liftM :: (a  $\rightarrow$  b)  $\rightarrow$  Gen a  $\rightarrow$  Gen b

liftM2 :: (a  $\rightarrow$  b  $\rightarrow$  c)  $\rightarrow$  Gen a  $\rightarrow$  Gen b  $\rightarrow$  Gen c



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Here:

liftM :: (a  $\rightarrow$  b)  $\rightarrow$  Gen a  $\rightarrow$  Gen b

liftM2 :: (a  $\rightarrow$  b  $\rightarrow$  c)  $\rightarrow$  Gen a  $\rightarrow$  Gen b  $\rightarrow$  Gen c

Termination is unlikely!



```
instance Arbitrary a  $\Rightarrow$  Arbitrary (Tree a) where  
  arbitrary = sized arbitraryTree  
  arbitraryTree :: Arbitrary a  $\Rightarrow$  Int  $\rightarrow$  Gen (Tree a)  
  arbitraryTree 0 = liftM Leaf arbitrary  
  arbitraryTree n = frequency [(1, liftM Leaf arbitrary),  
                                (4, liftM2 Node t t)]  
  where t = arbitraryTree (n 'div' 2)
```

Why a non-zero probability for Leaf in the second case of arbitraryTree?



The other method in Arbitrary is

| shrink :: (Arbitrary a) ⇒ a → [a]

- ▶ Maps each value to a number of structurally smaller values.
- ▶ Default definition returns [] and is always safe.
- ▶ When a failing test case is discovered, shrink is applied repeatedly until no smaller failing test case can be obtained.



- ▶ Both arbitrary and shrink are examples of **datatype-generic** functions – they can be defined for (almost) any Haskell datatype in a systematic way.
- ▶ Haskell does not provide any way to write down such an algorithm.
- ▶ Many extensions and tools do (cf. course on Generic Programming in block 4).



All lists are ordered?

```
Main> quickCheck ordered
+++ OK, passed 100 tests.
```





All lists are ordered?

```
Main> quickCheck ordered  
+++ OK, passed 100 tests.
```

Use type signatures in GHCi to make sure a sensible type is used!

```
Main> quickCheck (ordered :: [Int] → Bool)  
*** Failed! Falsifiable (after 3 tests and 2 shrinks):  
[0, -1]
```



- ▶ Haskell can deal with infinite values, and so can QuickCheck. However, properties must not inspect infinitely many values. For instance, we cannot compare two infinite values for equality and still expect tests to terminate. Solution: Only inspect finite parts.



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- ▶ Haskell can deal with infinite values, and so can QuickCheck. However, properties must not inspect infinitely many values. For instance, we cannot compare two infinite values for equality and still expect tests to terminate. Solution: Only inspect finite parts.
- ▶ QuickCheck can generate functional values automatically, but this requires defining an instance of another class `CoArbitrary`. Also, showing functional values is problematic.
- ▶ QuickCheck has facilities for testing properties that involve IO, but this is more difficult than testing pure properties.



- ▶ The next lecture (Wednesday) is on “Laziness and Debugging” .
- ▶ Lab session Wednesday at 09:00.
- ▶ Deadline for the first weekly assignments is Thursday.
- ▶ Programming project online today.

