

Space and Congruence Compression of Proofs

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Congruence

Example

Knowledge

- ① $f(a) = a$
- ② $a = b$
- ③ $b = f(b)$
- ④ $f(a) \neq f(b)$

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Unsatisfiable!

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Proof

Equality is transitive, therefore from $f(a) = a$, $a = b$ and $b = f(b)$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

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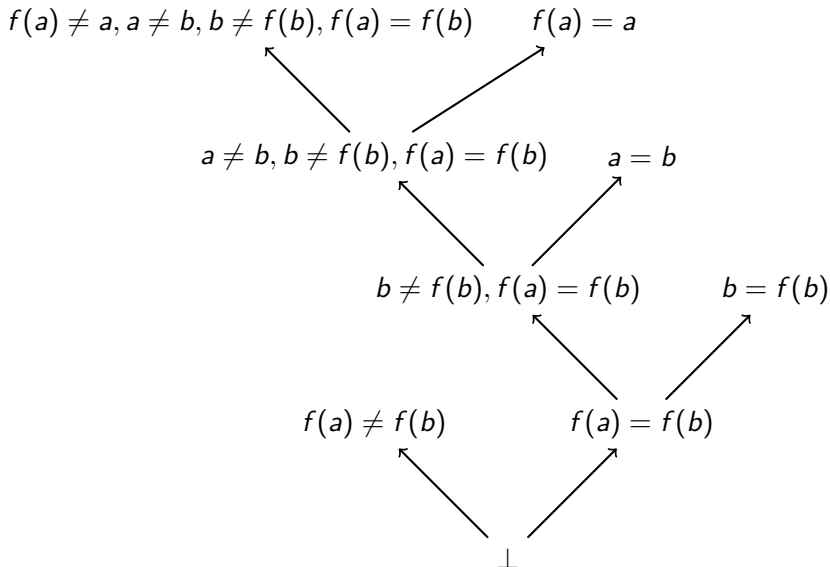
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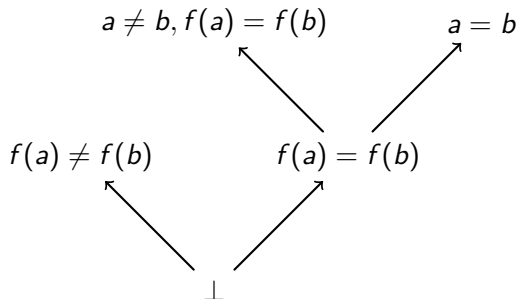
A different Proof

$f(.)$ is a function, therefore from $a = b$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

A proof



A different proof



Ground Terms

- Constants a, b, c, \dots
- Compound Terms $f(t_1, \dots, t_n)$

Definitions

Ground Terms

- Constants a, b, c, \dots
- Compound Terms $f(t_1, \dots, t_n)$

Congruence Relation R

- Reflexive: $\forall t (t, t) \in R$
- Symmetric: $(s, t) \in R \Rightarrow (t, s) \in R$
- Transitive: $(t_1, t_2) \in R \dots (t_{m-1}, t_m) \in R \Rightarrow (t_1, t_m) \in R$
- Compatible: $\forall i (t_i, s_i) \in R \Rightarrow (f(t_1, \dots, t_n), f(s_1, \dots, s_n)) \in R$

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Congruence Closure E^* of set of equations E

- Smallest Congruence Relation containing E
- Computable in $O(n \log(n))$
- E is explanation for $(s, t) \in E^*$

Example continued

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Explanation for $f(a) = f(b)$

$\{ f(a) = a, a = b, b = f(b) \}$

Example continued

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Explanation for $f(a) = f(b)$

{ $a = b, b = f(b)$ }

Example continued

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- ① $f(a) = a$
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Example continued

Knowledge

- ❶ $f(a) = a$
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- ❸ $b = f(b)$
- ❹ $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

{ $a = b$ }

Short explanation \rightsquigarrow short (sub)proof

Short Explanation Decision Problem

Given a set of input equations E , a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

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NP-complete

NP-completeness proof sketch

From a propositional logic formula Φ obtain ...

- a set of equations E_Φ
- a target equation $s_\Phi = t_\Phi$
- $k_\Phi \in \mathbb{N}$

such that ...

Φ is satisfiable if and only if there is an explanation $E' \subseteq E_\Phi$ of $s_\Phi = t_\Phi$ with $|E'| \leq k_\Phi$

NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

NP-completeness proof sketch example

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Translation to equations

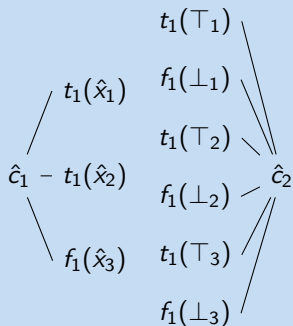
$$\begin{array}{c} t_1(\hat{x}_1) \\ \diagdown \\ \hat{c}_1 - t_1(\hat{x}_2) \\ \diagup \\ f_1(\hat{x}_3) \end{array}$$

NP-completeness proof sketch example

Formula

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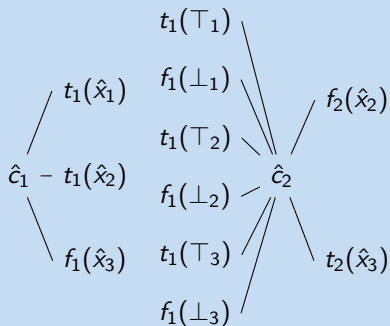


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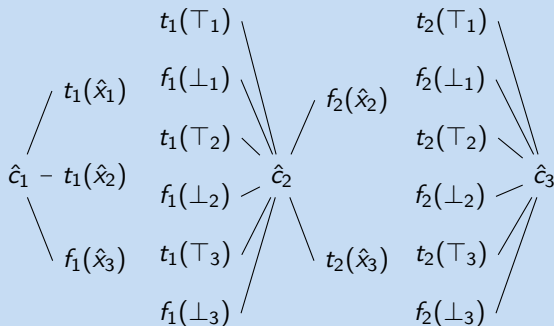


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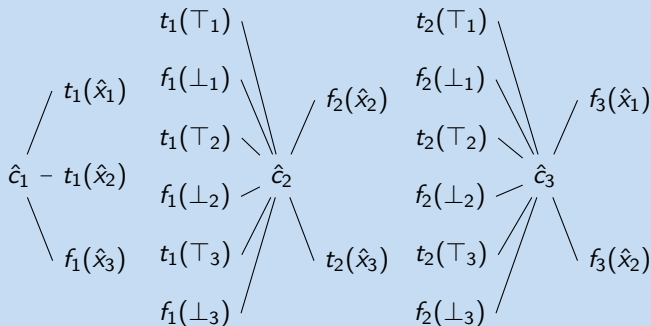


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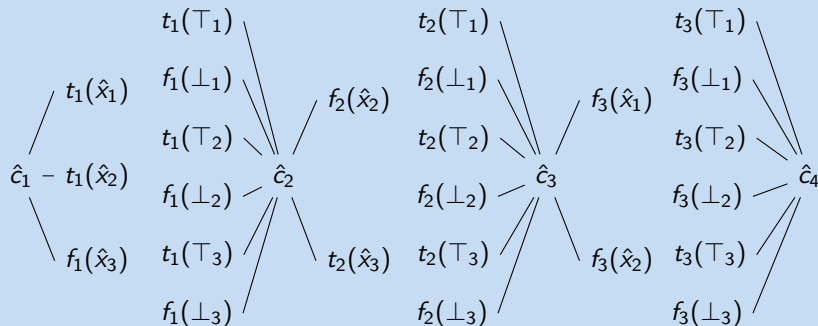


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Translation to equations

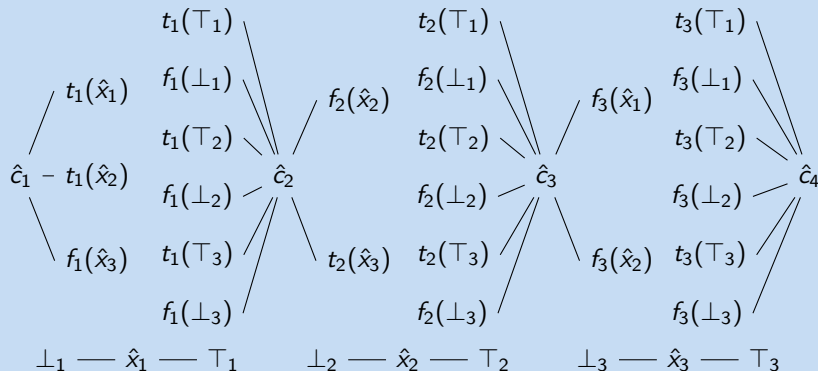


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Translation to equations



NP-completeness proof sketch example

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Small subset corresponding to satisfying assignment

$$\hat{c}_1 = t_1(\hat{x}_1) \quad t_1(\top_1) = \hat{c}_2 = f_2(\hat{x}_2) \quad f_2(\perp_2) = \hat{c}_3 = f_3(\hat{x}_2) \quad f_3(\perp_2) = \hat{c}_4$$

$$\hat{x}_1 \text{ --- } \top_1$$

$$\perp_2 \text{ --- } \hat{x}_2$$

$$\hat{x}_3 \text{ --- } \top_3$$

Space

Is it possible to remove parts of the proof from memory during proof processing?

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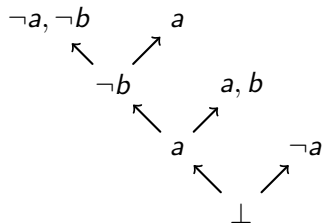
Which parts?

Is it possible to remove parts of the proof from memory during proof processing?

Space measure of a proof

Maximal amount of nodes that have to be kept in memory at once while processing the proof

Space Measure Example



Space Measure Example

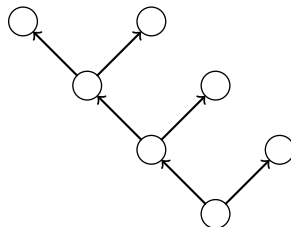
Maximum number of nodes in memory: 0



Not in memory



In memory



Space Measure Example

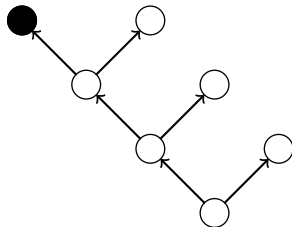
Maximum number of nodes in memory: 1



Not in memory



In memory



Space Measure Example

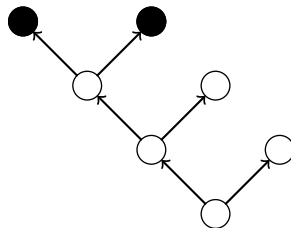
Maximum number of nodes in memory: 2



Not in memory



In memory



Space Measure Example

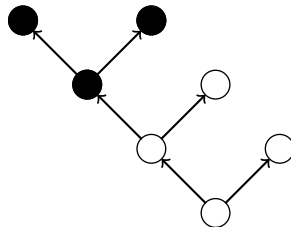
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

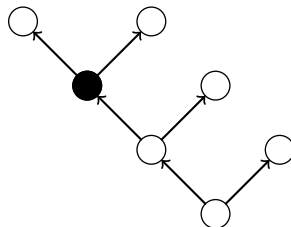
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

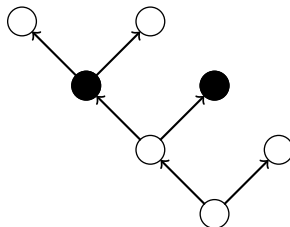
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

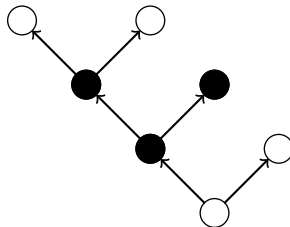
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

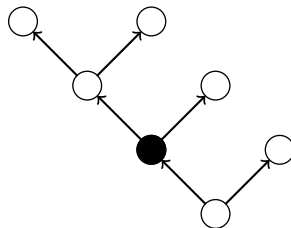
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

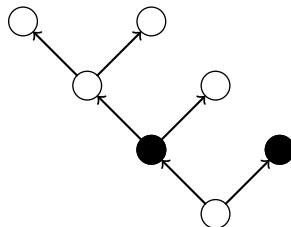
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

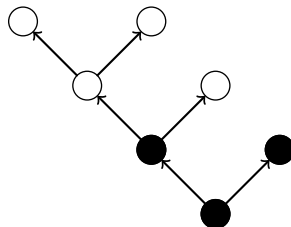
Maximum number of nodes in memory: 3



Not in memory



In memory



Space Measure Example

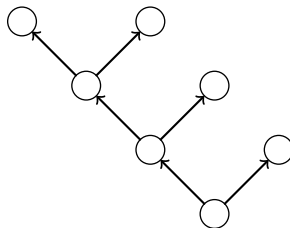
Maximum number of nodes in memory: 0



Not in memory



In memory



Space Measure Example

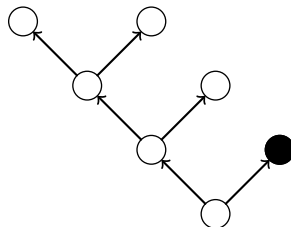
Maximum number of nodes in memory: 1



Not in memory



In memory



Space Measure Example

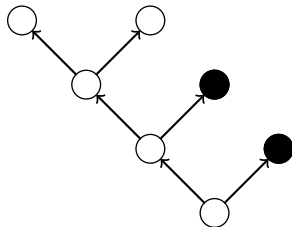
Maximum number of nodes in memory: 2



Not in memory

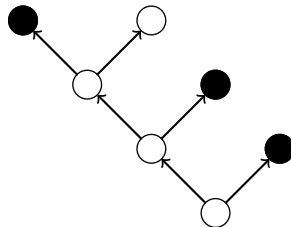
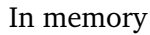
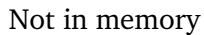


In memory



Space Measure Example

Maximum number of nodes in memory: 3



Space Measure Example

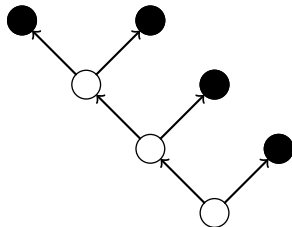
Maximum number of nodes in memory: 4



Not in memory



In memory



Space Measure Example

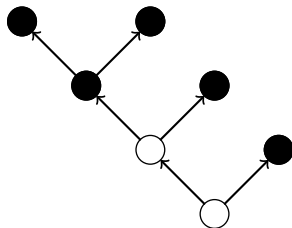
Maximum number of nodes in memory: 5



Not in memory



In memory



Space Measure Example

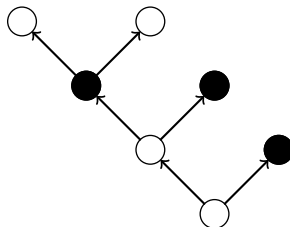
Maximum number of nodes in memory: 5



Not in memory



In memory



Space Measure Example

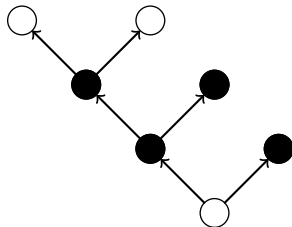
Maximum number of nodes in memory: 5



Not in memory



In memory



Space Measure Example

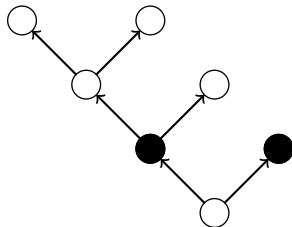
Maximum number of nodes in memory: 5



Not in memory



In memory



Space Measure Example

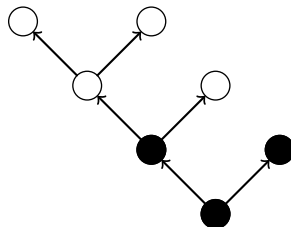
Maximum number of nodes in memory: 5



Not in memory



In memory



Space Measure Example

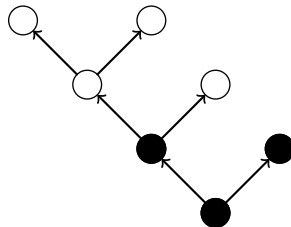
Maximum number of nodes in memory: 5



Not in memory



In memory



Good Traversal Orders are key!

Construct Traversal Orders

Construct Optimal Order

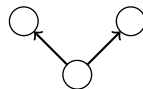
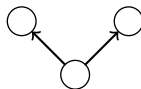
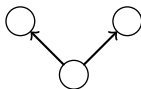
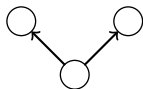
- NP-complete
- Optimal strategy in some pebbling game

Construct Good Order

- Top-Down
- Bottom-Up
- Heuristics

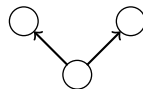
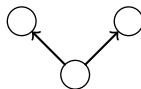
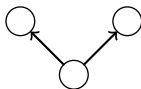
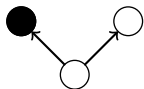
Construct Order Top-Down

Maximum number of nodes in memory: 0



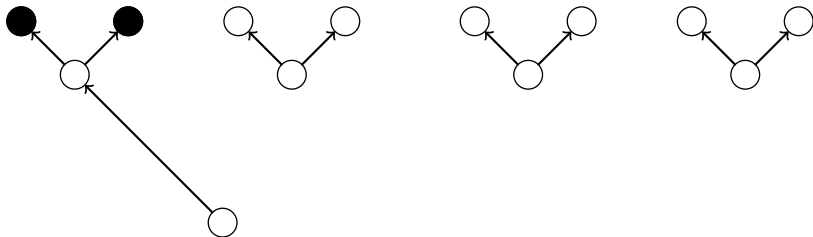
Construct Order Top-Down

Maximum number of nodes in memory: 1



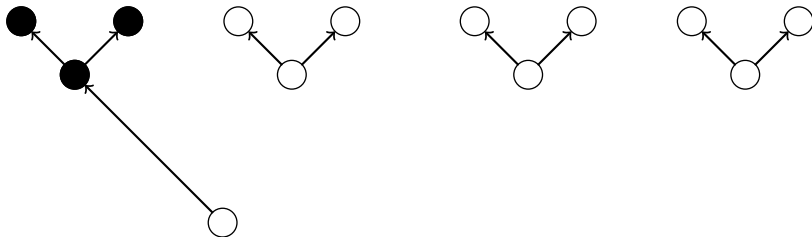
Construct Order Top-Down

Maximum number of nodes in memory: 2



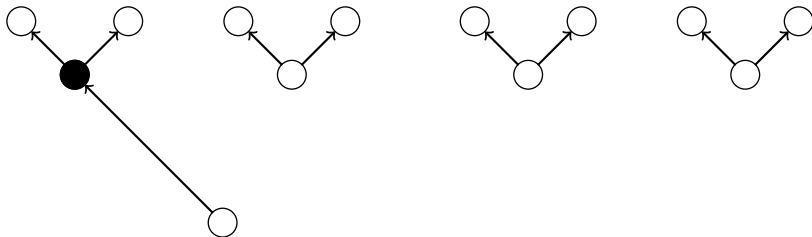
Construct Order Top-Down

Maximum number of nodes in memory: 3



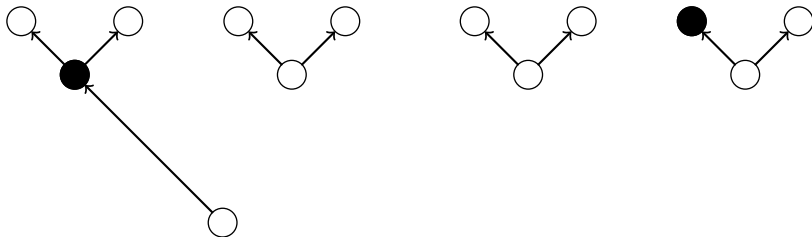
Construct Order Top-Down

Maximum number of nodes in memory: 3



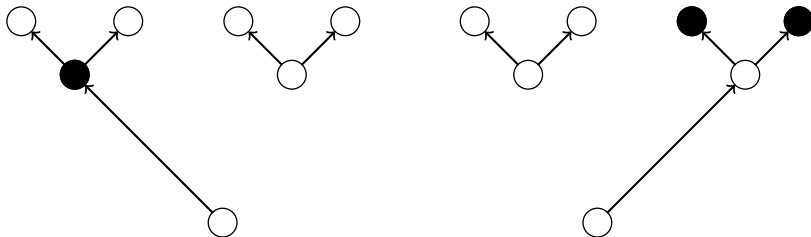
Construct Order Top-Down

Maximum number of nodes in memory: 3



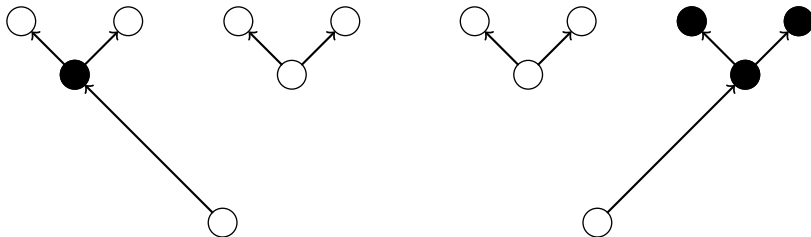
Construct Order Top-Down

Maximum number of nodes in memory: 3



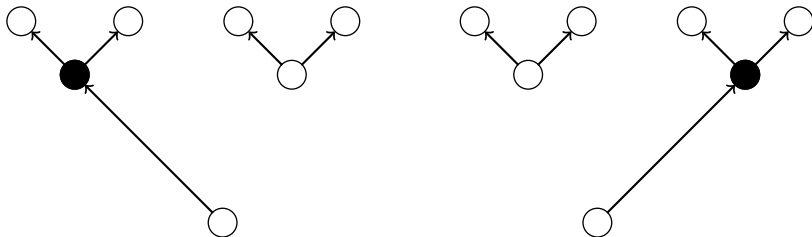
Construct Order Top-Down

Maximum number of nodes in memory: 4



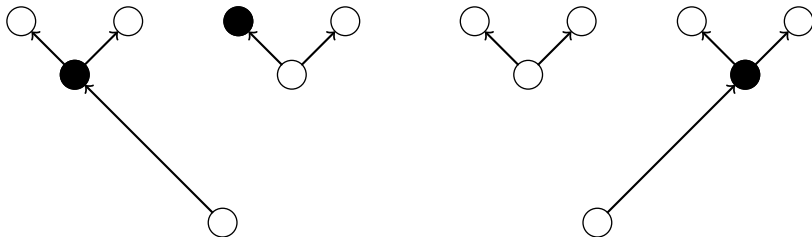
Construct Order Top-Down

Maximum number of nodes in memory: 4



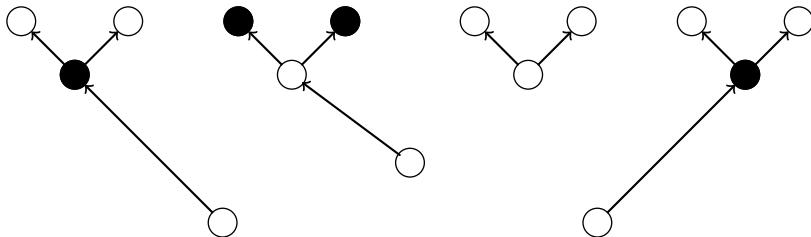
Construct Order Top-Down

Maximum number of nodes in memory: 4



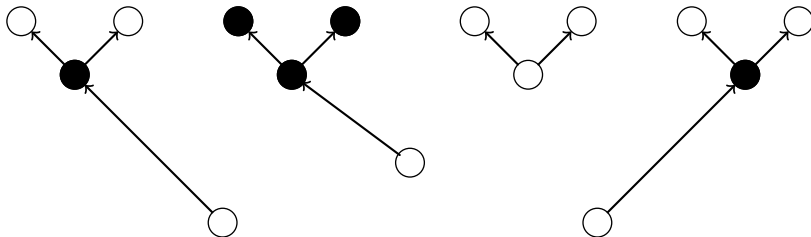
Construct Order Top-Down

Maximum number of nodes in memory: 4



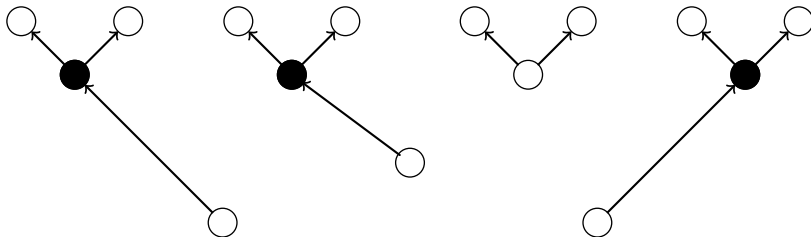
Construct Order Top-Down

Maximum number of nodes in memory: 5



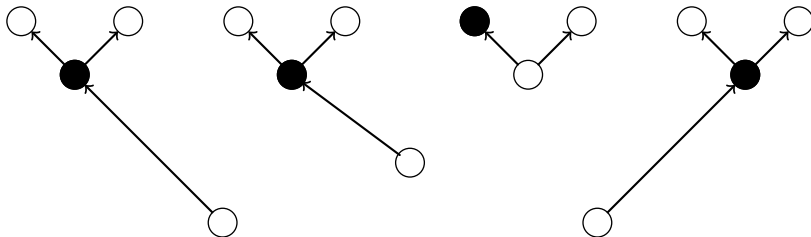
Construct Order Top-Down

Maximum number of nodes in memory: 5



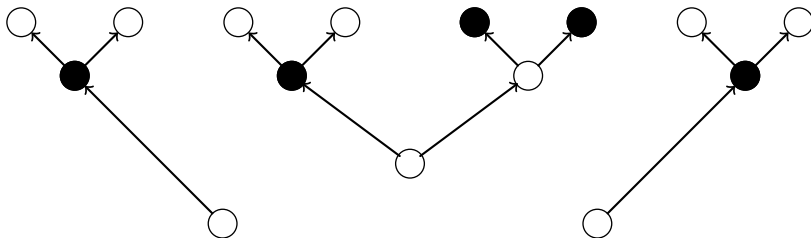
Construct Order Top-Down

Maximum number of nodes in memory: 5



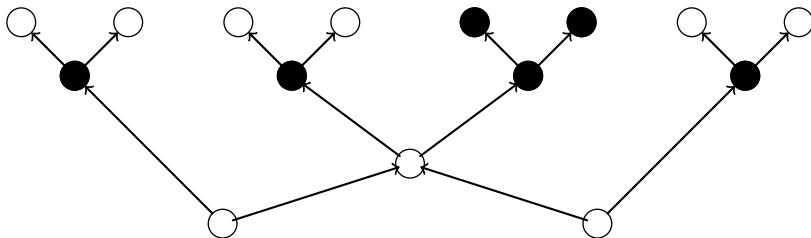
Construct Order Top-Down

Maximum number of nodes in memory: 5



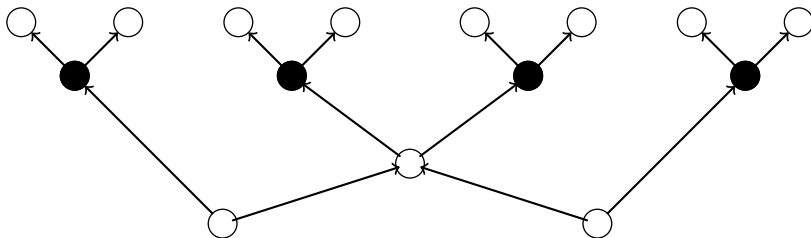
Construct Order Top-Down

Maximum number of nodes in memory: 6



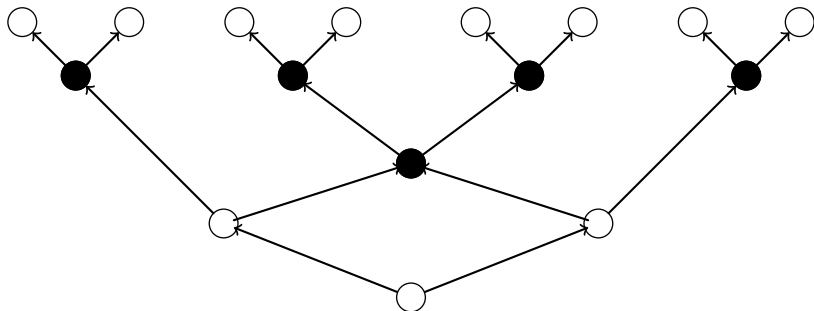
Construct Order Top-Down

Maximum number of nodes in memory: 6



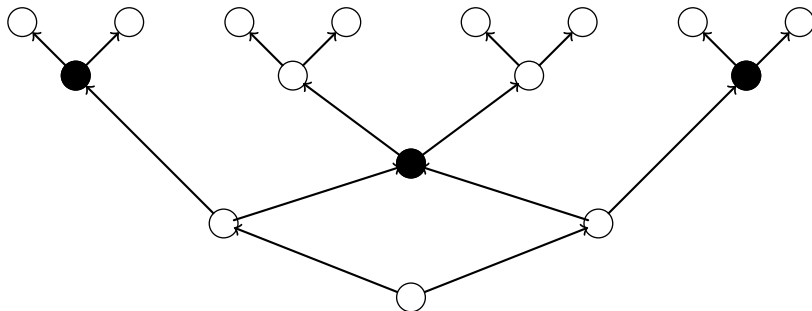
Construct Order Top-Down

Maximum number of nodes in memory: 6



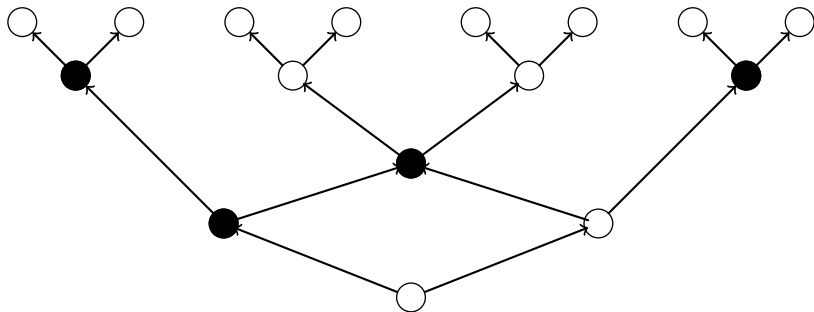
Construct Order Top-Down

Maximum number of nodes in memory: 6



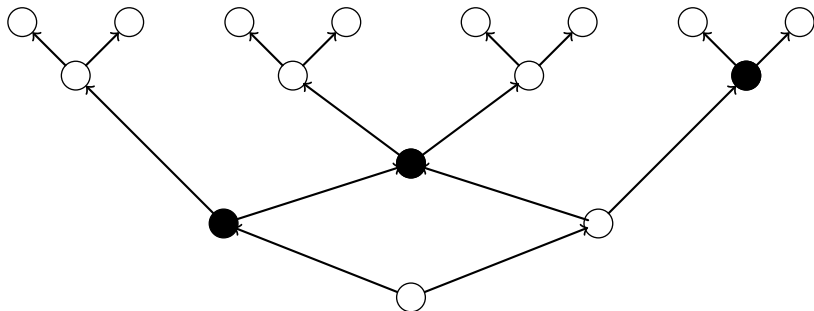
Construct Order Top-Down

Maximum number of nodes in memory: 6



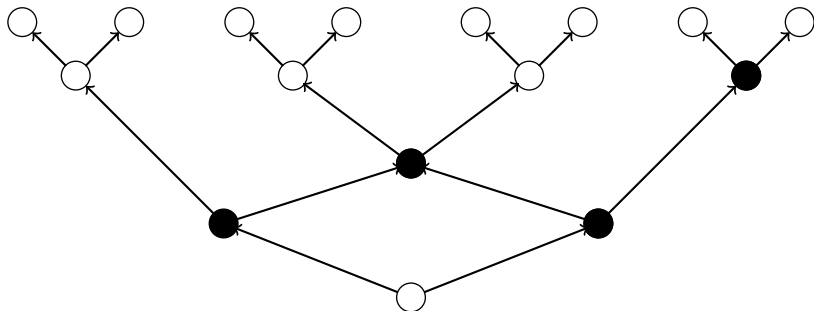
Construct Order Top-Down

Maximum number of nodes in memory: 6



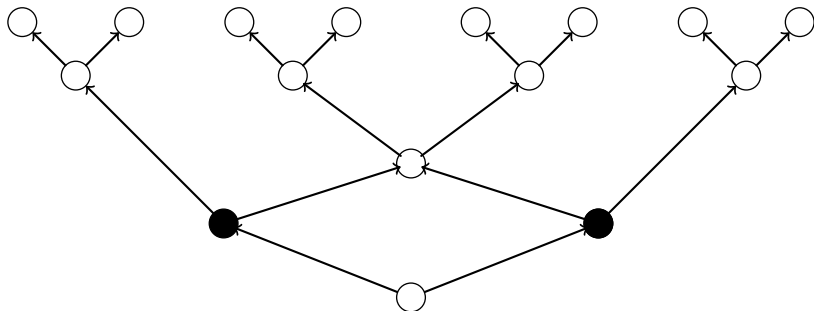
Construct Order Top-Down

Maximum number of nodes in memory: 6



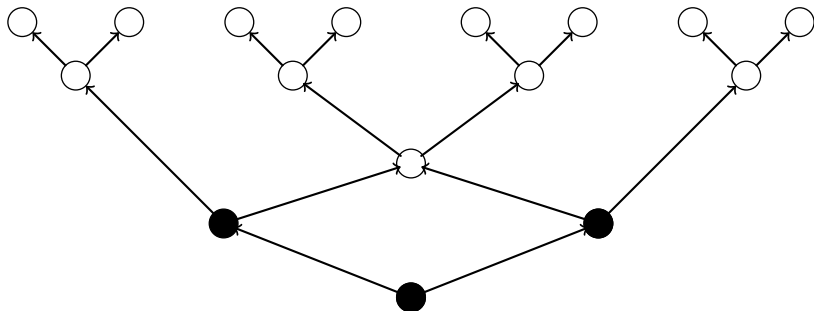
Construct Order Top-Down

Maximum number of nodes in memory: 6



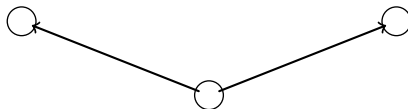
Construct Order Top-Down

Maximum number of nodes in memory: 6



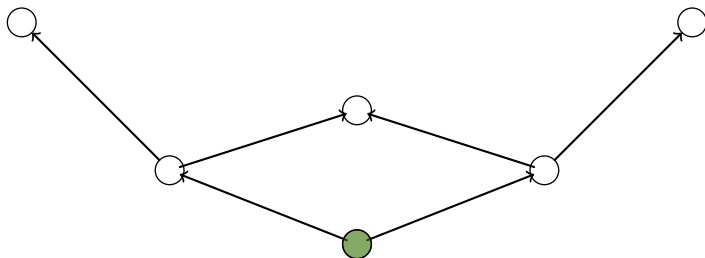
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



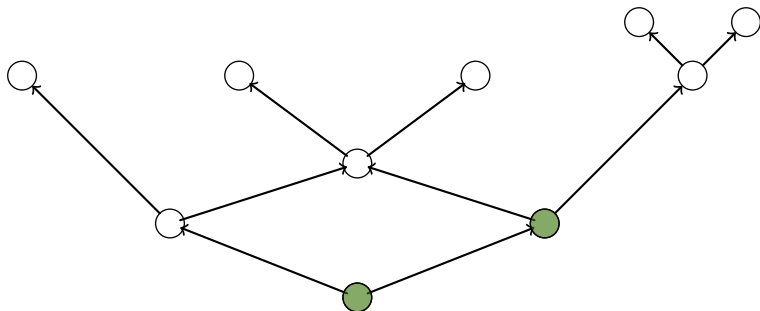
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



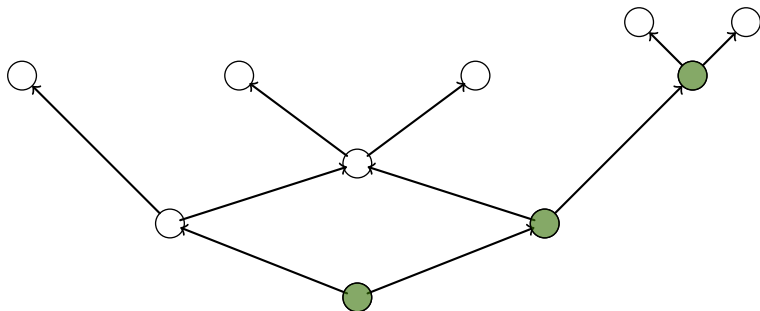
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



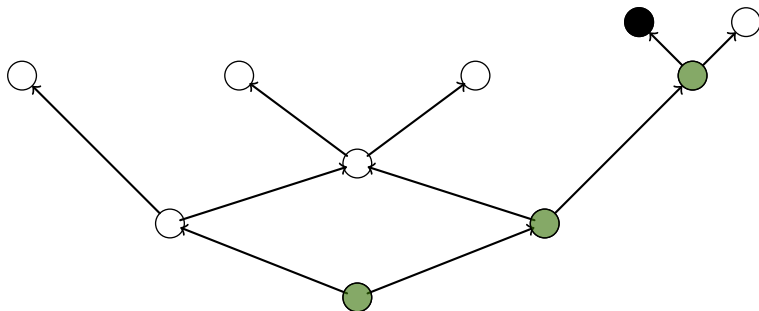
Construct Order Bottom-Up

Maximum number of nodes in memory: 0



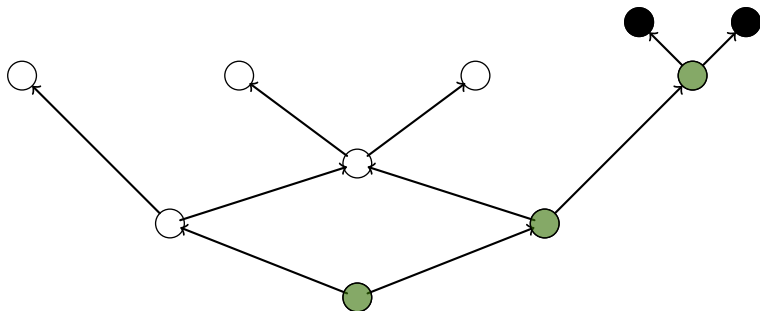
Construct Order Bottom-Up

Maximum number of nodes in memory: 1



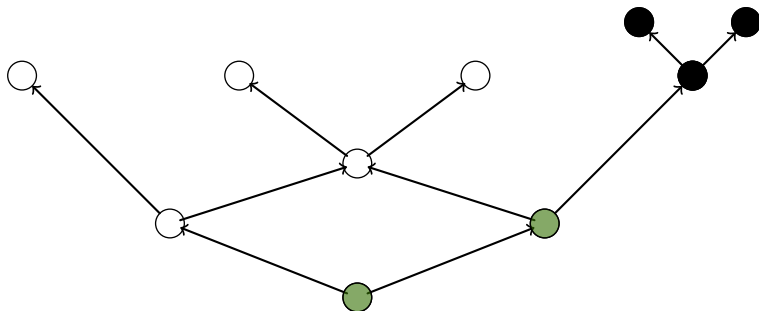
Construct Order Bottom-Up

Maximum number of nodes in memory: 2



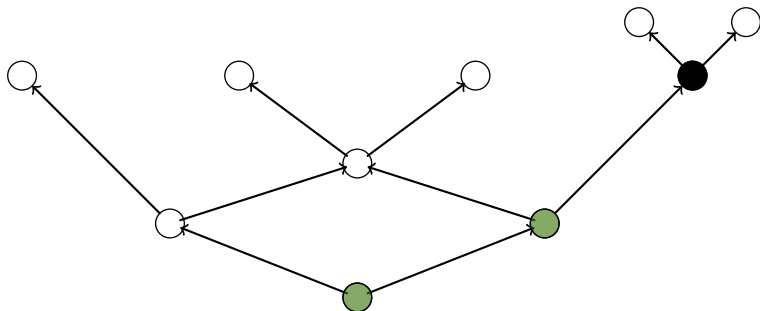
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



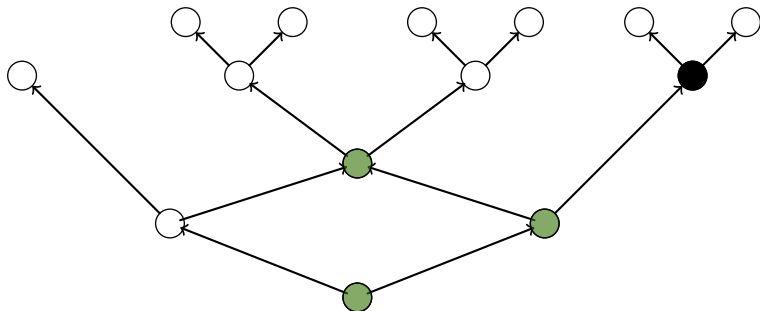
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



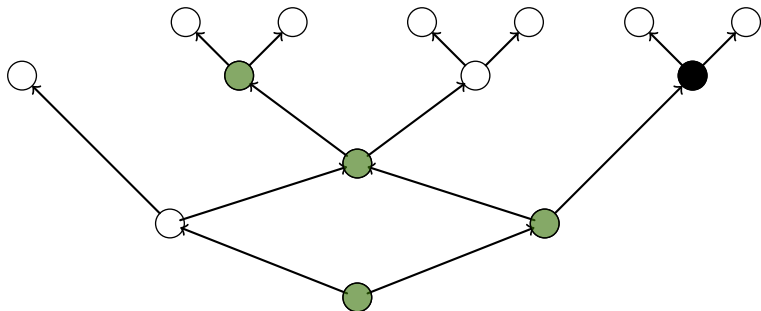
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



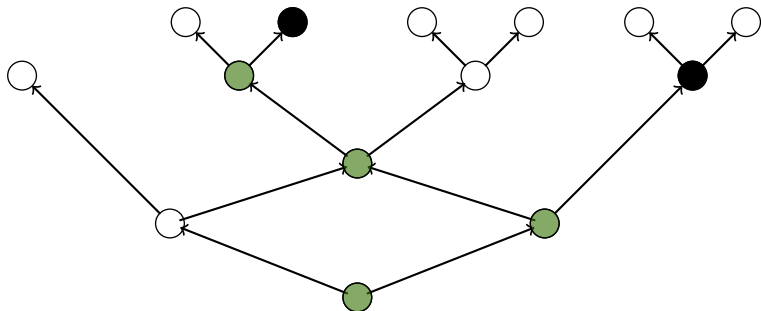
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



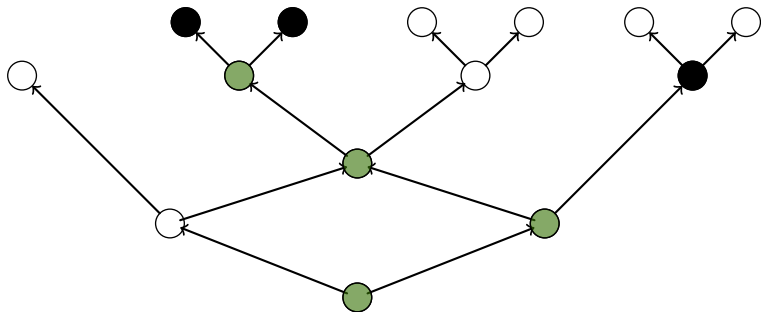
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



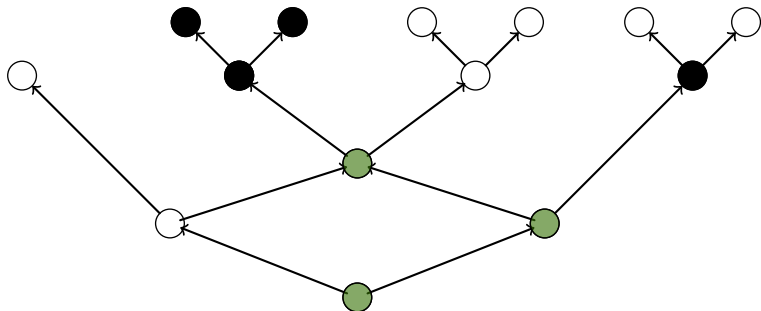
Construct Order Bottom-Up

Maximum number of nodes in memory: 3



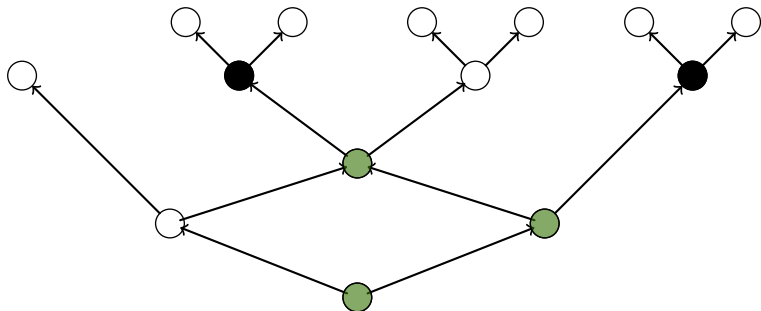
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



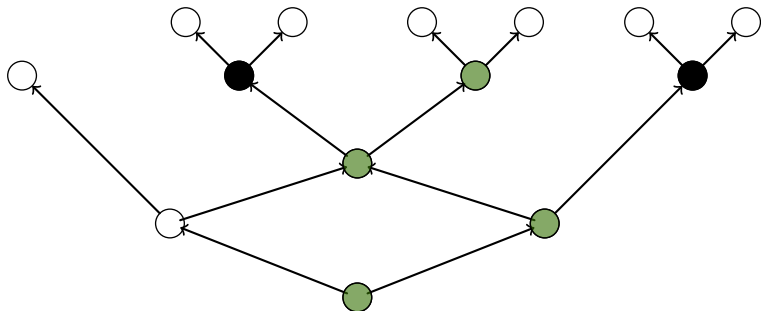
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



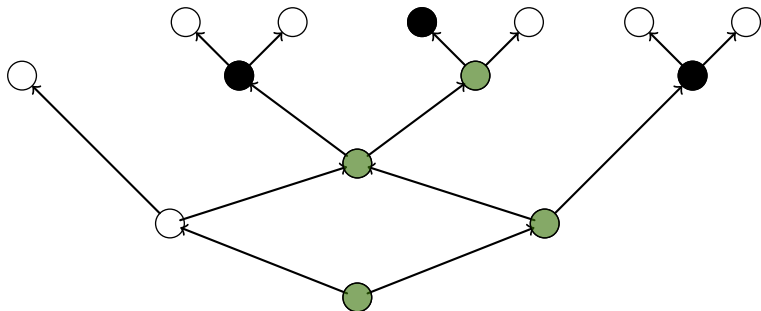
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



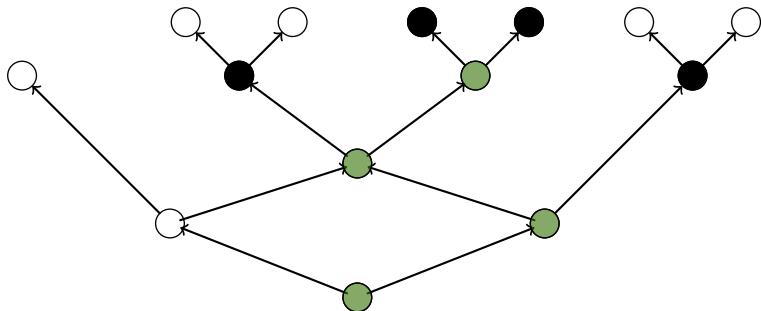
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



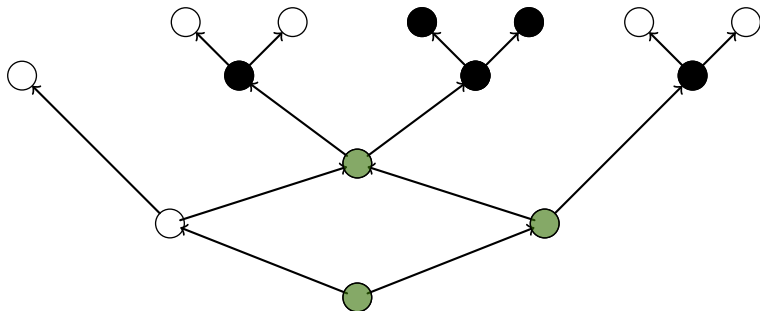
Construct Order Bottom-Up

Maximum number of nodes in memory: 4



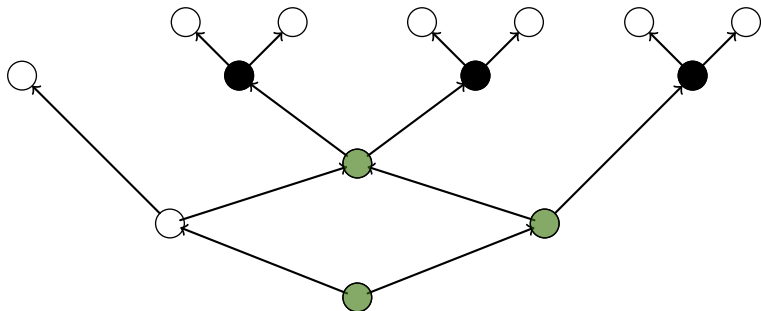
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



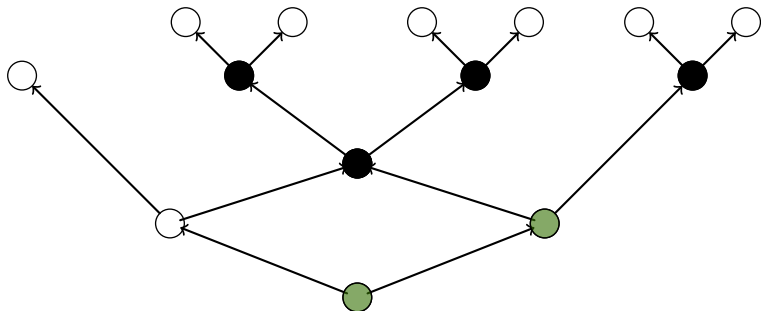
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



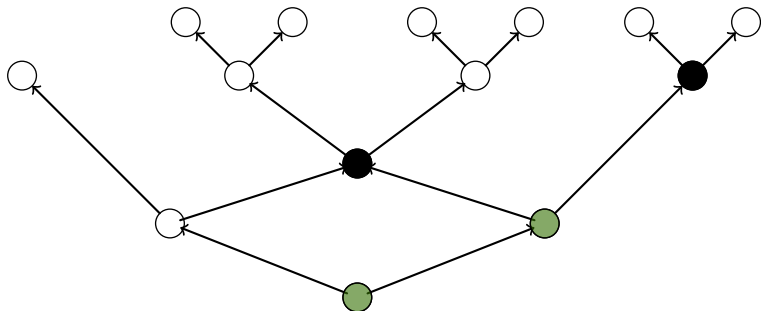
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



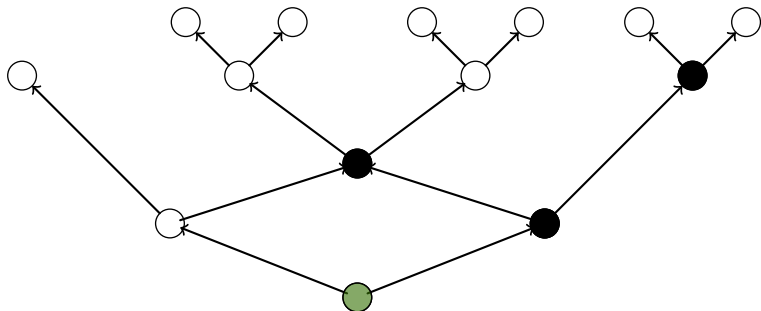
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



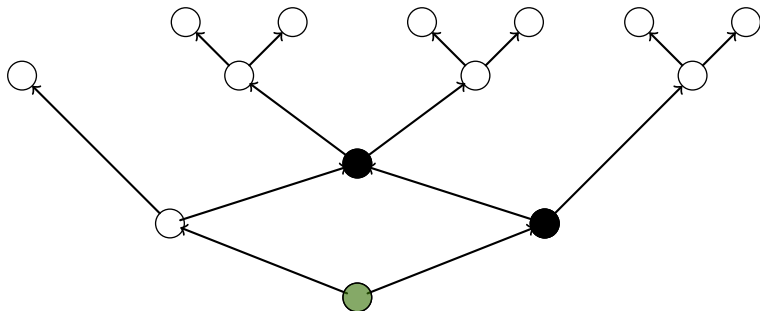
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



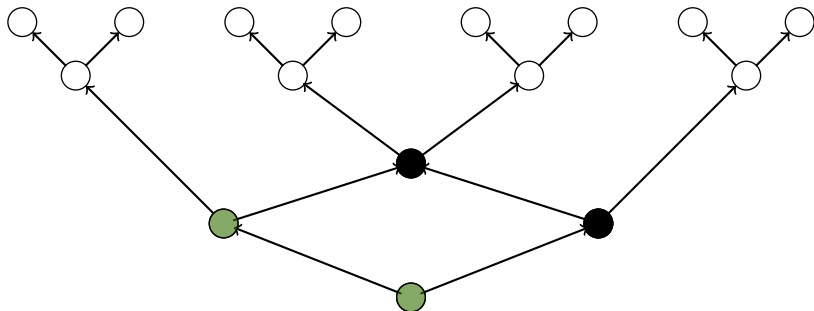
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



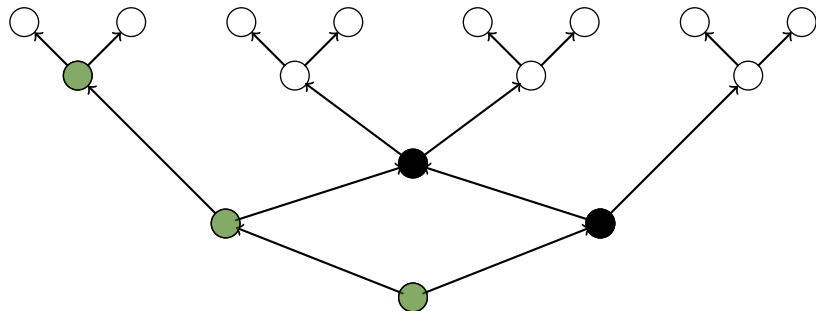
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



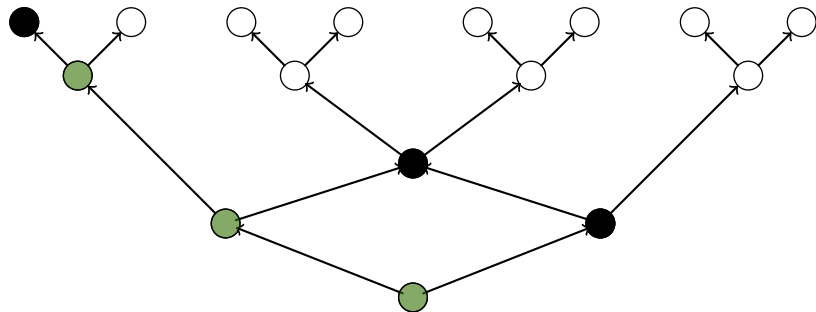
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



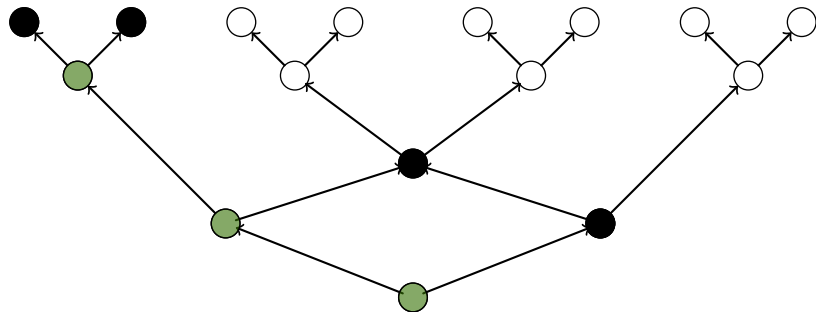
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



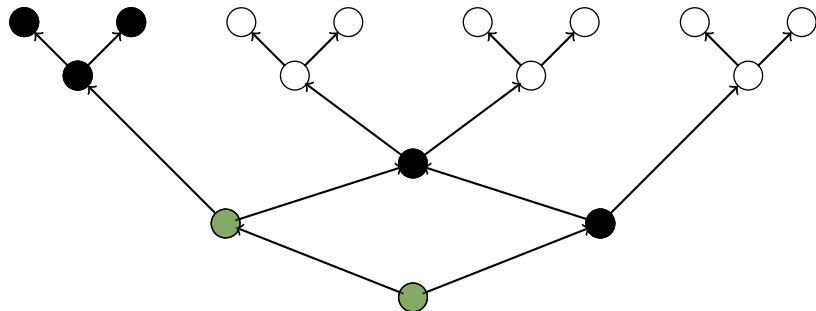
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



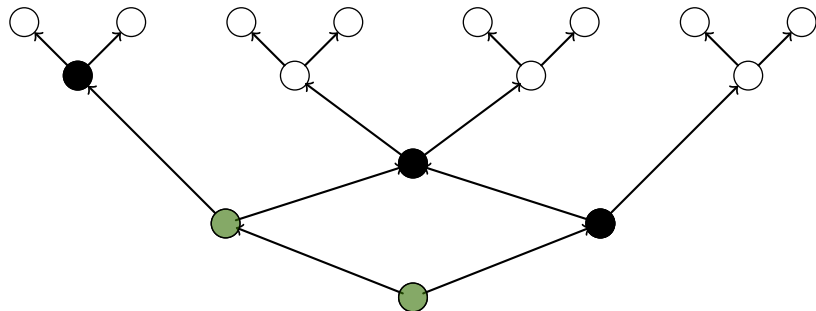
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



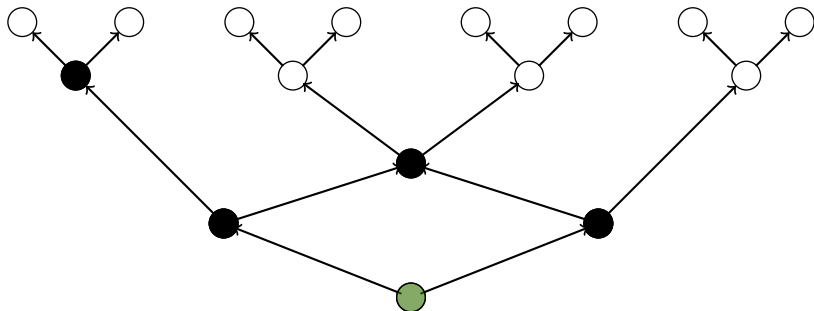
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



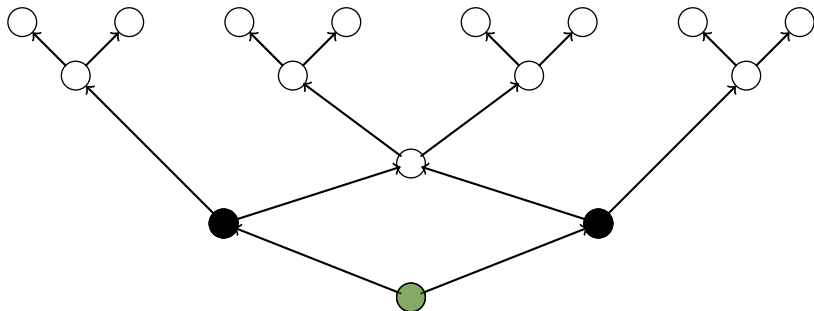
Construct Order Bottom-Up

Maximum number of nodes in memory: 5



Construct Order Bottom-Up

Maximum number of nodes in memory: 5



Construct Order Bottom-Up

Maximum number of nodes in memory: 5

