Space and Congruence Compression of Proofs

Andreas Fellner

European Master in Computational Logic

Master Thesis Presentation
Vienna, 23rd of September 2014

Congruence

Example

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Example

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Unsatisfiable!

Example

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Unsatisfiable!

Proof

Equality is transitive, therefore from $f(a) = a$, $a = b$ and $b = f(b)$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

Example

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Unsatisfiable!

Proof

Equality is transitive, therefore from $f(a) = a$, $a = b$ and $b = f(b)$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

A different Proof

$f(.)$ is a function, therefore from $a = b$ follows $f(a) = f(b)$, which contradicts $f(a) \neq f(b)$

A proof

$$
\begin{tikzcd}
f(a) \neq a, a \neq b, b \neq f(b), f(a) = f(b) & & f(a) = a & & \\
& a \neq b, b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[ur] & & a = b & \\
& & b \neq f(b), f(a) = f(b) \arrow[ul] \arrow[ur] & & b = f(b) \\
& f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
& & \perp \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

A different proof

$$\begin{tikzcd}
 & a \neq b, f(a) = f(b) & & a = b \\
f(a) \neq f(b) & & f(a) = f(b) \arrow[ul] \arrow[ur] & \\
 & \bot \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Definitions

Definitions

Ground Terms

- Constants $a, b, c, \ldots$
- Compound Terms $f(t_1, \ldots, t_n)$

Definitions

Ground Terms

- Constants $a, b, c, \ldots$
- Compound Terms $f(t_1, \ldots, t_n)$

Congruence Relation R

- Reflexive: $\forall t (t, t) \in R$
- Symmetric: $(s, t) \in R \Rightarrow (t, s) \in R$
- Transitive: $(t_1, t_2) \in R \ldots (t_{m-1}, t_m) \in R \Rightarrow (t_1, t_m) \in R$
- Compatible: $\forall i (t_i, s_i) \in R \Rightarrow (f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)) \in R$

Definitions

Ground Terms

- Constants $a, b, c, \ldots$
- Compound Terms $f(t_1, \ldots, t_n)$

Congruence Relation R

- Reflexive: $\forall t (t, t) \in R$
- Symmetric: $(s, t) \in R \Rightarrow (t, s) \in R$
- Transitive: $(t_1, t_2) \in R \ldots (t_{m-1}, t_m) \in R \Rightarrow (t_1, t_m) \in R$
- Compatible: $\forall i (t_i, s_i) \in R \Rightarrow (f(t_1, \ldots, t_n), f(s_1, \ldots, s_n)) \in R$

Congruence Closure E^* of set of equations E

- Smallest Congruence Relation containing E
- Computable in $O(n \log(n))$
- E is explanation for $(s, t) \in E^*$

Example continued

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Example continued

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ f(a) = a, a = b , b = f(b) \}$

Example continued

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ \quad a = b \, , b = f(b) \ \}$

Example continued

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ \quad a = b \quad \}$

Example continued

Knowledge

1. $f(a) = a$
2. $a = b$
3. $b = f(b)$
4. $f(a) \neq f(b)$

Explanation for $f(a) = f(b)$

$\{ \quad a = b \quad \}$

Short explanation $\rightsquigarrow$ short (sub)proof

Short Explanation Decision Problem

Given a set of input equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

Short Explanation Decision Problem

Given a set of input equations E, a target equation $s = t$ and $k \in \mathbb{N}$, does there exist an explanation $E' \subseteq E$ of $s = t$ with $|E'| \leq k$?

NP-complete

NP-completeness proof sketch

From a propositional logic formula Φ obtain ...

- a set of equations E_Φ
- a target equation $s_\Phi = t_\Phi$
- $k_\Phi \in \mathbb{N}$

such that ...

Φ is satisfiable if and only if there is an explanation $E' \subseteq E_\Phi$ of $s_\Phi = t_\Phi$ with $|E'| \leq k_\Phi$

NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations

NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations

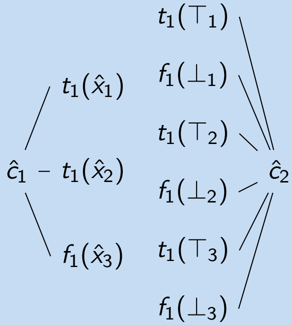
$$\hat{c}_1 \begin{array}{l} \diagup \; t_1(\hat{x}_1) \\ - \; t_1(\hat{x}_2) \\ \diagdown \; f_1(\hat{x}_3) \end{array}$$

NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations

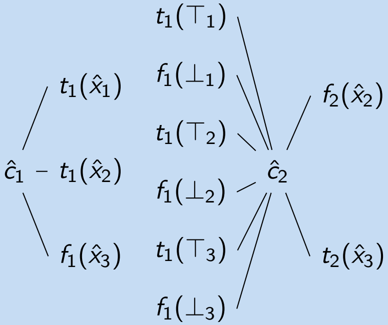


NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

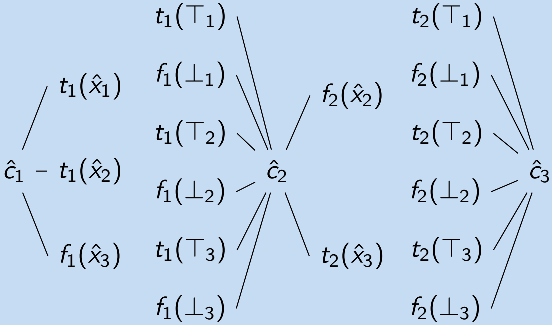


NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations

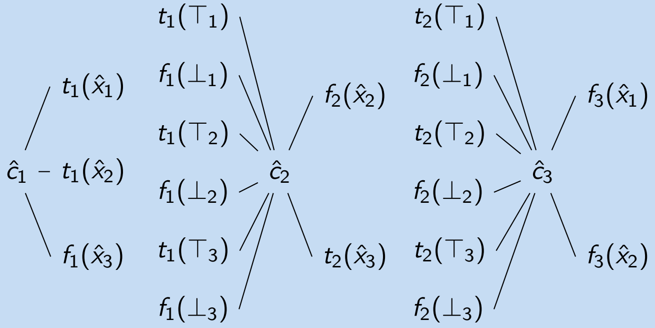


NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

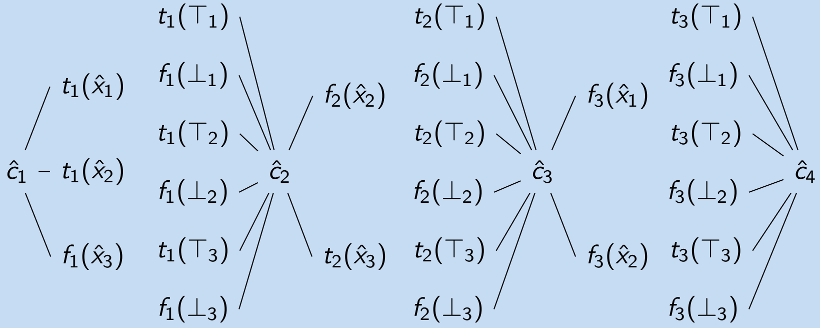


NP-completeness proof sketch example

Formula

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Translation to equations

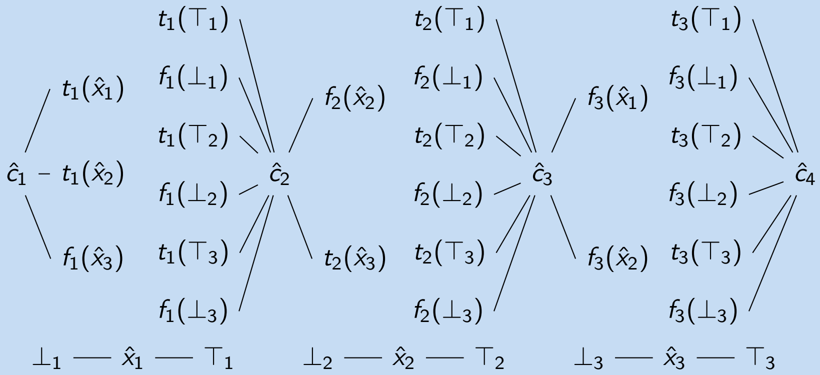


NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Translation to equations



NP-completeness proof sketch example

Formula

$$(x_1 \lor x_2 \lor \neg x_3) \land (\neg x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2)$$

Small subset corresponding to satisfying assignment

$\hat{c}_1 - t_1(\hat{x}_1) \quad t_1(\top_1) - \hat{c}_2 - f_2(\hat{x}_2) \quad f_2(\bot_2) - \hat{c}_3 - f_3(\hat{x}_2) \quad f_3(\bot_2) - \hat{c}_4$

$\hat{x}_1 — \top_1 \qquad \bot_2 — \hat{x}_2 \qquad \hat{x}_3 —— \top_3$

Space

Space Compression

Space measure of a proof

Maximal amount of nodes that have to be kept in memory at once while processing the proof

Deletion information

- Extra lines in proof output
- Example: y is the last child of x
  - Read and check node x

    ...
  - Read and check node y
  - Delete node x

    ...

Space Measure Example

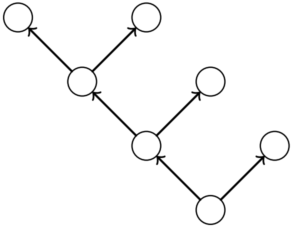
$$
\begin{tikzcd}
\neg a, \neg b & & a & & \\
& \neg b \arrow[ul] \arrow[ur] & & a, b & \\
& & a \arrow[ul] \arrow[ur] & & \neg a \\
& & & \perp \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Space Measure Example

Maximum pebbles used: 0

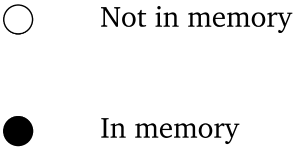
○ Not in memory

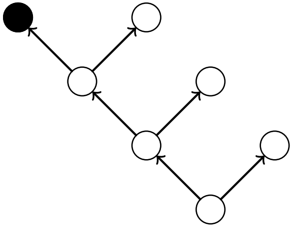
● In memory



Space Measure Example

Maximum pebbles used: 1



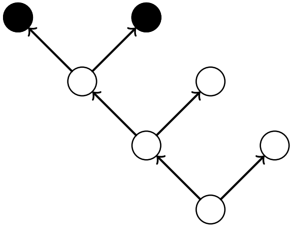


Space Measure Example

Maximum pebbles used: 2

Not in memory

In memory

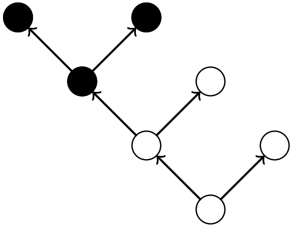


Space Measure Example

Maximum pebbles used: 3

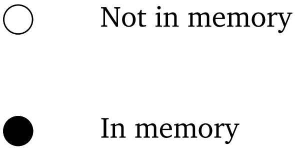
○ Not in memory

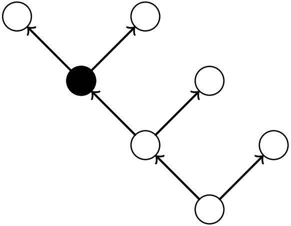
● In memory



Space Measure Example

Maximum pebbles used: 3



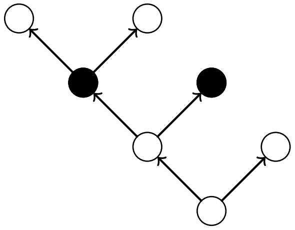


Space Measure Example

Maximum pebbles used: 3

Not in memory

In memory

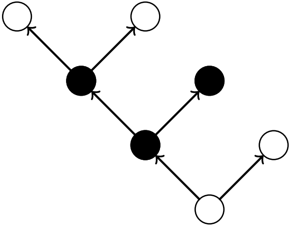


Space Measure Example

Maximum pebbles used: 3

○ Not in memory

● In memory

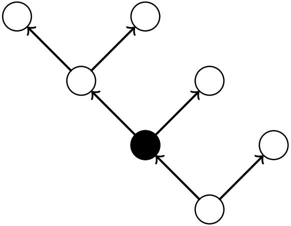


Space Measure Example

Maximum pebbles used: 3

Not in memory

In memory

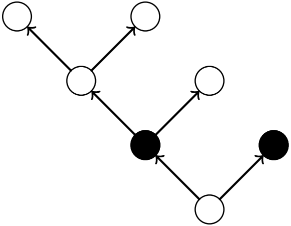


Space Measure Example

Maximum pebbles used: 3

Not in memory

In memory

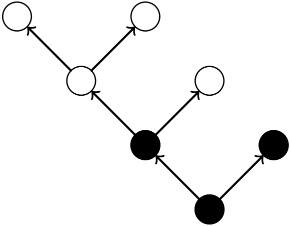


Space Measure Example

Maximum pebbles used: 3

Not in memory

In memory

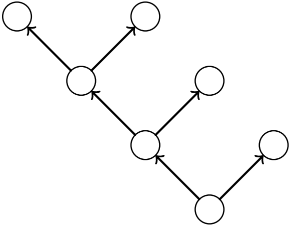


Space Measure Example

Maximum pebbles used: 0

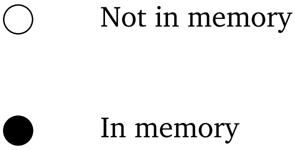
Not in memory

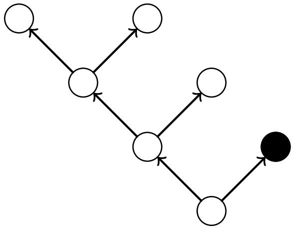
In memory



Space Measure Example

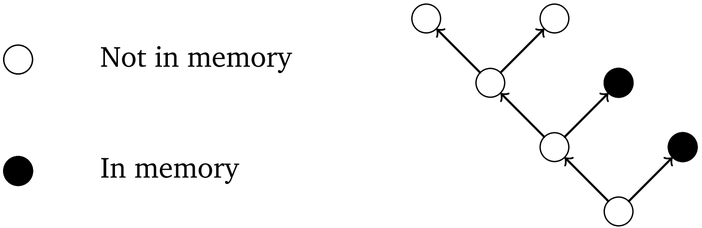
Maximum pebbles used: 1





Space Measure Example

Maximum pebbles used: 2

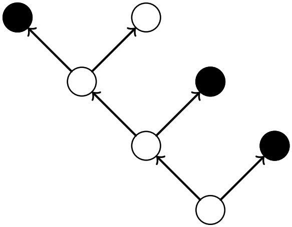


Space Measure Example

Maximum pebbles used: 3

Not in memory

In memory

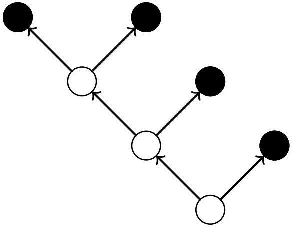


Space Measure Example

Maximum pebbles used: 4

○ Not in memory

● In memory

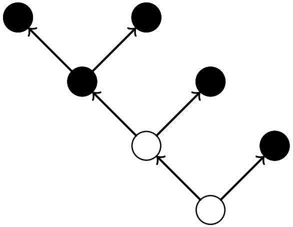


Space Measure Example

Maximum pebbles used: 5

○ Not in memory

● In memory

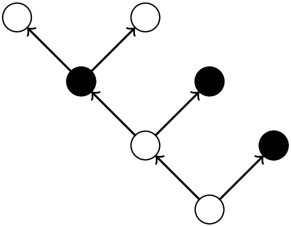


Space Measure Example

Maximum pebbles used: 5

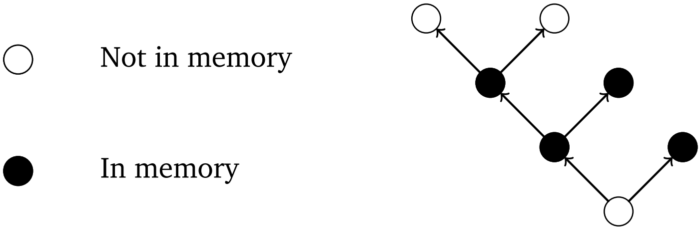
 Not in memory

 In memory



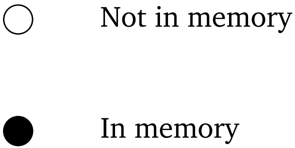
Space Measure Example

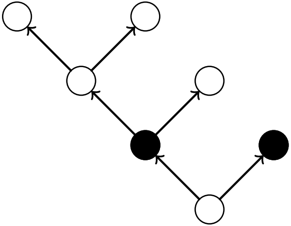
Maximum pebbles used: 5



Space Measure Example

Maximum pebbles used: 5



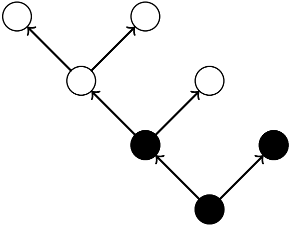


Space Measure Example

Maximum pebbles used: 5

 Not in memory

 In memory



Construct Traversal Orders

Construct Optimal Order

- NP-complete
- Optimal strategy in some pebbling game

Construct Good Order

- Top-Down
- Bottom-Up
- Heuristics

Construct Order Top-Down

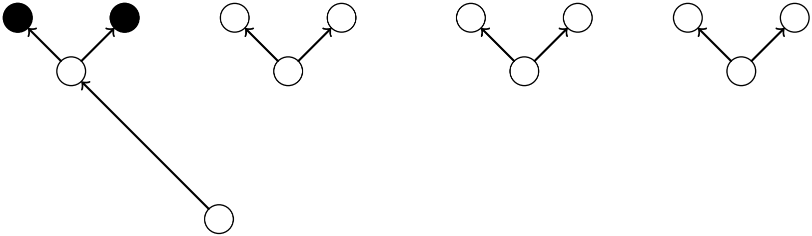
Maximum pebbles used: 0

Construct Order Top-Down

Maximum pebbles used: 1

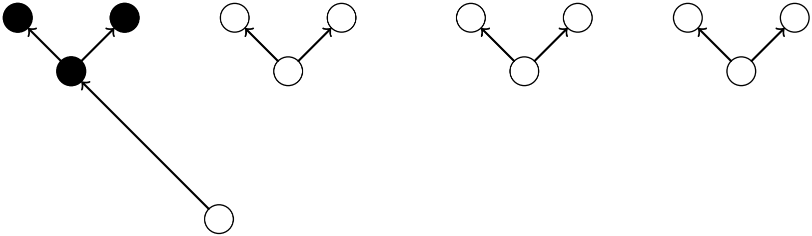
Construct Order Top-Down

Maximum pebbles used: 2



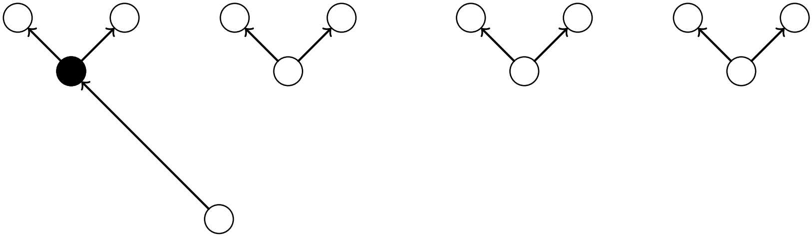
Construct Order Top-Down

Maximum pebbles used: 3



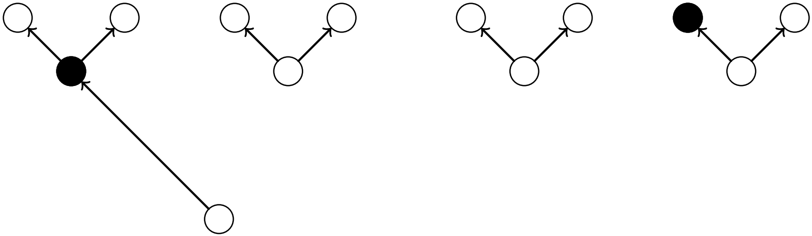
Construct Order Top-Down

Maximum pebbles used: 3



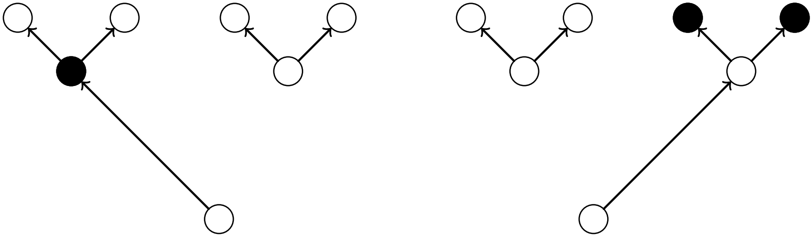
Construct Order Top-Down

Maximum pebbles used: 3



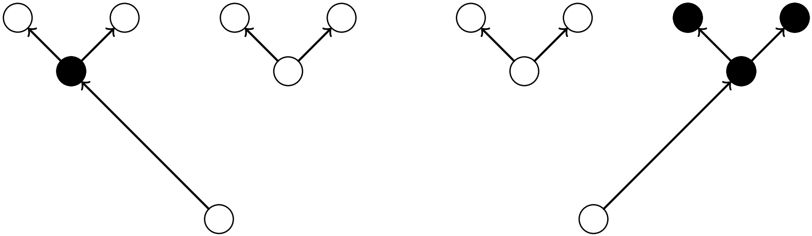
Construct Order Top-Down

Maximum pebbles used: 3



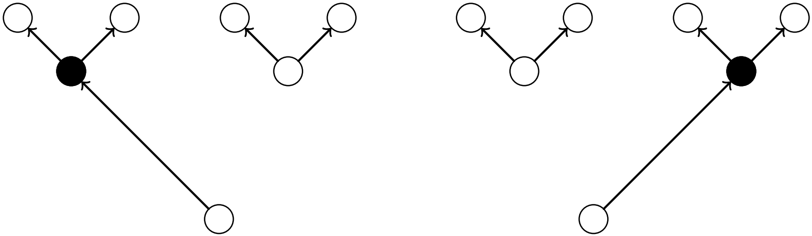
Construct Order Top-Down

Maximum pebbles used: 4



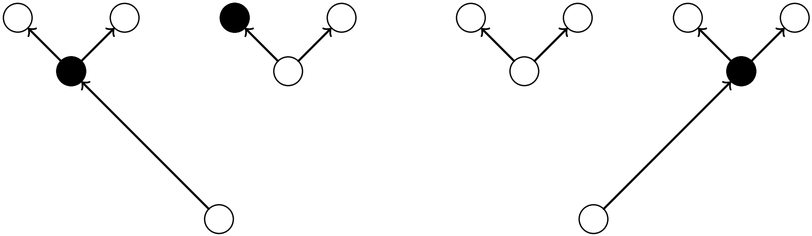
Construct Order Top-Down

Maximum pebbles used: 4



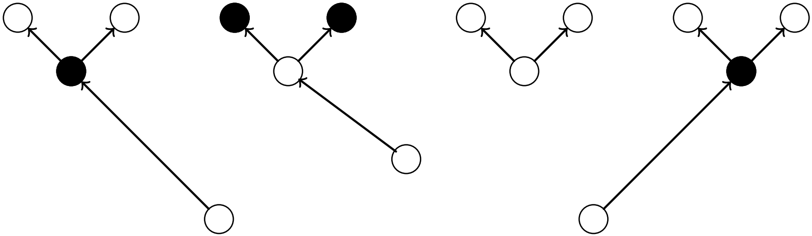
Construct Order Top-Down

Maximum pebbles used: 4



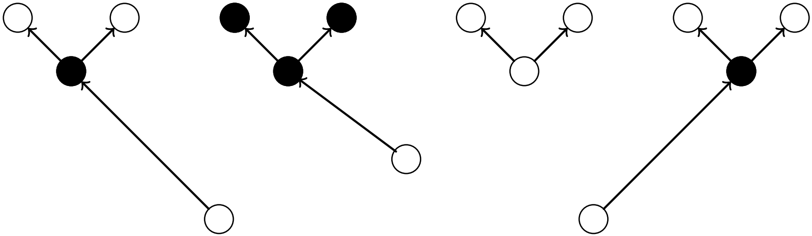
Construct Order Top-Down

Maximum pebbles used: 4



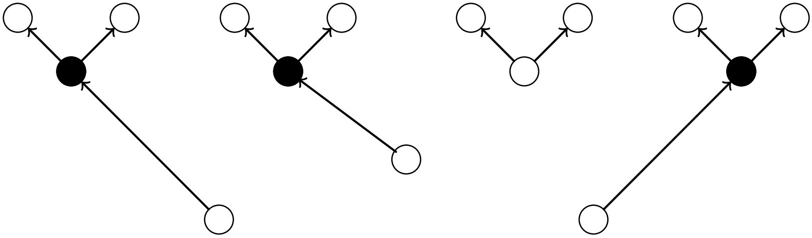
Construct Order Top-Down

Maximum pebbles used: 5



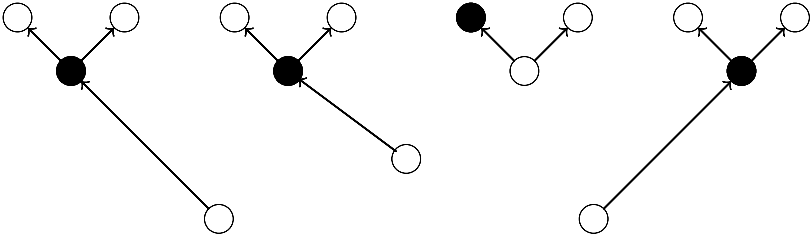
Construct Order Top-Down

Maximum pebbles used: 5



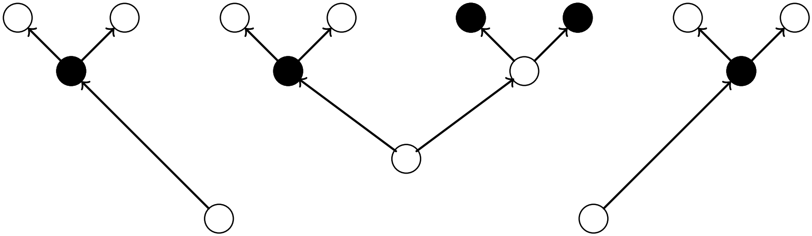
Construct Order Top-Down

Maximum pebbles used: 5



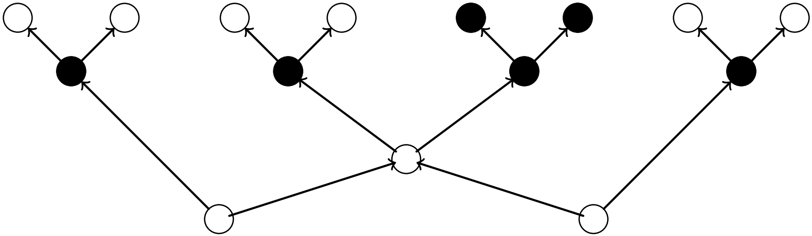
Construct Order Top-Down

Maximum pebbles used: 5



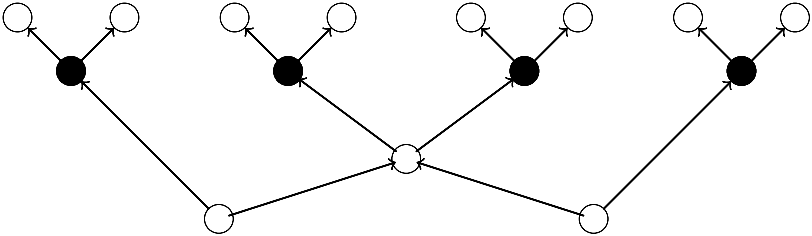
Construct Order Top-Down

Maximum pebbles used: 6



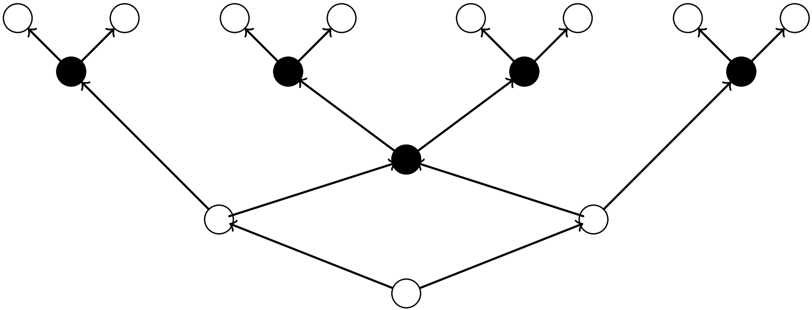
Construct Order Top-Down

Maximum pebbles used: 6



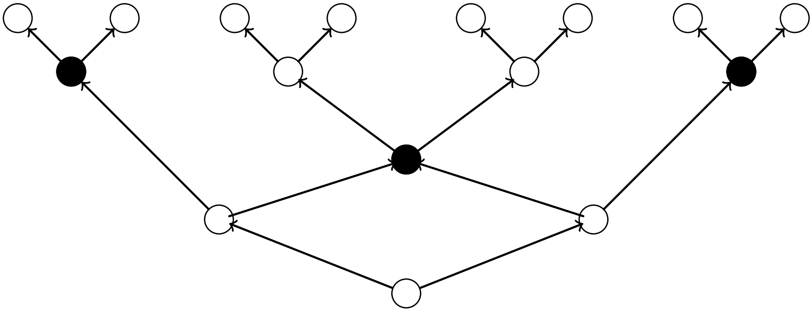
Construct Order Top-Down

Maximum pebbles used: 6



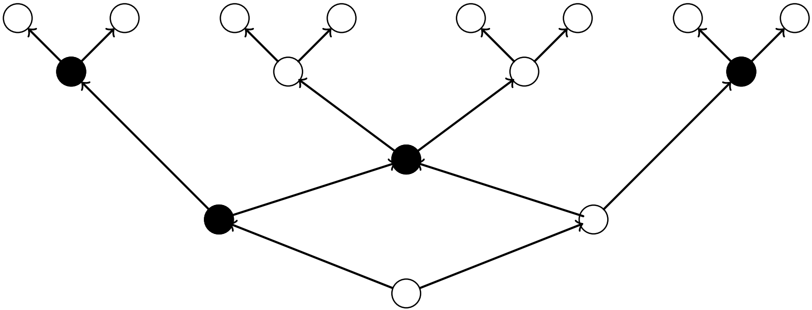
Construct Order Top-Down

Maximum pebbles used: 6



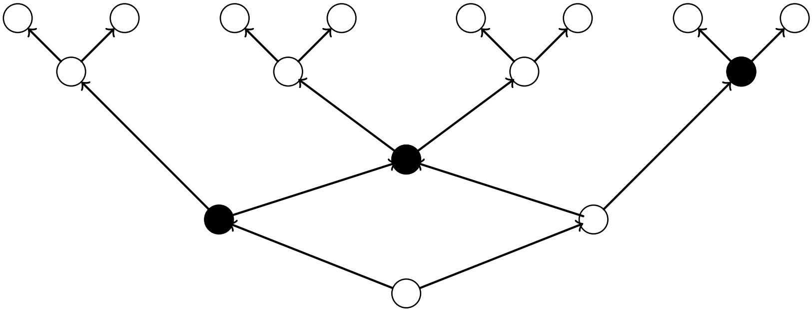
Construct Order Top-Down

Maximum pebbles used: 6



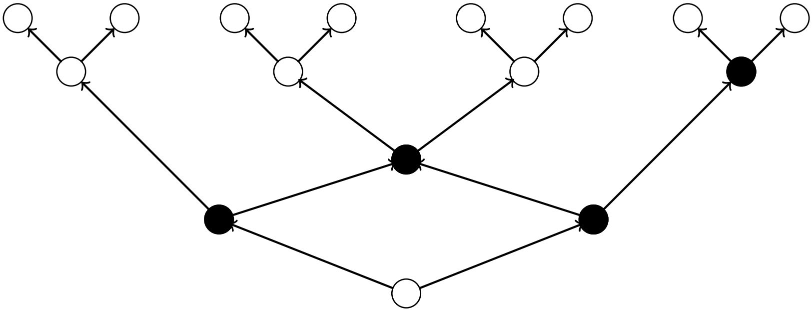
Construct Order Top-Down

Maximum pebbles used: 6



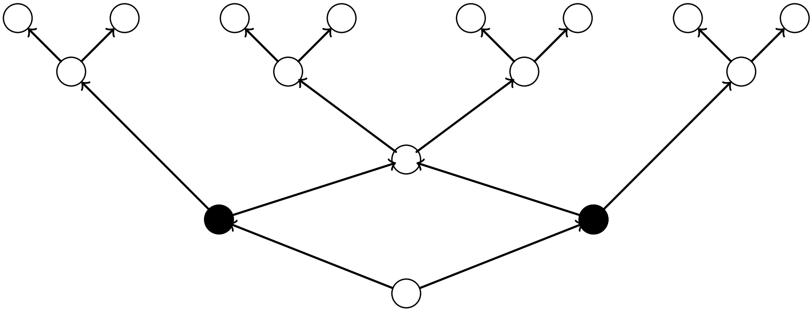
Construct Order Top-Down

Maximum pebbles used: 6



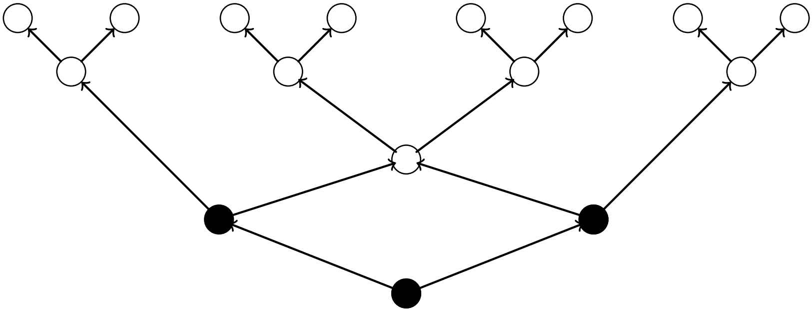
Construct Order Top-Down

Maximum pebbles used: 6



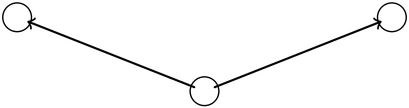
Construct Order Top-Down

Maximum pebbles used: 6



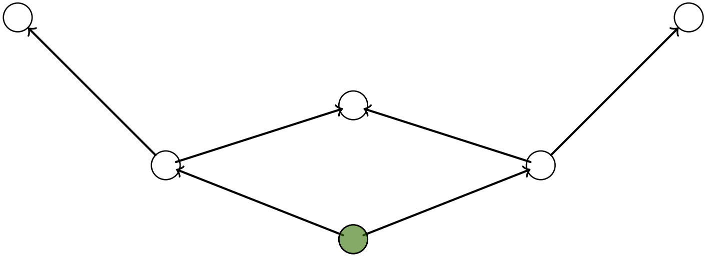
Construct Order Bottom-Up

Maximum pebbles used: 0



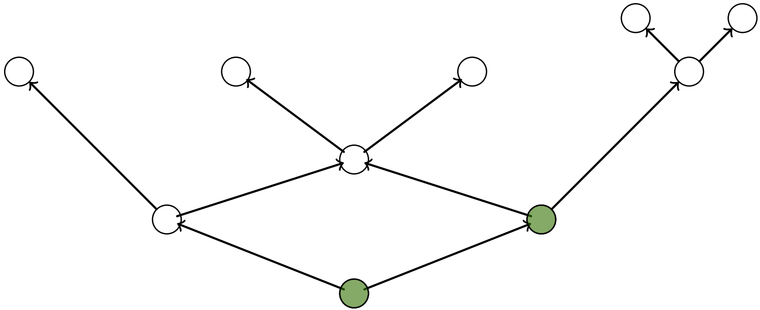
Construct Order Bottom-Up

Maximum pebbles used: 0



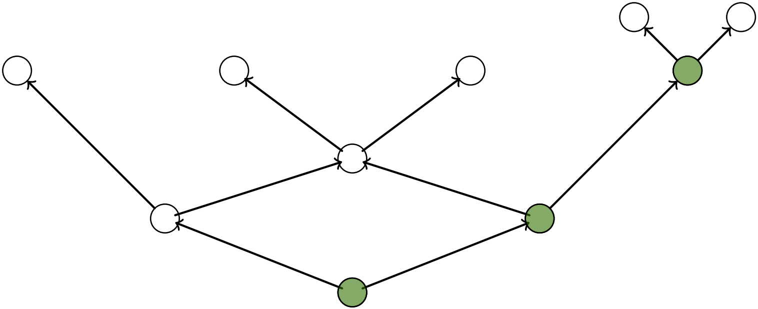
Construct Order Bottom-Up

Maximum pebbles used: 0



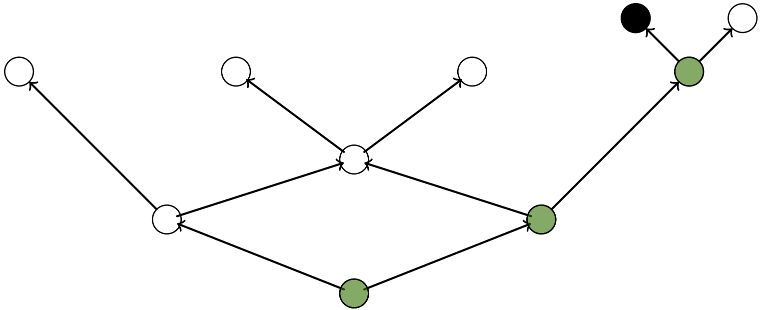
Construct Order Bottom-Up

Maximum pebbles used: 0



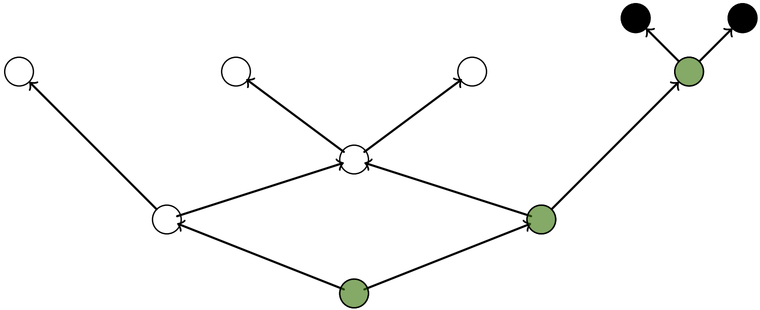
Construct Order Bottom-Up

Maximum pebbles used: 1



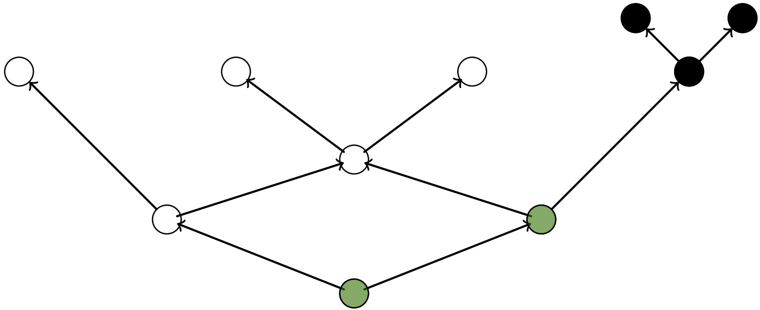
Construct Order Bottom-Up

Maximum pebbles used: 2



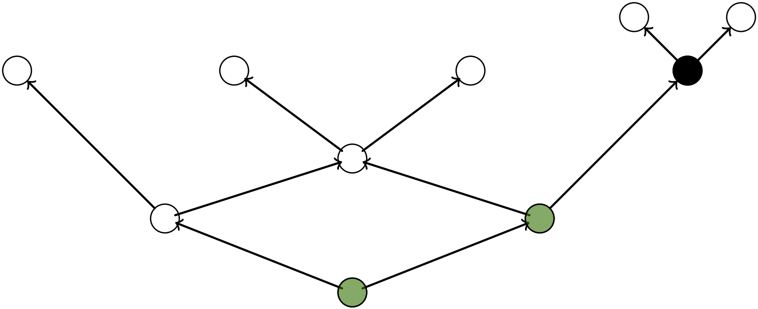
Construct Order Bottom-Up

Maximum pebbles used: 3



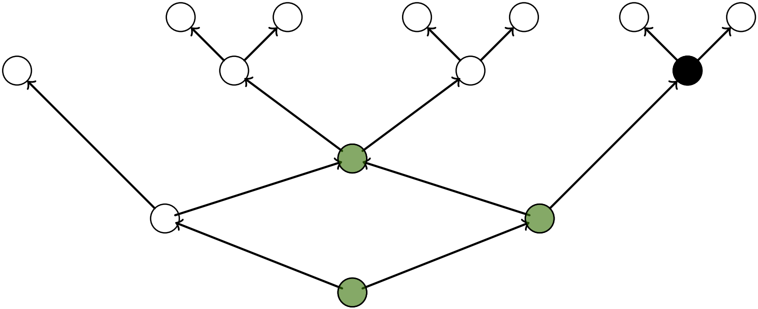
Construct Order Bottom-Up

Maximum pebbles used: 3



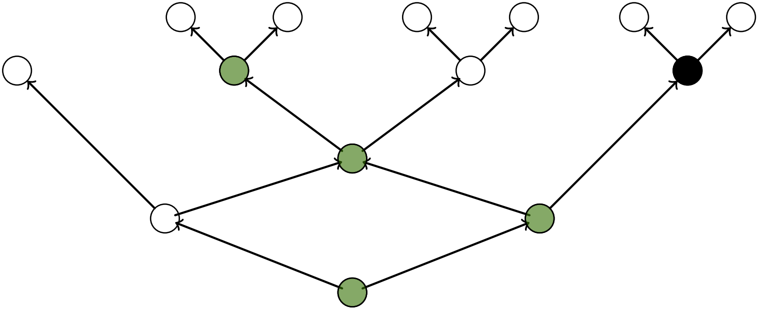
Construct Order Bottom-Up

Maximum pebbles used: 3



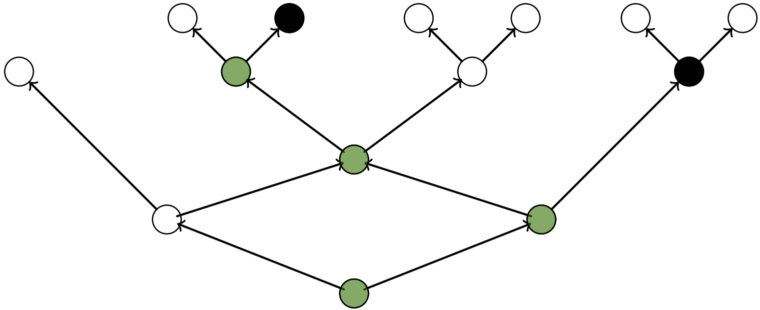
Construct Order Bottom-Up

Maximum pebbles used: 3



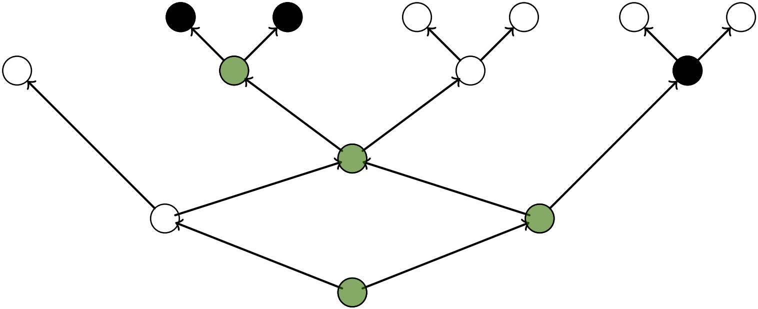
Construct Order Bottom-Up

Maximum pebbles used: 3



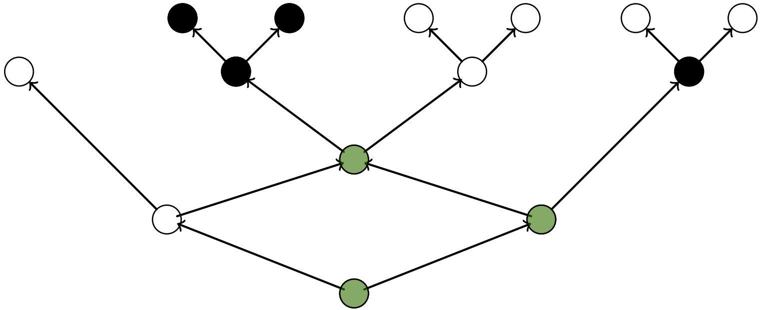
Construct Order Bottom-Up

Maximum pebbles used: 3



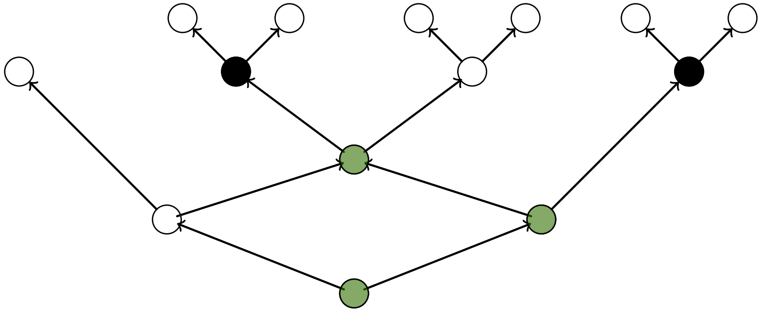
Construct Order Bottom-Up

Maximum pebbles used: 4



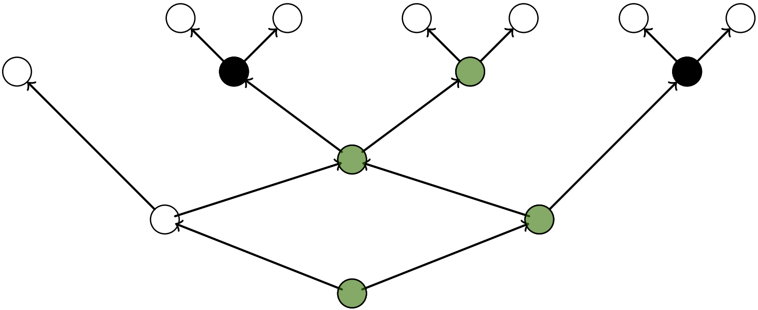
Construct Order Bottom-Up

Maximum pebbles used: 4



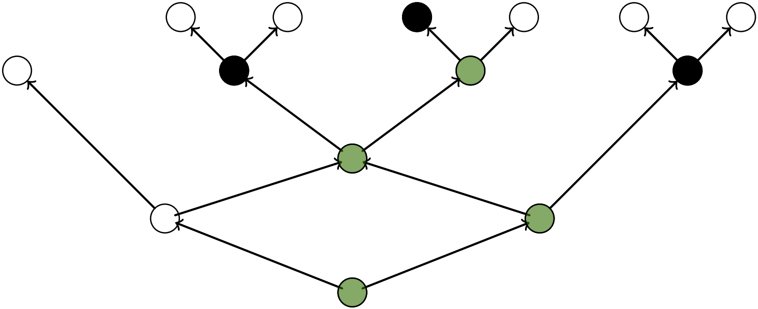
Construct Order Bottom-Up

Maximum pebbles used: 4



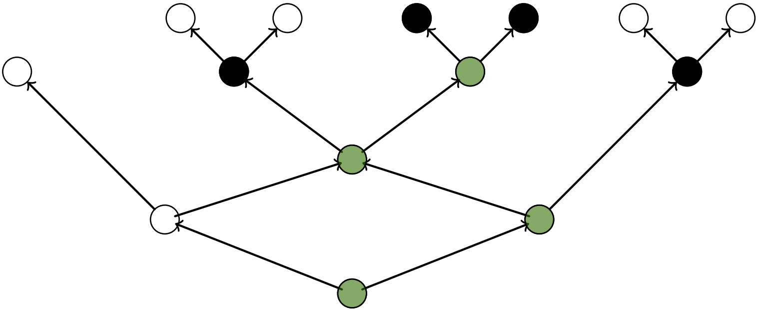
Construct Order Bottom-Up

Maximum pebbles used: 4



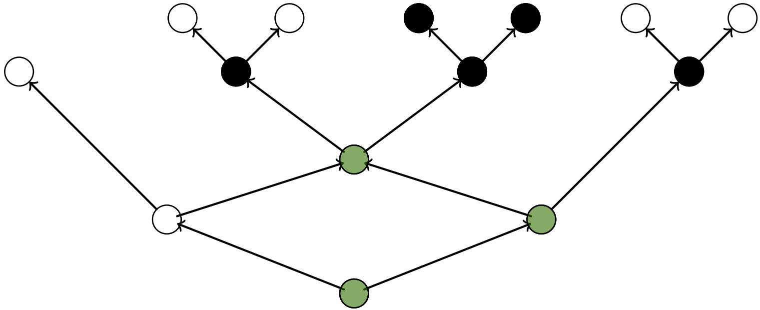
Construct Order Bottom-Up

Maximum pebbles used: 4



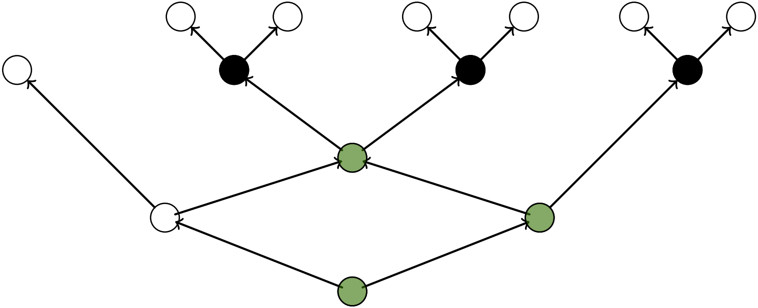
Construct Order Bottom-Up

Maximum pebbles used: 5



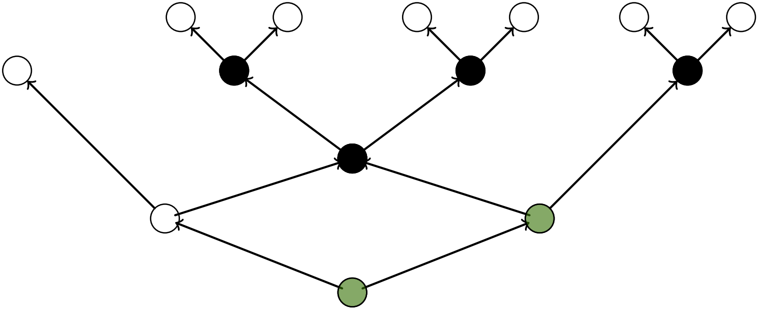
Construct Order Bottom-Up

Maximum pebbles used: 5



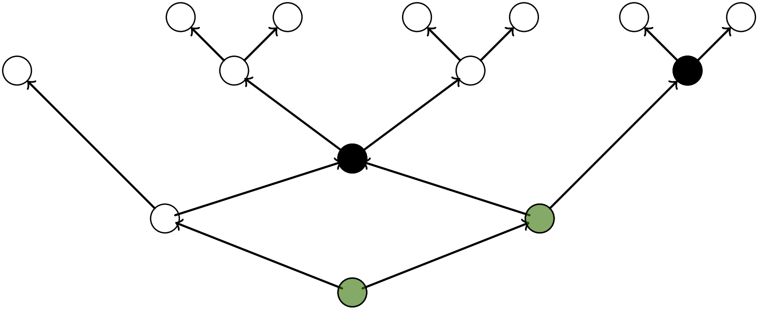
Construct Order Bottom-Up

Maximum pebbles used: 5



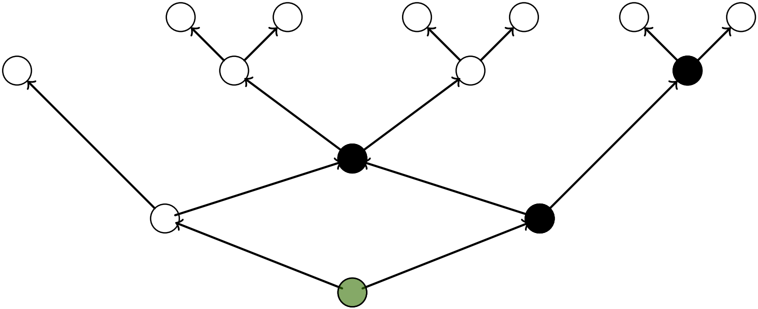
Construct Order Bottom-Up

Maximum pebbles used: 5



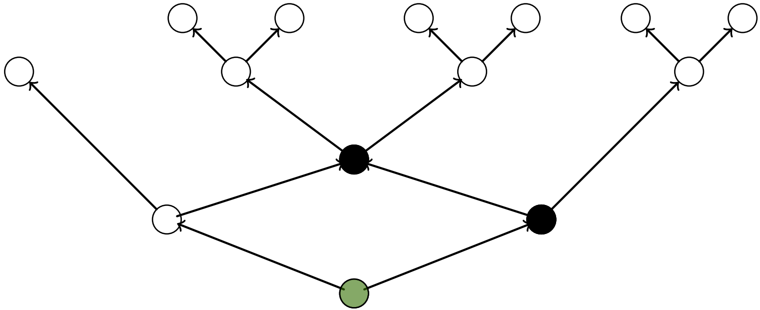
Construct Order Bottom-Up

Maximum pebbles used: 5



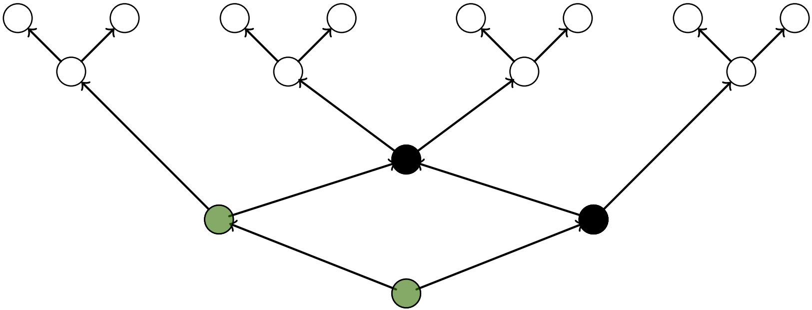
Construct Order Bottom-Up

Maximum pebbles used: 5



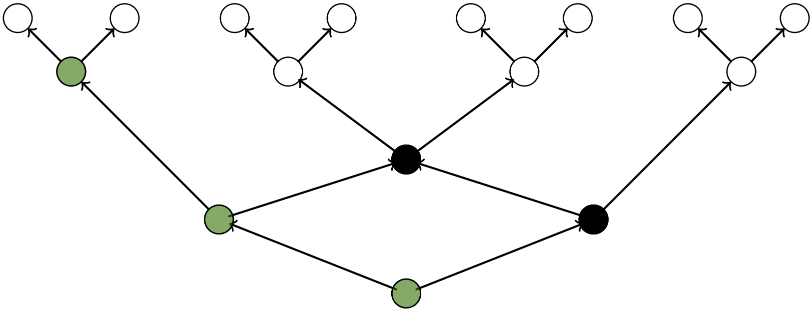
Construct Order Bottom-Up

Maximum pebbles used: 5



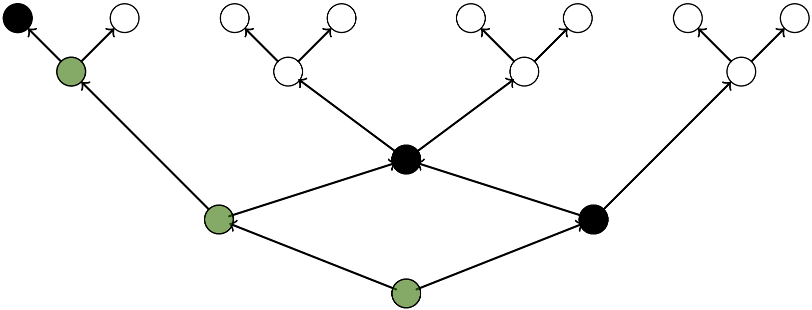
Construct Order Bottom-Up

Maximum pebbles used: 5



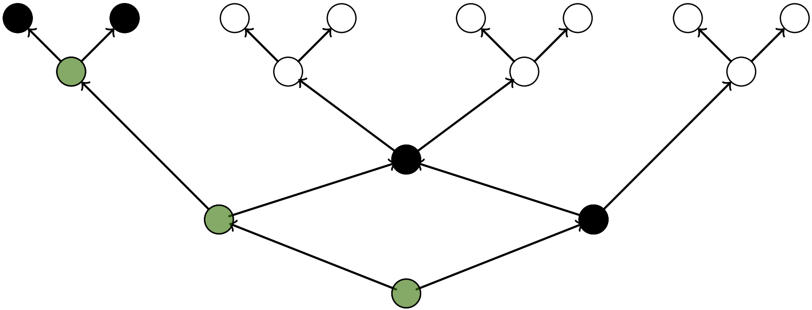
Construct Order Bottom-Up

Maximum pebbles used: 5



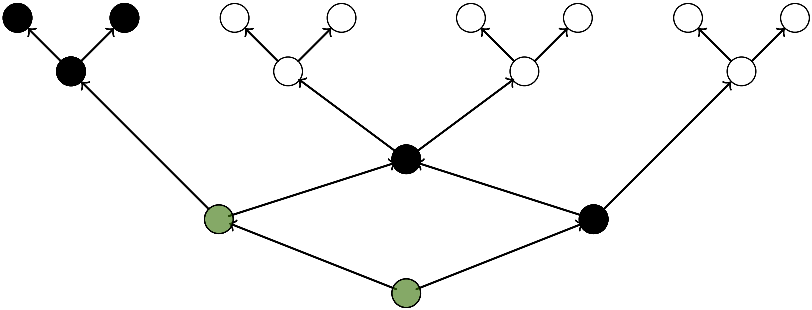
Construct Order Bottom-Up

Maximum pebbles used: 5



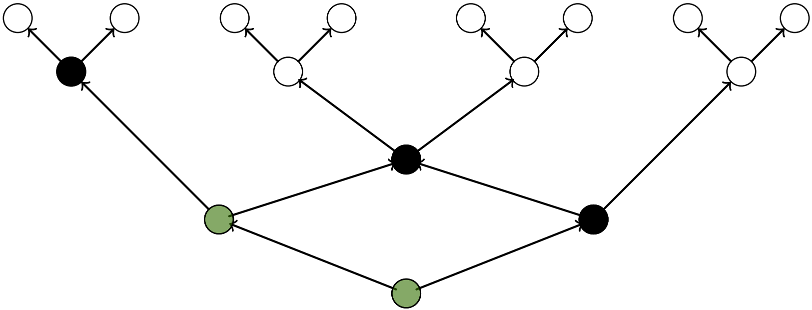
Construct Order Bottom-Up

Maximum pebbles used: 5



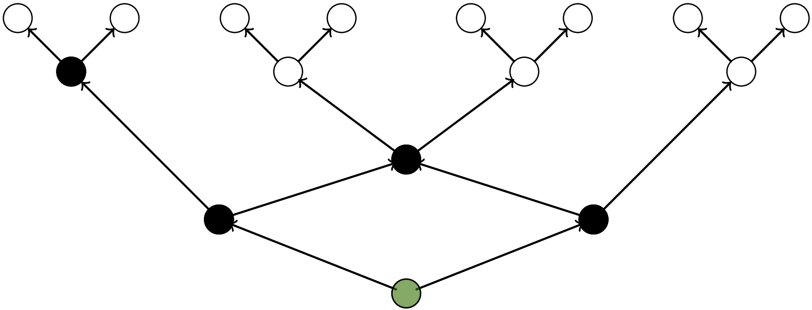
Construct Order Bottom-Up

Maximum pebbles used: 5



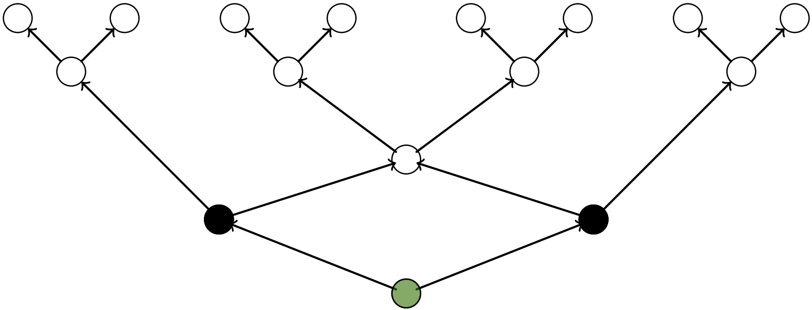
Construct Order Bottom-Up

Maximum pebbles used: 5



Construct Order Bottom-Up

Maximum pebbles used: 5



Construct Order Bottom-Up

Maximum pebbles used: 5