ADVANCES IN PEBBLING
(Preliminary Version)

Nicholas Pippenger
IBM Research Laboratory
San Jose, CA 95193 USA

1. Pebbling

The purpose of this paper is threefold: to survey some known results on pebbling, to sketch the proofs of some new results, and to present some open problems. All of this will be done in the context of a new setting for pebbling problems, but first we shall give an informal description of the pebble game and some of its variations.

The pebble game is played on the vertices of an acyclic directed graph. A play of the game is a sequence of moves according to the following two rules.

Rule 1. If the immediate predecessors of a vertex all have pebbles on them, a pebble may be put on that vertex.

Rule 2. A pebble may be taken off a vertex.

The game begins with no pebbles on the graph. A *simple* play is one in which Rule 1 is applied to each vertex at most once. A *complete* play is one in which Rule 1 is applied to each vertex at least once. The game abstracts certain properties of computations, especially those dealing with *time* (reckoned as the number of moves in a play) and *space* (reckoned as the maximum number of pebbles on the graph at any move of the play).

We shall deal frequently in this paper with a variant of the pebble game in which there are available one or more pushdown stacks capable of holding names of vertices in the graph. The pushdown stacks are manipulated according to the following two rules.

Rule 3. If there is a pebble on a vertex, the name of that vertex may be pushed onto a stack.

Rule 4. If the name of a vertex is at the top of a stack, it may be popped off that stack and a pebble may be placed on that vertex.

The game now begins with no pebbles on the graph and no names on the stacks. This variant allows us to consider trade-offs among not only time and space but an additional computational resource, which might be called pushdowns (reckoned as the number of different stacks manipulated during a play).

The pebble game was introduced by Paterson and Hewitt [13], who raised the question of the minimum space required by complete plays for a given graph. Sethi [21] raised the question of the minimum space required by complete simple plays for a given graph. The pebble game with auxiliary pushdowns was introduced by Pippenger [18,20]. Another variant of the pebble game (with which we shall not deal in this paper) is the black-and-white pebble game, introduced by Cook and Sethi [3]. For a general survey of results on the pebble game and its variants, see Pippenger [19].

2. Lifting

In this section we shall introduce a new setting for the study of pebbling problems. This setting accomodates most known results on pebbling in a natural way; it also suggests new problems and conjectures.

It will be convenient in this paper to employ a special notion of graph, in which the vertices are presented in a specific total order, compatible with the partial order induced by the direction of the edges. Let an N-graph be an acyclic directed graph with vertices $\{1, \ldots, N\}$ in which every vertex has at most two immediate predecessors (the number of immediate successors is not restricted) and in which every edge is directed out of a lower-numbered and into a higher-numbered vertex.

An N-graph will be called a pebble graph if, for every $1 \le M \le N$, there is at most one vertex $K \le M$ out of which is directed an edge to some vertex $L > M$. Let a B-pebble N-graph be an N-graph that is the union of at most B pebble graphs.

An N-graph will be called a pushdown graph if there is at most one edge directed into or out of each vertex and if, for every pair of edges (K, L) and (K', L'), either the intervals $\{K, \ldots, L\}$ and $\{K', \ldots, L'\}$ are disjoint, or one is included in the other. Let an A-pushdown B-pebble N-graph be an N-graph that is the union of at most A pushdown graphs and at most B pebble graphs.

An important notion for our setting is that of lifting. A lift from an N-graph G to and N'-graph G' is a map p from the vertices of G onto the vertices of G' such that, whenever $p(L)=L'$ and K' is an immediate predecessor of L', there exists an immediate predecessor K of L such that $p(K)=K'$ or $p(K)=L'$. If there is a lift from G to G', we shall say that G is a lift of G'. The relation of lifting is reflexive and transitive.

Another important notion for our setting is that of simple lifting. We shall say that a lift p from G to G' is simple if, for any vertex L' in G', the inverse image $p^{-1}(L')$ is connected (so that there is a directed path from the lowest-numbered vertex to any other vertex). If there is a simple lift from G to G', we shall say that G is a simple lift of G'. The relation of simple lifting is also reflexive and transitive.

It is not hard to see that if the N'-graph G' has a complete play of the pebble game with R pushdowns in space S and time T, then there exists a lift G of G' that is an R-pushdown B-pebble N-graph with B=O(S) and N=O(T). Conversely, if an N'-graph G' has a lift G that is an A-pushdown B-pebble N-graph, then it has a complete play of the pebble game with A pushdowns in space O(B) and time O(N). In particular, an A-pushdown B-pebble N-graph itself has a complete play of the pebble game with A pushdowns in space O(B) and time O(N).

It is also not hard to see that if the N'-graph G' has a complete simple play of the pebble game with R pushdowns in space S and time T, then there exists a simple lift G of G' that is an R-pushdown B-pebble N-graph with B=O(S) and N=O(T). Conversely, if an N'-graph G' has a simple lift G that is an A-pushdown B-pebble N-graph, then it has a complete simple play of the pebble game with A pushdowns in space O(B) and time O(N). In particular, an A-pushdown B-pebble N-graph itself has a complete simple play of the pebble game with A pushdowns in space O(B) and time O(N).

Thus, to within constant factors in time and space, questions about pebbling with auxiliary pushdowns can be reduced to relationships of lifting and simple lifting among graphs. These relationships concern only the embeddability of graphs in other graphs. In particular, we can discuss pebbling with auxiliary pushdowns without mentioning either pebbles or pushdowns!

A basic property of lifts (and *a fortiori* of simple lifts) is the following. Let p be a lift from G to G', let $B_1, \ldots, B_L$ be vertices of G with $p(B_1)=B_1', \ldots, p(B_L)=B_L'$ and let $P_1', \ldots, P_L'$ be vertex-disjoint paths in G' from vertices $A_1', \ldots, A_L'$ to $B_1', \ldots, B_L'$ (in the order indicated). Then there exist vertices $A_1, \ldots, A_L$ in G with $p(A_1)=A_1', \ldots, p(A_L)=A_L'$ and vertex-disjoint paths $P_1, \ldots, P_L$ in G from $A_1, \ldots, A_L$ to $B_1, \ldots, B_L$ (in the order indicated). Roughly speaking, vertex-disjoint paths can be lifted from G' to G.

The notion of A-pushdown B-pebble N-graphs defined above is closely related to some notions of "computation graphs" that have appeared in the literature, specifically, to the "m-tape Turing machine graphs" of Paul, Tarjan and Celoni [14] and the "m*-head graphs" of Paul and Reischuk [15]. This relationship can be expressed formally by observing that a m-tape Turing machine graph or m*-head graph with n vertices has a simple lift that is an A-pushdown B-pebble N-graph with $A=2m$, $B=O(m)$ and $N=O(mn)$. Conversely, an A-pushdown B-pebble N-graph has simple lifts that are m-tape Turing machine and m^*-head graph with n vertices, where $m=O(A+B)$ and $n=O(N)$. These relationships of mutual simple lifting between classes of graphs clearly show the equivalence of these classes as regards pebbling, and we shall see later that it implies a similar equivalence as regards separability.

3. Pebbling with Auxiliary Pushdowns

In this section we shall survey the known results on pebbling of general graphs, with and without auxiliary pushdowns, and sketch the proofs of the results concerning two or more pushdowns, which have not yet appeared in the literature.

First, let us consider space requirements. In the case of no auxiliary pushdowns, Hopcroft, Paul and Valiant [6] showed that every N-graph has a complete play of the pebble game in space $O(N/\log N)$, and Paul, Tarjan and Celoni [14] showed that there exist N-graphs that require space $\Omega(N/\log N)$. In the case of one or more auxiliary pushdowns, every N-graph has a complete play of the pebble game in space 3, and it is clear that there exist N-graphs that require space 3.

Next, for space in the range from N down to the lower bounds just indicated, let us consider time requirements. For the case of no auxiliary pushdowns, Lengauer and Tarjan [7] showed that every N-graph has a complete play of the pebble game in space S and time at most $S \exp\exp O(N/S)$, and that there exist N-graphs that require time $S \exp\exp \Omega(N/S)$. This time-space trade-off can be written

$$T/N = \exp\exp \Theta(N/S),$$

since a factor of N/S can be absorbed into the factor $\exp\exp\Theta(N/S)$.

For the case of one auxiliary pushdown, Pippenger [18,20] showed that

$$T/N = \exp \Theta(N/S).$$

We shall see below that for two auxiliary pushdowns,

$$T/N = \Theta(N/S),$$

and that for $R\geq 3$ auxiliary pushdowns,

$$T/N = \Theta(\log_R(N/S)).$$

In the case of two pushdowns, the upper bound

$$T/N = O(N/S)$$

is easy. Letting $K=\lfloor S/3 \rfloor$, we can deal with each successive interval of K vertices in time $O(N)$, keeping the names of all previously pebbled vertices on the stacks.

For the lower bound

$$T/N = \Omega(N/S),$$

we shall use superconcentrators. An M-superconcentrator is a graph G with the following property. There are M distinguished vertices called inputs and M other distinguished vertices called outputs such that if $A_1, \ldots, A_L$ are distinct inputs and $B_1, \ldots, B_L$ are distinct outputs, then there exist vertex-disjoint paths $P_1, \ldots, P_L$ from $A_1, \ldots, A_L$ to $B_1, \ldots, B_L$ (not necessarily in the order indicated). Let the N-graph G on which the game is to be played be obtained from a series of superconcentrators of geometrically increasing sizes: the output of a 1-superconcentrator is identified with one of the inputs of a 2-superconcentrator, and so forth until the outputs of an (M/2)-superconcentrator are identified with half of the inputs of an M-superconcentrator. We may take $M=\Omega(N)$, since there exist N'-graphs that are M'-superconcentrators for $M'=\Omega(N')$ (see Valiant [22] or Pippenger [16]).

If there is a play of the pebble game on G with two pushdowns in space S and time T, then there is a lift F of G that is a 2-pushdown B-pebble C-graph, where $B=O(S)$ and $C=O(T)$. It is not hard to see that such a graph can be immersed in the plane with $U=O(BC)=O(ST)$ "nodes" (where a node is either a vertex of F or a crossing of two edges of F). It will thus suffice to show that $U=\Omega(N^2)$.

We claim that if a subgraph F' of F contains at least L/2 inverse images of distinct outputs of the L-superconcentrator in G, and if F' has fewer than L/64 edges directed into it from outside of it, then F' contains $U' \geq cL^2$ nodes (for some suitable constant $c>0$). The proof is by induction on L. For $L=2$, the claim is trivial. For $L\geq 4$, by the Planar Separator Theorem (see Lipton and Tarjan [9]), F' can be partitioned by the removal of $O(U'^{1/2})$ nodes into 16 parts, each containing at most L/32 of the given inverse images of outputs. Unless $U' \geq cL^2$, at least 8 of these parts must each contain at least L/64 of the given inverse images of outputs, and at least 4 of these parts must each have at most L/128 edges directed into them from outside of them. By the basic properties of superconcentrators and lifts, each of these four parts must contain at least L/4 inverse images of distinct inputs of the L-superconcentrator in G, which are inverse images of distinct outputs of the (L/2)-superconcentrator in G. Thus, by applying the inductive hypothesis to these 4 parts, $U' \geq 4c(L/2)^2 = cL^2$, which completes the proof of the claim. Taking $L=M$ yields $U \geq cM^2 = \Omega(N^2)$, which completes the proof of the lower bound.

In the case of $R\geq 3$ auxiliary pushdowns, the upper bound

$$T/N = O(\log_R(N/S))$$

is obtained as a generalization of the Postman Algorithm of M. J. Fischer and M. S. Paterson (see Pippenger [17]). The Postman Algorithm is in fact the special case $R=3$, $S=3$ of the result we need. To obtain the required generalization, we modify the Postman Algorithm in two ways.

First, we terminate non-recursively whenever the interval with which we have to deal has length at most $K=\lfloor S/3 \rfloor$ (rather than length one). We then have enough pebbles for the at most 2K immediate predecessors of the vertices in the interval, as well as for these vertices themselves. Second, when we proceed recursively, we partition the interval with which we have to deal into R-1 subintervals (rather than two subintervals). The depth of the tree of recursive invocations is then $\log_{(R-1)}(N/K)=O(\log_R(N/S))$ and the upper bound follows as in the special case.

For the lower bound, we use a counting argument based on a special class of graphs called permutaion graphs, which were introduced by Lengauer and Tarjan [7]. An M-permutation graph is graph with vertices $\{1, \ldots, N\}$, where $N=2M$, having edges $(L, L+1)$ for $1 \le L < M$ and $M+1 \le L < 2M$ and edges $(p(L), M+L)$ for $1 \le L \le M$ and some permutation p of $\{1, \ldots, M\}$ There are M! permutation graphs G_p, so that $\Omega(M \log M)=\Omega(N \log N)$ bits are required to uniquely identify a permutation graph. It is not hard to show that the plays of the pebble game on these graphs with R pushdowns in space S and time T can be encoded using $O(T \log (RS))$ bits in such a way that the encoding of the play uniquely identifies the graph on which the play takes place. This yields the lower bound

$$T/N = \Omega(\log_{(RS)} N).$$

This lower bound is weaker than one we have claimed, in that the number R of pushdowns and the number S of pebbles enter symmetrically. It is intuitively clear that additional pushdowns are more powerful than additional pebbles, and this intuition is confirmed by the following argument.

Suppose for convenience that S divides M. Let us say that two permutation graphs G_p and G_q are S-equivalent if pq^{-1} stabilizes each of the successive blocks $\{1, \ldots, S\}$, $\ldots$, $\{M-S+1, \ldots, M\}$ of length S. There are $M!/S!^{M/S}$ equivalence classes of permutation graphs, so that $\Omega(M \log (M/S))=\Omega(N \log (N/S))$ bits are required to uniquely identify an equivalence class. It is not hard to show that plays of the pebble game on these graphs with R pushdowns in space S and time T can be encoded using $O(T \log R)$ bits in such a way that the encoding of the play uniquely identifies the equivalence class of the graph on which the play takes place. This yields the claimed lower bound

$$T/N = \Omega(\log_R (N/S)).$$

It should be noted that this lower bound differs from the other lower bounds described in this section in that it does not give an explicit construction for a graph that is hard to pebble. The other lower bounds are all based on expanders and superconcentrators, for which explicit constructions are available in the work of Margulis [11] and Gabber and Galil [4].

We should also mention the situation for simple plays of the pebble game, although this situation is indeed too simple to be very interesting. For no auxiliary pushdowns or one auxiliary pushdown, there exist N-graphs that require space $\Omega(N)$ for any complete simple play of the pebble game. Since no significant saving of space is possible in these cases, no significant time-space trade-offs exist. For two or more auxiliary pushdowns, the upper bounds described above in fact apply to complete simple plays of the pebble game. Thus, the time-space trade-offs are the same in these cases, whether or not the plays are required to be simple.

The pebble game as we have discussed it thus far models computations using certain data structures: registers (pebbles) and pushdown stacks. Let us conclude this section by mentioning the possibility of incorporating other data structures (such as queues, deques and tapes) into our setting. The case of $C\geq 1$ tapes is easily dealt with by the methods used for pushdown stacks: for $C=1$ tapes we have

$$T/N = \Theta(N/S)$$

and for $C\geq 2$ tapes we have

$$T/N = \Theta(\log_C(N/S))$$

The case of $D\geq 1$ deques exhibits exactly the same time-space trade-off as $C=D$ tapes. (For the upper bounds, observe that each deque can simulate two pushdowns. For the lower bounds, observe that 1-deque graphs enjoy the same immersions in the plane as 2-pushdown graphs, and that each deque can be simulated by two heads on one tape, which can be simulated by two heads on two tapes (see Hartmanis and Stearns [5]), which can be simulated by four pushdowns.) The case of $Q\geq 1$ queues, however, seems more difficult. It is not hard to show that

$$T/N = O(Q(N/S)^{1/Q}),$$

and we conjecture that

$$T/N = \Omega(Q(N/S)^{1/Q}),$$

at least for simple plays of the pebble game.

4. Pebbling Pushdown Graphs

In the preceding section, we considered A-pushdown B-pebble N-graphs as lifts of other graphs. In this section, we shall consider the question of what graphs are lifts of A-pushdown B-pebble N-graphs. In its full generality, this question appears to be very difficult, so we shall restrict our attention to a special class of A-pushdown 1-pebble N-graphs. Let an A-pushdown N-graph be an N-graph that is the union of at most A pushdown graphs together with the pebble graph that has edges (M, M+1) for $1\leq M<N$.

The study of the pebble game on 1-pushdown N-graphs was begun implicitly by Cook [2] (in his work on the time and space requirements for recognizing deterministic context-free languages) and was continued explicitly by Mehlhorn [12] (who called these graphs "mountain ranges"). They showed that every 1-pushdown N-graph has a complete play of the pebble game (without auxiliary pushdowns) in space $O(\log N)$, and that there exist 1-pushdown N-graphs that require space $\Omega(\log N)$. For space S in the range from N down to the lower bound just indicated, von Braunmuehl and Verbeek [1] showed that every 1-pushdown N-graph has a complete play of the pebble game in space S and time

$$T = N \exp O((\log N)/\log (S/\log N)),$$

and Verbeek [23] showed that there exist 1-pushdown N-graphs that require time

$$T = N \exp \Omega((\log N)/\log (S/\log N)).$$

(The bounds that they actually prove are somewhat sharper than these, but they fall short of establishing the time-space trade-off to within constant factors in time and space.)

The study of 2-pushdown N-graphs begins with the observation that they can be embedded in the plane (a property they share with "1-tape N-graphs" and "1-deque N-graphs", if these are defined in the obvious way). Lipton and Tarjan [10], using their Planar Separator Theorem (see Lipton and Tarjan [9]), have shown that every planar N-graph has a complete play of the pebble game (without auxiliary pushdowns) in space $O(N^{1/2})$, and it is not hard to see that there exist 2-pushdown N-graphs (or 1-tape N-graphs or 1-deque N-graphs) that require space $\Omega(N^{1/2})$. In this case the time-space trade-off remains to be determined.

The study of A-pushdown N-graphs for $A \geq 3$ has barely begun. Of course, every A-pushdown N-graph has a complete play of the pebble game (without auxiliary pushdowns) in space $O(N/\log N)$ (since every N-graph has, by the result of Hopcroft, Paul and Valiant [6]) and it is not hard to see that there exist A-pushdown N-graphs for which space $\Omega((N \log A)/(\log N)^2)$ is required (following a similar observation by Paul, Tarjan and Celoni [14] for m-tape Turing machine graphs). Thus in this case even the space requirements remain to be precisely determined.

Let us conclude this paper with some comments on the role that separator theorems seem to play in pebbling and with a separator conjecture that has some interesting consequences.

We shall say that an N-graph has an <u>M-separator</u> if there exists a set of at most M vertices whose removal allows the remainder of the graph to be partitioned into two parts, each containing at most 2N/3 vertices, with no edge directed out of one part into the other. We shall say that a graph with non-negative weights summing to 1 assigned to its vertices has a <u>strong weighted M-separator</u> if there exists a set of at most M vertices whose removal allows the remainder of the graph to be partitioned into two parts, each having weights summing to at most 1/2, with no edge directed out of one part into the other.

Let q be a non-negative non-decreasing function. We shall say that a class of graphs closed under taking subgraphs has q(N) separators if every N-graph in the class has an M-separator for some $M \leq q(N)$, and that it has strong weighted q(N) separators if every N-graph in the class has a strong weighted M-separator for some $M \leq q(N)$. Lipton and Tarjan have shown that if a class of graphs has q(N) separators, then it has strong weighted $q(N)+q(2N/3)+q(4N/9)+\ldots$ separators (see the proof of Corollary 3 in [9]).

Trees have O(1) separators (see Lewis, Stearns and Hartmanis [8]), and therefore have strong weighted O(log N) separators. This can be used to prove that 1-pushdown N-graphs can be pebbled in space O(log N), since these graphs are trees with some inessential extra edges.

Planar graphs have $O(N^{1/2})$ separators (see Lipton and Tarjan [9]), and therefore have strong weighted $O(N^{1/2})$ separators. This was used to prove that 2-pushdown N-graphs can be pebbled in space $O(N^{1/2})$, since these graphs are planar.

It is natural to conjecture that A-pushdown N-graphs have separation properties that would facilitate the determination of their space requirements and time-space trade-offs. We indeed conjecture that A-pushdown N-graphs have $O(N/\log_A N)$ separators, and therefore strong weighted $O(N/\log_A N)$ separators. It is not hard to see that no stronger conjecture is possible, since there exist A-pushdown N-graphs with only $\Omega(N/\log_A N)$ separators.

Aside from its possible consequences for pebbling, this conjecture has other computational consequences. First observe that if A-pushdown N-graphs have $O(N/\log_A N)$ separators, then A-pushdown B-pebble N-graphs have $O(BN/\log_A N)$ separators. Next observe that if G is a simple lift of G' and G has an M-separator, then G' has an M-separator.

Thus, by virtue of equivalence by mutual simple lifting, the multi-pushdown separator conjecture implies similar multi-tape and multi-head separator conjectures (and conversely, of course). These latter conjectures have several consequences for computations by multi-tape and multi-head machines. First, one could show that non-deterministic time T can be simulated by deterministic space O(T/log T) (strengthening the analogous result of Hopcroft, Paul and Valiant [6] for deterministic time T). Second, one could show that for some language, non-deterministic time T is more powerful than deterministic time T (using the technique of Paul and Reischuk [15], Theorem 2). Finally, one could show that for some language, non-oblivious time T is more powerful than oblivious time T (using the technique of Paul and Reischuk [15], Theorem 4).

5. References

[1] B. von Braunmuehl and R. Verbeek, "A Recognition Algorithm for DCFLs Optimal in Time and Space", IEEE Symp. on Foundations of Computer Science, 21 (1980) 411-420.

[2] S. A. Cook, "Deterministic CFL's Are Accepted Simultaneously in Polynomial Time and Log Squared Space", ACM Symp. on Theory of Computing, 11 (1979) 338-345.

[3] S. A. Cook and R. Sethi, "Storage Requirements for Deterministic Polynomial Time Recognizable Languages", J. Comp. and Sys. Sci., 13 (1976) 25-37.

[4] O. Gabber and Z. Galil, "Explicit Construction of Linear Size Superconcentrators", J. Comp. and Sys. Sci., 22 (1981) 407-420.

[5] J. Hartmanis and R. E. Stearns, "On the Computational Complexity of Algorithms", Trans. AMS, 117 (1965) 285-306.

[6] J. E. Hopcroft, W. J. Paul and L. G. Valiant, "On Time versus Space", J. ACM, 24 (1977) 332-337.

[7] T. Lengauer and R. E. Tarjan, "Upper and Lower Bounds on Time-Space Tradeoffs", ACM Symp. on Theory of Computing, 11 (1979) 262-277.

[8] P. M. Lewis, R. E. Stearns and J. Hartmanis, "Memory Bounds for the Recognition of Context-Free and Context-Sensitive Languages", IEEE Symp. on Switching Theory and Logical Design, 6 (1965) 191-202.

[9] R. J. Lipton and R. E. Tarjan, "A Separator Theorem for Planar Graphs", SIAM J. Appl. Math., 36 (1979) 177-189.

[10] R. J. Lipton and R. E. Tarjan, "Applications of a Planar Separator Theorem", SIAM J. Comp., 9 (1980) 615-627.

[11] G. A. Margulis, "Explicit Construction of Concentrators", Prob. Info. Trans., 9 (1973) 325-332.

[12] K. Mehlhorn, "Pebbling Mountain Ranges and Its Application to DCFL-Recognition", Internat. Coll. on Automata, Languages, and Programming, 7 (1980) 422-434.

[13] M. S. Paterson and C. E. Hewitt, "Comparative Schematology", Proj. MAC Conf. on Concurrent Systems and Parallel Computation, (1970) 119-127.

[14] W. J. Paul, R. E. Tarjan and J. R. Celoni, "Space Bounds for a Game on Graphs", Math. Sys. Theory, 10 (1977) 239-251.

[15] W. J. Paul and R. Reischuk, "On Alternation II", Acta Inf., 14 (1980) 391-403.

[16] N. Pippenger, "Superconcentrators", SIAM J. Comp., 6 (1977) 298-304.

[17] N. Pippenger, "Fast Simulation of Combinational Logic Networks by Machines without Random-Access Storage", Allerton Conf. on Communication, Control, and Computing, 15 (1977).

[18] N. Pippenger, "Comparative Schematology and Pebbling with Auxiliary Pushdowns", ACM Symp. on Theory of Computing, 12 (1980) 351-356.

[19] N. Pippenger, "Pebbling", IBM Japan Symp. on Mathematical Foundations of Computer Science, 5 (1980).

[20] N. Pippenger, "Pebbling with an Auxiliary Pushdown", J. Comp. and Sys. Sci., 23 (1981) 151-165.

[21] R. Sethi, "Complete Register Allocation Problems", SIAM J. Comp., 4 (1975) 226-248.

[22] L. G. Valiant, "Graph-Theoretic Properties in Computational Complexity", J. Comp. and Sys. Sci., 13 (1976) 278-285.

[23] R. Verbeek, "Time-Space Trade-Offs for General Recursion", IEEE Symp. on Foundations of Computer Science, 22 (1981) 228-234.