Chapter 11: Hash Tables

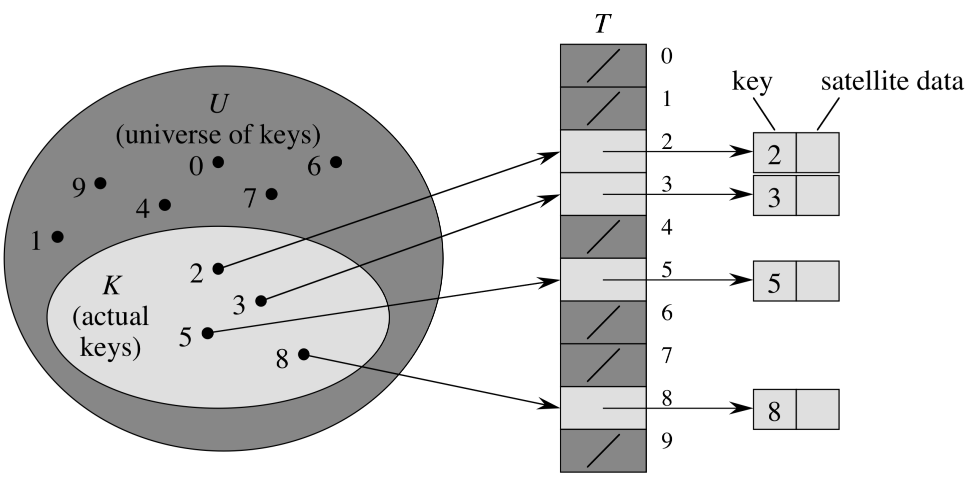
To search an element in a hash table:
Worst-case: $\Theta(n)$ (no better than a linked list).
Average-case: $O(1)$.

Direct-address Tables

- Each element to be stored has a unique key.
- Suitable for applications with small set U (the set of all possible keys).
- Suppose $U = \{0, 1, \ldots, m\}$. To store a dynamic set of elements, a direct-address table $T[0...m-1]$ is allocated, where

$$T[i] = \begin{cases} x & \text{if the element with key } i \text{ is stored, where } x \text{ points to the element} \\ nil & \text{otherwise} \end{cases}$$

Ex: $U = \{0, 1, \ldots, 9\}$ and the set of elements (keys) stored = $\{4, 7, 9\}$.




```
Direct-Address-Search(T, k)
1. return T(k)                // O(1)

Direct-Address-Insert(T, x)
<<<<<<< HEAD
1. T[x.key] = x            // O(1)

Direct-Address-Delete(T, x)
1. T[x.key] = nil          // O(1)
```

```
=======
1. T[key[x]] = x            // O(1)

Direct-Address-Delete(T, x)
1. T[key[x]] = nil          // O(1)
>>>>>>> fac36428d911c42afb28c4b9d34feb81c1776b9c
```

Hash Tables

- It's impractical for direct addressing if U is large.
  Ex: If U is the set of all possible SSN, then T requires 10^9 entries.

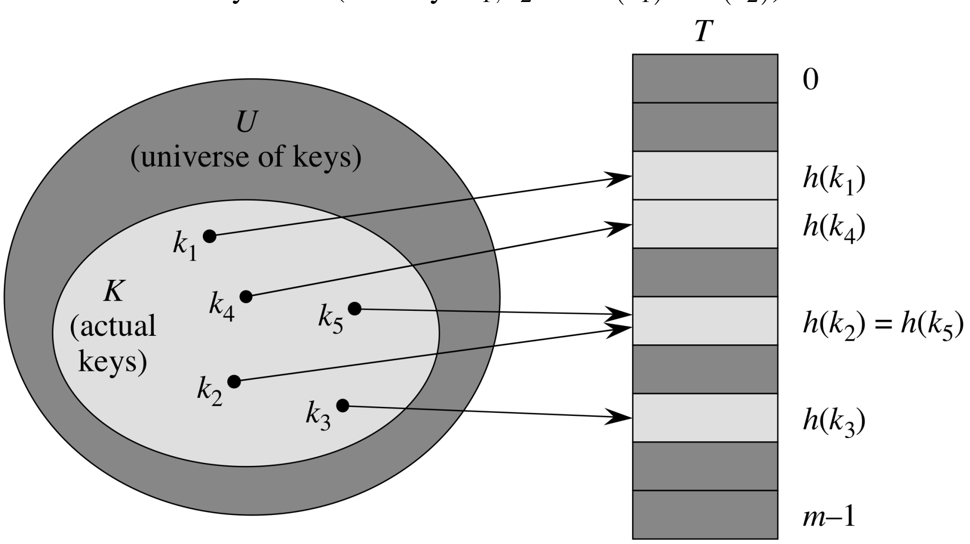
- We can reduce the memory requirement to $\Theta(n)$ and still have $O(1)$ average searching time by hashing technique, where n is the number of elements stored.

  ```
  Direct-addressing: an element with key k is stored in entry k.
  Hashing          : an element with key k is stored in entry h(k),
                     where h: hash function and h(k): hash value.
  ```

  For a hash table $T[0..m-1]$, the hash function $h : U \rightarrow \{0, 1, \ldots, m-1\}$.

- The hashing technique is for applications that $|U| >> m$ (the size of table T).

  $\Longrightarrow$ collisions may occur (two keys k_1, k_2 that $h(k_1) = h(k_2)$).



Solution for collision:

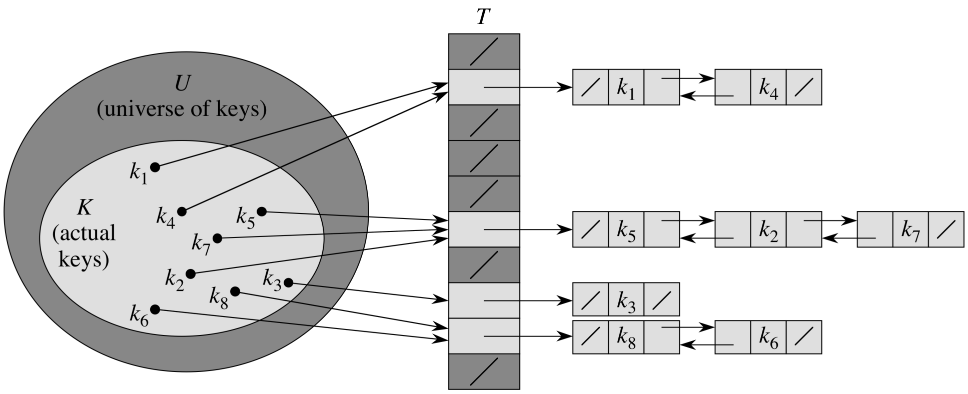
- avoid the collisions altogether: impossible.
- choose h to be "random" to minimize the # of collisions.

Two topics to research on:

1. 1. What is a good choice of h?
2. 2. How to resolve the collisions?

Collision resolution by chaining

We put all elements colliding on one entry into a linked list. For example:




```
Chained-Hash-Insert(T, x)
1. insert x at the front of list T[h(x.key)]        // O(1)


Chained-Hash-Search(T, k)
1. search for an element with key k in list T[h(k)]
```

Average time: $\Theta(1+\alpha)$, where $\alpha = n/m$ is the load factor of the table T. Based on *Theorem 11.1* and *Theorem 11.2* in the textbook. We will skip the proofs of those theorems though for this class.

```
Chained-Hash-Delete(T, x)                  // O(1): if doubly linked list.
1. delete x from the list T[h(x.key)]    // Same running time as search
                                           //      if singly linked list.
```

Hash Functions

- Good hash function: **simple uniform hashing**.
  Each key is equally likely to hash to any of the m entries.
- Assumption: Let the domain of keys is the set of natural numbers.
  If the keys are not numbers (e.g., strings), they should be mapped to numbers before hashing.

The division method

$h(k) = k \bmod m$.

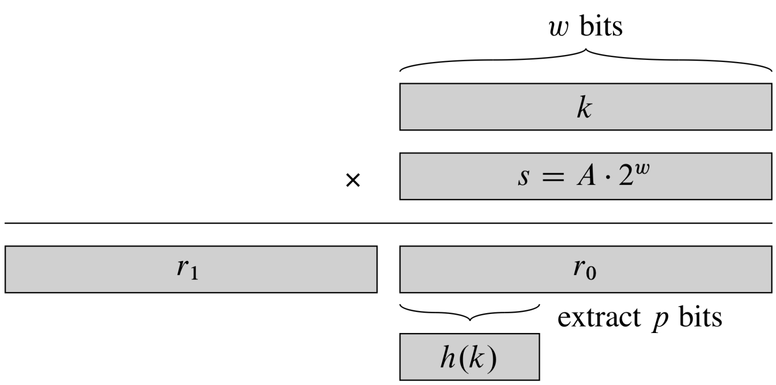
- the division method is almost simple uniform.
- m should not be a power of 2 (if $m = 2^p$, then $h(k)$ is the p low-order bits of k. It is better to make hash function depends on all bits of the key).
- Good choice for m: primes that are not too close to exact powers of 2.

Example: To hold 2000 keys and 3 elements examined in average for an unsuccessful search. The collision is resolved by chaining. What the table size m should be?
Sol: $m = 701$, since 701 is a prime and $701 \simeq 2000/\alpha$, and also 701 is not close to any power of 2.

Multiplication method

$$h(k) = \lfloor m(kA \bmod 1) \rfloor$$

- The value of m isn't critical so we could choose to be a power of 2 ($m = 2^p$ for some integer p) to make the calculation of the hash function easier.



Recommended Exercises: 11.3-1, 11.3-2.

Open Addressing

- Another collision resolution technique.
- Each entry in the table T can store at most one element, rather than a linked list of elements in chaining. Thus, $\alpha \leq 1$.
- To perform insertion, we successively examine, or probe, the hash table until we find an empty entry.
- The probe sequence (the sequence of entries examined) depends on the probing techniques used. There are 3 popular probing techniques.

Pseudocode:

```
Hash-Search(T, k)
1. i = 0
2. repeat
3          j = h(k, i)
4.         if T[j] == k
5.                 return j
6.            else i = i+1
7. until T[j] == nil or i == m
8. return nil

Hash-Insert(T, k)
1. i = 0
2. repeat
3.       j = h(k, i)
3.       if T[j] == nil or DELETED
4.            T[j] = k
5.            return j
6.       else i = i+1
7. until i == m
8. error "hash table overflow"


//If an entry is occupied by an element but the element
//has been deleted later, we need to set a flag ``deleted'' to this entry.

Hash-Delete(T, k)
1. i = 0
2. repeat
```

```
    j = h(k, i)
    if T[j] == k
        T[j] = DELETED
        return j
    else i = i+1
until T[j] == nil or i == m
error  "k is not in the table"
```

Linear Probing: The probe sequence is:

$$h(k,i) = (h'(k)+i) \bmod m,\ for\ i = 0,1,\ldots,m-1$$

That is, $T[h'(k)],\ T[h'(k)+1],\ldots,T[m-1],\ T[0],\ T[1],\ldots,T[h'(k)-1]$.

This technique suffers on the **primary clustering** problem: long runs of occupied entries.

- Why does linear probing usually results in primary clustering?
- What's wrong with the primary clustering?
- An empty slot preceded by i full slots gets filled next with probability $(i+1)/m$. So long runs tend to get longer, thus decreasing the average search time.

Quadratic Probing: The probe sequence is

$$h(k,i) = (h'(k)+c_1 i + c_2 i^2) \bmod m,\ \ where\ c_1, c_2 > 0, i = 0,1,\ldots,m-1$$

c_1, c_2 and m need to be chosen carefully to **fully utilize** the table T (see Problem 11.3).

- To fully utilize the table T, we mean that for given any key k, the probe sequence of k can check the entire table.

If two keys k_1, k_2 have the same initial probe position, then they will have the same probe sequence. That is, $h(k_1,0) = h(k_2,0)$ implies $h(k_1,i) = h(k_2,i), \forall\ i$.

This phenomenon leads to a milder form of clustering: **secondary clustering**.

Double Hashing: The probe sequence is

$$h(k,i) = (h_1(k)+ih_2(k)) \bmod m,\ for\ i = 0,1,\ldots,m-1$$

- The probe sequence depends in two ways upon the key k. In this case, if two keys have the same initial probe position, they may not have the same probe sequence. Thus, double hashing does not suffer from secondary clustering.
- In order to fully utilize the table, given any k, $h_2(k)$ **must be always relatively prime to** m.
  If $d = \gcd\{h_2(k), m\}$, then only $\frac{1}{d}$ th of table will be searched.
  Two different approaches to make $h_2(k)$ always relatively prime to m.

1. Let $m = 2^p$, where p is some positive integers.
   Design h_2 so that it always produces an odd number.
2. Let m be a prime number.
   Design h_2 so that it always produces a positive integer less than m.

   Ex: Let m be a prime and let

$$\begin{cases} h_1(k) = k \bmod m \\ h_2(k) = 1 + (k \bmod m'), \text{where } m' = m-1 \text{ or } m-2 \end{cases}$$

Recommended Exercise: 11.4-1, 11.4-2.

Analysis of hashing by chaining and open address hashing:

Theorem 11.1 *For hashing by chaining, an unsuccessful search takes time* $\Theta(1+\alpha)$ *in average, under the simple uniform hashing assumption.*

Theorem 11.2 *For hashing by chaining, a successful search takes time* $\Theta(1+\alpha)$ *in average, under the simple uniform hashing assumption.*

Theorem 11.6 *For open-address hashing, the expected # of probes in an unsuccessful search is at most* $1/(1-\alpha)$*, assuming uniform hashing.*

Theorem 11.8 *For open-address hashing, the expected # of probes in a successful search is at most* $\frac{1}{\alpha}\ln\frac{1}{1-\alpha}$*, assuming uniform hashing.*