

# Chapter 2.3: Designing Algorithms and Merge sort

## Techniques for Designing Algorithms

- Insertion sort, selection sort, linear search use an **incremental** algorithm design techniques. These usually result in *iterative* algorithms.
- **Recursive** algorithms are useful and common and provide a different way of tackling problems. It goes hand-in-hand with the **divide-and-conquer** algorithm design technique.
- **Divide-and-Conquer** is an useful recursive technique for designing algorithms. It consists of three steps:

**Divide** the problem into a number of subproblems that are smaller instances of the same problem.

**Conquer** the subproblems by solving them recursively. If the subproblems are small enough, just solve the problems directly.

**Combine** the solutions of the subproblems into the solution for the original problem.

## Merge sort: a divide and conquer algorithm

- **Merge sort** is an example of a divide-and-conquer algorithm.

**Divide:** the  $n$ -element sequence to be sorted into two subsequences of  $n/2$  elements each.

**Conquer:** sort the two subsequences recursively using merge sort.

**Combine:** by **merging** the two sorted subsequences to produce a sorted answer.

- The base case for the recursion is when the sequence to be sorted has length 1.
- The key part is to merge to two sorted subsequence. Let's examine the merge procedure shown below.

```

MERGE( $A, p, q, r$ )
1   $n_L = q - p + 1$            // length of  $A[p : q]$ 
2   $n_R = r - q$                // length of  $A[q + 1 : r]$ 
3  let  $L[0 : n_L - 1]$  and  $R[0 : n_R - 1]$  be new arrays
4  for  $i = 0$  to  $n_L - 1$  // copy  $A[p : q]$  into  $L[0 : n_L - 1]$ 
5       $L[i] = A[p + i]$ 
6  for  $j = 0$  to  $n_R - 1$  // copy  $A[q + 1 : r]$  into  $R[0 : n_R - 1]$ 
7       $R[j] = A[q + j + 1]$ 
8   $i = 0$                      //  $i$  indexes the smallest remaining element in  $L$ 
9   $j = 0$                      //  $j$  indexes the smallest remaining element in  $R$ 
10  $k = p$                      //  $k$  indexes the location in  $A$  to fill
11 // As long as each of the arrays  $L$  and  $R$  contains an unmerged element,
    // copy the smallest unmerged element back into  $A[p : r]$ .
12 while  $i < n_L$  and  $j < n_R$ 
13     if  $L[i] \leq R[j]$ 
14          $A[k] = L[i]$ 
15          $i = i + 1$ 
16     else  $A[k] = R[j]$ 
17          $j = j + 1$ 
18      $k = k + 1$ 
19 // Having gone through one of  $L$  and  $R$  entirely, copy the
    // remainder of the other to the end of  $A[p : r]$ .
20 while  $i < n_L$ 
21      $A[k] = L[i]$ 
22      $i = i + 1$ 
23      $k = k + 1$ 
24 while  $j < n_R$ 
25      $A[k] = R[j]$ 
26      $j = j + 1$ 
27      $k = k + 1$ 

```

- Observations about the Merge procedure
  - The worst-case run-time is  $\Theta(n)$ .
  - The algorithm is **oblivious** in that its run-time doesn't change due to the instance of the problem.

- Now we can write out the pseudo-code for the merge sort algorithm:

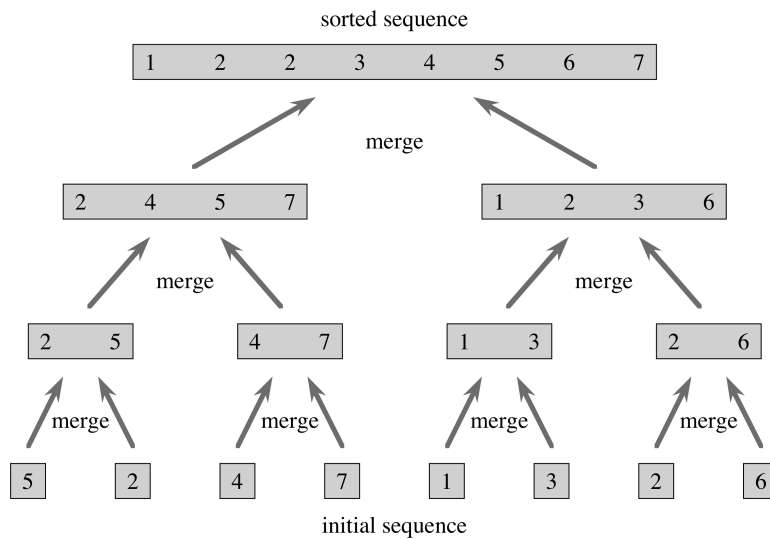
MERGE-SORT( $A, p, r$ )

```

1  if  $p \geq r$                                 // zero or one element?
2      return
3   $q = \lfloor (p + r)/2 \rfloor$                     // midpoint of  $A[p:r]$ 
4  MERGE-SORT( $A, p, q$ )                        // recursively sort  $A[p:q]$ 
5  MERGE-SORT( $A, q + 1, r$ )                    // recursively sort  $A[q + 1:r]$ 
6  // Merge  $A[p:q]$  and  $A[q + 1:r]$  into  $A[p:r]$ .
7  MERGE( $A, p, q, r$ )

```

- Here is an example of running merge sort on the input  $A = \langle 5, 2, 4, 6, 1, 3, 2, 6 \rangle$



- Try Exercise 2.3-1 (on your own): Try merge sort on the input  $A = \langle 3, 41, 52, 26, 38, 57, 9, 49 \rangle$ .

# Analysis of Divide and Conquer Algorithms

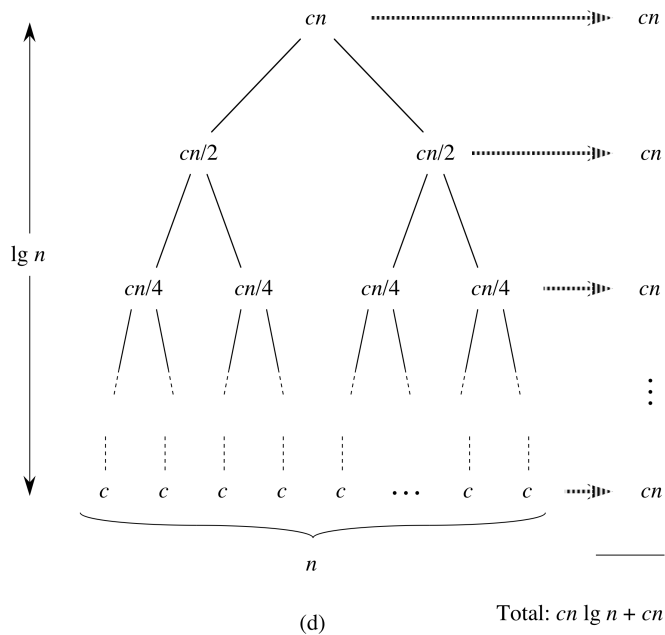
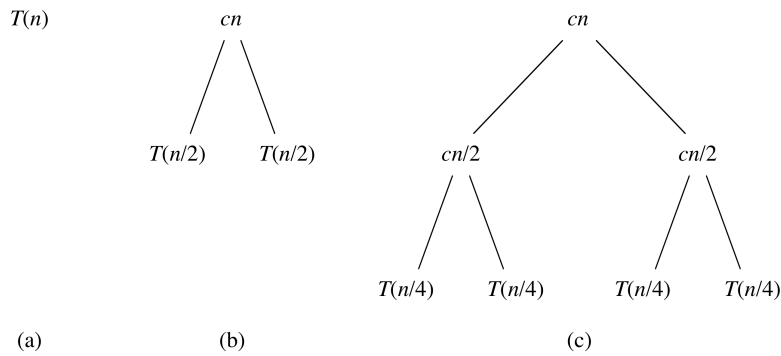
- The run time of a divide-and-conquer algorithm can be described by a **recurrence equation** (or just **recurrence**). Here is the general form:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

- For merge sort,  $a = 2$  and  $b = 2$ .

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1, \\ 2T(n/2) + cn & \text{otherwise} \end{cases}$$

- Techniques for solving recurrence equations:
  - *guess* and then prove by mathematical induction
  - *substitution method* → keep expanding the recurrence until we can come up with the closed form. To be thorough, we would also have to prove using induction that the closed form is correct.
  - *draw a recurrence tree* and then use the tree to add up the run-time
- Running time analysis, by drawing the recurrence tree:



- $\lceil \log_2 n \rceil$  levels of merging in the tree
- Each level takes linear (to  $n$ ) time
- Thus, total running time is  $\Theta(n \log n)$

• **Recommended Exercises:**

- Ex 2.3-6: Binary Search
- Ex 2.3-7. Insertion sort combined with binary search
- Problem 2.1: Merge sort combined with insertion sort

- **Extra reading (or listening!):** Check out the song “*There’s a hole in my bucket*” for recursion without a base case! It even has a wikipedia page for it: [https://en.wikipedia.org/wiki/There%27s\\_a\\_Hole\\_in\\_My\\_Bucket](https://en.wikipedia.org/wiki/There%27s_a_Hole_in_My_Bucket)

Here is a classic performance of the song!

[https://www.youtube.com/watch?v=xVAvMIhvqfk&ab\\_channel=HarryBelafonteTe](https://www.youtube.com/watch?v=xVAvMIhvqfk&ab_channel=HarryBelafonteTe)