Minimally Adaptive BH

a tiny but uniform improvement of the procedure of Benjamini and Hochberg

Aldo Solari

Joint work with Jelle Goeman

Adaptive Designs and Multiple Testing Procedures

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Outline

Simes test

- $m \geq 2$ hypotheses, $m_0 \leq m$ true hypotheses
- $p_1 \leq \ldots \leq p_m$ ordered p-values
- Assumption: Simes inequality holds for p-values corresponding to true hypotheses

- Global hypothesis $\mathrm{G}_0 : \pi_0 = \dfrac{m_0}{m} = 1$
- $S(\alpha) = \bigcap_{i=1}^{m} \left\{ p_i > \dfrac{i\alpha}{m} \right\}$: "Simes test doesn't reject G_0 at α"
- $\bar{S}(\alpha) = \bigcup_{i=1}^{m} \left\{ p_i \leq \dfrac{i\alpha}{m} \right\}$: "Simes test rejects G_0 at α"

Benjamini and Hochberg procedure

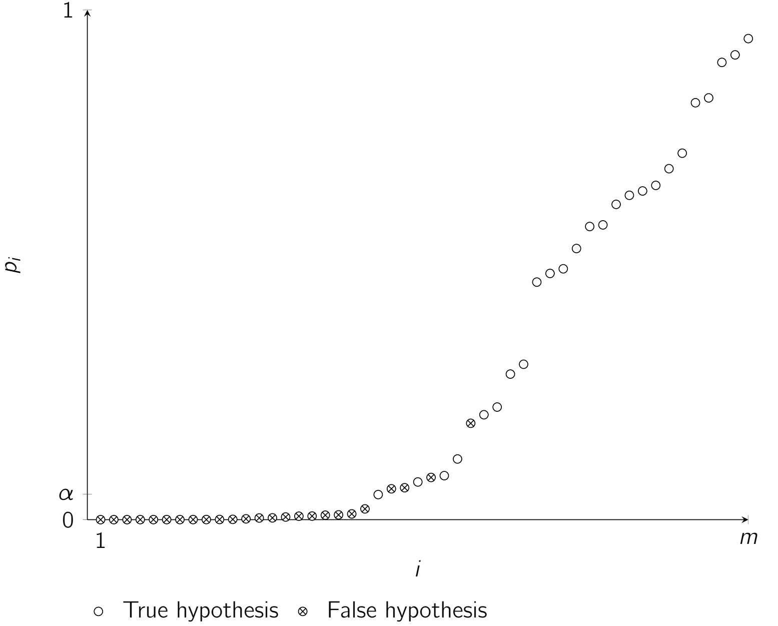
BH(α) rejects the $R(\alpha)$ hypotheses with smallest p-values:

$$R(\alpha) = \begin{cases} 0 & \text{if } S(\alpha) \\ \max\left\{i : p_i \leq \frac{i\alpha}{m}\right\} & \text{if } \bar{S}(\alpha) \end{cases}$$

Very popular: BH95 has $> 32K$ citations

?p.adjust example

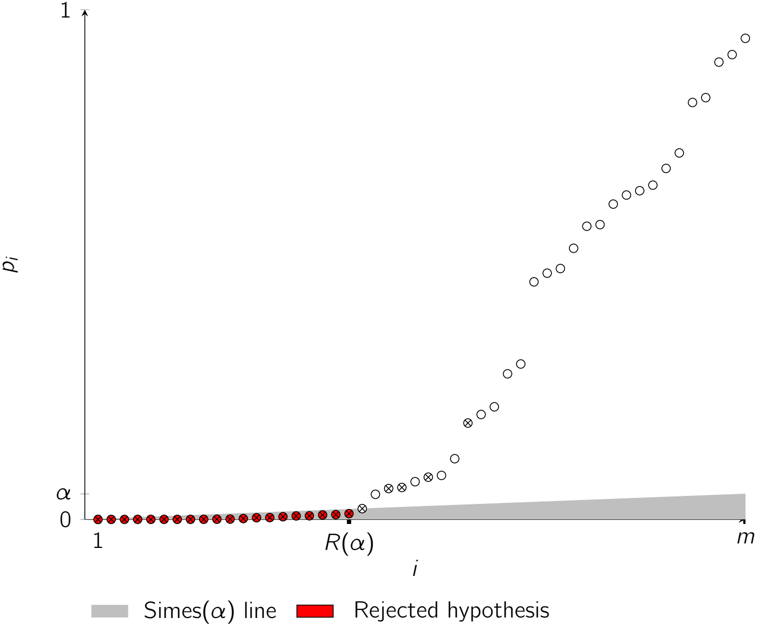
```
set.seed(123)
x <- rnorm(50, mean = c(rep(0, 25), rep(3, 25)))
p <- 2*pnorm(sort(-abs(x)))
```


1
p_i
α
0
1
m
i
True hypothesis
False hypothesis

BH(5%)

```
p.adjust(p, "BH")

 [1] 0.0000 0.0004 0.0004 0.0009 0.0009
 [6] 0.0009 0.0009 0.0010 0.0013 0.0019
[11] 0.0028 0.0067 0.0126 0.0126 0.0175
[16] 0.0208 0.0208 0.0245 0.0247 0.0282
[21] 0.0504 oh,no!
```



1
α
0
p_i
1
R(α)
m
i
Simes(α) line
Rejected hypothesis

FDR control for BH

Under the assumption of independence or PRDS,
BH(α) controls the FDR at level $\pi_0\alpha \leq \alpha$, i.e.

$$\mathrm{E}[Q(\alpha)] \leq \pi_0\alpha \leq \alpha$$

- $Q(\alpha)$ is the FPD of BH(α), i.e.

$$Q(\alpha) = \begin{cases} 0 & \text{if } S(\alpha) \\ \frac{T(\alpha)}{R(\alpha)} & \text{if } \bar{S}(\alpha) \end{cases}$$

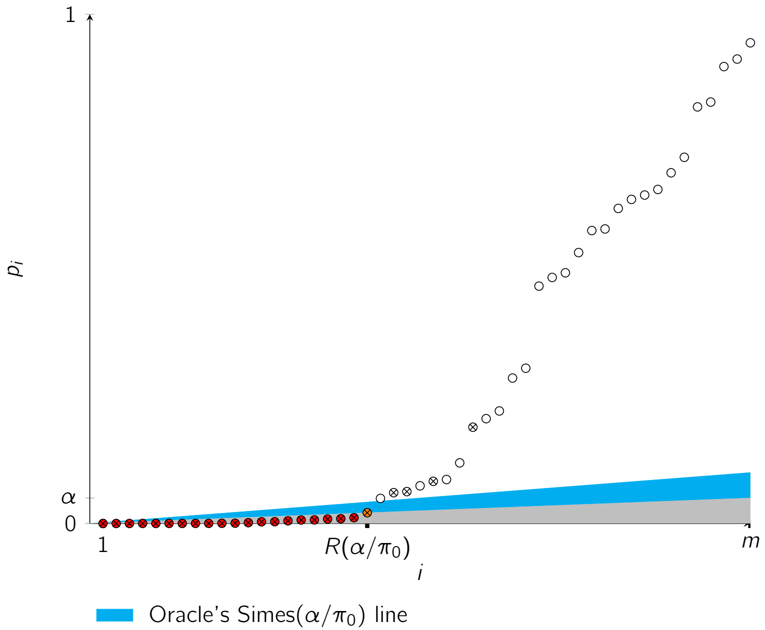
- $T(\alpha)$ is number of true hypotheses rejected by BH(α)

Oracle BH procedure

- The Oracle knows the true value of π_0
- OBH(α) = BH(α/π_0):

$$R(\alpha/\pi_0) = \begin{cases} 0 & \text{if } S(\alpha/\pi_0) \\ \max\left\{i : p_i \leq \frac{i\alpha}{m_0}\right\} & \text{if } \bar{S}(\alpha/\pi_0) \end{cases}$$

- If BH(α) controls FDR at $\pi_0\alpha$ for all α, then OBH(α) controls FDR at α for all α
- OBH(α) is uniformly more powerful than BH(α)



1
α
0
p_i
1
$R(\alpha/\pi_0)$
m
i
Oracle's Simes(α/π_0) line

Two-stages Adaptive BH procedures

1. Estimate π_0 by $\hat{\pi}_0$
2. $\mathrm{ABH}(\alpha) = \mathrm{BH}(\alpha^*/\hat{\pi}_0)$

Existing procedures

- Existing $\mathrm{ABH}(\alpha)$ are not uniformly more powerful than $\mathrm{BH}(\alpha)$ because $\alpha^*/\hat{\pi}_0$ can be strictly smaller than α
- Usually proven FDR control under independence only
- Storey, Taylor and Siegmund (2004)

  $\alpha^* = \alpha$ but $\hat{\pi}_0(\lambda) = \dfrac{\#\{p_i > \lambda\} + 1}{m(1-\lambda)}$ can be larger than 1
- Benjamini, Krieger and Yekutieli (2006)

  $\hat{\pi}_0 = \dfrac{m - R(\alpha^*)}{m}$ with $\alpha^* = \dfrac{\alpha}{1+\alpha} < \alpha$

Outline

1. Introduction
2. **The MABH procedure**
3. The case of two hypotheses

Minimally Adaptive BH procedure

1. $\hat{\pi}_0 = \begin{cases} 1 & \text{if } S(\alpha) \\ (m-1)/m & \text{if } \bar{S}(\alpha) \end{cases}$
2. $\mathsf{MABH}(\alpha) = \mathsf{BH}(\alpha/\hat{\pi}_0) = \begin{cases} \mathrm{BH}(\alpha) & \text{if } S(\alpha) \\ \mathrm{BH}\left(\frac{m\alpha}{m-1}\right) & \text{if } \bar{S}(\alpha) \end{cases}$

MABH(α) rejects the $R'(\alpha)$ hypotheses with smallest p-values:

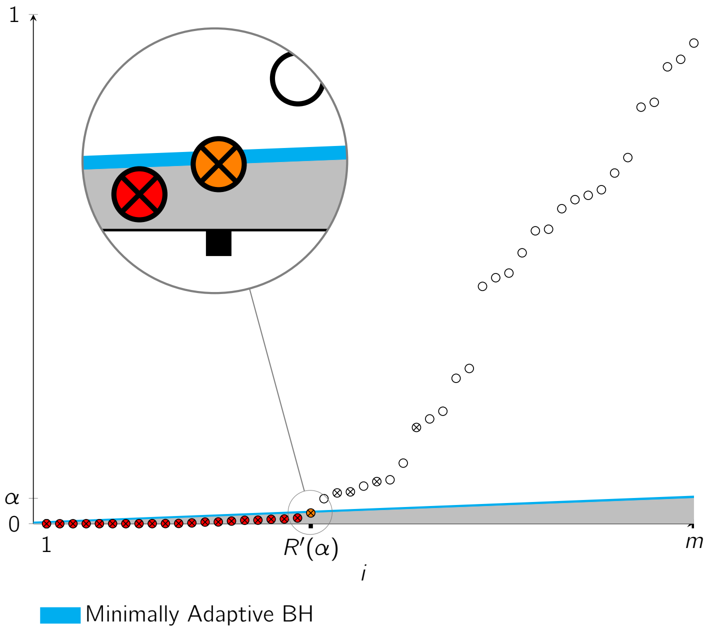
$$R'(\alpha) = \begin{cases} 0 & \text{if } S(\alpha) \\ \max\left\{i : p_i \leq \frac{i\alpha}{m-1}\right\} & \text{if } \bar{S}(\alpha) \end{cases}$$

MABH(5%)

```
p.adjust(p, "MABH")

 [1] 0.0000 0.0004 0.0004 0.0009 0.0009
 [6] 0.0009 0.0009 0.0010 0.0012 0.0019
[11] 0.0027 0.0066 0.0124 0.0124 0.0171
[16] 0.0203 0.0203 0.0240 0.0242 0.0277
[21] 0.0494
```

$$\tilde{p}_i = \max\left(\min_j \frac{mp_j}{j},\ \min_{k \geq i} \frac{(m-1)p_k}{k}\right)$$



1
α
0
1
R′(α)
m
i
Minimally Adaptive BH

FDR control for MABH

If BH(α) controls FDR at $\pi_0\alpha$ for all α, then MABH(α) controls FDR at α for all α, i.e.

$$\mathrm{E}[Q(\alpha)] \leq \pi_0\alpha \Rightarrow \mathrm{E}[Q'(\alpha)] \leq \alpha \quad \forall\alpha$$

- $\pi_0 = 1$

$$\mathrm{E}[Q'(\alpha)] = \mathrm{P}[R'(\alpha) > 0] = \mathrm{P}[\bar{S}(\alpha)] = \mathrm{P}[R(\alpha) > 0] = \mathrm{E}[Q(\alpha)] \leq \alpha$$

- $\pi_0 \leq (m-1)/m$

$$Q'(\alpha) = \begin{cases} 0 \leq Q\left(\frac{m\alpha}{m-1}\right) & \text{if } S(\alpha) \\ Q\left(\frac{m\alpha}{m-1}\right) & \text{if } \bar{S}(\alpha) \end{cases}$$

$$\mathrm{E}[Q'(\alpha)] \leq \mathrm{E}\left[Q\left(\frac{m\alpha}{m-1}\right)\right] \leq \pi_0 \frac{m\alpha}{m-1} \leq \alpha$$

Outline

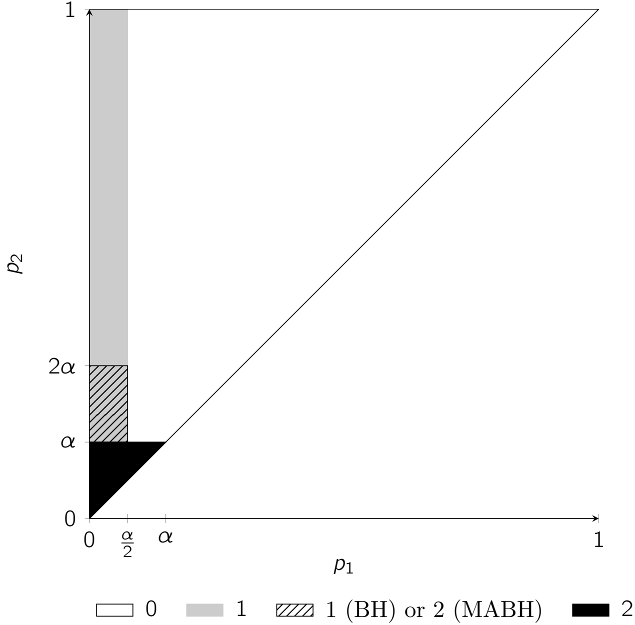
Case $m = 2$

- BH(α)

$$R(\alpha) = \begin{cases} 0 & \text{if } \{p_1 > \alpha/2\} \cap \{p_2 > \alpha\} \\ 1 & \text{if } \{p_1 \leq \alpha/2\} \cap \{p_2 > \alpha\} \\ 2 & \text{if } p_2 \leq \alpha \end{cases}$$

- MABH(α)

$$R'(\alpha) = \begin{cases} 0 & \text{if } \{p_1 > \alpha/2\} \cap \{p_2 > \alpha\} \\ 1 & \text{if } \{p_1 \leq \alpha/2\} \cap \{p_2 > 2\alpha\} \\ 2 & \text{if } \{\{p_1 \leq \alpha/2\} \cap \{p_2 \leq 2\alpha\}\} \cup \{p_2 \leq \alpha\} \end{cases}$$


1
2α
α
0
0
α/2
α
1
p2
p1
0
1
1 (BH) or 2 (MABH)
2

Case $m = 2$: FWER and FDR control

- Hochberg(α) = Hommel(α) controls FWER at α under Simes inequality
- BH(α) = Hochberg(α) = Hommel(α), thus BH(α) controls both FWER and FDR at α under PRDS
- MABH(α) controls FDR at α but not FWER under PRDS
- Counter-example (Dirac-Uniform configuration):
  H_1 false with p-value 0, H_2 true with p-value Uniform
  Then FWER for MABH(α) is $2\alpha > \alpha$

Conclusions

- $\oplus$ MABH is uniformly more powerful than BH
- $\oplus$ When $m = 2$, MABH is the first procedure that controls FDR but not FWER under PRDS
- $\ominus$ Power gain of MABH over BH is negligible when m is large
- $\ominus$ MABH is less elegant than BH

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