Weighted Kolmogorov-Smirnov Tests in One- and Two-Sample Problems

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One-Sample Weighted Kolmogorov-Smirnov (W-KS) Tests

Let $X_1, \ldots, X_m \sim F$ be iid. W.l.o.g. we test

$$H_0 : F(t) = t, \quad t \in [0, 1].$$

For $\nu \in [0, 1]$ one-sample W-KS statistics are defined by

$$\boxed{KS_m^\nu = \sup_{t \in (0,1)} \sqrt{m} \frac{|\hat{F}_m(t) - t|}{(t(1-t))^\nu}.}$$

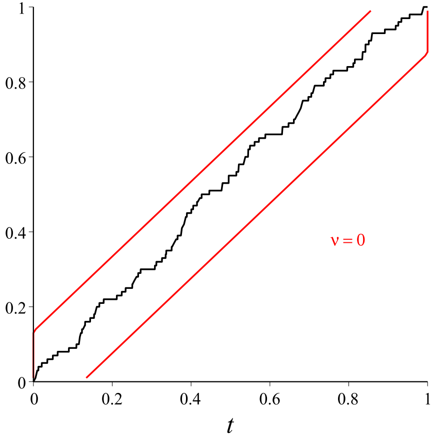
Reject H_0 if $KS_m^\nu > b_m$, where b_m denotes a suitable critical value. Let b_m be such that $\mathbb{P}(KS_m^\nu > b_m | H_0) = \alpha$.

For $\nu = 0$ we get the original KS test and for $\nu = 1/2$ we get the supremum version of the Anderson-Darling test, also known as higher criticism (HC) test.

One-Sample W-KS Tests:

Graphical Representation

Ecdf of m iid $U(0,1)$-random variables and rejection curves related to level α W-KS tests with $m = 100$ and $\alpha = 0.05$



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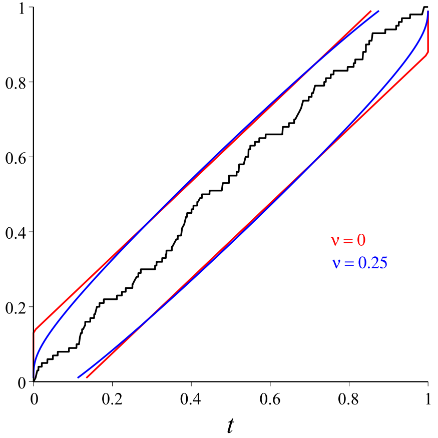

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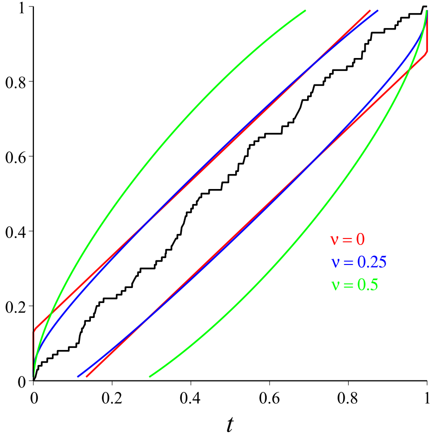

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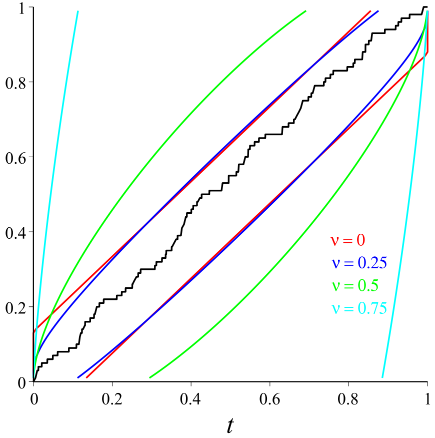
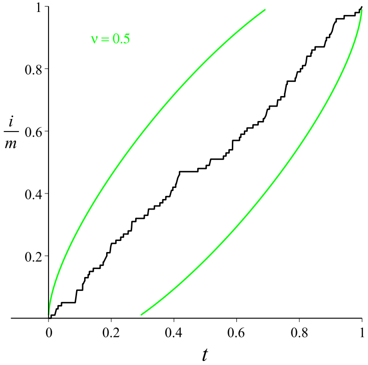


A New Point of View: Local Levels



Local levels α_i, $i = 1, \ldots, m$, are defined as probabilities that $X_{i:m}$ leads to the rejection of true H_0, i.e.,

$$\boxed{\alpha_i^{low} = \mathbb{P}(X_{i:m} < c_i^\nu | H_0), \quad \alpha_i^{up} = \mathbb{P}(X_{i:m} > d_i^\nu | H_0),}$$

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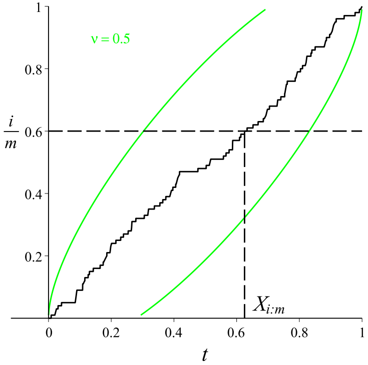

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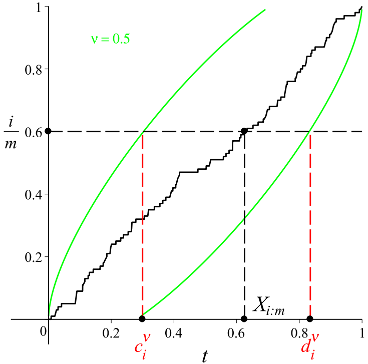

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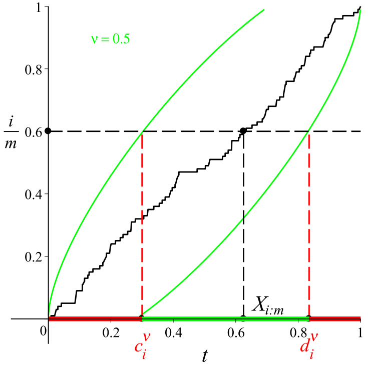

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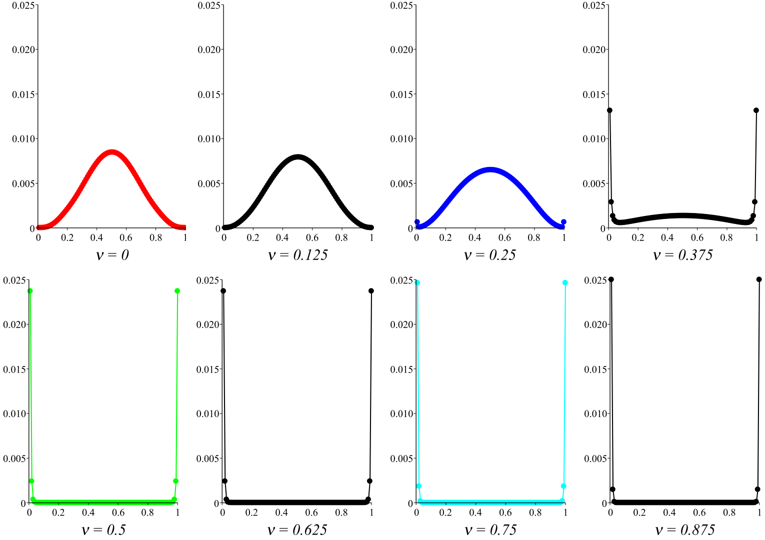
One-Sample W-KS Local Levels: Examples

Local levels of level α W-KS tests with $\alpha = 0.05$, $m = 100$ and $\nu = 0(0.125)0.875$



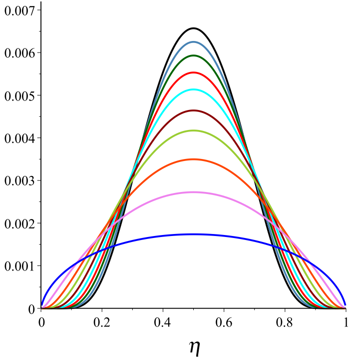
Asymptotics of One-Sample W-KS Local Levels

The asymptotics of the W-KS statistics is examined in many papers and can be applied to derive the asymptotics of W-KS local levels.

- For $\nu \in [0, 1/2)$ we get $\lim_{m\to\infty} \alpha_i > 0$ for central i-values, i.e., $\lim_{m\to\infty} i/m \in (0, 1)$, and $\lim_{m\to\infty} \alpha_i = 0$ else. The asymptotic sensitivity range is the central range.
- For $\nu = 1/2$ (HC) we get that $\lim_{m\to\infty} \alpha_i = 0$ for all i and the most HC local levels are asymptotically equal, i.e., $\lim_{m\to\infty} \alpha_i/\alpha_j = 1$, cf. **Gontscharuk, Landwehr & Finner [2015,2016]**. The asymptotic sensitivity range consists of intermediates.
- For $\nu \in (1/2, 1]$ we get $\lim_{m\to\infty} \alpha_i > 0$ and $\lim_{m\to\infty} \alpha_{m-i} > 0$ only for fixed $i \in \mathbb{N}$. The asymptotic sensitivity range coincides with the extreme range.

Asymptotic Local Levels for $\nu < 1/2$

Asymptotic local levels $\lim_{m\to\infty} \alpha_i$ with $\lim_{m\to\infty} i/m = \eta$ for level α W-KS tests with $\alpha = 0.05$, $\nu = 0(0.05)0.45$ (from top to bottom)



$$\lim_{m\to\infty} \alpha_i = 2\Phi\left(-\frac{b_{\nu,\alpha}}{(\eta(1-\eta))^{1/2-\nu}}\right) \text{ for } \eta \in (0,1)$$

$$\lim_{m\to\infty} \alpha_i = 0 \text{ for } \eta = 0, 1$$

Two-Sample W-KS Tests

Let $X_1, \ldots, X_m \sim F$ be iid and let $Y_1, \ldots, Y_n \sim G$ be iid with continuous F, G. We test

$$H_0 : F(t) = G(t), \quad t \in \mathbb{R}.$$

For $\nu \in [0, 1]$, two-sample W-KS statistics are defined by

$$\boxed{\mathit{KS}^{\nu}_{m,n} = \sup_{t\in\mathbb{R}} \sqrt{\frac{mn}{m+n}} \frac{|\hat{G}_n(t) - \hat{F}_m(t)|}{(\hat{H}_{m+n}(t)(1 - \hat{H}_{m+n}(t)))^{\nu}}.}$$

Reject H_0 if $\mathit{KS}^{\nu}_{m,n} > b_{m,n}$, where $b_{m,n}$ denotes a suitable critical value.

Two-Sample W-KS Local Levels

Let $t_1 \leqslant \ldots \leqslant t_{m+n}$ denote the ordered combined sample and let $V_{m,s} = m\hat{F}_m(t_s)$. For suitable c_s^ν, d_s^ν, $s = 1, \ldots, m+n$, we get

$$KS^\nu_{m,n} \leqslant b_{m,n} \text{ iff } c_s^\nu \leqslant V_{m,s} \leqslant d_s^\nu \text{ for all } s = 1, \ldots, m+n.$$

Assumptions:

- Critical values c_s^ν, d_s^ν, $s = 1, \ldots, m+n$, are proper;
- $\mathbb{P}(KS^\nu_{m,n} > b_{m,n} | H_0) \leqslant \alpha$;
- $\mathbb{P}(KS^\nu_{m,n} > b_{m,n} | H_0)$ is as large as possible.

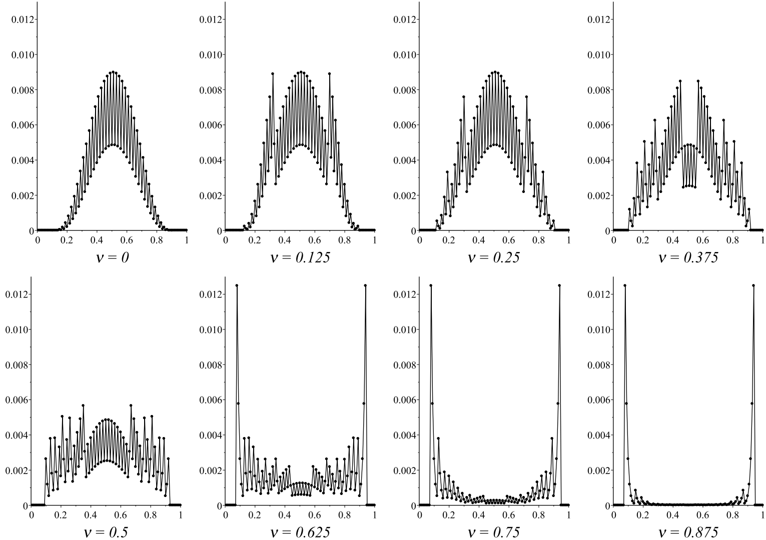
Similar as in the one-sample case, W-KS local levels are defined by

$$\boxed{\alpha_s^{low} = \mathbb{P}(V_{m,s} < c_s^\nu | H_0), \quad \alpha_s^{up} = \mathbb{P}(V_{m,s} > d_s^\nu | H_0),}$$

$$\boxed{\alpha_s = \alpha_s^{low} + \alpha_s^{up}.}$$

Two-Sample W-KS Local Levels: $m = n$

Local levels of level α W-KS tests with $\alpha = 0.05$, $m = n = 50$ and $\nu = 0(0.125)0.875$



Asymptotics of Two-Sample W-KS Local Levels

In contrast to the one-sample case, it seems that the asymptotics of two-sided W-KS test statistics for $\nu \in (0,1]$ has not been considered in the literature.

- For $\nu \in [0, 1/2)$ the two-sample W-KS asymptotics coincides with the one-sample case.
- For $\nu = 1/2$ (HC) the two-sample W-KS asymptotics coincides with the related one-sample HC asymptotics with sample size $\min(m, n)$.
- For $\nu \in (1/2, 1]$ the two-sample W-KS asymptotics differs from the one-sample counterpart if $\lim_{m,n\to\infty} m/n = \zeta \in (0, \infty)$.

Key tools: Weighted Brownian bridges, weighted standardized Poisson and Binomial processes.

Asymptotics for $\nu < 1/2$

Let $\mathbb{B}$ denote a standard Brownian bridge on $[0,1]$ and let

$$\mathbb{B}^{\nu}(t) = \frac{\mathbb{B}(t)}{(t(1-t))^{\nu}}, \quad t \in (0,1), \quad \nu \in [0,1/2].$$

Theorem 1: For $\nu \in [0,1/2)$ it holds

$$\boxed{\lim_{m,n\to\infty} KS^{\nu}_{m,n} \overset{\mathcal{D}}{=} \sup_{t\in(0,1)} |\mathbb{B}^{\nu}(t)|.}$$

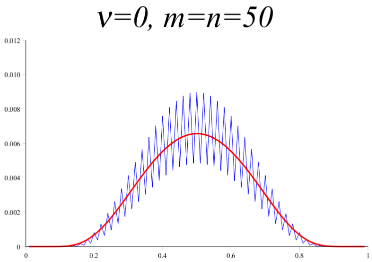
Moreover, let $b_{\nu,\alpha}$ be the asymptotic critical value, i.e., $\mathbb{P}(\sup_{t\in(0,1)} |\mathbb{B}^{\nu}(t)| > b_{\nu,\alpha}) = \alpha$, then

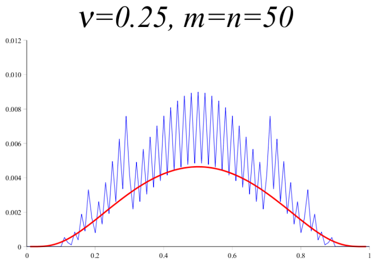
$$\boxed{\lim_{m,n\to\infty} \alpha_s = 2\Phi\Big(- \frac{b_{\nu,\alpha}}{(\eta(1-\eta))^{1/2-\nu}}\Big)}$$

for s such that $\lim_{m,n\to\infty} s/(m+n) = \eta \in (0,1)$.

Asymptotic and Finite Local Levels

Illustration for $\nu = 0$ and $\nu = 0.25$





The W-KS asymptotics seems to be faster for smaller ν-values.

Asymptotics for $\nu = 1/2$

Theorem 2: Let $m = \min(m, n)$ and let $b_{m,\alpha}$ be such that

$$\lim_{m,n\to\infty} \mathbb{P}\Big(\sup_{t\in(1/m, 1-1/m)} |\mathbb{B}^{1/2}(t)| > b_{m,\alpha} \Big) = \alpha.$$

Then

$$\boxed{\lim_{m,n\to\infty} \mathbb{P}(\mathit{KS}^{1/2}_{m,n} > b_{m,\alpha}) = \alpha.}$$

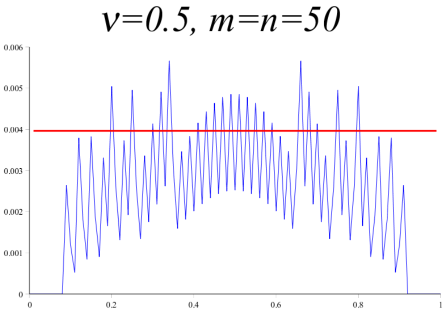
Moreover, almost all HC local levels are asymptotically equal to

$$\boxed{\alpha^*_m = \frac{-\log(1-\alpha)}{2\log(m)\log(\log(m))}}$$

in the sense $\lim_{m,n\to\infty} \alpha_s/\alpha^*_m = 1$.

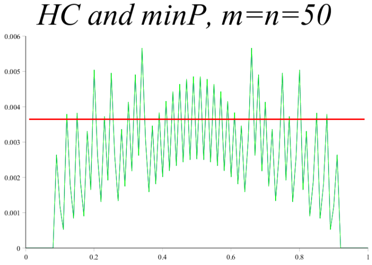
Asymptotic and Finite HC Local Levels

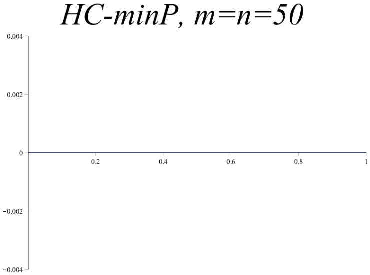
Illustration for $\nu = 1/2$



The two-sample HC asymptotics seems to be much better than the related one-sample asymptotics.

Comparison of HC and Minimum p-Value Tests





The HC and minP tests coincide asymptotically under H_0 and differ not much even for a finite setting.

Asymptotics for $\nu > 1/2$

Let Z_i, $i \in \mathbb{N}$, be iid Bernoulli distributed with parameter $p \in (0,1)$ and let $Y_s = \sum_{i=1}^{s} Z_i$, $s \in \mathbb{N}$. Let $\widetilde{Y}_s$ be an independent copy of Y_s.

Theorem 3: Let $\lim_{m,n\to\infty} m/(m+n) = p \in (0,1)$, then

$$\boxed{\lim_{m,n\to\infty} \left(\frac{m+n}{mn}\right)^{\nu-1/2} KS^{\nu}_{m,n} \stackrel{\mathcal{D}}{=} Q_{\nu,p},}$$

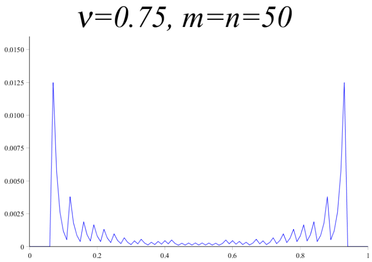
where

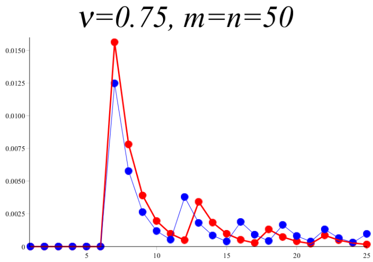
$$\boxed{Q_{\nu,p} = \frac{1}{(p(1-p))^{\nu}} \max\left(\sup_{s\in\mathbb{N}} \frac{|sp - Y_s|}{s^{\nu}}, \sup_{s\in\mathbb{N}} \frac{|sp - \widetilde{Y}_s|}{s^{\nu}}\right).}$$

The asymptotic W-KS distribution is typically discrete for $\nu > 1/2$.

Asymptotic and Finite Local Levels

Illustration for $\nu = 0.75$





The W-KS asymptotics is fast at least for larger ν-values.

Summary

- Local levels are a measure for local sensitivity of goodness-of-fit tests.
- We provided asymptotic local levels of one- and two-sample W-KS tests.
- The one-sample HC asymptotics is extremely slow, while the asymptotics is much better for two-sample HC tests.
- HC and minimum p-value tests coincide asymptotically under H_0.
- In the two-sample case, HC and minimum p-value tests differ not much even for finite samples.

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- . . . thousands of papers on Kolmogorov-Smirnov tests