The ModularGroup Package

Finite-index subgroups of $(P)SL_2(\mathbb{Z})$

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Chapter 1

Introduction

1.1 General aims of the ModularGroup package

This GAP package provides methods for computing with finite-index subgroups of the modular groups $SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$. This includes, but is not limited to, computation of the generalized level, index or cusp widths. It also implements algorithms described in [Hsu96] and [HL14] for testing if a given group is a congruence subgroup. Hence it differs from the Congruence package [DJKV18], which can be used - among other things - to construct canonical congruence subgroups of $SL_2(\mathbb{Z})$.

1.2 Technicalities

A convenient way to represent finite-index subgroups of $SL_2(\mathbb{Z})$ is by specifying the action of generator matrices of $SL_2(\mathbb{Z})$ on the right cosets by right multiplication. For example, one could choose the generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and represent a subgroup as a tuple of transitive permutations (σ_S, σ_T) describing the action of S and T. This is exactly the way this package internally treats such subgroups. We use the convention that 1 corresponds to the coset of the identity matrix. Note that such a representation as a tuple of permutations is only unique up to relabelling of the cosets, i.e. up to simultaneous conjugation (fixing the 1 coset by our convention).

Chapter 2

Subgroups of $SL_2(\mathbb{Z})$

For representing finite-index subgroups of $SL_2(\mathbb{Z})$, this package introduces the new object `ModularSubgroup`. As stated in the introduction, a `ModularSubgroup` essentially consists of the two permutations σ_S and σ_T describing the coset graph with respect to the generators S and T (with the convention that 1 corresponds to the identity coset). So explicitly specifying these permutations is the canonical way to construct a `ModularSubgroup`.
Though you might not always have a coset graph of your subgroup at hand, but rather a list of generating matrices. Therefore we implement multiple constructors for `ModularSubgroup`: three that take as input two permutations describing the coset graph with respect to different pairs of generators of $SL_2(\mathbb{Z})$, and one that takes a list of $SL_2(\mathbb{Z})$ matrices as generators.

2.1 Construction of modular subgroups

2.1.1 Constructors

▷ `ModularSubgroup(s, t)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ described by the permutations `s` and `t`.
This constructor tests if the given permutations actually describe the coset action of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

by checking that they act transitively and satisfy the relations

$$s^4 = (s^3 t)^3 = s^2 t s^{-2} t^{-1} = 1$$

Upon creation, the cosets are renamed in a standardized way to make the internal interaction with existing GAP methods easier. (The fact that 1 corresponds to the identity coset is not changed by this)

Example

```
gap> G := ModularSubgroup(
> (1,2)(3,4)(5,6)(7,8)(9,10),
> (1,4)(2,5,9,10,8)(3,7,6));
<modular subgroup of index 10>
```

▷ `ModularSubgroupST(s, t)` (operation)

Returns: A modular subgroup.

Synonymous for `ModularSubgroup` (see above). ▷ `ModularSubgroupRT(r, t)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ determined by the permutations `r` and `t` which describe the action of the matrices

$$R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on the right cosets.

A check is performed if the permutations actually describe such an action on the cosets of some subgroup.

Upon creation, the cosets are renamed in a standardized way to make the internal interaction with existing GAP methods easier. (The fact that 1 corresponds to the identity coset is not changed by this)

Example
```
gap> G := ModularSubgroupRT(
> (1,9,8,10,7)(2,6)(3,4,5),
> (1,4)(2,5,9,10,8)(3,7,6));
<modular subgroup of index 10>
```

▷ `ModularSubgrouSJ(s, j)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ determined by the permutations `s` and `j` which describe the action of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

on the right cosets.

A check is performed if the permutations actually describe such an action on the cosets of some subgroup.

Upon creation, the cosets are renamed in a standardized way to make the internal interaction with existing GAP methods easier. (The fact that 1 corresponds to the identity coset is not changed by this)

Example
```
gap> G := ModularSubgroupSJ(
> (1,2)(3,6)(4,7)(5,9)(8,10),
> (1,5,6)(2,3,7)(4,9,10));
<modular subgroup of index 10>
```

▷ `ModularSubgroup(gens)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ generated by the matrices in *gens*.
No test is performed to check if the generated subgroup actually has finite index!
This constructor implicitly computes a coset table of the subgroup. Hence it might be slow for very large index subgroups.

Example
```
gap> G := ModularSubgroup([
> [[1,2], [0,1]],
> [[1,0], [2,1]],
> [[-1,0], [0,-1]]
> ]);
<modular subgroup of index 6>
```

2.1.2 Getters for the coset action

▷ `SAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_S describing the action of the matrix S on the cosets of *G*. ▷ `TAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_T describing the action of the matrix T on the cosets of *G*.
▷ `RAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_R describing the action of the matrix R on the cosets of *G*.
▷ `JAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_J describing the action of the matrix J on the cosets of *G*.
▷ `CosetActionOf(A, G)` (operation)

Returns: A permutation.

Returns the permutation σ_A describing the action of the matrix $A \in SL_2(\mathbb{Z})$ on the cosets of *G*.

2.2 Computing with modular subgroups

2.2.1 Index (IndexSL2Z)

▷ `Index(G)` (attribute)

Returns: A natural number.

For a given modular subgroup *G* this method returns its index in $SL_2(\mathbb{Z})$. As *G* is internally stored as permutations (s,t) this is just

```
LargestMovedPoint(s,t)
```

(or 1 if the permutations are trivial).

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> Index(G);
6
```

2.2.2 GeneralizedLevel (GeneralizedLevelSL2Z)

▷ `GeneralizedLevel(G)` (attribute)

Returns: A natural number.

This method calculates the general Wohlfahrt level (i.e. the lowest common multiple of all cusp widths) of `G` as defined in [Woh64].

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> GeneralizedLevel(G);
2
```

2.2.3 RightCosetRepresentatives (RightCosetRepresentativesSL2Z)

▷ `RightCosetRepresentatives(G)` (attribute)

Returns: A list of words.

This function returns a list of representatives of the (right) cosets of `G` as words in S and T.

Example

```
gap> G := ModularSubgroup((1,2),(2,3));
<modular subgroup of index 3>
gap> RightCosetRepresentatives(G);
[ <identity ...>, S, S*T ]
```

2.2.4 GeneratorsOfGroup (GeneratorsOfGroupSL2Z)

▷ `GeneratorsOfGroup(G)` (attribute)

Returns: A list of words.

Calculates a list of generators (as words in S and T) of `G`. This list might include redundant generators (or even duplicates).

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> GeneratorsOfGroup(G);
[ S^-2, T^-2, S*T^-2*S^-1 ]
```

2.2.5 MatrixGeneratorsOfGroup

▷ `MatrixGeneratorsOfGroup(G)` (attribute)

Returns: A list of matrices.

Calculates a list of generator matrices of `G`. This list might include redundant generators (or even duplicates).

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> MatrixGeneratorsOfGroup(G);
[ [ [ -1, 0 ], [ 0, -1 ] ], [ [ 1, -2 ], [ 0, 1 ] ], [ [ 1, 0 ], [ 2, 1 ] ] ]
```

2.2.6 IsCongruence (IsCongruenceSL2Z)

▷ `IsCongruence(G)` (attribute)

Returns: True or false.

This method test whether a given modular subgroup `G` is a congruence subgroup. It is essentially an implementation of an algorithm described in [HL14].

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> IsCongruence(G);
true
```

2.2.7 Cusps (CuspsSL2Z)

▷ `Cusps(G)` (attribute)

Returns: A list of rational numbers and infinity.

This method computes a list of inequivalent cusp representatives with respect to `G`.

Example

```
gap> G := ModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<modular subgroup of index 24>
gap> Cusps(G);
[ infinity, 0, 1, 2, 3/2, 5/3 ]
```

2.2.8 CuspWidth (CuspWidthSL2Z)

▷ `CuspWidth(c, G)` (operation)

Returns: A natural number.

This method takes as input a cusp `c` (a rational number or infinity) and a modular group `G` and calculates the width of this cusp with respect to `G`.

Example
```
gap> G := ModularSubgroup(
> (1,2,6,3)(4,11,15,12)(5,13,16,14)(7,17,9,18)(8,19,10,20)(21,24,22,23),
> (1,4,5)(2,7,8)(3,9,10)(6,15,16)(11,20,21)(12,19,22)(13,23,17)(14,24,18)
> );
<modular subgroup of index 24>
gap> CuspWidth(-1, G);
3
gap> CuspWidth(infinity, G);
3
```

2.2.9 CuspsEquivalent (CuspsEquivalentSL2Z)

▷ `CuspsEquivalent(p, q, G)` (operation)

Returns: True or false.

Takes two cusps `p` and `q` and a modular subgroup `G` and checks if they are equivalent modulo `G`, i.e. if there exists a matrix $A \in G$ with $Ap = q$.

Example
```
gap> G := ModularSubgroup(
> (1,2,6,3)(4,11,15,12)(5,13,16,14)(7,17,9,18)(8,19,10,20)(21,24,22,23),
> (1,4,5)(2,7,8)(3,9,10)(6,15,16)(11,20,21)(12,19,22)(13,23,17)(14,24,18)
> );
<modular subgroup of index 24>
gap> CuspsEquivalent(infinity, 1, G);
false
gap> CuspsEquivalent(-1, 1/2, G);
true
```

2.2.10 CosetRepresentativeOfCusp (CosetRepresentativeOfCuspSL2Z)

▷ `CosetRepresentativeOfCusp(c, G)` (operation)

Returns: A word in S and T.

For a cusp `c` this function returns a right coset representative A of `G` such that $A\infty$ and `c` are equivalent with respect to `G`.

Example
```
gap> G := ModularSubgroup(
> (1,2,6,3)(4,11,15,12)(5,13,16,14)(7,17,9,18)(8,19,10,20)(21,24,22,23),
> (1,4,5)(2,7,8)(3,9,10)(6,15,16)(11,20,21)(12,19,22)(13,23,17)(14,24,18)
> );
```

```
<modular subgroup of index 24>
gap> CosetRepresentativeOfCusp(4, G);
T*S
```

2.2.11 IndexModN (IndexModNSL2Z)

▷ IndexModN(*G*, *N*) (operation)

Returns: A natural number.

For a modular subgroup `G` and a natural number `N` this method calculates the index of the projection $\bar{G}$ of G in $SL_2(\mathbb{Z}/N\mathbb{Z})$.

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> IndexModN(G, 2);
6
```

2.2.12 Deficiency (DeficiencySL2Z)

▷ Deficiency(*G*, *N*) (operation)

Returns: A natural number.

For a modular subgroup `G` and a natural number `N` this method calculates the so-called *deficiency* of `G` from being a congruence subgroup of level `N`.
The deficiency of a finite-index subgroup Γ of $SL_2(\mathbb{Z})$ was introduced in [WS15]. It is defined as the index $[\Gamma(N){:}\Gamma(N)\cap\Gamma]$ where $\Gamma(N)$ is the principal congruence subgroup of level N.

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> Deficiency(G, 2);
2
gap> Deficiency(G, 4);
1
```

2.2.13 Deficiency (DeficiencySL2ZAttr)

▷ Deficiency(*G*) (attribute)

Returns: A natural number.

Shorthand for `Deficiency(G, GeneralizedLevel(G))`.

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> Deficiency(G);
2
gap> Deficiency(G, GeneralizedLevel(G));
2
```

2.2.14 Projection

▷ `Projection(G)` (operation)

Returns: A projective modular subgroup.

For a given modular subgroup G this function calculates its image $\bar{G}$ under the projection $\pi: SL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z})$.

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> Projection(G);
<projective modular subgroup of index 6>
```

2.2.15 Conjugate (ConjugateSL2Z)

▷ `Conjugate(G, A)` (operation)

Returns: A ModularSubgroup.

Conjugates the group G by A and returns the group $A^{-1} * G * A$.

2.2.16 NormalCore (NormalCoreSL2Z)

▷ `NormalCore(G)` (attribute)

Returns: A modular subgroup.

Calculates the normal core of G in $SL_2(\mathbb{Z})$, i.e. the maximal subgroup of G that is normal in $SL_2(\mathbb{Z})$.

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> NormalCore(G);
<modular subgroup of index 48>
```

2.2.17 QuotientByNormalCore (QuotientByNormalCoreSL2Z)

▷ `QuotientByNormalCore(G)` (attribute)

Returns: A finite group.

Calculates the quotient of $SL_2(\mathbb{Z})$ by the normal core of `G`.

Example
```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> QuotientByNormalCore(G);
<permutation group with 2 generators>
```

2.2.18 AssociatedCharacterTable (AssociatedCharacterTableSL2Z)

▷ `AssociatedCharacterTable(G)` (attribute)

Returns: A character table.

Returns the character table of $SL_2(\mathbb{Z})/N$ where N is the normal core of `G`.

Example
```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> AssociatedCharacterTable(G);
CharacterTable( <permutation group of size 48 with 2 generators> )
```

2.2.19 IsElementOf (IsElementOfSL2Z)

▷ `IsElementOf(A, G)` (operation)

Returns: True or false.

This function checks if a given matrix `A` is an element of the modular subgroup `G`.

Example
```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> IsElementOf([[-1,0],[0,-1]], G);
false
gap> IsElementOf([[1,4],[0,1]], G);
true
```

2.2.20 Genus (GenusSL2Z)

▷ `Genus(G)` (attribute)

Returns: A non-negative integer.

Computes the genus of the quotient $G \backslash \mathbb{H}$ via an algorithm described in [Sch04].

Example

```
gap> G := ModularSubgroup((1,2),(2,3));
<modular subgroup of index 3>
gap> Genus(G);
0
```

2.3 Miscellaneous

The following functions are mostly helper functions used internally and are only documented for sake of completeness.

2.3.1 DefinesCosetActionST

▷ `DefinesCosetActionST(s, t)` (operation)

Returns: True or false.

Checks if two given permutations `s` and `t` describe the action of the generator matrices S and T on the cosets of some subgroup. This is the case if they satisfy the relations

$$s^4 = (s^3t)^3 = s^2ts^{-2}t^{-1} = 1$$

and act transitively.

Example

```
gap> s := (1,2)(3,4)(5,6)(7,8)(9,10);;
gap> t := (1,4)(2,5,9,10,8)(3,7,6);;
gap> DefinesCosetActionST(s,t);
true
```

2.3.2 DefinesCosetActionRT

▷ `DefinesCosetActionRT(r, t)` (operation)

Returns: True or false.

Checks if two given permutations `r` and `t` describe the action of the generator matrices R and T on the cosets of some subgroup. This is the case if they satisfy the relations

$$(rt^{-1}r)^4 = ((rt^{-1}r)^3t)^3 = (rt^{-1}r)^2t(rt^{-1}r)^{-2}t^{-1} = 1$$

and act transitively.

Example

```
gap> r := (1,9,8,10,7)(2,6)(3,4,5);;
gap> t := (1,4)(2,5,9,10,8)(3,7,6);;
gap> DefinesCosetActionRT(r,t);
true
```

2.3.3 DefinesCosetActionSJ

▷ `DefinesCosetActionSJ(s, j)` (operation)

Returns: True or false.

Checks if two given permutations `s` and `j` describe the action of the generator matrices S and J on the cosets of some subgroup. This is the case if they satisfy the relations

$$s^4 = (s^3 j^{-1} s^{-1})^3 = s^2 j^{-1} s^{-2} j = 1$$

and act transitively.

Example

```
gap> s := (1,2)(3,4)(5,6)(7,8)(9,10);;
gap> j := (1,5,6)(2,3,7)(4,9,10);;
gap> DefinesCosetActionSJ(s,j);
true
```

2.3.4 CosetActionFromGenerators

▷ `CosetActionFromGenerators(gens)` (operation)

Returns: A tuple of permutations.

Takes a list of generator matrices and calculates the coset graph (as two permutations σ_S and σ_T) of the generated subgroup of $SL_2(\mathbb{Z})$.

Example

```
gap> CosetActionFromGenerators([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
[ (1,2,5,3)(4,8,10,9)(6,11,7,12), (1,4)(2,6)(3,7)(5,10)(8,12,9,11) ]
```

2.3.5 STDecomposition

▷ `STDecomposition(A)` (operation)

Returns: A word in S and T.

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and T.

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> STDecomposition(M);
S^2*T^-1*S^-1*T^2*S^-1*T^-1*S^-1
```

2.3.6 RTDecomposition

▷ `RTDecomposition(A)` (operation)

Returns: A word in R and T.

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices R and T.

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> RTDecomposition(M);
(R*T^-1*R)^2*T^-1*R^-1*(T*R^-1*T)^2*R^-1*T^-1*R^-1*T*R^-1
```

2.3.7 SJDecomposition

▷ `SJDecomposition(A)` (operation)

Returns: A word in S and J.

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and J.

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> SJDecomposition(M);
S^3*J*(S^-1*J^-1)^2*S^-1*J*S^-1
```

2.3.8 STDecompositionAsList

▷ `STDecompositionAsList(A)` (operation)

Returns: A list representing a word in S and T.

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and T. The word is represented as a list in the format [[generator, exponent], ...]

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> STDecompositionAsList(M);
[ [ "S", 2 ], [ "T", -1 ], [ "S", -1 ], [ "T", 2 ], [ "S", -1 ], [ "T", -1 ],
  [ "S", -1 ], [ "T", 0 ] ]
```

Chapter 3

Subgroups of $PSL_2(\mathbb{Z})$

Analogous to finite-index subgroups of $SL_2(\mathbb{Z})$, we define a new type `ProjectiveModularSubgroup` for representing subgroups of $PSL_2(\mathbb{Z})$. It consists essentially of two permutations $\sigma_{\overline{S}}$ and $\sigma_{\overline{T}}$ describing the action of $\overline{S}$ and $\overline{T}$ on the cosets of the given subgroup, where $\overline{S}$ and $\overline{T}$ are the images of S and T in $PSL_2(\mathbb{Z})$.
The methods implemented for $PSL_2(\mathbb{Z})$ subgroups are mostly the same as for $SL_2(\mathbb{Z})$ subgroups and behave more or less identically. Nevertheless we list them here.

3.1 Construction of projective modular subgroups

3.1.1 Constructors

▷ `ProjectiveModularSubgroup(`*s*`,` *t*`)` (operation)

Returns: A projective modular subgroup.

Constructs a `ProjectiveModularSubgroup` object corresponding to the finite-index subgroup of $PSL_2(\mathbb{Z})$ described by the permutations `s` and `t`.
This constructor tests if the given permutations actually describe the coset action of $\overline{S}$ and $\overline{T}$ on some subgroup by checking that they act transitively and satisfy the relations

$$s^2 = (st)^3 = 1$$

Upon creation, the cosets are renamed in a standardized way to make the internal interaction with extisting GAP methods easier.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,4)(5,6)(7,8)(9,10),
> (1,4)(2,5,9,10,8)(3,7,6));
<projective modular subgroup of index 10>
```

If you want to construct a `ProjectiveModularSubgroup` from a list of generators, you can lift each generator to a matrix in $SL_2(\mathbb{Z})$, construct from these a `ModularSubgroup`, and then project it to $PSL_2(\mathbb{Z})$ via `Projection`.

3.1.2 Getters for the coset action

▷ `SAction(G)` (operation)

Returns: A permutation.

Returns the permutation $\sigma_{\overline{S}}$ describing the action of $\overline{S}$ on the cosets of `G`.

▷ `TAction(G)` (operation)

Returns: A permutation.

Returns the permutation $\sigma_{\overline{T}}$ describing the action of $\overline{T}$ on the cosets of `G`.

3.2 Computing with projective modular subgroups

3.2.1 Index (IndexPSL2Z)

▷ `Index(G)` (attribute)

Returns: A natural number.

For a given projective modular subgroup `G` this method returns its index in $PSL_2(\mathbb{Z})$. As `G` is internally stored as permutations (s,t) this is just

```
LargestMovedPoint(s,t)
```

(or 1 if the permutations are trivial).

Example

```
gap> G := ProjectiveModularSubgroup((1,2),(2,3));
<projective modular subgroup of index 3>
gap> Index(G);
3
```

3.2.2 GeneralizedLevel (GeneralizedLevelPSL2Z)

▷ `GeneralizedLevel(G)` (attribute)

Returns: A natural number.

This method calculates the general Wohlfahrt level (i.e. the lowest common multiple of all cusp widths) of `G` as defined in [Woh64].

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> GeneralizedLevel(G);
12
```

3.2.3 RightCosetRepresentatives (RightCosetRepresentativesPSL2Z)

▷ `RightCosetRepresentatives(G)` (attribute)

Returns: A list of words.

This function returns a list of representatives of the (right) cosets of `G` as words in $\overline{S}$ and $\overline{T}$.

Example
```
gap> G := ProjectiveModularSubgroup((1,2),(2,3));
<projective modular subgroup of index 3>
gap> RightCosetRepresentatives(G);
[ <identity ...>, S, S*T ]
```

3.2.4 GeneratorsOfGroup (GeneratorsOfGroupPSL2Z)

▷ `GeneratorsOfGroup(G)` (attribute)

Returns: A list of words.

Calculates a list of generators (as words in $\overline{S}$ and $\overline{T}$) of `G`. This list might include redundant generators.

Example
```
gap> G := ProjectiveModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<projective modular subgroup of index 6>
gap> GeneratorsOfGroup(G);
[ T^-2, S*T^-2*S^-1 ]
```

3.2.5 IsCongruence (IsCongruencePSL2Z)

▷ `IsCongruence(G)` (attribute)

Returns: True or false.

This method test whether a given modular subgroup `G` is a congruence subgroup. It is essentially an implementation of an algorithm described in [Hsu96].

Example
```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,5)(4,6),
> (1,3)(2,4)(5,6)
> );
<projective modular subgroup of index 6>
gap> IsCongruence(G);
true
```

3.2.6 Cusps (CuspsPSL2Z)

▷ `Cusps(G)` (attribute)

Returns: A list of rational numbers and infinity.

This method computes a list of inequivalent cusp representatives with respect to `G`.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> Cusps(G);
[ infinity, 0, 1, 2, 3/2, 5/3 ]
```

3.2.7 CuspWidth (CuspWidthPSL2Z)

▷ CuspWidth(*c*, *G*) (operation)

Returns: A natural number.

This method takes as input a cusp *c* (a rational number or infinity) and a modular group *G* and calculates the width of this cusp with respect to *G*.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,7)(4,8)(5,9)(6,10)(11,12),
> (1,3,4)(2,5,6)(7,10,11)(8,12,9)
> );
<projective modular subgroup of index 12>
gap> CuspWidth(-1, G);
3
gap> CuspWidth(infinity, G);
3
```

3.2.8 CuspsEquivalent (CuspsEquivalentPSL2Z)

▷ CuspsEquivalent(*p*, *q*, *G*) (operation)

Returns: True or false.

Takes two cusps *p* and *q* and a projective modular subgroup *G* and checks if they are equivalent modulo *G*, i.e. if there exists $A \in G$ with $Ap = q$.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,7)(4,8)(5,9)(6,10)(11,12),
> (1,3,4)(2,5,6)(7,10,11)(8,12,9)
> );
<projective modular subgroup of index 12>
gap> CuspsEquivalent(infinity, 1, G);
false
gap> CuspsEquivalent(-1, 1/2, G);
true
```

3.2.9 CosetRepresentativeOfCusp (CosetRepresentativeOfCuspPSL2Z)

▷ `CosetRepresentativeOfCusp(c, G)` (operation)

Returns: A word in S and T.

For a cusp `c` this function returns a right coset representative A of `G` such that $A\infty$ and `c` are equivalent with respect to `G`.

Example
```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,7)(4,8)(5,9)(6,10)(11,12),
> (1,3,4)(2,5,6)(7,10,11)(8,12,9)
> );
<projective modular subgroup of index 12>
gap> CosetRepresentativeOfCusp(4, G);
T*S
```

3.2.10 LiftToSL2ZEven

▷ `LiftToSL2ZEven(G)` (operation)

Returns: A modular subgroup.

Lifts a given subgroup `G` of $PSL_2(\mathbb{Z})$ to an even subgroup of $SL_2(\mathbb{Z})$, i.e. a group that contains -1 and whose projection to $PSL_2(\mathbb{Z})$ is `G`.

Example
```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,7)(4,8)(5,9)(6,10)(11,12),
> (1,3,4)(2,5,6)(7,10,11)(8,12,9)
> );
<projective modular subgroup of index 12>
gap> LiftToSL2ZOdd(G);
<modular subgroup of index 12>
```

3.2.11 LiftToSL2ZOdd

▷ `LiftToSL2ZOdd(G)` (operation)

Returns: A modular subgroup.

Lifts a given subgroup `G` of $PSL_2(\mathbb{Z})$ to an odd subgroup of $SL_2(\mathbb{Z})$, i.e. a group that does not contain -1 and whose projection to $PSL_2(\mathbb{Z})$ is `G`.

Example
```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,7)(4,8)(5,9)(6,10)(11,12),
> (1,3,4)(2,5,6)(7,10,11)(8,12,9)
> );
<projective modular subgroup of index 12>
gap> LiftToSL2ZOdd(G);
<modular subgroup of index 24>
```

3.2.12 IndexModN (IndexModNPSL2Z)

▷ `IndexModN(G, N)` (operation)

Returns: A natural number.

For a projective modular subgroup `G` and a natural number `N` this method calculates the index of the projection $\bar{G}$ of G in $PSL_2(\mathbb{Z}/N\mathbb{Z})$.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> IndexModN(G, 2);
6
```

3.2.13 Deficiency (DeficiencyPSL2Z)

▷ `Deficiency(G, N)` (operation)

Returns: A natural number.

For a projective modular subgroup `G` and a natural number `N` this method calculates the so-called *deficiency* of `G` from being a congruence subgroup of level `N`.
The deficiency of a finite-index subgroup Γ of $PSL_2(\mathbb{Z})$ was introduced in [WS15]. It is defined as the index $[\Gamma(N)\colon\Gamma(N)\cap\Gamma]$ where $\Gamma(N)$ is the principal congruence subgroup of level N.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> Deficiency(G, 4);
4
```

3.2.14 Deficiency (DeficiencyPSL2ZAttr)

▷ `Deficiency(G)` (attribute)

Returns: A natural number.

Shorthand for `Deficiency(G, GeneralizedLevel(G))`.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> Deficiency(G);
4
```

```
gap> Deficiency(G, GeneralizedLevel(G));
4
```

3.2.15 Conjugate (ConjugatePSL2Z)

▷ `Conjugate(G, A)` (operation)

Returns: A ProjectiveModularSubgroup.

Conjugates the group `G` by (the redidue class in $PSL_2(\mathbb{Z})$ of) `A` and returns the group $A^{-1} * G * A$.

3.2.16 NormalCore (NormalCorePSL2Z)

▷ `NormalCore(G)` (attribute)

Returns: A projective modular subgroup.

Calculates the normal core of `G` in $PSL_2(\mathbb{Z})$, i.e. the maximal subgroup of `G` that is normal in $PSL_2(\mathbb{Z})$.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> NormalCore(G);
<projective modular subgroup of index 3456>
```

3.2.17 QuotientByNormalCore (QuotientByNormalCorePSL2Z)

▷ `QuotientByNormalCore(G)` (attribute)

Returns: A finite group.

Calculates the quotient of $PSL_2(\mathbb{Z})$ by the normal core of `G`.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> QuotientByNormalCore(G);
<permutation group with 2 generators>
```

3.2.18 AssociatedCharacterTable (AssociatedCharacterTablePSL2Z)

▷ `AssociatedCharacterTable(G)` (attribute)

Returns: A character table.

Returns the character table of $PSL_2(\mathbb{Z})/N$ where N is the normal core of `G`.

Example

```
gap> G := ProjectiveModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<projective modular subgroup of index 24>
gap> AssociatedCharacterTable(G);
CharacterTable( <permutation group of size 3456 with 2 generators> )
```

3.2.19 IsElementOf (IsElementOfPSL2Z)

▷ `IsElementOf(`*`A, G`*`)` (operation)

Returns: True or false.

This function checks if the image of a given matrix *A* in $PSL_2(\mathbb{Z})$ is an element of the group *G*.

Example

```
gap> G := ProjectiveModularSubgroup((1,2),(2,3));
<modular subgroup of index 3>
gap> IsElementOf([[1,1],[0,1]], G);
true
gap> IsElementOf([[0,-1],[1,0]], G);
false
```

3.2.20 Genus (GenusPSL2Z)

▷ `Genus(`*`G`*`)` (attribute)

Returns: A non-negative integer.

Computes the genus of the quotient $G\backslash\mathbb{H}$ via an algorithm described in [Sch04].

Example

```
gap> G := ProjectiveModularSubgroup((1,2),(2,3));
<modular subgroup of index 3>
gap> Genus(G);
0
```

References

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