

The ModularGroup Package

Finite-index subgroups of $(P)SL_2(\mathbb{Z})$

Version 0.0.1

16 February 2018

Luca Leon Junk

Luca Leon Junk

Email: junk@math.uni-sb.de

Homepage: <http://www.math.uni-sb.de/ag/weitze/>

Address: AG Weitze-Schmithüsen

FR 6.1 Mathematik

Universität des Saarlandes

D-66041 Saarbrücken

Copyright

© 2018 by Luca L. Junk

Acknowledgements

Supported by Project I.8 of SFB-TRR 195 'Symbolic Tools in Mathematics and their Application' of the German Research Foundation (DFG).

Contents

1	Introduction	4
1.1	General aims of the ModularGroup package	4
1.2	Installation	4
1.3	Technicalities	4
2	Subgroups of $SL_2(\mathbb{Z})$	5
2.1	Construction of modular subgroups	5
2.2	Computing with modular subgroups	7
2.3	Miscellaneous	11
3	Subgroups of $PSL_2(\mathbb{Z})$	14
	References	15
	Index	16

Chapter 1

Introduction

1.1 General aims of the ModularGroup package

This `GAP` package provides methods for computing with finite-index subgroups of the modular groups $SL_2(\mathbb{Z})$ and $PSL_2(\mathbb{Z})$. This includes, but is not limited to, computation of the generalized level, index or cusp widths. It also implements algorithms described in [Hsu96] and [HL14] for testing if a given group is a congruence subgroup.

1.2 Installation

1.3 Technicalities

A convenient way to represent finite-index subgroups of $SL_2(\mathbb{Z})$ is by specifying the action of generator matrices on the right cosets by right multiplication. For example, one could choose the generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and represent a subgroup as a tuple of transitive permutations (σ_S, σ_T) describing the action of S and T . This is exactly the way this package internally treats such subgroups. We employ the convention that 1 corresponds to the coset of the identity matrix.

Chapter 2

Subgroups of $SL_2(\mathbb{Z})$

For representing finite-index subgroups of $SL_2(\mathbb{Z})$, this package introduces the new type `ModularSubgroup`. As stated in the introduction, a `ModularSubgroup` essentially consists of the two permutations σ_S and σ_T describing the coset graph with respect to the generator matrices S and T (with the convention that 1 corresponds to the identity coset). So explicitly specifying these permutations is the canonical way to construct a `ModularSubgroup`.

Though you might not always have a coset graph of your subgroup at hand, but rather a list of generator matrices. Therefore we implement multiple constructors for `ModularSubgroup`: three that take as input two permutations describing the coset graph with respect to different pairs of generators, and one that takes a list of $SL_2(\mathbb{Z})$ matrices as generators.

2.1 Construction of modular subgroups

2.1.1 Constructors

▷ `ModularSubgroup(s, t)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ described by the permutations s and t .

This constructor tests if the given permutations actually describe the coset action of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

by checking that they act transitively and satisfy the relations

$$s^4 = (s^3 t)^3 = s^2 t s^{-2} t^{-1} = 1$$

Upon creation, the cosets are renamed in a [standardized way](#) to make the internal interaction with existing **GAP** methods easier.

Example

```
gap> G := ModularSubgroup(
> (1,2)(3,4)(5,6)(7,8)(9,10),
> (1,4)(2,5,9,10,8)(3,7,6));
<modular subgroup of index 10>
```

▷ `ModularSubgroupST(s, t)` (operation)

Returns: A modular subgroup.

Synonymous for `ModularSubgroup` (see above). ▷ `ModularSubgroupRT(r, t)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ determined by the permutations *r* and *t* which describe the action of the matrices

$$R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on the right cosets.

A check is performed if the permutations actually describe such an action on the cosets of some subgroup.

Upon creation, the cosets are renamed in a [standardized way](#) to make the internal interaction with existing **GAP** methods easier.

Example

```
gap> G := ModularSubgroupRT(
> (1,9,8,10,7)(2,6)(3,4,5),
> (1,4)(2,5,9,10,8)(3,7,6));
<modular subgroup of index 10>
```

▷ `ModularSubgroupSJ(s, j)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ determined by the permutations *s* and *j* which describe the action of the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

on the right cosets.

A check is performed if the permutations actually describe such an action on the cosets of some subgroup.

Upon creation, the cosets are renamed in a [standardized way](#) to make the internal interaction with existing **GAP** methods easier.

Example

```
gap> G := ModularSubgroupSJ(
> (1,2)(3,6)(4,7)(5,9)(8,10),
> (1,5,6)(2,3,7)(4,9,10));
<modular subgroup of index 10>
```

▷ `ModularSubgroup(gens)` (operation)

Returns: A modular subgroup.

Constructs a `ModularSubgroup` object corresponding to the finite-index subgroup of $SL_2(\mathbb{Z})$ generated by the matrices in `gens`.

No test is performed to check if the generated subgroup actually has finite index!

This constructor implicitly computes a coset table of the subgroup. Hence it might be slow for very large index subgroups.

Example

```
gap> G := ModularSubgroup([
> [[1,2], [0,1]],
> [[1,0], [2,1]],
> [[-1,0], [0,-1]]
> ]);
<modular subgroup of index 6>
```

2.1.2 Getters for the coset action

▷ `SAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_S describing the action of the matrix S on the cosets of G .

▷ `TAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_T describing the action of the matrix T on the cosets of G .

▷ `RAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_R describing the action of the matrix R on the cosets of G .

▷ `JAction(G)` (operation)

Returns: A permutation.

Returns the permutation σ_J describing the action of the matrix J on the cosets of G .

▷ `CosetActionOf(A, G)` (operation)

Returns: A permutation.

Returns the permutation σ_A describing the action of the matrix $A \in SL_2(\mathbb{Z})$ on the cosets of G .

2.2 Computing with modular subgroups

2.2.1 Index

▷ `Index(G)` (attribute)

Returns: A natural number.

For a given modular subgroup G this method returns its index in $SL_2(\mathbb{Z})$. As G is internally stored as permutations (s, t) this is just

`LargestMovedPoint(s, t)`

(or 1 if the permutations are trivial).

2.2.2 GeneralizedLevel

▷ `GeneralizedLevel(G)` (attribute)

Returns: A natural number.

This method calculates the general Wohlfahrt level (i.e. the lowest common multiple of all cusp widths) of G .

2.2.3 RightCosetRepresentatives

▷ `RightCosetRepresentatives(G)` (attribute)

Returns: A list of words.

This function returns a list of representatives of the (right) cosets of G as words in S and T .

Example

```
gap> G := ModularSubgroup((1,2),(2,3));
<modular subgroup of index 3>
gap> RightCosetRepresentatives(G);
[ <identity ...>, S, S*T ]
```

2.2.4 GeneratorsOfGroup

▷ `GeneratorsOfGroup(G)` (attribute)

Returns: A list of words.

Calculates a list of generators (as words in S and T) of G . This list might include redundant generators (or even duplicates).

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> GeneratorsOfGroup(G);
[ S^-2, T^-2, S*T^-2*S^-1 ]
```

2.2.5 MatrixGeneratorsOfGroup

▷ `MatrixGeneratorsOfGroup(G)` (attribute)

Returns: A list of matrices.

Calculates a list of generator matrices of G . This list might include redundant generators (or even duplicates).

Example

```
gap> G := ModularSubgroup((1,2)(3,5)(4,6), (1,3)(2,4)(5,6));
<modular subgroup of index 6>
gap> MatrixGeneratorsOfGroup(G);
[ [ [ -1, 0 ], [ 0, -1 ] ], [ [ 1, -2 ], [ 0, 1 ] ], [ [ 1, 0 ], [ 2, 1 ] ] ]
```


2.2.6 IsCongruence

▷ `IsCongruence(G)` (attribute)

Returns: True or false.

This method test whether a given modular subgroup G is a congruence subgroup. It is essentially an implementation of an algorithm described in [HL14].

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> IsCongruence(G);
true
```

2.2.7 Cusps

▷ `Cusps(G)` (attribute)

Returns: A list of rational numbers and infinity.

This method computes a list of inequivalent cusp representatives with respect to G .

Example

```
gap> G := ModularSubgroup(
> (1,2)(3,6)(4,8)(5,9)(7,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)(22,24),
> (1,3,7,4)(2,5)(6,9,8,12,14,10)(11,13,16,20,18,15)(17,21,22,19)(23,24)
> );
<modular subgroup of index 24>
gap> Cusps(G);
[ infinity, 0, 1, 2, 3/2, 5/3 ]
```

2.2.8 CuspWidth

▷ `CuspWidth(c , G)` (operation)

Returns: A natural number.

This method takes as input a cusp c (a rational number or infinity) and a modular group G and calculates the width of this cusp with respect to G .

Example

```
gap> G := ModularSubgroup(
> (1,2,6,3)(4,11,15,12)(5,13,16,14)(7,17,9,18)(8,19,10,20)(21,24,22,23),
> (1,4,5)(2,7,8)(3,9,10)(6,15,16)(11,20,21)(12,19,22)(13,23,17)(14,24,18)
> );
<modular subgroup of index 24>
gap> CuspWidth(-1, G);
3
gap> CuspWidth(infinity, G);
3
```

2.2.9 CuspsEquivalent

▷ `CuspsEquivalent(p, q, G)` (operation)

Returns: True or false.

Takes two cusps p and q and a modular subgroup G and checks if they are equivalent modulo G , i.e. if there exists a matrix $A \in G$ with $Ap = q$.

Example

```
gap> G := ModularSubgroup(
> (1,2,6,3)(4,11,15,12)(5,13,16,14)(7,17,9,18)(8,19,10,20)(21,24,22,23),
> (1,4,5)(2,7,8)(3,9,10)(6,15,16)(11,20,21)(12,19,22)(13,23,17)(14,24,18)
> );
<modular subgroup of index 24>
gap> CuspsEquivalent(infinity, 1, G);
false
gap> CuspsEquivalent(-1, 1/2, G);
true
```

2.2.10 IndexModN

▷ `IndexModN(G, N)` (operation)

Returns: A natural number.

For a modular subgroup G and a natural number N this method calculates the index of the projection \bar{G} of G in $SL_2(\mathbb{Z}/N\mathbb{Z})$.

2.2.11 Deficiency

▷ `Deficiency(G, N)` (operation)

Returns: A natural number.

For a modular subgroup G and a natural number N this method calculates the so-called *deficiency* of G from being a congruence subgroup of level N .

The deficiency of a finite-index subgroup Γ of $SL_2(\mathbb{Z})$ was introduced in [WS15]. It is defined as the index $[\Gamma(N) : \Gamma(N) \cap \Gamma]$ where $\Gamma(N)$ is the principal congruence subgroup of level N .

Example

```
gap> G := ModularSubgroup([
> [[1,2],[0,1]],
> [[1,0],[2,1]]
> ]);
<modular subgroup of index 12>
gap> Deficiency(G, 2);
2
gap> Deficiency(G, 4);
1
```

2.2.12 Projection

▷ `Projection(G)` (operation)

Returns: A projective modular subgroup.

For a given modular subgroup G this function calculates its image \tilde{G} under the projection $\pi: SL_2(\mathbb{Z}) \rightarrow PSL_2(\mathbb{Z})$.

2.2.13 NormalCore

▷ `NormalCore(G)` (attribute)

Returns: A modular subgroup.

Calculates the normal core of G in $SL_2(\mathbb{Z})$, i.e. the maximal subgroup of G that is normal in $SL_2(\mathbb{Z})$.

2.2.14 QuotientByNormalCore

▷ `QuotientByNormalCore(G)` (attribute)

Returns: A finite group.

Calculates the quotient of $SL_2(\mathbb{Z})$ by the normal core of G .

2.2.15 AssociatedCharacterTable

▷ `AssociatedCharacterTable(G)` (attribute)

Returns: A character table.

Returns the character table of $SL_2(\mathbb{Z})/N$ where N is the normal core of G .

2.2.16 IsElementOf

▷ `IsElementOf(A, G)` (operation)

Returns: True or false.

This function checks if a given matrix A is an element of the modular subgroup G .

2.3 Miscellaneous

The following functions are mostly helper functions used internally and are only documented for sake of completeness.

2.3.1 DefinesCosetActionST

▷ `DefinesCosetActionST(s, t)` (operation)

Returns: True or false.

Checks if two given permutations s and t describe the action of the generator matrices S and T on the cosets of some subgroup. This is the case if they satisfy the relations

$$s^4 = (s^3 t)^3 = s^2 t s^{-2} t^{-1} = 1$$

and act transitively.

2.3.2 DefinesCosetActionRT

▷ DefinesCosetActionRT(r , t) (operation)

Returns: True or false.

Checks if two given permutations r and t describe the action of the generator matrices R and T on the cosets of some subgroup. This is the case if they satisfy the relations

$$(rt^{-1}r)^4 = ((rt^{-1}r)^3t)^3 = (rt^{-1}r)^2t(rt^{-1}r)^{-2}t^{-1} = 1$$

and act transitively.

2.3.3 DefinesCosetActionSJ

▷ DefinesCosetActionSJ(s , j) (operation)

Returns: True or false.

Checks if two given permutations s and j describe the action of the generator matrices S and J on the cosets of some subgroup. This is the case if they satisfy the relations

$$s^4 = (s^3j^{-1}s^{-1})^3 = s^2j^{-1}s^{-2}j = 1$$

and act transitively.

2.3.4 CosetActionFromGenerators

▷ CosetActionFromGenerators($gens$) (operation)

Returns: A tuple of permutations.

Takes a list of generator matrices and calculates the coset graph (as two permutations σ_S and σ_T) of the generated subgroup of $SL_2(\mathbb{Z})$.

2.3.5 STDecomposition

▷ STDecomposition(A) (operation)

Returns: A word in S and T .

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and T .

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> STDecomposition(M);
S^2*T^-1*S^-1*T^2*S^-1*T^-1*S^-1
```

2.3.6 RTDecomposition

▷ RTDecomposition(A) (operation)

Returns: A word in R and T .

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices R and T .

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> RTDecomposition(M);
(R*T^-1*R)^2*T^-1*R^-1*(T*R^-1*T)^2*R^-1*T^-1*R^-1*T*R^-1
```

2.3.7 SJDecomposition

▷ `SJDecomposition(A)` (operation)

Returns: A word in S and J .

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and J .

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> SJDecomposition(M);
S^3*J*(S^-1*J^-1)^2*S^-1*J*S^-1
```

2.3.8 STDecompositionAsList

▷ `STDecompositionAsList(A)` (operation)

Returns: A list representing a word in S and T .

Takes a matrix $A \in SL_2(\mathbb{Z})$ and decomposes it into a word in the generator matrices S and T . The word is represented as a list in the format `[[generator, exponent], ...]`

Example

```
gap> M := [ [ 4, 3 ], [ -3, -2 ] ];;
gap> STDecompositionAsList(M);
[ [ "S", 2 ], [ "T", -1 ], [ "S", -1 ], [ "T", 2 ], [ "S", -1 ], [ "T", -1 ],
  [ "S", -1 ], [ "T", 0 ] ]
```

Chapter 3

Subgroups of $PSL_2(\mathbb{Z})$

References

- [HL14] Thomas Hamilton and David Loeffler. Congruence testing for odd subgroups of the modular group. *LMS Journal of Computation and Mathematics*, 17(1):206–208, 2014. [4](#), [9](#)
- [Hsu96] Tim Hsu. Identifying congruence subgroups of the modular group. *Proceedings of the American Mathematical Society*, 124(5), April 1996. [4](#)
- [WS15] Gabriela Weitze-Schmithuesen. The deficiency of being a congruence group for veech groups of origamis. *International Mathematics Research Notices*, 2015(6):1613–1637, 2015. [10](#)

Index

AssociatedCharacterTable, 11

CosetActionFromGenerators, 12

CosetActionOf, 7

Cusps, 9

CuspsEquivalent, 10

CuspWidth, 9

Deficiency, 10

DefinesCosetActionRT, 12

DefinesCosetActionSJ, 12

DefinesCosetActionST, 11

GeneralizedLevel, 8

GeneratorsOfGroup, 8

Index, 7

IndexModN, 10

IsCongruence, 9

IsElementOf, 11

JAction, 7

MatrixGeneratorsOfGroup, 8

ModularSubgroup, 5, 7

ModularSubgroupRT, 6

ModularSubgroupST, 6

ModularSubgrouSJ, 6

NormalCore, 11

Projection, 11

QuotientByNormalCore, 11

RAction, 7

RightCosetRepresentatives, 8

RTDecomposition, 12

SAction, 7

SJDecomposition, 13

STDecomposition, 12

STDecompositionAsList, 13

TAction, 7