ESC195 W23 Notes

Andy Gong

April 3, 2023

Contents

28 Partial Derivatives

March 23, 2023

- Continuity:

$$\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$

> **Theorem**: The continuity of composite functions are defined as, for any function $g(\vec{x_0})$, if it is continuous at $\vec{x_0}$ and function f is continuous at the NUMBER $g(\vec{x_0})$ m then we can say $f(g(\vec{x}))$ is continuous at $\vec{x_0}$

- if $f(\vec{x})$ is xontinuous at $\vec{x_0}$, then:

$$\lim_{x \to x_0} f(x, y_0) = f(x_0, y_0)$$
$$\lim_{y \to y_0} f(x_0, y) = f(x_0, y_0).$$

- If we have the top half of a sphere with radius 5, we have:

$$f = \sqrt{25 - x^2 - y^2}.$$

Suppose we are interested in what is happening when we are moving along the line y = 2:

> **Definition**: Partial derivative of $f(x, y)$ is given by:
>
> $$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \quad (1)$$
>
> or the partial derivative with respect to y:
>
> $$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h} \quad (2)$$
>
> This can be extended to an arbitrary number of dimensions.

Example 1: Suppose we have a function $f(x, y) = e^{x^2 y^3}$:

$$f_x = y^3 \cdot 2x e^{x^2 y^3}$$
$$f_y = 3y^2 x^2 e^{x^2 y^3}$$
$$.$$

At $y = 2$, we have:

$$f_x(x, 2) = 16x e^{8x^2}$$
$$f_x(1, 2) = 16e^8$$
$$.$$

This is equivalent if we take a cross section of this equation on the $y = 2$ plane, and look at the derivative or the slope of tangent at that point.

Example 2: Now we have a 3-D function, $f(x,y,z) = \ln\left(\frac{x}{y}\right) - ye^{xz}$. The partial derivatives are:

$$\begin{aligned} f_x &= \frac{1}{x} - yze^{xz} \\ f_y &= -\frac{1}{y} - e^{xz} \\ f_z &= -xye^{xz} \end{aligned}$$

.

Example 3: Supplse we have $h(r,\theta,\phi) = r^2 \sin\theta\cos\phi$:

$$\begin{aligned} h_r &= 2r\sin\theta\cos\phi \\ h_\theta &= r^2\cos\theta\cos\phi \\ h_\phi &= -r^2\sin\theta\sin\phi \end{aligned}$$

.

- We can also have Mixed Partials:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &\rightarrow \frac{\partial^2 f}{\partial x^2} = f_{xx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &\rightarrow \frac{\partial^2 f}{\partial y^2} = f_{yy} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \end{aligned}$$

Theorem: Clairaut's Theorem says that:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \tag{3}$$

on every open set on which f and its partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are continuous, so we have:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial z \partial y} &= \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} &= \frac{\partial^2 f}{\partial z \partial x} \end{aligned}$$

.

Example 4: We have $f(x,y) = \cos\left(xy^2\right)$

$$\begin{aligned} f_x &= -y^2\sin\left(xy^2\right) \\ f_y &= -\sin\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial y \partial x} &= -2y\sin\left(xy^2\right) - y^2\cos\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial x \partial y} &= -2y\sin\left(xy^2\right) - 2xy\cos\left(xy^2\right) \cdot y^2 = \frac{\partial^2 f}{\partial y \partial x} \end{aligned}$$

.

- partial derivatives can be used to desribe differential equations with multiple variables:

Example 5: Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{4}$$

One-dimensional wave equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2} \tag{5}$$

Here a represents the speed of the wave.

29 Directional Derivatives and Gradient Functions

March 24, 2023

- A partial derivative:

Definition: A partial derivative of f on x, denoted as $f_x = \frac{\partial f}{\partial x}$ is defined as:

$$\lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} \tag{6}$$

only if $\lim_{h \to 0} \frac{g(h)}{h} = 0 \implies g(h) = o(h)$

Example 6: For a function $f(x) = x^2$, $f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2$ Note here that

$$\lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \tag{7}$$

since the function of h, $g(h)$ satisfies the previous definition, we say that h^2 is an $o(h)$, which further shows that $f'(x) = 2x$

Idea: With this, we can write out the derivative definition using just the numerator part of the derivative

$$f(x+h) - f(x) \tag{8}$$

And the result of this expression will leave us with two types of equations.

$$\text{derivative} \cdot h \tag{9}$$

or

$$\text{something} \cdot o(h) \tag{10}$$

anything that is multipled by $o(h)$ will be reduced to zero due to the definition of $o(h)$, which is why the first expression gives us the derivative.
This is especially useful when we are trying to find the derivative of a multivariable function.

- differentiability of a multivariable function

Definition: we say f is differentiable at $\vec{x}$ iff there exists $\vec{y}$ s.t.

$$f(\vec{x} + \vec{h}) - f(\vec{x}) = \vec{y} \cdot \vec{h} + o(\vec{h})\vec{y} = Df(\vec{x}) = \text{the gradient of } f.$$

Example 7: Here we have the function $f(x, y) = x + y^2$ and the h function $\vec{h} = (h_1, h_2)$

$$\begin{aligned} f(\vec{x} + \vec{h}) - f(\vec{x}) &= f(x + h_1, y + h_2) - f(x, y) \\ &= x + h_1 + (y + h_2)^2 - x - y^2 \\ &= h_1 + 2yh_2 + h_2^2 \\ &= (1\hat{i} + 2y\hat{j}) \cdot \vec{h} + h_2^2 \end{aligned}$$

After this, we need to demonstrate that the remaining part of the function, that is h_2^2, is actually a $o\left(\vec{h}\right)$ function.

To do that, we define $g(\vec{h}) = h^2 = (h_2\hat{j}) \cdot (h_1\hat{i} + h_2\hat{j}) = h_2\hat{j} \cdot \vec{h}$. So we can write $g\left(\vec{h}\right) = h_2\vec{j} \cdot \vec{h}$

$$\frac{|g\left(\vec{h}\right)|}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0}|xh_2| = 0 \implies xh_2h_3 = o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{i} + xy\hat{k}.$$

We know that $h_2 \to 0$ as $\vec{h} \to \vec{0}$ So $g(\vec{h})$ is $o(\vec{h})$ and we can claim that:

$$\nabla f(\vec{x}) = 1\hat{i} + wy\hat{j} \tag{11}$$

Example 8: For this example we have $f\left(x, y, z\right) = xyz$

$$\begin{aligned} f\left(\vec{x} + \vec{h}\right) - f\left(\vec{x}\right) &= (x + h_1)\left(y + h_2\right)\left(z + h_3\right) - xyz \\ &= xyz + xyh_3 + xh_2z + xh_2h_3 + h_1yz + h_1yh_3 + h_1h_2z + h_1h_2h_3 - xyz \\ &= \left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right) \cdot \vec{h}. \end{aligned}$$

We consider the case of $xh_2h_3 = g(\vec{h}) = xh_2\hat{k} \cdot \vec{h}$

$$\frac{g\left(\vec{h}\right)}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0}|xh_2| = 0 \implies xh_2h_3 \text{ is } o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{j} + xy\hat{k}.$$
$$\nabla f\left(x, y, z\right) = \left(f_x, f_y, f_z\right) \tag{12}$$

Theorem: For cartesian coordinates:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{13}$$

Or equally written as:

$$\nabla f(x, y, z) = (f_x, f_y, f_z) \tag{14}$$

- Gradients are the vectors that points steepest way "up the hill"
- $\vec{x}$ is vector
- $f(\vec{x})$ is not vector
- $\nabla f(\vec{x})$ is a vector

Example 9: Suppose we have a temperature function with respect to x and y, and we have $\dfrac{\partial T}{\partial x} = 3\dfrac{{}^\circ C}{m}$ and $\dfrac{\partial T}{\partial y} = 4\dfrac{{}^\circ C}{m}$. From this, we can conclude that:

$$\nabla T = 3\hat{i} + 4\hat{j} \tag{15}$$

And that:

$$|\nabla T| = \sqrt{3^2 + 4^2} = 5 \tag{16}$$

Example 10: For this example we have $f(\vec{x}) = xy^2z^3$, and we can compute their partial derivatives:

$$f_x = y^2z^3$$
$$f_y = 2xyz^3$$
$$f_z = 3xy^2z^2$$
$$.$$

And we have

$$\nabla f = \left(y^2z^3, 2xyz^3, 3xy^2z^2\right) \tag{17}$$

Example 11: We have function $\vec{r}(x, y, z) \implies r = \sqrt{x^2+y^2+z^2}$ So we then have

$$\begin{aligned}\nabla r &= \nabla\sqrt{x^2+y^2+z^2} \\ &= \frac{\frac{1}{2}2x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{\frac{1}{2}2y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{\frac{1}{2}2z}{\sqrt{x^2+y^2+z^2}}\hat{k}\end{aligned}$$

rewriting this gives us:

$$\nabla r = \frac{\vec{r}}{r} \tag{18}$$

- For any directional derivative, we can define it being:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f\left(\vec{x_0} + h\hat{i}\right) - f(\vec{x_0})}{h}.$$

Expanding this to any arbituary direction is:

Definition: Directional derivative of function f at $\vec{x_0}$ in direction $\hat{u}$

$$f_{\hat{u}}(\vec{x_0}) = \lim_{h\to 0}\frac{f(\vec{x_0} + h\hat{u}) - f(\vec{x_0})}{h} \tag{19}$$

We also have

$$f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0})\cdot\hat{u} \tag{20}$$

Proof. Proving that $f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0})\cdot\hat{u}$:

$$f\left(\vec{x} + \vec{h}\right) - f(\vec{x}) = \nabla f(\vec{x})\cdot\vec{h} + o\left(\vec{h}\right) \tag{21}$$

Where $\vec{h} = h\hat{i}$. Using this, we can show the above is equal to:

$$\nabla f(\vec{x})\cdot h\hat{u} + o\left(\vec{h}\right) \tag{22}$$

and therefore:

$$\frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \nabla f\cdot\hat{u} + \frac{o\left(\vec{h}\right)}{h} \tag{23}$$

and taking the limit as $h \to 0$, we have the desired result. $\square$

Example 12: Using the same temperature example, we have $T(x, y)$ where $\frac{\partial T}{\partial y} = 4\frac{°C}{m}$ and $\frac{\partial T}{\partial x} = 3\frac{°C}{m}$. If we want to move in a direction of $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$$T_{\hat{u}} = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}\right)\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = 3\cos\theta + 4\sin\theta \tag{24}$$

Example 13: Suppose we have a parabolic hill described by $z(x,y) = 20 - x^2 - y^2$ and we move straight up, or we can say that $\hat{u} = (0,1)$.

$$\begin{gathered}\frac{\partial f}{\partial x} = -2x \\ \frac{\partial f}{\partial y} = -2y \\ .\end{gathered}$$

$$\therefore z_{\hat{u}} = (-2x, -2y) \cdot (0,-1) = 2y \tag{25}$$

(The following is not on that lecture, but from Xue Qilin's notes)

- Note that:

$$\begin{aligned}|f_{\hat{k}}(\vec{x})| &= |\nabla f \cdot \hat{u}| \\ &= \|\nabla f\| \|\hat{u}\| |\cos\theta| \\ &\leq \|\nabla f\|.\end{aligned}$$

Example 14: Suppose that $z = f(x,y) = A + x + 2y + x^2 - 2y^2$ and we wish to find the steepest path down starting from $(0,0,A)$. We know that:

$$\begin{gathered}\frac{\partial f}{\partial x} = 1 - 2x \\ \frac{\partial f}{\partial y} = 2 - 6y \\ .\end{gathered}$$

such that:

$$\nabla f = (1-2x)\hat{i} + (2-6y)\hat{j} \implies -\nabla f = (2x-1)\hat{i} + (6y-2)\hat{j}.$$

The curve is given by:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}.$$

Where $x'(t) = 2x(t) - 1$ and $y'(t) = 6y(t) - 2$. This is in parametric form and we can convert to cartesian form by writing the derivatives as:

$$\frac{dy}{dx} = \frac{6y-2}{2x-1} \tag{26}$$

and solving this differential equation to get:

$$3y = (2x-1)^3 + 1 \tag{27}$$

30 Chain Rule

March 27, 2023

- For gradient functions:

$$f\left(\vec{x}+\vec{h}\right)-f\left(\vec{x}\right)=\vec{y}\cdot\vec{h}+o\left(\vec{h}\right)$$
$$\implies \vec{y}=\nabla f=\text{gradient of} f$$
$$\text{which is } \left(f_x, f_y, f_z\right)$$

- directional derivative, allow us to take the derivative with any direction we choose:

$$f_{\hat{u}}=\nabla f\cdot\hat{u} \tag{28}$$

Example 15: We have $z=f\left(x,y\right)=A+x+2y-x^2-3y^2$, an eliptical parabolic hill. With the starting point of $(0,0,A)$, our job is to get down the hill, following the path of steepest descent. We will calculate a few things, and put together the process. We have partial derivatives:

$$f_x=1-2x$$
$$f_y=2-6y$$
$$.$$

And we have:

$$\nabla f=\left(1-2x\right)\hat{\imath}+\left(2-6y\right)\hat{\jmath} \tag{29}$$

This is the gradient, pointing UPHILL, to get the down hill:

$$-\nabla f=\left(2x-1\right)\hat{\imath}+\left(6y-2\right)\hat{\jmath} \tag{30}$$

We have the path of steepest descent, we call it curve $C: r\left(t\right)=x\left(t\right)\hat{\imath}+y\left(t\right)\hat{\jmath}$, Note: This path will have direction same as the steepest descent path.

$$x'\left(t\right)=2x\left(t\right)-1$$
$$y'\left(t\right)=6y\left(t\right)-2$$
$$\therefore \frac{dy}{dx}=\frac{6y-2}{2x-1}$$
$$.$$

This is a separable equation, we can solve it to find the actual path from the directions. After separating we can integrate directly.

$$\frac{dy}{6y-2}=\frac{dx}{2x-1}$$
$$\implies \frac{1}{6}\ln\left|6y-2\right|=\frac{1}{2}\ln\left|2x-1\right|+C$$
$$\therefore 6y-2=\left(2x-1\right)^3e^C$$
$$.$$

We need to find what the integration constant: e^C is, so we start from our initial condition: $x=0, y=0$

$$-2=\left(-1\right)^3e^C$$
$$\therefore e^c=2$$
$$\therefore 3y=\left(2x-1\right)^3+1.$$

And this is our steepest descent path. Everywhere along the curve C, the direction is the steepest downhill descent, and the path itself is the steepest descent

- The Chain Rule:

Theorem: Chain Rule Along a Curve:

$$\begin{aligned}\frac{d}{dt}\left[f\left(\vec{r}\left(t\right)\right)\right] &= \nabla f\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right)\\ &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\\ &= \nabla f\cdot\vec{T}\cdot\left(\frac{ds}{dt}\right)\end{aligned}$$

which is unit tangent $\times$ change in length

Example 16: Assume we have function $\vec{r}\left(t\right)=x\left(t\right)\hat{i}+y\left(t\right)\hat{j}$ which describes path or position, say:

$$\vec{r}\left(t\right)=\left(t^3,\cos t\right) \tag{31}$$

and we have a Temeprature function that's dependent on position

$$T\left(x,y\right)=xy^2 \tag{32}$$

What is the change in temperature as I go along the path $\vec{r}$ We have:

$$\begin{aligned}\nabla T &= \left(y^2,2xy\right)\\ \vec{r}\,' &= \left(3t^2,-\sin t\right)\end{aligned}$$
.

Solving for chain rule:

$$\begin{aligned}\therefore\frac{dT}{dt} &= \nabla T\cdot\vec{r}\,'\\ &= y^2\cdot 3t^2-2xy\sin t\\ &= 3t^2\cos^2 t-2t^3\cos t\sin t\end{aligned}$$
.

Or we can solve using the old fashioned method, by subbing x,y from $\vec{r}$

$$T=xy^2=t^3\cos^2 t \tag{33}$$

$$\therefore\frac{dT}{dt}=3t^2\cos^2 t-2t^3\cos t\sin t \tag{34}$$

Example 17: We have a room, $V=\ell\cdot h\cdot d$. Say:

$$\begin{aligned}\ell\text{ is increasing at:}\frac{d\ell}{dt} &= 3\frac{\text{m}}{\text{s}}\\ h\text{ dec }\frac{dh}{dt} &= -2\frac{\text{m}}{\text{s}}\\ d\text{ inc }\frac{dd}{dt} &= 5\frac{\text{m}}{\text{s}}\end{aligned}$$
.

starting from $\ell=2,h=3,d=4$, and we create a q vector that is the vector containing these 3 components.

$$V\left(t\right)=\ell\left(t\right)\cdot h\left(t\right)\cdot d\left(t\right)\ \text{ and }\vec{q}\left(t\right)=\left(\ell,h,d\right) \tag{35}$$

Using chain rule for multivariable functions:

$$\begin{aligned}\frac{dV(t)}{dt} &= \nabla V(\vec{q}(t)) \cdot \vec{q}\,'(t) \\ \implies \nabla V &= \left(\frac{\partial V}{\partial \ell}, \frac{\partial V}{\partial h}, \frac{\partial V}{\partial d}\right) = (hd, \ell d, h\ell) \\ \implies \vec{q}\,'(t) &= \left(\frac{d\ell}{dt}, \frac{dh}{dt}, \frac{dd}{dt}\right) = (3, -2, 5)\end{aligned}$$

Therefore, at the initial point of $(3, -2, 5)$

$$\frac{dV}{dt} = 3hd - 2\ell d + 5\ell h = 3\cdot 3\cdot 4 - 2\cdot 2\cdot 4 + 5\cdot 2\cdot 3 = 50\frac{\mathrm{m}^3}{\mathrm{s}} \tag{36}$$

As a comparison, we can also solve this using the traditional method of subbing in variables:

$$\begin{aligned}\ell &= 2 + 3t \\ h &= 3 - 2t \\ d &= 4 + 5t\end{aligned}$$

We can also solve using the old fashioned method:

$$\therefore V = (2 + 3t)(3 - 2t)(4 + 5t) \tag{37}$$

Idea: We can have x,y given as functions with two parameters, this will represent a SURFACE in 3D, not a curve anymore.

$$\begin{aligned}x &= x(t, s) \\ y &= y(t, s) \\ f &= f(x, y) \\ &.\end{aligned}$$

But we can still follow the same process of applying chain rule, we just have two rates of changes now

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &.\end{aligned}$$

- **Implicit Differentiation:**
  Let us have this function

$$u(x, y) = 0 \implies \frac{dy}{dx} = ?.$$

Let $x = t$, $y = y(t)$, Then we have $u = u(t, y(t))$ because we chose $x = t$, so y must be a function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Following the original statement that $u(x, y) = 0$, $u(t, u(t)) = 0$, $\therefore \frac{du}{dt} = 0$ since u is constant.

$$x = t \therefore \frac{dx}{dt} = 1 \text{ and } \frac{dy}{dt} = \frac{dy}{dx}.$$

Therefore

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} \\ \implies \frac{dy}{dx} &= -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \end{aligned}$$

Example 18: We have:

$$x^4 + 4x^3y + y^4 = 1 \implies u = x^4 + 4x^3y + y^4 - 1 = 0 \tag{38}$$

Rearranged into u, now we just solve for derivative of u with respect to each variables

$$\begin{aligned} \frac{du}{dx} &= 4x^3 + 12x^2y \\ \frac{du}{dy} &= 4x^3 + 4y^3 \\ &. \end{aligned}$$

Partial derivatives are easy to find, and we just have to find the ratio

$$\therefore \frac{dy}{dx} = -\frac{4x^3 + 12x^2y}{4x^3 + 4y^3} = -\frac{x^2(x+3y)}{x^3+y^3} \tag{39}$$

- **14.4 and 14.6: Tangent Plane and Linear Approximation**

Definition: Level curve is the curve on a plane where the "height" doesn't change.

Example 19:
Returning to the parabolic hill example:
Say we have $z(x,y) = 20 - x^2 - y^2$, we start from $P(1,2)$. The level curve at this point, that retains the "height" value of '15' will be:

$$C = 20 - x^2 - y^2, P(1,2) \therefore C = 20 - 1^2 - 2^2 = 15.$$

$C = 15 \implies 20 - x^2 - y^2 = 15 \implies$ level curve: $x^2 + y^2 = 5$
We have the tangent vector $\vec{t}$. (not the unit tangent, but also perpendicular to $\vec{r}$):

$$\begin{aligned} &\text{Starting point: } \vec{r}(1,2) \\ \therefore \vec{t} \cdot \vec{r} &= (t_1, t_2) \cdot (1,2) = 0 \\ &\implies t_1 \cdot 1 + t_2 \cdot 2 = 0 \end{aligned}$$

Choose $t_1 = 2, t_2 = -1 \therefore \vec{t} = (2,-1)$ is the tangent vector

$$\begin{aligned} \nabla z \cdot \vec{t} &= (-2x, -2y) \cdot (2,-1) \\ &= -4x + 2y \\ \text{at } (1,2) &= -4 + 4 = 0 \end{aligned}$$

Theorem: For level curve, the gradient will be perpendicular to the tangent at a specific point.
Proof: let's define a function f to represent the curve:

$$\begin{aligned} f(x,y) &= C, \text{ let } \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} \qquad \vec{t} = \vec{r}\,'(t) \\ \implies f(\vec{r}(t)) &= C \\ &. \end{aligned}$$

$$\begin{aligned}\frac{d}{dt}f(\vec{r}(t)) &= \nabla f(\vec{r})\cdot\vec{r}\,' \\ &= \frac{dC}{dt} && \text{[level curve: } C \text{ is constant]} \\ &= 0\end{aligned}$$
.

along the level curve, the tangent vector dot the gradient is 0, so we can conclude that gradient is NORMAL to the normal curve:

$$\nabla f\cdot\vec{r}\,' = 0 \text{ or } \nabla f(\vec{r}) \text{ is perpendicular to } \vec{r}\,'.$$

Since gradient function is perpendicular to tangent, we can define tangent to be:

$$\nabla f\cdot\vec{r} = \nabla f\cdot\vec{t} = 0.$$

$$\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \tag{40}$$

This way we always satisify the "perpendicular" property between gradient and tangent.

$$\begin{aligned}\implies \nabla f\cdot\vec{t} &= \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot\left(\frac{\partial f}{\partial y},-\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ &= 0\end{aligned}$$

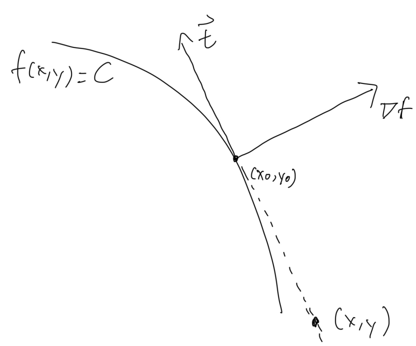

Figure 1: normal and tangent lines

- Refer to the above figure, we can see that for every level curve C at every point, there will be a gradient ∇f and a tangent $\vec{t}$. Most importantly, they are orthagonal to each other:

$$\nabla f\cdot\vec{t} = 0 \tag{41}$$

Using this, we can draw normal and parallel lines to the curve.

Theorem: To draw parallel line to point (x_0, y_0), we want to describe any point that lies on the line of $\vec{t}$. The vector $(x_0, y_0) \to (x, y) = (x - x_0, y - y_0)$ and is orthagonal to ∇f:

$$\begin{aligned}(x-x_0, y-y_0)\cdot\nabla f &= 0 \\ (x-x_0, y-y_0)\cdot\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right) &= 0 \\ \therefore (x-x_0)\cdot\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\cdot\frac{\partial f}{\partial y}(x_0,y_0) &= 0\end{aligned}$$

To draw normal line, we apply the same procedure, but now (x, y) lies on ∇f and is orthogonal to $\vec{t}$:

$$(x - x_0, y - y_0) \cdot \vec{t} = 0$$
$$(x - x_0, y - y_0) \cdot \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) = 0$$
$$\therefore (x - x_0) \cdot \frac{\partial f}{\partial y}(x_0, y_0) - (y - y_0) \cdot \frac{\partial f}{\partial x}(x_0, y_0) = 0$$

31 Tangent Planes and Linear Approximations

March 30, 2023

- We were working with 2D curves, we are thinking these curves as LEVEL CURVES of 3D surfaces. We find that: Gradient vector is always normal to the tangent. We can use this property to find normal lines using gradient vector, and find tangent line using tangent vector.
$$\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right).$$

Example 20: We have a circle $c^2 + y^2 = 9$, or $f(x, y) = c$. We then have $f_x = 2x, f_y = 2y$
Therefore, tangent line is:

$$(x - x_0) 2x_0 + (y - y_0) 2y_0 = 0 \tag{42}$$

USING

$$(x_0, y_0) = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \tag{43}$$

Tangent line:

$$\left(x - \frac{3}{\sqrt{2}}\right) 2\frac{3}{\sqrt{2}} + \left(y - \frac{3}{\sqrt{2}}\right) 2\frac{3}{\sqrt{2}} = 0 \tag{44}$$

$$y = \frac{6}{\sqrt{2}} - x \tag{45}$$

This gives us a tangent line to the circle.
At the same time, the normal line at that point is: $y = x$

- Function of 3 variables: Level surfaces is $f(x, y, z) = C$. The level surface will be a surface in 3D plane:

Idea: Gradient vector ∇f is perpendicular to the surface, or we can say it's perpendicular to the tangent plane

Example 21: we have $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ This is any curve on surface

$$\therefore f(x(t), y(t), z(t)) = f(\vec{r}(t)) = C$$

we have: (Note here's some equation about chain rule, and how gradient dot tangent to this curve is 0
From this we can conclude that the gradient will be perpendicular to any tangent line, showing it's normal to the whole surface

Idea: tangent plane is formed by the tangent vector of any curve on the surface at that point

Example 22: Suppose we now have $f(x,y,z) = x^2 + y^2 + z^2 = 25 = C$ (this is a sphere)

$$\nabla f = (2x, 2y, 2z) = 2(x,y,z) \tag{46}$$

Drawing a 3D curve showing the level surface, we have $\vec{x_0} = (x_0, y_0, z_0)$ and we form a level plane. Any vector $\vec{x}$ on this level plane will:

$$\begin{aligned} (\vec{x} - \vec{x_0}) &\text{ is perpendicular to } \nabla f \\ (\vec{x} - \vec{x_0}) \cdot \nabla f(\vec{x_0}) &= 0 \end{aligned}$$
.

This is the equation of tangent plane to surface $f(x,y,z) = C$ at $\vec{x_0}$

Example 23: We have $x^2 + y^2 + z^2 = 25$, $\nabla f = 2(x,y,z)$. At point $(0,5,0)$ on this sphere, the tangent plane to that point will be a plane parallel to the xy plane, $y = 5$
We check this:

$$\begin{aligned} (x - x_0) 2x_0 + (y - y_0) 2y_0 + (z - z_0) 2z_0 &= 0 \\ (y - 5) 10 &= 0 \\ y &= 5 \end{aligned}$$
.

Checking the normal line, we expect a line that's the y axis:

$$\begin{aligned} \vec{r}(q) &= \vec{x_0} + q\nabla f(\vec{x_0}) \\ x &= x_0 + qf_x \\ y &= y_0 + qf_y \\ z &= z_0 + qf_z \\ &\text{so} \\ x &= 0 \\ y &= 5 + 10q \\ z &= 0 \end{aligned}$$
.

Example 24: Another example: $xy^2 + 2z^2 = 12$ at $(1,2,2)$

$$\begin{aligned} f_x &= y^2 = 4 \\ f_y &= 2xy = 4 \\ f_z &= 4z = 8 \end{aligned}$$
.

therefore the normal line is:

$$\begin{aligned} x &= 1 + 4q \\ y &= 2 + 4q \\ z &= 2 + 8q \end{aligned}$$
.

Warning: Make sure the point actually lies on the plane!!!

Example 25: Having 2 elipsoid:

$$f = x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$$

$$g = x^2 + 3y^2 + 2z^2 = 9$$
$$P\left(2, 1, 1\right).$$

Gradient of sphere:

$$\begin{aligned}\nabla f &= \left(2x - 8\right)\hat{i} + \left(2y - 8\right)\hat{j} + \left(2z - 6\right)\hat{k}\\ \nabla f\left(2, 1, 1\right) &= \left(-4, -6, -4\right)\\ \nabla g &= 2x\hat{i} + 6y\hat{j} + 4z\hat{k}\\ \nabla g\left(2, 1, 1\right) &= \left(4, 6, 4\right) = -\nabla f.\end{aligned}$$

This shows that f and g are parallel at that point, and that they touch at that point. So they SHARE a tangent plane

Example 26: We now have a sphere and a parabaloid:

$$f = x^2 + y^2 + z^2 - 4y - 2z + 2 = 0$$

$$g = 3x^2 + 2y^2 - 2z = 1$$
$$P\left(1, 1, 2\right).$$

$$\begin{aligned}\nabla f &= 2x\hat{i} + \left(2y - 4\right)\hat{j} + \left(2z - 2\right)\hat{k}\\ \nabla f\left(1, 1, 2\right) &= \left(2, -2, 2\right)\\ \nabla g &= 6x\hat{i} + 4y\hat{j} - 2\hat{k}\\ \nabla g\left(1, 1, 2\right) &= \left(6, 4, -2\right)\end{aligned}.$$

Check their relationship, are they perpendicular?

$$\nabla f\left(1, 1, 2\right) \cdot \nabla g\left(1, 1, 2\right) = 12 - 8 - 4 = 0 \tag{47}$$

So these two surfaces are perpendicular at that point

Example 27: A space curve:

$$\vec{r}(t) = \frac{3}{2}\left(t^2 + 1\right)\hat{i} + \left(t^4 + 1\right)\hat{j} + t^3\hat{k}.$$

Ellipsoid:

$$x^2 + 2y^2 + 3z^2 = 20.$$

Point:

$$P(3, 2, 1).$$

So the line intersects the curve at that point, we want to show that the line is perpendicular to the curve at the intersection:
For the surface:

$$\begin{aligned}\nabla f &= \left(2x, 4y, 6z\right)\\ \nabla f\left(3, 2, 1\right) &= \left(6, 8, 6\right)\end{aligned}.$$

For the curve:

$$\vec{r}(t=1) = (3,2,1)$$
$$\vec{r}\,'(t) = \left(3t, 4t^3, 3t^2\right)$$
$$\vec{r}\,'(t=1) = (3,4,3) = \frac{1}{2}\nabla f$$
$$.$$

So the direction vector of the curve is parallel to the normal vector of the surface, showing that this curve is normal to the plane at ths intersection

Example 28: Ellipsoid:

$$x^2 + y^2 + 2z^2 = 7.$$

On ths ellipsoid there's a bee at point $(1,2,1)$ and the bee is flying away normal to the surface, but hits a plane. Let's first find the normal line / direction vector:

$$\nabla f = (2x, 2y, 4z)$$
$$\implies \nabla f\,(1,2,1) = (2,4,4)\,.$$

Normal line will be:

$$x = 1 + 2q$$
$$y = 2 + 4q$$
$$z = 1 + 4q$$
$$.$$

Position vector follows this line:

$$\vec{r}(q) = (1+2q)\,\hat{\imath} + (2+4q)\,\hat{\jmath} + (1+4q)\,\hat{k} \tag{48}$$

we know the speed is 4 m/s, $\|\vec{r}\,'(t)\| = 4$ but $\|\vec{r}\,'(q)\| = 6$. We can scale it down a bit, $q = \frac{2}{3}t$ so you have $\|\vec{r}\,'\left(\frac{2}{3}q\right)\| = 4$

$$\therefore \vec{r}(t) = \left(1 + \frac{4}{3}t\right)\hat{\imath} + \left(2 + \frac{8}{5}t\right)\hat{\jmath} + \left(1 + \frac{8}{3}t\right)\hat{k} \tag{49}$$

Intersection of this line with the plane (when the bee hits the plane). The plane is $2x + 3y + z = 49$. Sub in x, y, z into the equation for the line:

$$2\left(1 + \frac{4}{3}t\right) + 3\left(2 + \frac{8}{5}t\right) + \left(1 + \frac{8}{3}t\right) = 49$$
$$\frac{40}{3}t = 40$$
$$t = 3$$
$$.$$

So the bee hits the plane after 3s, and it ends up at: $(5, 10, 9)$

- Last example for normal planes

Example 29: We have two surfaces: $S_1 : xy = az^2$ and $S_2 : z^2 = b - x^2 - y^2$. Putting them in function form on level plane:

$$f(x,y,z) = xy - az^2 = 0$$
$$g(x,y,z) = x^2 + y^2 + z^2 = b$$
$$.$$

Then we have

$$\nabla f = (y, x, -2az)$$
$$\nabla g = (2x, 2y, 2z)$$

We have dot product of those two:

$$\nabla f \cdot \nabla g = 4sy - 4az^2 = 4\left(xy - az^2\right) = 0 \tag{50}$$

Note here, this tells us that the two surfaces are always normal to each other everywhere they intersect, since their tangent planes' normal vectors are normal.

32 Maximum and Minimum Values

March 31, 2023

- **14.7 Maximum and Minimum Values**

Definition: f is said to have a local maximum at $\vec{x_0} \iff f(\vec{x_0}) \geq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x_0}$
This is almost identical to the wording in single variable case, just replacing the original x with $\vec{x}$
f is said to have a local mimimum at $\vec{x_0} \iff f(\vec{x_0}) \leq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x_0}$

Theorem: if f has a local extreme value at $\vec{x_0}$, then either $\nabla f(\vec{x_0}) = 0$ or $\nabla f(x_0)$ DNE.
This is just like single variable case.

Proof. let $g(x) = f(x, y_0)$ be function with a fixed y_0 value
If f has a maximum at x_0, then g has a maximum at x_0. In this case, the derivative of g will just be a partial derivative of f with respect to x

$$\therefore \frac{dg}{dx}(x_0) = 0 = \frac{\partial f}{\partial x}(x_0, y_0) \tag{51}$$

From this, lets suppose we have function $z = f(x,y)$ and make a function $h(x,y,z) = z - f(x,y)$.
This implies $\nabla h = (-f_x, -f_y, 1) = \hat{k}$. This shows that the normal is perpendicular to the vector k.

□

Idea: if the planes have zero slope, than the normal is zero since $\nabla f = \vec{0}$

Definition: points where $\nabla f = \vec{0}$ or DNE are called critical points

Definition: Points where $\nabla f = \vec{0}$ called stationary points (at that point in either direction you are not going up or down)

Definition: stationary points where are not local extremes are called saddle points

Idea: just because the tangent plane is horizontal, doesn't mean you have a maximum or a minimum

Example 30: Lets begin with a simple function $f(x, y,) = 20 - x^2 - y^2$, $\nabla f = (-2x, -2y)$.
We set $\nabla f = \vec{0} \implies -2x = 0, -2y = 0 \implies x = 0, y = 0$

$$\therefore (0,0) \text{ is a stationary point.} \tag{52}$$

set $x = h, y = k$ where h, k are some tiny numbers. At origin, $f(0,0) = 20$, $f(h,k) = 20 - h^2 - k^2 < 20\ \forall h, k > 0$

Example 31: another surface $f(x, y) = xy$, $\nabla f = (y, x)$
Set $\nabla f = (0,0) \implies y = 0, x = 0$
$f(0,0) = 0$, $f(h,k) = hk$
For $h, k > 0$ or $h, k < 0$, $hk > 0$
But if we have them alternating signs, $hk < 0$, showing that this is a saddle point

Example 32: $f(x, y) = 2x^2 + y^2 - xy - 7y \implies \nabla f = (4x - y, 2y - x - 7)$
$\nabla f = \vec{0} \implies y = 4x, yx - x - 7 = 0 \implies x = 1, y = 4$
$f(1,4) = 2 + 16 - 4 - 28 = 14$

$$\begin{aligned} f(1.01, 4.01) &= -13.9998 \\ f(1,01, 3.99) &= -13.9996 \\ f(0.99, 4.01) &= -13.9996 \\ f(0.99, 3.99) &= -13.9998 \end{aligned}$$

This seems to be a minimum point

- It's perfectly possible for a maximum or a minimum that occurs along a line!
  Using a donut shape as an example, the equation is

$$f(x, y) = z = \left(a^2 - \left(\sqrt{x^2 + y^2} - R\right)^2\right)^{\frac{1}{2}}.$$

$$\nabla f = \left[\frac{1}{2}(\ldots)^{-\frac{1}{2}}(-2)\left(\sqrt{\ldots} - R\right)\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x, \frac{1}{2}(\ldots)^{-\frac{1}{2}}(-2)\left(\sqrt{\ldots} - R\right)\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y\right].$$

$$\nabla f = 0 \implies \sqrt{x^2 + y^2 = R} f\left(x^2 + y^2 = R^2\right) = \left(a^2 - \left(\sqrt{R^2} - R\right)^2\right)^{\frac{1}{2}} = a.$$

This is the top surface of the donut shape, and shows that the critical point CAN be a curve.
(question): what about neighboring point, aren't they the "same" height?

Example 33: cone $f(x, y) = -\sqrt{x^2 + y^2}$, $\nabla f = \left(-\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x, -\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y\right)$ This is DNE at $(0,0)$

Theorem: Second Derivatives Test:
Note the strategic "s" at the end of the "derivatives"?
For $f(x, y)$ with continuous second order partial, and $\nabla f(x_0, y_0) = 0$, set

$$A = \frac{\partial^2 f(x_0, y_0)}{\partial x^2}, B = \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}, C = \frac{\partial^2 f(x_0, y_0)}{\partial y^2}.$$

and form the discriminant $D = AC - B^2$

1. if $D < 0$, then (x_0, y_0) is a saddle point
2. if $D > 0$, $A, C > 0$ (x_0, y_0) is a local minimum

3. if $D > 0$, $A, C < 0$ (x_0, y_0) is a local maximum

4. if $D = 0$, you know nothing

(question) can $D = 0$?

Example 34: $f(x, y) = xy$

$$f_x = y, f_{xy} = 0 = A f_y = x, f_{yy} = 0 = C f_{xy} = 1 = B.$$

$\therefore D = AC - B^2 = -1 < 0$, therefore $(0, 0)$ is a saddle point.

Example 35: $f(x, y, z) = 2x^2 + y^2 - xy - 7y, \nabla f = 0$ at $(1, 4), f(1, 4) = -14$

$$\begin{aligned} f_x &= 4x - y \\ f_y &= 2y - x - 7 \\ f_{xx} &= 4 = A \\ f_{yy} &= 2 = C \\ f_{xy} &= -1 = B \end{aligned}$$

.

$\therefore D = 8 - 1 = 7 > 0$ and $A = 4 > 0$ which shows it's a local minimum

- Absolute Extreme Values

Theorem: if f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set.
Basically: if domain is not infinite, we have both abs max and abs min

Example 36: $f(x, y) = (x - 4)^2 + y^2$, we want to find abs max and min
On set $\{(x, y) : 0 \leq x \leq 2, x^3 \leq y \leq 4x\}$. Geometrically, this function's input is the area between the $y = 4x$ and $y = x^3$ curves for $x \geq 0$
We first start looking for INTERNAL extreme values:

$$\nabla f = (2(x - 4), -2y)\ \nabla f = 0 \implies x = 4, y = 0. \text{ This is not within the set. }.$$

(This example continues next lecture)