ESC195 W23 Notes

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Contents

28 Partial Derivatives

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- Continuity:

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$

Theorem: The continuity of composite functions are defined as, for any function $g(\vec{x_0})$, if it is continuous at $\vec{x_0}$ and function f is continuous at the NUMBER $g(\vec{x_0})$ m then we can say $f(g(\vec{x}))$ is continuous at $\vec{x_0}$

- if $f(\vec{x})$ is xontinuous at $\vec{x_0}$, then:

$$\lim_{x\to x_0} f(x, y_0) = f(x_0, y_0)$$
$$\lim_{y\to y_0} f(x_0, y) = f(x_0, y_0).$$

- If we have the top half of a sphere with radius 5, we have:

$$f = \sqrt{25 - x^2 - y^2}.$$

Suppose we are interested in what is happening when we are moving along the line y = 2:

Definition: Partial derivative of $f(x, y)$ is given by:

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{1}$$

or the partial derivative with respect to y:

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{2}$$

This can be extended to an arbitrary number of dimensions.

Example 1: Suppose we have a function $f(x, y) = e^{x^2y^3}$:

$$f_x = y^3 \cdot 2xe^{x^2y^3}$$
$$f_y = 3y^2x^2e^{x^2y^3}$$
$$.$$

At $y = 2$, we have:

$$f_x(x, 2) = 16xe^{8x^2}$$
$$f_x(1, 2) = 16e^8$$
$$.$$

This is equivalent if we take a cross section of this equation on the $y = 2$ plane, and look at the derivative or the slope of tangent at that point.

Example 2: Now we have a 3-D function, $f(x, y, z) = \ln\left(\frac{x}{y}\right) - ye^{xz}$. The partial derivatives are:

$$\begin{aligned} f_x &= \frac{1}{x} - yze^{xz} \\ f_y &= -\frac{1}{y} - e^{xz} \\ f_z &= -xye^{xz} \end{aligned}$$
.

Example 3: Supplse we have $h(r, \theta, \phi) = r^2 \sin\theta \cos\phi$:

$$\begin{aligned} h_r &= 2r \sin\theta \cos\phi \\ h_\theta &= r^2 \cos\theta \cos\phi \\ h_\phi &= -r^2 \sin\theta \sin\phi \end{aligned}$$
.

- We can also have Mixed Partials:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &\to \frac{\partial^2 f}{\partial x^2} = f_{xx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &\to \frac{\partial^2 f}{\partial y^2} = f_{yy} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \end{aligned}$$

Theorem: Clairaut's Theorem says that:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \tag{3}$$

on every open set on which f and its partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are continuous, so we have:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial z \partial y} &= \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} &= \frac{\partial^2 f}{\partial z \partial x} \end{aligned}$$
.

Example 4: We have $f(x, y) = \cos\left(xy^2\right)$

$$\begin{aligned} f_x &= -y^2 \sin\left(xy^2\right) \\ f_y &= -\sin\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial y \partial x} &= -2y \sin\left(xy^2\right) - y^2 \cos\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial x \partial y} &= -2y \sin\left(xy^2\right) - 2xy \cos\left(xy^2\right) \cdot y^2 = \frac{\partial^2 f}{\partial y \partial x} \end{aligned}$$
.

- partial derivatives can be used to desribe differential equations with multiple variables:

Example 5: Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{4}$$

One-dimensional wave equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2} \tag{5}$$

Here a represents the speed of the wave.

29 Directional Derivatives and Gradient Functions

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- A partial derivative:

Definition: A partial derivative of f on x, denoted as $f_x = \frac{\partial f}{\partial x}$ is defined as:

$$\lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} \tag{6}$$

only if $\lim_{h \to 0} \frac{g(h)}{h} = 0 \implies g(h) = o(h)$

Example 6: For a function $f(x) = x^2$, $f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2$ Note here that

$$\lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \tag{7}$$

since the function of h, $g(h)$ satisfies the previous definition, we say that h^2 is an $o(h)$, which further shows that $f'(x) = 2x$

Idea: With this, we can write out the derivative definition using just the numerator part of the derivative

$$f(x+h) - f(x) \tag{8}$$

And the result of this expression will leave us with two types of equations.

$$\text{derivative} \cdot h \tag{9}$$

or

$$\text{something} \cdot o(h) \tag{10}$$

anything that is multipled by $o(h)$ will be reduced to zero due to the definition of $o(h)$, which is why the first expression gives us the derivative.
This is especially useful when we are trying to find the derivative of a multivariable function.

- differentiability of a multivariable function

Definition: we say f is differentiable at $\vec{x}$ iff there exists $\vec{y}$ s.t.

$$f(\vec{x} + \vec{h}) - f(\vec{x}) = \vec{y} \cdot \vec{h} + o(\vec{h})\vec{y} = Df(\vec{x}) = \text{the gradient of } f.$$

Example 7: Here we have the function $f(x, y) = x + y^2$ and the h function $\vec{h} = (h_1, h_2)$

$$\begin{aligned} f(\vec{x} + \vec{h}) - f(\vec{x}) &= f(x + h_1, y + h_2) - f(x, y) \\ &= x + h_1 + (y + h_2)^2 - x - y^2 \\ &= h_1 + 2yh_2 + h_2^2 \\ &= (1\hat{i} + 2y\hat{j}) \cdot \vec{h} + h_2^2 \end{aligned}$$

After this, we need to demonstrate that the remaining part of the function, that is h_2^2, is actually a $o\left(\vec{h}\right)$ function.

To do that, we define $g(\vec{h}) = h^2 = (h_2\hat{j}) \cdot (h_1\hat{i} + h_2\hat{j}) = h_2\hat{j} \cdot \vec{h}$. So we can write $g\left(\vec{h}\right) = h_2\vec{j} \cdot \vec{h}$

$$\frac{|g\left(\vec{h}\right)|}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0} |xh_2| = 0 \implies xh_2h_3 = o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{i} + xy\hat{k}.$$

We know that $h_2 \to 0$ as $\vec{h} \to \vec{0}$ So $g(\vec{h})$ is $o(\vec{h})$ and we can claim that:

$$\nabla f(\vec{x}) = 1\hat{i} + wy\hat{j} \tag{11}$$

Example 8: For this example we have $f\left(x, y, z\right) = xyz$

$$\begin{aligned} f\left(\vec{x} + \vec{h}\right) - f\left(\vec{x}\right) &= \left(x + h_1\right)\left(y + h_2\right)\left(z + h_3\right) - xyz \\ &= xyz + xyh_3 + xh_2z + xh_2h_3 + h_1yz + h_1yh_3 + h_1h_2z + h_1h_2h_3 - xyz \\ &= \left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right) \cdot \vec{h}. \end{aligned}$$

We consider the case of $xh_2h_3 = g(\vec{h}) = xh_2\hat{k} \cdot \vec{h}$

$$\frac{g\left(\vec{h}\right)}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0} |xh_2| = 0 \implies xh_2h_3 \text{ is } o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{j} + xy\hat{k}.$$
$$\nabla f\left(x, y, z\right) = \left(f_x, f_y, f_z\right) \tag{12}$$

Theorem: For cartesian coordinates:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{13}$$

Or equally written as:

$$\nabla f(x, y, z) = (f_x, f_y, f_z) \tag{14}$$

- Gradients are the vectors that points steepest way "up the hill"
- $\vec{x}$ is vector
- $f(\vec{x})$ is not vector
- $\nabla f(\vec{x})$ is a vector

Example 9: Suppose we have a temperature function with respect to x and y, and we have $\frac{\partial T}{\partial x} = 3\frac{°C}{m}$ and $\frac{\partial T}{\partial y} = 4\frac{°C}{m}$. From this, we can conclude that:

$$\nabla T = 3\hat{i} + 4\hat{j} \tag{15}$$

And that:

$$|\nabla T| = \sqrt{3^2 + 4^2} = 5 \tag{16}$$

Example 10: For this example we have $f(\vec{x}) = xy^2z^3$, and we can compute their partial derivatives:

$$\begin{aligned} f_x &= y^2z^3 \\ f_y &= 2xyz^3 \\ f_z &= 3xy^2z^2 \\ &. \end{aligned}$$

And we have

$$\nabla f = \left(y^2z^3, 2xyz^3, 3xy^2z^2\right) \tag{17}$$

Example 11: We have function $\vec{r}(x,y,z) \implies r = \sqrt{x^2+y^2+z^2}$ So we then have

$$\begin{aligned} \nabla r &= \nabla\sqrt{x^2+y^2+z^2} \\ &= \frac{\frac{1}{2}2x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{\frac{1}{2}2y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{\frac{1}{2}2z}{\sqrt{x^2+y^2+z^2}}\hat{k} \end{aligned}$$

rewriting this gives us:

$$\nabla r = \frac{\vec{r}}{r} \tag{18}$$

- For any directional derivative, we can define it being:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f\left(\vec{x_0}+h\hat{i}\right) - f(\vec{x_0})}{h}.$$

Expanding this to any arbituary direction is:

Definition: Directional derivative of function f at $\vec{x_0}$ in direction $\hat{u}$

$$f_{\hat{u}}(\vec{x_0}) = \lim_{h\to 0}\frac{f(\vec{x_0}+h\hat{u}) - f(\vec{x_0})}{h} \tag{19}$$

We also have

$$f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0})\cdot\hat{u} \tag{20}$$

Proof. Proving that $f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0})\cdot\hat{u}$:

$$f\left(\vec{x}+\vec{h}\right) - f(\vec{x}) = \nabla f(\vec{x})\cdot\vec{h} + o\left(\vec{h}\right) \tag{21}$$

Where $\vec{h} = h\hat{i}$. Using this, we can show the above is equal to:

$$\nabla f(\vec{x})\cdot h\hat{u} + o\left(\vec{h}\right) \tag{22}$$

and therefore:

$$\frac{f(\vec{x}+h\hat{u}) - f(\vec{x})}{h} = \nabla f\cdot\hat{u} + \frac{o\left(\vec{h}\right)}{h} \tag{23}$$

and taking the limit as $h\to 0$, we have the desired result. □

Example 12: Using the same temperature example, we have $T(x,y)$ where $\frac{\partial T}{\partial y} = 4\frac{°C}{m}$ and $\frac{\partial T}{\partial x} = 3\frac{°C}{m}$. If we want to move in a direction of $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$$T_{\hat{u}} = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}\right)\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = 3\cos\theta + 4\sin\theta \tag{24}$$

Example 13: Suppose we have a parabolic hill described by $z(x,y) = 20 - x^2 - y^2$ and we move straight up, or we can say that $\hat{u} = (0,1)$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= -2x \\ \frac{\partial f}{\partial y} &= -2y \\ &.\end{aligned}$$

$$\therefore z_{\hat{u}} = (-2x, -2y) \cdot (0,-1) = 2y \tag{25}$$

(The following is not on that lecture, but from Xue Qilin's notes)

- Note that:

$$\begin{aligned}|f_{\hat{k}}(\vec{x})| &= |\nabla f \cdot \hat{u}| \\ &= \|\nabla f\| \|\hat{u}\| |\cos\theta| \\ &\leq \|\nabla f\|.\end{aligned}$$

Example 14: Suppose that $z = f(x,y) = A + x + 2y + x^2 - 2y^2$ and we wish to find the steepest path down starting from $(0,0,A)$. We know that:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 1 - 2x \\ \frac{\partial f}{\partial y} &= 2 - 6y \\ &.\end{aligned}$$

such that:

$$\nabla f = (1-2x)\,\hat{i} + (2-6y)\,\hat{j} \implies -\nabla f = (2x-1)\,\hat{i} + (6y-2)\,\hat{j}.$$

The curve is given by:

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}.$$

Where $x'(t) = 2x(t) - 1$ and $y'(t) = 6y(t) - 2$. This is in parametric form and we can convert to cartesian form by writing the derivatives as:

$$\frac{dy}{dx} = \frac{6y-2}{2x-1} \tag{26}$$

and solving this differential equation to get:

$$3y = (2x-1)^3 + 1 \tag{27}$$

30 Chain Rule

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- For gradient functions:

$$f\left(\vec{x}+\vec{h}\right)-f\left(\vec{x}\right)=\vec{y}\cdot\vec{h}+o\left(\vec{h}\right)$$
$$\implies \vec{y}=\nabla f=\text{gradient of} f$$
$$\text{which is } \left(f_x, f_y, f_z\right)$$

- directional derivative, allow us to take the derivative with any direction we choose:

$$f_{\hat{u}}=\nabla f\cdot\hat{u} \tag{28}$$

Example 15: We have $z=f\left(x,y\right)=A+x+2y-x^2-3y^2$, an eliptical parabolic hill. With the starting point of $(0,0,A)$, our job is to get down the hill, following the path of steepest descent. We will calculate a few things, and put together the process. We have partial derivatives:

$$f_x=1-2x$$
$$f_y=2-6y$$
$$.$$

And we have:

$$\nabla f=\left(1-2x\right)\hat{\imath}+\left(2-6y\right)\hat{\jmath} \tag{29}$$

This is the gradient, pointing UPHILL, to get the down hill:

$$-\nabla f=\left(2x-1\right)\hat{\imath}+\left(6y-2\right)\hat{\jmath} \tag{30}$$

We have the path of steepest descent, we call it curve $C: r\left(t\right)=x\left(t\right)\hat{\imath}+y\left(t\right)\hat{\jmath}$, Note: This path will have direction same as the steepest descent path.

$$x'\left(t\right)=2x\left(t\right)-1$$
$$y'\left(t\right)=6y\left(t\right)-2$$
$$\therefore \frac{dy}{dx}=\frac{6y-2}{2x-1}$$
$$.$$

This is a separable equation, we can solve it to find the actual path from the directions. After separating we can integrate directly.

$$\frac{dy}{6y-2}=\frac{dx}{2x-1}$$
$$\implies \frac{1}{6}\ln\left|6y-2\right|=\frac{1}{2}\ln\left|2x-1\right|+C$$
$$\therefore 6y-2=\left(2x-1\right)^3e^C$$
$$.$$

We need to find what e^C is, so we start from our initial condition: $x=0, y=0$

$$-2=\left(-1\right)^3e^C$$
$$\therefore e^c=2$$
$$\therefore 3y=\left(2x-1\right)^3+1.$$

And this is our steepest descent path. Everywhere along the curve C, the direction is the steepest downhill descent.

- The Chain Rule:

Theorem: Chain Rule Along a Curve:

$$\begin{aligned}\frac{d}{dt}\left[f\left(\vec{r}\left(t\right)\right)\right] &= \nabla f\left(\vec{r}\left(t\right)\right)\cdot\vec{r}\,'\left(t\right)\\ &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}\\ &= \nabla f\cdot\vec{T}\cdot\left(\frac{ds}{dt}\right)\end{aligned}$$

which is unit tangent * change in length

Example 16: Assume we have function $\vec{r}\left(t\right)=x\left(t\right)\hat{\imath}+y\left(t\right)\hat{\jmath}$ which describes path or position, say:

$$\vec{r}\left(t\right)=\left(t^3,\cos t\right) \tag{31}$$

and we have a Temeprature function that's dependent on position

$$T\left(x,y\right)=xy^2 \tag{32}$$

What is the change in temperature as I go along the path $\vec{r}$ We have:

$$\begin{aligned}\nabla T &= \left(y^2,2xy\right)\\ \vec{r}\,' &= \left(3t^2,-\sin t\right)\end{aligned}.$$

Solving for chain rule:

$$\begin{aligned}\therefore\frac{dT}{dt} &= \nabla T\cdot\vec{r}\,'\\ &= y^2\cdot 3t^2-2xy\sin t\\ &= 3t^2\cos^2 t-2t^3\cos t\sin t\end{aligned}.$$

Or we can solve using the old fashioned method, by subbing x,y from $\vec{r}$

$$T=xy^2=t^3\cos^2 t \tag{33}$$

$$\therefore\frac{dT}{dt}=3t^2\cos^2 t-2t^3\cos t\sin t \tag{34}$$

Example 17: We have a room, $V=\ell\cdot h\cdot d$. Say:

$$\begin{aligned}\ell\text{ is increasing at:}\frac{d\ell}{dt} &= 3\frac{\text{m}}{\text{s}}\\ h\text{ dec }\frac{dh}{dt} &= -2\frac{\text{m}}{\text{s}}\\ d\text{ inc }\frac{dd}{dt} &= 5\frac{\text{m}}{\text{s}}\end{aligned}.$$

starting from $\ell=2,h=3,d=4$, and we create a q vector that is the vector containing these 3 components.

$$V\left(t\right)=\ell\left(t\right)\cdot h\left(t\right)\cdot d\left(t\right)\ \text{ and }\ \vec{q}\left(t\right)=\left(\ell,h,d\right) \tag{35}$$

Using chain rule for multivariable functions:

$$\frac{dV(t)}{dt} = \nabla V(\vec{q}(t)) \cdot \vec{q}\,'(t) \implies \nabla V = \left(\frac{\partial V}{\partial \ell}, \frac{\partial V}{\partial h}, \frac{\partial V}{\partial d}\right) = (hd, \ell d, h\ell) \tag{36}$$

From this we get:

$$\implies \vec{q}\,'(t) = \left(\frac{d\ell}{dt}, \frac{dh}{dt}, \frac{dd}{dt}\right) = (3, -2, 5) \tag{37}$$

Therefore, at the initial point of $(3, -2, 5)$

$$\frac{dV}{dt} = 3hd - 2\ell d + 5\ell h = 3 \cdot 3 \cdot 4 - 2 \cdot 2 \cdot 4 + 5 \cdot 2 \cdot 3 = 50\frac{\text{m}^3}{\text{s}} \tag{38}$$

From this, expanding to the 3 dimensions so we can try solving using the old fashioned method of subbing in each variable

$$\begin{aligned} \ell &= 2 + 3t \\ h &= 3 - 2t \\ d &= 4 + 5t \end{aligned}$$

We can also solve using the old fashioned method:

$$\therefore V = (2 + 3t)(3 - 2t)(4 + 5t) \tag{39}$$

Idea: We can have x,y given as functions with two parameters, this will represent a SURFACE in 3D, not a curve anymore.

$$\begin{aligned} x &= x(t, s) \\ y &= y(t, s) \\ . \end{aligned}$$

But we can still follow the same process of applying chain rule, we just have two rates of changes now

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ . \end{aligned}$$

- **Implicit Differentiation:**

$$u(x, y) = 0 \implies \frac{dy}{dx} = 0.$$

Let $x = t$, $y = y(t)$, Then we have $u = u(t, y(t))$ because we chose $x = t$, so y must be a function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Following the original statement that $u(x, y) = 0$, $u(t, u(t)) = 0$, $\therefore \frac{du}{dt} = 0$ since u is constant.

$$x = t \therefore \frac{dx}{dt} = 1 \& \frac{dy}{dt} = \frac{dy}{dx}.$$

Therefore

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} \\ \implies \frac{dy}{dx} &= -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \end{aligned}$$

Example 18: We have:

$$x^4 + 4x^3y + y^4 = 1 \implies u = x^4 + 4x^3y + y^4 - 1 = 0 \tag{40}$$

Rearranged into u, now we just solve for derivative of u with respect to each variables

$$\begin{aligned} \frac{du}{dx} &= 4x^3 + 12x^2y \\ \frac{du}{dy} &= 4x^3 + 4y^3 \\ . \end{aligned}$$

Partial derivatives are easy to find, and we just have to find the ratio

$$\therefore \frac{dy}{dx} = -\frac{4x^3 + 12x^2y}{4x^3 + 4y^3} = -\frac{x^2(x+3y)}{x^3+y^3} \tag{41}$$

- **14.4 and 14.6: Tangent Plane and Linear Approximation**

Definition: Level curve is the curve on a plane where the "height" doesn't change.

Example 19:
Returning to the parabolic hill example:
Say we have $z(x, y) = 20 - x^2 - y^2$, we start from $P(1, 2)$. The level curve at this point, that retains the "height" value of '15' will be:

$$C = 20 - x^2 - y^2, P(1, 2) \therefore C = 20 - 1^2 - 2^2 = 15.$$

$C = 15 \implies 20 - x^2 - y^2 = 15 \implies$ level curve: $x^2 + y^2 = 5$
We have the tangent vector $\vec{t}$. (not the unit tangent)

$$\begin{aligned} &\vec{r}(1, 2) \\ \therefore \vec{t} \cdot \vec{r} &= (t_1, y_2) \cdot (2, -1) = 0 \\ \implies &t_1 \cdot 1 + t_2 \cdot 2 = 0 \end{aligned}$$

Choose $t_1 = 2, t_2 = -1 \therefore \vec{t} = (2, -1)$ is the tangent vector

$$\begin{aligned} \nabla z \cdot \vec{t} &= (-2x, -2y) \cdot (2, -1) \\ &= -4x + 2y \\ \text{at } (1, 2) &= -4 + 4 = 0 \end{aligned}$$

Theorem: For level curve: Since the gradient function produces a zero result when dot product with tangent, we say gradient function is normal to tangent let's define a function f to represent the curve:

$$\begin{aligned} & f(x,y) = C, \text{ let } \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} && \vec{t} = \vec{r}\,'(t) \\ \implies & f(\vec{r}(t)) = C \end{aligned}$$

.

$$\begin{aligned} \frac{d}{dt} f(\vec{r}(t)) &= \nabla f(\vec{r}) \cdot \vec{r}\,' \\ &= \frac{dC}{dt} && \text{[level curve: } C \text{ is constant]} \\ &= 0 \end{aligned}$$

.

along the level curve, the tangent vector dot the gradient is 0, so we can conclude that gradient is NORMAL to the normal curve:

$$\nabla f \cdot \vec{r}\,' = 0 \text{ or } \nabla f(\vec{r}) \text{ is perpendicular to } \vec{r}\,'.$$

Since gradient function is perpendicular to tangent, we can define tangent to be:

$$\nabla f \cdot \vec{r} = \nabla f \cdot \vec{t} = 0.$$

$$\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \tag{42}$$

This way we always satisify the "perpendicular" property between gradient and tangent.

$$\begin{aligned} \implies \nabla f \cdot \vec{t} &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ &= 0 \end{aligned}$$

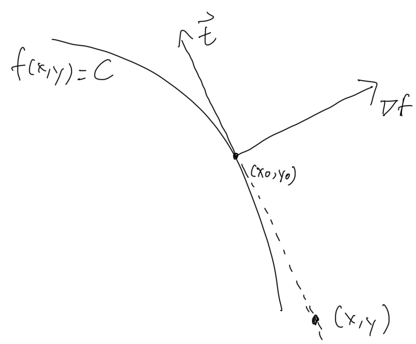

Figure 1: normal and tangent lines

- Refer to the above figure, we can see that for every level curve C at every point, there will be a gradient ∇f and a tangent $\vec{t}$. Most importantly, they are orthagonal to each other:

$$\nabla f \cdot \vec{t} = 0 \tag{43}$$

Using this, we can draw normal and parallel lines to the curve.

Theorem: To draw parallel line to point (x_0, y_0), we want to describe any point that lies on the line of $\vec{t}$. The vector $(x_0, y_0) \to (x, y) = (x - x_0, y - y_0)$ and is orthagonal to ∇f:

$$(x - x_0, y - y_0) \cdot \nabla f = 0$$
$$(x - x_0, y - y_0) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0$$
$$\therefore (x - x_0) \cdot \frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0) \cdot \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

To draw normal line, we apply the same procedure, but now (x, y) lies on ∇f and is orthagonal to $\vec{t}$:

$$(x - x_0, y - y_0) \cdot \vec{t} = 0$$
$$(x - x_0, y - y_0) \cdot \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) = 0$$
$$\therefore (x - x_0) \cdot \frac{\partial f}{\partial y}(x_0, y_0) - (y - y_0) \cdot \frac{\partial f}{\partial x}(x_0, y_0) = 0$$