# ESC195 W23 Notes

Andy Gong

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## 29 Partial Derivatives

- Continuity:

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$

> **Theorem**: The continuity of composite functions are defined as, for any function $g(\vec{x_0})$, if it is continuous at $\vec{x_0}$ and function $f$ is continuous at the NUMBER $g(\vec{x_0})$ m then we can say $f(g(\vec{x}))$ is continuous at $\vec{x_0}$

- if $f(\vec{x})$ is xontinuous at $\vec{x_0}$, then:

$$\lim_{x\to x_0} f(x, y_0) = f(x_0, y_0)$$
$$\lim_{y\to y_0} f(x_0, y) = f(x_0, y_0).$$

- If we have the top half of a sphere with radius 5, we have:

$$f = \sqrt{25 - x^2 - y^2}.$$

Suppose we are interested in what is happening when we are moving along the line y = 2:

> **Definition**: Partial derivative of $f(x, y)$ is given by:
>
> $$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \tag{1}$$
>
> or the partial derivative with respect to $y$:
>
> $$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \tag{2}$$
>
> This can be extended to an arbitrary number of dimensions.

**Example 1:** Suppose we have a function $f(x, y) = e^{x^2y^3}$ :

$$f_x = y^3 \cdot 2xe^{x^2y^3}$$
$$f_y = 3y^2x^2e^{x^2y^3}$$
$$.$$

At $y = 2$, we have:

$$f_x(x, 2) = 16xe^{8x^2}$$
$$f_x(1, 2) = 16e^8$$
$$.$$

This is equivalent if we take a cross section of this equation on the $y = 2$ plane, and look at the derivative or the slope of tangent at that point.

**Example 2:** Now we have a 3-D function, $f(x, y, z) = \ln\left(\frac{x}{y}\right) - ye^{xz}$. The partial derivatives are:

$$f_x = \frac{1}{x} - yze^{xz}$$
$$f_y = -\frac{1}{y} - e^{xz}$$
$$f_z = -xye^{xz}$$
$$.$$

**Example 3:** Supplse we have $h(r, \theta, \phi) = r^2 \sin\theta \cos\phi$:

$$\begin{aligned} h_r &= 2r \sin\theta \cos\phi \\ h_\theta &= r^2 \cos\theta \cos\phi \\ h_\phi &= -r^2 \sin\theta \sin\phi \end{aligned}$$

.

- We can also have Mixed Partials:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &\to \frac{\partial^2 f}{\partial x^2} = f_{xx} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &\to \frac{\partial^2 f}{\partial y^2} = f_{yy} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \end{aligned}$$

**Theorem**: Clairaut's Theorem says that:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \tag{3}$$

on every open set on which $f$ and its partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are continuous, so we have:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial z \partial y} &= \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} &= \frac{\partial^2 f}{\partial z \partial x} \end{aligned}$$

.

**Example 4:** We have $f(x, y) = \cos\left(xy^2\right)$

$$\begin{aligned} f_x &= -y^2 \sin\left(xy^2\right) \\ f_y &= -\sin\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial y \partial x} &= -2y \sin\left(xy^2\right) - y^2 \cos\left(xy^2\right) \cdot 2xy \\ \frac{\partial^2 f}{\partial x \partial y} &= -2y \sin\left(xy^2\right) - 2xy \cos\left(xy^2\right) \cdot y^2 = \frac{\partial^2 f}{\partial y \partial x} \end{aligned}$$

.

- partial derivatives can be used to desribe differential equations with multiple variables:

**Example 5:** Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{4}$$

One-dimensional wave equation:

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2} \tag{5}$$

Here $a$ represents the speed of the wave.

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## 30 Directional Derivatives and Gradient Functions

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- A partial derivative:

**Definition**: A partial derivative of $f$ on $x$, denoted as $f_x = \frac{\partial f}{\partial x}$ is defined as:

$$\lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} \tag{6}$$

only if $\lim_{h \to 0} \frac{g(h)}{h} = 0 \implies g(h) = o(h)$

**Example 6:** For a function $f(x) = x^2$, $f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2$ Note here that

$$\lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \tag{7}$$

since the function of h, $g(h)$ satisfies the previous definition, we say that $h^2$ is an $o(h)$, which further shows that $f'(x) = 2x$

**Idea**: With this, we can write out the derivative definition using just the numerator part of the derivative

$$f(x+h) - f(x) \tag{8}$$

And the result of this expression will leave us with two types of equations.

$$\text{derivative} \cdot h \tag{9}$$

or

$$\text{something} \cdot o(h) \tag{10}$$

anything that is multipled by $o(h)$ will be reduced to zero due to the definition of $o(h)$, which is why the first expression gives us the derivative.
This is especially useful when we are trying to find the derivative of a multivariable function.

- differentiability of a multivariable function

**Definition**: we say $f$ is differentiable at $\vec{x}$ iff there exists $\vec{y}$ s.t.

$$f(\vec{x} + \vec{h}) - f(\vec{x}) = \vec{y} \cdot \vec{h} + o(\vec{h})\vec{y} = Df(\vec{x}) = \text{the gradient of } f.$$

**Example 7:** Here we have the function $f(x, y) = x + y^2$ and the h function $\vec{h} = (h_1, h_2)$

$$\begin{aligned} f(\vec{x} + \vec{h}) - f(\vec{x}) &= f(x + h_1, y + h_2) - f(x, y) \\ &= x + h_1 + (y + h_2)^2 - x - y^2 \\ &= h_1 + 2yh_2 + h_2^2 \\ &= (1\hat{\imath} + 2y\hat{\jmath}) \cdot \vec{h} + h_2^2 \end{aligned}$$

After this, we need to demonstrate that the remaining part of the function, that is $h_2^2$, is actually a $o\left(\vec{h}\right)$ function.

To do that, we define $g(\vec{h}) = h^2 = (h_2\hat{j}) \cdot (h_1\hat{i} + h_2\hat{j}) = h_2\hat{j} \cdot \vec{h}$. So we can write $g\left(\vec{h}\right) = h_2\vec{j} \cdot \vec{h}$

$$\frac{|g\left(\vec{h}\right)|}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0} |xh_2| = 0 \implies xh_2h_3 = o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{i} + xy\hat{k}.$$

We know that $h_2 \to 0$ as $\vec{h} \to \vec{0}$ So $g(\vec{h})$ is $o(\vec{h})$ and we can claim that:

$$\nabla f(\vec{x}) = 1\hat{i} + wy\hat{j} \tag{11}$$

**Example 8:** For this example we have $f\left(x, y, z\right) = xyz$

$$\begin{aligned} f\left(\vec{x} + \vec{h}\right) - f\left(\vec{x}\right) &= \left(x + h_1\right)\left(y + h_2\right)\left(z + h_3\right) - xyz \\ &= xyz + xyh_3 + xh_2z + xh_2h_3 + h_1yz + h_1yh_3 + h_1h_2z + h_1h_2h_3 - xyz \\ &= \left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right) \cdot \vec{h}. \end{aligned}$$

We consider the case of $xh_2h_3 = g(\vec{h}) = xh_2\hat{k} \cdot \vec{h}$

$$\frac{g\left(\vec{h}\right)}{\|\vec{h}\|} = \frac{|x||h_2|\|\vec{h}\||\cos\theta|}{\|\vec{h}\|} \leq |xh_2|$$
$$\lim_{h\to 0} |xh_2| = 0 \implies xh_2h_3 \text{ is } o\left(\vec{h}\right)$$
$$\therefore \nabla f\left(\vec{x}\right) = yz\hat{i} + xz\hat{j} + xy\hat{k}.$$
$$\nabla f\left(x, y, z\right) = \left(f_x, f_y, f_z\right) \tag{12}$$

**Theorem**: For cartesian coordinates:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{13}$$

Or equally written as:

$$\nabla f(x, y, z) = (f_x, f_y, f_z) \tag{14}$$

- Gradients are the vectors that points steepest way "up the hill"
- $\vec{x}$ is vector
- $f(\vec{x})$ is not vector
- $\nabla f(\vec{x})$ is a vector

**Example 9:** Suppose we have a temperature function with respect to $x$ and $y$, and we have $\frac{\partial T}{\partial x} = 3\frac{°C}{m}$ and $\frac{\partial T}{\partial y} = 4\frac{°C}{m}$. From this, we can conclude that:

$$\nabla T = 3\hat{i} + 4\hat{j} \tag{15}$$

And that:

$$|\nabla T| = \sqrt{3^2 + 4^2} = 5 \tag{16}$$

**Example 10:** For this example we have $f(\vec{x}) = xy^2z^3$, and we can compute their partial derivatives:

$$\begin{aligned} f_x &= y^2z^3 \\ f_y &= 2xyz^3 \\ f_z &= 3xy^2z^2 \\ &. \end{aligned}$$

And we have

$$\nabla f = \left(y^2z^3, 2xyz^3, 3xy^2z^2\right) \tag{17}$$

**Example 11:** We have function $\vec{r}(x, y, z) \implies r = \sqrt{x^2 + y^2 + z^2}$ So we then have

$$\begin{aligned} \nabla r &= \nabla\sqrt{x^2 + y^2 + z^2} \\ &= \frac{\frac{1}{2}2x}{\sqrt{x^2 + y^2 + z^2}}\hat{i} + \frac{\frac{1}{2}2y}{\sqrt{x^2 + y^2 + z^2}}\hat{j} + \frac{\frac{1}{2}2z}{\sqrt{x^2 + y^2 + z^2}}\hat{k} \end{aligned}$$

rewriting this gives us:

$$\nabla r = \frac{\vec{r}}{r} \tag{18}$$

- For any directional derivative, we can define it being:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f\left(\vec{x_0} + h\hat{i}\right) - f(\vec{x_0})}{h}.$$

Expanding this to any arbituary direction is:

**Definition**: Directional derivative of function $f$ at $\vec{x_0}$ in direction $\hat{u}$

$$f_{\hat{u}}(\vec{x_0}) = \lim_{h \to 0} \frac{f(\vec{x_0} + h\hat{u}) - f(\vec{x_0})}{h} \tag{19}$$

We also have

$$f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \hat{u} \tag{20}$$

*Proof.* Proving that $f_{\hat{u}}(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \hat{u}$:

$$f\left(\vec{x} + \vec{h}\right) - f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{h} + o\left(\vec{h}\right) \tag{21}$$

Where $\vec{h} = h\hat{i}$. Using this, we can show the above is equal to:

$$\nabla f(\vec{x}) \cdot h\hat{u} + o\left(\vec{h}\right) \tag{22}$$

and therefore:

$$\frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \nabla f \cdot \hat{u} + \frac{o\left(\vec{h}\right)}{h} \tag{23}$$

and taking the limit as $h \to 0$, we have the desired result. □

**Example 12:** Using the same temperature example, we have $T(x, y)$ where $\frac{\partial T}{\partial y} = 4\frac{°C}{m}$ and $\frac{\partial T}{\partial x} = 3\frac{°C}{m}$. If we want to move in a direction of $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$

$$T_{\hat{u}} = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}\right)\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = 3\cos\theta + 4\sin\theta \tag{24}$$

**Example 13:** Suppose we have a parabolic hill described by $z(x, y) = 20 - x^2 - y^2$ and we move straight up, or we can say that $\hat{u} = (0, 1)$.

$$\frac{\partial f}{\partial x} = -2x$$
$$\frac{\partial f}{\partial y} = -2y$$
$$.$$

$$\therefore z_{\hat{u}} = (-2x, -2y) \cdot (0, -1) = 2y \tag{25}$$

(The following is not on that lecture, but from Xue Qilin's notes)

- Note that:

$$\begin{aligned} |f_{\hat{k}}(\vec{x})| &= |\nabla f \cdot \hat{u}| \\ &= \|\nabla f\| \|\hat{u}\| |\cos\theta| \\ &\leq \|\nabla f\|. \end{aligned}$$

**Example 14:** Suppose that $z = f(x, y) = A + x + 2y + x^2 - 2y^2$ and we wish to find the steepest path down starting from $(0, 0, A)$. We know that:

$$\frac{\partial f}{\partial x} = 1 - 2x$$
$$\frac{\partial f}{\partial y} = 2 - 6y$$
$$.$$

such that:

$$\nabla f = (1 - 2x)\,\hat{i} + (2 - 6y)\,\hat{j} \implies -\nabla f = (2x - 1)\,\hat{i} + (6y - 2)\,\hat{j}.$$

The curve is given by:

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}.$$

Where $x'(t) = 2x(t) - 1$ and $y'(t) = 6y(t) - 2$. This is in parametric form and we can convert to cartesian form by writing the derivatives as:

$$\frac{dy}{dx} = \frac{6y - 2}{2x - 1} \tag{26}$$

and solving this differential equation to get:

$$3y = (2x - 1)^3 + 1 \tag{27}$$

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