

## 1.6 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u; \quad y(0) = 0 \quad (20)$$

Consider a proportional control law  $u(t) = K_p(r(t) - y(t))$  where  $r(t)$  is the reference command. Consider a unit step command:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \text{ sec} \\ 1 & \text{for } t \geq 0 \text{ sec} \end{cases} \quad (2)$$

For what gains  $K_p$  is  $|u(t)| \leq 1$  for all time? Please format your answer as follows: If there is a specific range, answer with 'Range: [lower bound, upper bound]'. If there is no range that satisfies the condition, answer with 'No valid range'. If there is a specific value, answer with 'Value: [value]'."

### Solution :

First, note that we must ensure that the closed-loop is stable otherwise the control will grow unbounded. The closed-loop is modeled by:

$$\dot{y}(t) + (0.5 + 2K_p)y(t) = 2K_p r(t) \quad (21)$$

Thus,  $K_p > -0.25$  is required for stability. There are two cases to consider: (i)  $K_p \geq 0$  and (ii)  $-0.25 < K_p < 0$ .

First consider the case  $K_p \geq 0$ . In this case, the output starts at  $y(0) = 0$  and converges exponentially to the final value  $\bar{y} = \frac{2K_p}{0.5+2K_p}$ . The error  $e(t) = r(t) - y(t)$  thus starts at  $e(0) = 1$  and converges to the following final value:

$$\bar{e} = 1 - \frac{2K_p}{0.5 + 2K_p} = \frac{0.5}{0.5 + 2K_p} \leq 1 \quad (22)$$

In other words,  $e$  achieves its largest value at  $t = 0$  and decays to a smaller value. Thus the control effort  $u(t) = K_p e(t)$  achieves its largest value at  $t = 0$ . To ensure  $|u(t)| \leq 1$  it is sufficient to require  $u(0) = K_p e(0) = K_p \leq 1$ .

The alternative case  $-0.25 < K_p < 0$  actually causes the error to grow from  $e(0) = 1$  to its final value. In this case, the controller is acting to degrade tracking, which would not be typical. The hint given in the problem statement is incorrect for this case. It is fine if you only considered positive gains. However, to analyze this case you can use the expression above for the final value  $\bar{e}$ . Use this expression to show that  $K_p \geq -0.2$  is required to ensure  $|u(t)| \leq 1$ .

Combining these results, the input satisfies  $|u(t)| \leq 1$  for all time if the gain is selected in the range  $-0.2 \leq K_p \leq 1$ .

### GPT Answer :

Given the analysis and the considerations involved in ensuring  $|u(t)| \leq 1$  for all time, it is clear that a precise range for  $K_p$  cannot be provided without further numerical simulation or detailed mathematical manipulation. Therefore, the confidence level in providing a specific range or value based on the initial abstract approach is low.

**Confidence Level: 20%**

### Reasoning Error

#### Claude Answer :

We want to find the range of gains  $K_p$  for which  $|u(t)| \leq 1$  holds for all time. The control input  $u(t)$  is given by:

$$u(t) = K_p(1 - y(t)) = K_p \left( 1 - \frac{2K_p}{0.5 + 2K_p} (1 - e^{-(0.5+2K_p)t}) \right) \quad (23)$$

First, let's consider the stability condition. For the closed-loop system to be stable, the pole must be in the left half-plane. The pole is located at  $s = -0.5 - 2K_p$ , so for stability, we require  $K_p > -0.25$ .

Now, let's analyze the boundedness of  $u(t)$  for different ranges of  $K_p$ .

Case 1:  $K_p > 0$

1. At  $t = 0$ ,  $u(0) = K_p$ . To satisfy  $|u(0)| \leq 1$ , we need  $0 < K_p \leq 1$ .
2. As  $t \rightarrow \infty$ ,  $u_{ss} = \frac{0.5K_p}{0.5+2K_p}$ . To satisfy  $|u_{ss}| \leq 1$ , we need  $0 < K_p \leq 1$ .
3. The derivative of  $u(t)$  with respect to  $t$  is:

$$\frac{du}{dt} = -2K_p^2 e^{-(0.5+2K_p)t} \quad (24)$$

For  $K_p > 0$ , the derivative is always negative, indicating that  $u(t)$  is monotonically decreasing. Therefore, the maximum value of  $u(t)$  occurs at  $t = 0$ , which is  $u_{max} = K_p$ , and the minimum value occurs as  $t \rightarrow \infty$ , which is  $u_{min} = \frac{0.5K_p}{0.5+2K_p}$ .

Since both  $|u_{max}| \leq 1$  and  $|u_{min}| \leq 1$  for  $0 < K_p \leq 1$ , we can conclude that  $|u(t)| \leq 1$  holds for all time in this range.

Case 2:  $-0.25 < K_p \leq 0$

1. At  $t = 0$ ,  $u(0) = K_p$ . Since  $-0.25 < K_p \leq 0$ , we have  $|u(0)| \leq 0.25 < 1$ , satisfying the condition.

2. As  $t \rightarrow \infty$ ,  $u_{ss} = \frac{0.5K_p}{0.5+2K_p}$ . To satisfy  $|u_{ss}| \leq 1$ , we need:

$$-1 \leq \frac{0.5K_p}{0.5 + 2K_p} \leq 1 \quad (25)$$

Solving this inequality for  $K_p$  in the range  $-0.25 < K_p \leq 0$  yields  $-0.2 < K_p \leq 0$ .

3. For  $-0.2 < K_p \leq 0$ , the derivative of  $u(t)$  with respect to  $t$  is:

$$\frac{du}{dt} = -2K_p^2 e^{-(0.5+2K_p)t} \quad (26)$$

The derivative is always positive, indicating that  $u(t)$  is monotonically increasing. Therefore, the minimum value of  $u(t)$  occurs at  $t = 0$ , which is  $u_{min} = K_p$ , and the maximum value occurs as  $t \rightarrow \infty$ , which is  $u_{max} = \frac{0.5K_p}{0.5+2K_p}$ .

Since both  $|u_{min}| \leq 0.2 < 1$  and  $|u_{max}| \leq 1$  for  $-0.2 < K_p \leq 0$ , we can conclude that  $|u(t)| \leq 1$  holds for all time in this range.

Combining the results from Case 1 and Case 2, we have proven that  $|u(t)| \leq 1$  holds for all time when  $-0.2 < K_p \leq 1$ .

Final answer with LaTeX code:

Range:  $-0.2 < K_p \leq 1$

Confidence: 99

**Correct**