

5.17 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \quad y(0) = 0$$

with a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. Assume a unit step for $r(t)$. For what gains K_p is $|u(t)| \leq 1$ for all time?

Solution :

Note that the magnitude $|u(t)|$ of the control law is directly proportional to the difference between the state value and the reference value. i.e. $|r(t) - y(t)|$ via the gain K_p . Notice that the largest value of $|u(t)|$ will occur at $t = 0$ which gives us

$$\max_t |u(t)| = |K_p| |r(0) - y(0)| = |K_p| |1 - 0| = |K_p|$$

Therefore a preliminary condition for $|u(t)| \leq 1$ for all $t \in \mathbb{R}_+$ is that $|u| \leq 1$. However, note that we can write:

$$\dot{y} = -0.5y + 2u = -y(0.5 + 2K_p) + 2K_p r$$

and so when $K_p = 0$ we get a stable autonomous system with eigenvalue 0.5. On the other hand, for any constant reference signal $r(t) \equiv \text{const}$ we have that $K_p < -0.25$ will result in an unstable system. But note, that by solving for $y(t)$ we see that further we need,

$$|u(t)| \leq \left| K_p \left(1 - \frac{2K_p}{0.5 + 2K_p} \right) \right| \leq 1$$

Therefore, the acceptable range is $K_p \in [-0.2, 1]$.

GPT Answer :

To find the values of K_p that satisfy $|u(t)| \leq 1$ for all time given the system $\dot{y} = -0.5y + 2u$ and the proportional control law $u(t) = K_p(r(t) - y(t))$ with a unit step reference command $r(t) = 1$, we first substitute $u(t)$ into the differential equation and then analyze the resulting system to find the constraints on K_p .

The system equation becomes:

$$\dot{y} = -0.5y + 2K_p(1 - y)$$

Simplifying, we get:

$$\dot{y} = (-0.5 - 2K_p)y + 2K_p$$

Considering the steady-state condition ($\dot{y} = 0$) for $t \rightarrow \infty$, we solve for $y(\infty)$:

$$y(\infty) = \frac{2K_p}{0.5 + 2K_p}$$

Given the proportional control law $u(t) = K_p(1 - y(t))$, the condition $|u(t)| \leq 1$ becomes:

$$|K_p(1 - \frac{2K_p}{0.5 + 2K_p})| \leq 1$$

This simplifies to a condition on K_p . To solve for K_p , we must ensure that the magnitude of $u(t)$ does not exceed 1 for all time, which involves algebraically manipulating the above inequality.

Confidence Level: 75%

Calculation complexity

Claude Answer :

Great point. Let's investigate the case when K_p is negative more thoroughly.

1) From our previous work, we found that for stability, we need:

$$K_p > -0.25$$

2) Now, let's look at the steady-state value of $u(t)$ when K_p is negative. As $t \rightarrow \infty$, the exponential term in the expression for $u(t)$ goes to zero:

$$\lim_{t \rightarrow \infty} u(t) = K_p \left(1 - \frac{2K_p}{0.5 + 2K_p} \right)$$

3) Simplifying this:

$$\lim_{t \rightarrow \infty} u(t) = K_p \left(\frac{0.5 + 2K_p - 2K_p}{0.5 + 2K_p} \right) = K_p \left(\frac{0.5}{0.5 + 2K_p} \right)$$

4) For $|u(t)| < 1$ in steady state, we need:

$$\left| K_p \left(\frac{0.5}{0.5 + 2K_p} \right) \right| < 1$$

5) Given that K_p is negative, this condition can be written as:

$$-K_p \left(\frac{0.5}{0.5 + 2K_p} \right) < 1$$

$$\frac{-0.5K_p}{0.5 + 2K_p} < 1$$

$$-0.5K_p < 0.5 + 2K_p$$

$$-0.5K_p - 2K_p < 0.5$$

$$-2.5K_p < 0.5$$

$$K_p > -0.2$$

6) Combining this with the stability condition:

$$-0.25 < K_p < 0 \text{ (for stability)}$$

$$-0.2 < K_p < 0 \text{ (for } |u(t)| < 1 \text{ in steady state)}$$

7) The intersection of these two conditions is:

$$-0.2 < K_p < 0$$

8) Therefore, for negative K_p , the system will be stable and have $|u(t)| < 1$ in steady state if and only if $-0.2 < K_p < 0$.

9) However, remember that for $-0.25 < K_p \leq -0.2$, although the system is stable and $|u(t)| \rightarrow 1$ as $t \rightarrow \infty$, there will be an initial transient where $|u(t)| > 1$.

In conclusion, the complete conditions for stability and $|u(t)| \leq 1$ for all time are:

$$-0.2 < K_p \leq 1$$

Confidence Level: 95%

Correct