

## 10.2 Controllability

Consider the second-order system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} x(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} u(t)$$

and the output equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t).$$

The parameters  $a$ ,  $b$ ,  $c_1$ , and  $c_2$  are unknown a priori. Under what conditions is the system completely controllable? Select valid values of  $a$ ,  $b$ ,  $c_1$ , and  $c_2$  to ensure controllability and plot the step response.

### Solution :

The controllability matrix is

$$P_c = \begin{bmatrix} c_1 & c_2 \\ c_2 & -ac_1 - bc_2 \end{bmatrix}$$

and  $\det(P_c) = c_2^2 + [bc_1]c_2 + ac_1^2$ . For controllability, we require  $\det(P_c) \neq 0$ , hence

$$c_2^2 + [bc_1]c_2 + ac_1^2 \neq 0$$

implies

$$\frac{c_2}{c_1} \neq -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - a}$$

where  $\left(\frac{b}{2}\right)^2 - a \geq 0$ . For real-valued  $c_1$  and  $c_2$ , if  $\left(\frac{b}{2}\right)^2 - a < 0$ , all real values of  $c_1$  and  $c_2$  are valid. Valid values of the constants are  $c_1 = 0$ ,  $c_2 = 10$ ,  $a = 10$ , and  $b = 3$ . The step response is shown below:

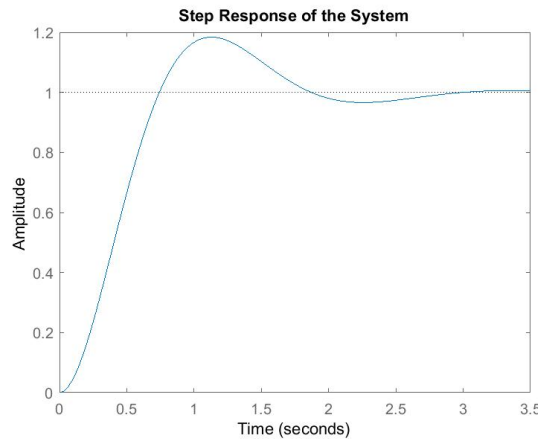


Figure 25: The system response over time.

**GPT Answer :**

To determine the conditions under which the system is completely controllable, we examine the controllability of the system described by the state-space equations:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} x(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t).$$

A system is said to be completely controllable if it is possible to move the system's state from any initial state to any final state within a finite time interval, through the application of appropriate inputs. For a linear time-invariant (LTI) system like the one above, a necessary and sufficient condition for complete controllability is that the controllability matrix  $\mathcal{C}$  has full rank. The controllability matrix  $\mathcal{C}$  for a system given by  $\dot{x} = Ax + Bu$  is calculated as:

$$\mathcal{C} = [B, AB, A^2B, \dots, A^{n-1}B]$$

where  $A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$  and  $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . In our case, for a second-order system ( $n=2$ ),  $\mathcal{C}$  simplifies to:

$$\mathcal{C} = [B, AB] = \left[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right] = \left[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \begin{bmatrix} c_2 \\ -ac_1 - bc_2 \end{bmatrix} \right]$$

For  $\mathcal{C}$  to have full rank (in this case, rank 2), its columns must be linearly independent. This condition will be satisfied if and only if the determinant of  $\mathcal{C}$  is non-zero:

$$\det(\mathcal{C}) = c_1(-ac_1 - bc_2) - c_2(c_2) \neq 0$$

Hence, the system is completely controllable if the above condition is met, implying that  $c_1$ ,  $c_2$ ,  $a$ , and  $b$  must be chosen such that the determinant of  $\mathcal{C}$  is non-zero. This is generally achievable as long as both  $c_1$  and  $c_2$  are not zero and the parameters do not result in the determinant equating to zero.

Now, let's choose valid values for  $a$ ,  $b$ ,  $c_1$ , and  $c_2$  that satisfy this condition and plot the step response. A simple choice could be  $a = 1$ ,  $b = 1$ ,  $c_1 = 1$ , and  $c_2 = 1$ , ensuring  $\det(\mathcal{C}) \neq 0$ .

**Confidence Level: 90%**

**Correct**

**Claude Answer :**

To determine the conditions for complete controllability, we need to check the rank of the controllability matrix. The system is completely controllable if the controllability matrix has full rank, which is equal to the dimension of the state vector.

Given:

System matrix:  $A = \begin{bmatrix} 0 & 1 & -a & -b \end{bmatrix}$  Input matrix:  $B = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$  The controllability matrix is given by:  $C = [B \quad AB]$

$$C = \begin{bmatrix} c_1 & c_2 \\ c_2 & -ac_1 - bc_2 \end{bmatrix}$$

For the system to be completely controllable, the determinant of the controllability matrix should be non-zero:

$$\det(C) = c_1(-ac_1 - bc_2) - c_2^2 \neq 0$$

This condition can be satisfied by selecting appropriate values for  $a$ ,  $b$ ,  $c_1$ , and  $c_2$ . One possible set of values that ensures controllability is:

$$a = 2, \quad b = 3, \quad c_1 = 1, \quad c_2 = 1$$

With these values, the determinant of the controllability matrix is:

$$\det(C) = 1(-2 \cdot 1 - 3 \cdot 1) - 1^2 = -4 \neq 0$$

**Confidence Level: 95%**

**Correct**