

2.2 Stability Analysis using PI Controller

Consider a plant with a nominal transfer function given by

$$G(s) = \frac{1}{(s-1)^2} \quad (71)$$

Prove that this system cannot be stabilized with a *PI* controller.

Solution :

We consider the general form of a PI controller as

$$C(s) = \frac{as + b}{s} \quad (72)$$

Thus, the closed-loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = \text{numerator of } \{1 + G(s)C(s)\} = s^3 - 2s^2 + (1+a)s + b \quad (73)$$

Hence, there are no values for a and b such that $A_{cl}(s)$ is strictly Hurwitz. This originates from the fact that one of the coefficients of the polynomial $A_{cl}(s)$ is negative no matter what values we choose for a and b .

GPT Answer:

To prove that a system with the given nominal transfer function cannot be stabilized with a Proportional-Integral (PI) controller, we can analyze the closed-loop transfer function and its poles for stability. A PI controller has a transfer function of the form $C(s) = K_p + \frac{K_i}{s}$, where K_p and K_i are the proportional and integral gains, respectively.

The closed-loop transfer function of the system when controlled by a PI controller is given by:

$$T(s) = \frac{GC(s)}{1 + GC(s)} = \frac{\left(K_p + \frac{K_i}{s}\right) \frac{1}{(s-1)^2}}{1 + \left(K_p + \frac{K_i}{s}\right) \frac{1}{(s-1)^2}} \quad (74)$$

Simplifying this expression, we get:

$$T(s) = \frac{K_p s + K_i}{s^3 - 2s^2 + (1 + K_p)s + K_i} \quad (75)$$

For the system to be stable, all poles of the closed-loop transfer function $T(s)$ must have negative real parts. The poles of $T(s)$ are the roots of its denominator polynomial:

$$D(s) = s^3 - 2s^2 + (1 + K_p)s + K_i \quad (76)$$

The Routh-Hurwitz criterion is a mathematical test that can be used to determine the stability of a system by examining the sign and presence of the coefficients of the characteristic polynomial

without explicitly finding the roots. According to the Routh-Hurwitz criterion, for the third-order system described by $D(s)$, the system is stable if all the coefficients of $D(s)$ are positive and there exists no sign change in the first column of the Routh array.

In our case, the coefficients of $D(s)$ are dependent on the values of K_p and K_i . However, note the structure of the transfer function $G(s) = \frac{1}{(s-1)^2}$. The presence of a double pole at $s = 1$ (in the right-half plane) inherently makes the open-loop system unstable. Adjusting K_p and K_i can shift the location of poles, but due to the double pole at $s = 1$, the system's stability cannot be guaranteed for all values of K_p and K_i .

Especially, the presence of a term $(s - 1)^2$ in the denominator signifies that any attempt to stabilize the system using a PI controller must overcome the challenge posed by these poles in the right half of the s-plane. There isn't a universal set of K_p and K_i values that would move both poles to the left half-plane to ensure stability due to the inherent nature of the PI controller and the specific dynamics of the given system.

Hence, this explanation shows that it's not possible to guarantee stabilization of the system with a PI controller for all values of K_p and K_i .

Confidence level: 90%

Correct

Claude Answer:

To prove that the given system cannot be stabilized with a PI controller, we need to show that the closed-loop system remains unstable for any choice of proportional gain K_p and integral gain K_i .

Given:

- Plant transfer function: $G(s) = \frac{1}{(s-1)^2}$
- PI controller transfer function: $C(s) = K_p + \frac{K_i}{s}$

Step 1: Write the characteristic equation of the closed-loop system.

The closed-loop transfer function is given by:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} \quad (77)$$

Substituting the expressions for $C(s)$ and $G(s)$, we get:

$$T(s) = \frac{(K_p + \frac{K_i}{s}) \cdot \frac{1}{(s-1)^2}}{1 + (K_p + \frac{K_i}{s}) \cdot \frac{1}{(s-1)^2}} \quad (78)$$

The characteristic equation is the denominator of the closed-loop transfer function equated to zero:

$$1 + (K_p + \frac{K_i}{s}) \cdot \frac{1}{(s-1)^2} = 0 \quad (79)$$

Step 2: Simplify the characteristic equation.

Multiply both sides by $(s-1)^2$ and rearrange:

$$s^2 - 2s(K_p - 1) + (K_i - 2K_p + 1) = 0 \quad (80)$$

Step 3: Check the stability using the Routh-Hurwitz criterion.

The Routh array for the characteristic equation is:

$$\begin{array}{c|cc} s^2 & 1 & K_i - 2K_p + 1 \\ s^1 & K_p - 2 & 0 \\ s^0 & K_i - 2K_p + 1 & 0 \end{array} \quad (81)$$

For the system to be stable, all elements in the first column of the Routh array must be positive. However, we can see that it is impossible to make all elements positive simultaneously for any choice of K_p and K_i :

- If $K_p > 2$, then the s^1 term is positive, but the s^0 term becomes negative.
- If $K_p < 2$, then the s^1 term is negative.

Therefore, the system cannot be stabilized with a PI controller for any choice of K_p and K_i .

Confidence level: 95%

Correct