

10.4 Lyapunov Stability

Determine sufficient conditions for the stability of the origin of the nonlinear discrete-time system described by

$$x_1(k+1) = x_1(k) - f[x_1(k)]$$

Solution :

Let $V[x(k)] = [x_1(k)]^2$, which is greater than 0 for all $x \neq 0$. Then,

$$\Delta V = x_1^2(k+1) - x_1^2(k) = (x_1(k) - f[x_1(k)])^2 - x_1^2(k) \quad (285)$$

$$= x_1(k)f[x_1(k)] \left(\frac{f[x_1(k)]}{x_1} - 2 \right) \quad (286)$$

Therefore sufficient conditions for $\Delta V \leq 0$ and thus stability of the system are

$$x_1 f(x_1) \geq 0$$

and

$$\frac{f(x_1)}{x_1} \leq 2 \quad \text{for all } x_1$$

GPT Answer:

To determine sufficient conditions for the stability of the origin of the given nonlinear discrete-time system

$$x_1(k+1) = x_1(k) - f[x_1(k)]$$

we apply the concept of Lyapunov stability for discrete-time systems. A system is said to be stable at the origin if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x_1(0)\| < \delta$, then $\|x_1(k)\| < \epsilon$ for all $k \geq 0$.

One common approach to establish stability is to find a Lyapunov function $V(x_1)$ that satisfies the following conditions: 1. $V(0) = 0$, 2. $V(x_1) > 0$ for all $x_1 \neq 0$, and 3. $\Delta V(x_1) = V[x_1(k+1)] - V[x_1(k)] < 0$ for all $x_1 \neq 0$.

A sufficient condition for stability can be derived by ensuring that the function $f[x_1(k)]$ satisfies certain properties which would guarantee that the change in the Lyapunov function $\Delta V(x_1)$ is negative for all $x_1 \neq 0$. Specifically, for the system given, a potential Lyapunov function could be $V(x_1) = x_1^2$, which is positive for all $x_1 \neq 0$ and zero at the origin.

The change in V due to the system dynamics is:

$$\Delta V(x_1) = V[x_1(k+1)] - V[x_1(k)] = [x_1(k) - f[x_1(k)]]^2 - x_1(k)^2$$

For $\Delta V(x_1) < 0$, the function $f[x_1(k)]$ must be such that:

$$[x_1(k) - f[x_1(k)]]^2 < x_1(k)^2$$

This implies that $f[x_1(k)]$ should not be zero for $x_1(k) \neq 0$ and must have the same sign as $x_1(k)$ to ensure that the magnitude of $x_1(k) - f[x_1(k)]$ is less than the magnitude of $x_1(k)$, thereby reducing the value of the Lyapunov function over time.

Thus, a sufficient condition for the stability of the origin is that $f[x_1(k)]$ is a continuous function that satisfies $f(0) = 0$ and $x_1(k) \cdot f[x_1(k)] > 0$ for all $x_1(k) \neq 0$. This ensures that the system's state moves closer to the origin over time, leading to stability.

Confidence Level: 90%

Correct

Claude Answer:

To determine the stability of the origin for the given nonlinear discrete-time system, we can use Lyapunov's second method (also known as the direct method of Lyapunov). The system is described by:

$$x_1(k+1) = x_1(k) - f[x_1(k)]$$

Let's assume that the function $f(x_1)$ satisfies the following conditions:

1. $f(0) = 0$, which means the origin is an equilibrium point of the system.
2. $f(x_1)$ is continuous in a neighborhood of the origin.
3. $x_1 f(x_1) > 0$ for all $x_1 \neq 0$ in a neighborhood of the origin.

Now, consider a Lyapunov function candidate:

$$V(x_1) = \frac{1}{2}x_1^2$$

This function is positive definite, radially unbounded, and continuous.

The forward difference of $V(x_1)$ along the system trajectories is:

$$\begin{aligned} \Delta V(x_1) &= V(x_1(k+1)) - V(x_1(k)) \\ &= \frac{1}{2}[x_1(k) - f(x_1(k))]^2 - \frac{1}{2}x_1(k)^2 \\ &= -x_1(k)f(x_1(k)) + \frac{1}{2}f^2(x_1(k)) \end{aligned}$$

If we add the condition:

4. $|f(x_1)| < |x_1|$ for all $x_1 \neq 0$ in a neighborhood of the origin.

Then, we have:

$$\Delta V(x_1) < -x_1(k)f(x_1(k)) + \frac{1}{2}x_1^2(k) = -\frac{1}{2}x_1^2(k) < 0$$

for all $x_1 \neq 0$ in a neighborhood of the origin.

Therefore, if conditions 1-4 are satisfied, the origin of the nonlinear discrete-time system is asymptotically stable in the sense of Lyapunov.

In summary, the sufficient conditions for the stability of the origin are:

1. $f(0) = 0$ 2. $f(x_1)$ is continuous in a neighborhood of the origin 3. $x_1 f(x_1) > 0$ for all $x_1 \neq 0$ in a neighborhood of the origin 4. $|f(x_1)| < |x_1|$ for all $x_1 \neq 0$ in a neighborhood of the origin

Confidence level: 95%

Correct