

9.6 Maximum Phase Lead

Show that the maximum phase lead of the lead compensator

$$P_{Lead}(j\omega) = \frac{(a/b)(1 + j\omega/a)}{1 + j\omega/b} \quad (274)$$

occurs at $\omega_m = \sqrt{ab}$ and find the maximum phase. What magnitude is produced at the maximum phase lead?

Solution :

The phase angle of the lead compensator is $\phi = \arg P_{Lead}(j\omega) = \tan^{-1} \omega/a - \tan^{-1} \omega/b$. Then

$$\frac{d\phi}{d\omega} = \frac{1}{a[1 + (\omega/a)^2]} - \frac{1}{b[1 + (\omega/b)^2]} \quad (275)$$

Setting $d\phi/d\omega$ equal to zero yields $\omega^2 = ab$. Thus the maximum phase lead occurs at $\omega_m = \sqrt{ab}$. Hence $\phi_{\max} = \tan^{-1} \sqrt{b/a} - \tan^{-1} \sqrt{a/b}$. But since $\tan^{-1} \sqrt{b/a} = \pi/2 - \tan^{-1} \sqrt{a/b}$, we have $\phi_{\max} = (90 - 2 \tan^{-1} \sqrt{a/b})$ degrees.

The magnitude factor is given by

$$|P_{Lead}(j\sqrt{ab})| = \left| \frac{(a/b)(1 + j\sqrt{b/a})}{1 + j\sqrt{a/b}} \right| = \sqrt{\frac{a}{b}}$$

GPT Answer :

To find the maximum phase lead of the lead compensator given by

$$P_{Lead}(j\omega) = \frac{(a/b)(1 + j\omega/a)}{1 + j\omega/b}$$

and to determine the frequency ω_m at which this maximum phase lead occurs, along with the magnitude at this frequency, we'll follow a series of steps:

Step 1: Phase of the Lead Compensator The phase lead $\phi(\omega)$ is given by the difference in phase angles of the numerator and the denominator:

$$\phi(\omega) = \tan^{-1} \left(\frac{\omega}{a} \right) - \tan^{-1} \left(\frac{\omega}{b} \right)$$

Step 2: Frequency for Maximum Phase Lead The maximum phase lead occurs where the derivative of $\phi(\omega)$ with respect to ω is zero. We can also use the property of symmetry in the phase-lead compensator to directly state that the maximum phase lead occurs at $\omega_m = \sqrt{ab}$.

Step 3: Maximum Phase Lead The maximum phase lead, ϕ_{max} , occurs at $\omega_m = \sqrt{ab}$. Substituting ω_m into the phase equation:

$$\phi_{max} = \tan^{-1} \left(\frac{\sqrt{ab}}{a} \right) - \tan^{-1} \left(\frac{\sqrt{ab}}{b} \right)$$

Simplifying, we get:

$$\phi_{max} = \tan^{-1} \left(\sqrt{\frac{b}{a}} \right) - \tan^{-1} \left(\sqrt{\frac{a}{b}} \right)$$

Step 4: Magnitude at Maximum Phase Lead The magnitude of $P_{Lead}(j\omega)$ at $\omega = \omega_m$ is:

$$|P_{Lead}(j\omega_m)| = \left| \frac{(a/b)(1 + j\omega_m/a)}{1 + j\omega_m/b} \right|$$

Substituting $\omega_m = \sqrt{ab}$:

$$|P_{Lead}(j\sqrt{ab})| = \left| \frac{a/b(1 + j\sqrt{ab}/a)}{1 + j\sqrt{ab}/b} \right|$$

After simplification, the magnitude is:

$$|P_{Lead}(j\sqrt{ab})| = \left| \frac{(a/b)\sqrt{1 + b/a}}{\sqrt{1 + a/b}} \right|$$

Confidence Level: 95%

Correct

Claude Answer:

Analysis of Maximum Phase Lead in Lead Compensator

To show that the maximum phase lead of the lead compensator occurs at $\omega_m = \sqrt{ab}$ and find the maximum phase, we will follow these steps:

Step 1: Express the lead compensator in polar form

$$\begin{aligned}
 P_{\text{Lead}}(j\omega) &= \frac{(a/b)(1 + j\omega/a)}{1 + j\omega/b} \\
 &= \frac{a}{b} \cdot \frac{1 + j\omega/a}{1 + j\omega/b} \\
 &= \frac{a}{b} \cdot \frac{\sqrt{1 + (\omega/a)^2}}{\sqrt{1 + (\omega/b)^2}} \cdot e^{j(\tan^{-1}(\omega/a) - \tan^{-1}(\omega/b))}
 \end{aligned}$$

Step 2: Determine the phase of the lead compensator The phase of the lead compensator is:

$$\phi(\omega) = \tan^{-1}(\omega/a) - \tan^{-1}(\omega/b)$$

To find the maximum phase lead, we differentiate $\phi(\omega)$ with respect to ω and set it to zero:

$$\begin{aligned}
 \frac{d\phi(\omega)}{d\omega} &= \frac{1}{1 + (\omega/a)^2} \cdot \frac{1}{a} - \frac{1}{1 + (\omega/b)^2} \cdot \frac{1}{b} = 0 \\
 \frac{b}{1 + (\omega/a)^2} &= \frac{a}{1 + (\omega/b)^2} \\
 b(1 + (\omega/b)^2) &= a(1 + (\omega/a)^2) \\
 b + \frac{\omega^2}{b} &= a + \frac{\omega^2}{a} \\
 \omega^2 \left(\frac{1}{b} - \frac{1}{a} \right) &= a - b \\
 \omega^2 &= \frac{a - b}{1/b - 1/a} = \frac{a - b}{(a - b)/ab} = ab \\
 \omega_m &= \sqrt{ab}
 \end{aligned}$$

Step 3: Calculate the maximum phase lead The maximum phase lead is:

$$\begin{aligned}
 \phi_{\text{max}} &= \tan^{-1}(\sqrt{ab}/a) - \tan^{-1}(\sqrt{ab}/b) \\
 &= \tan^{-1}(\sqrt{b/a}) - \tan^{-1}(\sqrt{a/b}) \\
 &= \tan^{-1}(\sqrt{b/a}) - \cot^{-1}(\sqrt{b/a})
 \end{aligned}$$

Step 4: Evaluate the magnitude at the maximum phase lead

$$\begin{aligned}
 |P_{\text{Lead}}(j\omega_m)| &= \frac{a}{b} \cdot \frac{\sqrt{1 + (\sqrt{ab}/a)^2}}{\sqrt{1 + (\sqrt{ab}/b)^2}} \\
 &= \frac{a}{b} \cdot \frac{\sqrt{1 + b/a}}{\sqrt{1 + a/b}} \\
 &= \sqrt{\frac{a}{b}}
 \end{aligned}$$

Therefore, the maximum phase lead of the lead compensator occurs at $\omega_m = \sqrt{ab}$, and the maximum phase is $\phi_{\max} = \tan^{-1}(\sqrt{b/a}) - \cot^{-1}(\sqrt{b/a})$. The magnitude produced at the maximum phase lead is $\sqrt{a/b}$.

Confidence level: 95%

Correct