

2.19 Discrete-Time Implementation

Consider the following plant:

$$\dot{y} + 5y = 3u$$

Suppose a PI controller is used with P gain K_p and I gain K_i . Assume the closed-loop system is stable. Then what sampling time would you recommend for a discrete-time implementation? Write out Δt as a function of K_p and K_i . Use $u(t) = k_p e(t) + K_i \int_0^t e(\tau) d\tau$ with $e(t) = r(t) - y(t)$.

Solution :

Since we are using PI control, $u(t) = k_p e(t) + K_i \int_0^t e(\tau) d\tau$ with $e(t) = r(t) - y(t)$. The closed loop system can be expressed as:

$$\ddot{y} + (5 + 3K_p) \dot{y} + 3K_i y = 3K_p \dot{r} + 3K_i r.$$

Performing Laplace transform on both sides, we have the closed-loop transfer function

$$T(s) = \frac{3K_p s + 3K_i}{s^2 + (5 + 3K_p)s + 3K_i}.$$

The poles of the closed-loop system is

$$s_{1,2} = \frac{-(5 + 3K_p) \pm \sqrt{(5 + 3K_p)^2 - 12K_i}}{2}.$$

Now we need to consider the following two scenarios: (i) If $(5 + 3K_p)^2 - 12K_i \geq 0$, s_1 and s_2 are both real, the corresponding time constant is $\tau_1 = \frac{2}{-(5+3K_p)+\sqrt{(5+3K_p)^2-12K_i}}$, $\tau_2 = \frac{2}{(5+3K_p)+\sqrt{(5+3K_p)^2-12K_i}}$.

For τ_1 , we have

$$\Delta t_1 = \frac{1}{5 | (5 + 3K_p) - \sqrt{(5 + 3K_p)^2 - 12K_i} |},$$

For τ_2 , we have

$$\Delta t_2 = \frac{1}{5(5 + 3K_p) + 5\sqrt{(5 + 3K_p)^2 - 12K_i}}.$$

Since $\Delta t_2 \leq \Delta t_1$, we can choose $\Delta t = \Delta t_2$. (ii) If $(5 + 3K_p)^2 - 12K_i < 0$, s_1 and s_2 are a complex conjugate, which has same real part and hence same time constant $\tau = \frac{2}{5+3K_p}$. In this case, we should design $\Delta t = \frac{1}{10} \frac{2}{5+3K_p} = \frac{1}{25+15K_p}$.

GPT Answer :

To recommend a sampling time (Δt) for a discrete-time implementation of a PI controller for the

given system, we first need to understand the system's dynamics and the effect of the PI controller on it. The given plant can be represented in the Laplace domain as:

$$sY(s) + 5Y(s) = 3U(s)$$

which simplifies to the plant's transfer function:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{3}{s+5}$$

The PI controller's transfer function is:

$$G_c(s) = K_p + \frac{K_i}{s}$$

The closed-loop transfer function $G_{cl}(s)$ of the system with the PI controller can be found by:

$$G_{cl}(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)}$$

Substituting $G_p(s)$ and $G_c(s)$ into the equation, we get:

$$G_{cl}(s) = \frac{3K_p + \frac{3K_i}{s}}{s+5 + 3K_p + \frac{3K_i}{s}}$$

This simplifies to:

$$G_{cl}(s) = \frac{3K_p + \frac{3K_i}{s}}{s^2 + (5 + 3K_p)s + 3K_i}$$

The poles of the system are given by:

$$s = -\frac{3K_p}{2} \pm \frac{\sqrt{-12K_i + 9K_p^2 + 30K_p + 25}}{2} - \frac{5}{2}$$

Given the complexity of the pole expressions, determining a precise formula for Δt as a function of K_p and K_i requires further simplification based on the specific system requirements and stability analysis. However, the general guidance for choosing Δt would be to ensure it's significantly smaller than the time constant associated with the slowest dynamics of the closed-loop system.

Confidence Level: 75%

Calculation Complexity

Claude Answer :

To determine an appropriate sampling time Δt for the discrete-time implementation of the PI controller, we need to consider the closed-loop system dynamics and the controller gains. The closed-loop transfer function can be derived using the following steps:

1. Laplace transform the plant equation: $sY(s) + 5Y(s) = 3U(s)$ $Y(s) = \frac{3}{s+5}U(s)$
2. The PI controller in the Laplace domain is: $U(s) = K_p E(s) + \frac{K_i}{s} E(s)$, where $E(s) = R(s) - Y(s)$

3. The closed-loop transfer function is: $\frac{Y(s)}{R(s)} = \frac{\frac{3}{s+5}(K_p + \frac{K_i}{s})}{1 + \frac{3}{s+5}(K_p + \frac{K_i}{s})}$

4. Simplifying the closed-loop transfer function: $\frac{Y(s)}{R(s)} = \frac{3K_p s + 3K_i}{s^2 + (5+3K_p)s + 3K_i}$

To ensure a stable and accurate discrete-time implementation, the sampling time Δt should be chosen such that it captures the dynamics of the closed-loop system. A rule of thumb is to select Δt to be approximately one-tenth of the fastest time constant in the closed-loop system.

The characteristic equation of the closed-loop system is: $s^2 + (5 + 3K_p)s + 3K_i = 0$

The roots of this equation determine the time constants of the system. The roots are: $s_{1,2} = \frac{-(5+3K_p) \pm \sqrt{(5+3K_p)^2 - 12K_i}}{2}$

The fastest time constant is given by the smallest absolute value of the real parts of the roots. Let's denote this time constant as τ : $\tau = \min \left(\left| \frac{2}{-(5+3K_p) \pm \sqrt{(5+3K_p)^2 - 12K_i}} \right| \right)$

Therefore, the recommended sampling time Δt as a function of K_p and K_i is: $\Delta t \approx \frac{\tau}{10} = \frac{1}{10} \min \left(\left| \frac{2}{-(5+3K_p) \pm \sqrt{(5+3K_p)^2 - 12K_i}} \right| \right)$

This expression provides a guideline for selecting the sampling time based on the PI controller gains, ensuring that the discrete-time implementation captures the relevant dynamics of the closed-loop system.

Confidence Level: 95%

Correct