

### 3 Time Response of Dynamical Systems

#### 3.1 Steady-State Response

Consider a second-order linear system described by the following ordinary differential equation (ODE):

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = \cos(3t) + 2\sin(4t), \quad x(0) = 0, \quad \dot{x}(0) = 0,$$

where  $a$ ,  $b$ , and  $c$  are positive real numbers. The system is known to have a steady-state response of the form:

$$x_{ss}(t) = A \cos(3t - \phi_1) + B \sin(4t - \phi_2),$$

where  $A$ ,  $B$ ,  $\phi_1$ , and  $\phi_2$  are constants.

Determine the values of  $a$ ,  $b$ , and  $c$  given that  $A = \frac{1}{3\sqrt{2}}$ ,  $B = \frac{1}{2\sqrt{2}}$ ,  $\phi_1 = \frac{\pi}{4}$ , and  $\phi_2 = \frac{3\pi}{4}$ .

**Solution :**

Transfer function of the system is

$$G(s) = \frac{1}{as^2 + bs + c} \quad (131)$$

and

$$M(\omega) = \frac{1}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}},$$

$$\phi(\omega) = -\tan^{-1} \left( \frac{b\omega}{c - a\omega^2} \right)$$

The steady-state response  $x_{ss}(t)$  is given by:

$$\begin{aligned} x_{ss}(t) &= 1 \times M(3) \times \cos(3t + \phi(3)) + 2 \times M(4) \times \sin(4t + \phi(4)) \\ &= 1 \times \frac{1}{\sqrt{(c - 9a)^2 + (3b)^2}} \times \cos \left( 3t - \tan^{-1} \left( \frac{3b}{c - 9a} \right) \right) \\ &\quad + 2 \times \frac{1}{\sqrt{(c - 16a)^2 + (4b)^2}} \times \sin \left( 4t - \tan^{-1} \left( \frac{4b}{c - 16a} \right) \right) \end{aligned}$$

Given the following conditions:

$$\begin{aligned}(c - 9a)^2 + (3b)^2 &= 18 \\(c - 16a)^2 + (4b)^2 &= 32 \\ \tan^{-1} \left( \frac{3b}{c - 9a} \right) &= \frac{\pi}{4} \Rightarrow c - 9a = 3b \\ \tan^{-1} \left( \frac{4b}{c - 16a} \right) &= \frac{3\pi}{4} \Rightarrow c - 16a = -4b\end{aligned}$$

Solving these equations, we find that  $b = 1$ ,  $c = 12$ , and  $a = 1$ .

**GPT Answer :**

To determine the values of  $a$ ,  $b$ , and  $c$  for the given second-order linear system, we analyze the system's response to a sinusoidal input. The steady-state response of a linear system to a sinusoidal input can be characterized using the system's transfer function,  $H(s) = \frac{Y(s)}{X(s)}$ , where  $s = j\omega$ , and  $\omega$  is the angular frequency of the input.

Given the differential equation

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = \cos(3t) + 2\sin(4t),$$

the corresponding transfer function is

$$H(s) = \frac{1}{as^2 + bs + c}.$$

The steady-state response to  $\cos(3t)$  and  $\sin(4t)$  can be determined by evaluating  $H(j\omega)$  at  $\omega = 3$  and  $\omega = 4$ , respectively.

For  $\cos(3t)$ , the input is represented as  $X_1(s) = \frac{s}{s^2+9}$ , and the steady-state amplitude and phase shift can be determined from  $H(j3)$ . Similarly, for  $\sin(4t)$ , the input is  $X_2(s) = \frac{4}{s^2+16}$ , and the steady-state response is determined from  $H(j4)$ .

Given the steady-state response

$$x_{ss}(t) = A \cos(3t - \phi_1) + B \sin(4t - \phi_2),$$

where  $A = \frac{1}{3\sqrt{2}}$ ,  $B = \frac{1}{2\sqrt{2}}$ ,  $\phi_1 = \frac{\pi}{4}$ , and  $\phi_2 = \frac{3\pi}{4}$ ,

the magnitude and phase of  $H(j3)$  and  $H(j4)$  can be determined from these given values, as they directly relate to the amplitude and phase shift of the system's response.

The magnitude of  $H(j\omega)$  at  $\omega = 3$  and  $\omega = 4$  is given by the amplitude of the response to the respective inputs, and the phase shift of  $H(j\omega)$  corresponds to  $\phi_1$  and  $\phi_2$ .

By solving for  $a$ ,  $b$ , and  $c$  using the conditions

$$\begin{aligned} |H(j3)| &= A, & \angle H(j3) &= -\phi_1, \\ |H(j4)| &= B, & \angle H(j4) &= -\phi_2, \end{aligned}$$

we can determine the system parameters. However, solving these equations directly requires manipulation and substitution that is complex without specific numerical values for  $a$ ,  $b$ , and  $c$ .

Given the complexity of the analysis and the lack of explicit equations for  $a$ ,  $b$ , and  $c$  without further algebraic manipulation and the use of the provided values of  $A$ ,  $B$ ,  $\phi_1$ , and  $\phi_2$ , the exact determination of  $a$ ,  $b$ , and  $c$  is not straightforwardly provided here. Therefore, the confidence level in providing a precise numerical answer without additional calculations or assumptions is low.

**Confidence Level: 30%**

### Calculation Complexity

#### Claude Answer :

To determine the values of  $a$ ,  $b$ , and  $c$ , we can substitute the given steady-state response into the ODE:

$$\begin{aligned} a(-9A \cos(3t - \phi_1) - 16B \sin(4t - \phi_2)) + b(-3A \sin(3t - \phi_1) + 4B \cos(4t - \phi_2)) \\ + c(A \cos(3t - \phi_1) + B \sin(4t - \phi_2)) = \cos(3t) + 2 \sin(4t) \end{aligned}$$

Equating the coefficients of  $\cos(3t)$ ,  $\sin(3t)$ ,  $\cos(4t)$ , and  $\sin(4t)$  on both sides, we get:

$$\begin{aligned} a(-9A \cos \phi_1) + b(3A \sin \phi_1) + c(A \cos \phi_1) &= 1 \\ a(9A \sin \phi_1) + b(3A \cos \phi_1) + c(A \sin \phi_1) &= 0 \\ a(16B \sin \phi_2) + b(4B \cos \phi_2) + c(B \sin \phi_2) &= 0 \\ a(-16B \cos \phi_2) + b(4B \sin \phi_2) + c(B \cos \phi_2) &= 2 \end{aligned}$$

Substituting the given values of  $A$ ,  $B$ ,  $\phi_1$ , and  $\phi_2$ , and solving the system of equations, we get:

$$a = 18, \quad b = 12, \quad c = 18$$

**Confidence level: 95%**

### Calculation Error