

## 5.20 PI Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \quad y(0) = 0$$

with a proportional-integral (PI) control law:  $u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$  with  $e(t) = r(t) - y(t)$ . Let  $K_p = 1$  and choose  $K_i$  to obtain a damping ratio of  $\zeta = 0.7$ . For these PI gains, what are the estimated maximum overshoot and 5% settling time (neglecting the integral effect of the zero)?

**Solution :**

The characteristic polynomial for the system with PI controller is given by:

$$s^2 + s(2K_p + \frac{1}{2}) + 2K_i \implies \omega_n = \sqrt{2K_i} \quad \text{and} \quad \zeta = \frac{4K_p + 1}{4\sqrt{2K_i}}$$

Plugging in  $K_p = 1$  and  $\zeta = 0.7$  into the last equation above yields that  $K_i = \frac{625}{392}$  and  $\omega_n = 1.7857$ . We know,

$$M_p = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \approx 4.6\%$$

and for 5% settling time,

$$T_s = \frac{-\ln(5/100)}{\zeta\omega_n} = \frac{-\ln(.05)}{0.7 \times 1.7857} = 2.4s$$

**GPT Answer :**

To answer your question, let's analyze the given system and its control law to determine the estimated maximum overshoot and the 5% settling time. The system is defined as follows:

$$\dot{y} = -0.5y + 2u \quad \text{with} \quad y(0) = 0$$

and the PI control law is:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

where  $e(t) = r(t) - y(t)$ ,  $K_p = 1$ , and we need to find  $K_i$  that results in a damping ratio  $\zeta = 0.7$ .

The transfer function of the system can be represented as:

$$G(s) = \frac{2}{s + 0.5}$$

The PI controller's transfer function is:

$$C(s) = K_p + \frac{K_i}{s} = 1 + \frac{K_i}{s}$$

The closed-loop transfer function becomes:

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{2(1 + \frac{K_i}{s})}{s + 0.5 + 2(1 + \frac{K_i}{s})}$$

To find  $K_i$  that yields a damping ratio of  $\zeta = 0.7$ , we typically would look into the characteristic equation of the closed-loop system. However, without directly solving for  $K_i$ , we focus on the question's request regarding maximum overshoot and settling time.

The maximum overshoot (MO) and settling time ( $T_s$ ) can be related to the damping ratio  $\zeta$  and the natural frequency  $\omega_n$  of the system by the following formulas:

$$\text{MO} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

However, to use these formulas, we need the system's natural frequency, which requires more detailed analysis or assumptions regarding the integral effect of the zero introduced by the PI controller. Since we're neglecting the integral effect of the zero, we're asked to focus on the damping ratio's effect on the overshoot and settling time, without directly computing from the specific system parameters.

Given these constraints and the typical behavior of a PI-controlled system, we can estimate the qualitative effects:

- The damping ratio  $\zeta = 0.7$  suggests a moderately damped system, which will have a certain amount of overshoot.
- The exact values for maximum overshoot and settling time depend on the natural frequency, which is influenced by  $K_i$ .

Without precise calculations of  $K_i$  and  $\omega_n$ , we cannot accurately determine the maximum overshoot and settling time. However, based on the damping ratio, we can infer that the system is designed to be stable with a moderate overshoot and a reasonably quick settling time, given typical control system responses.

Given the lack of specific calculation details due to the constraints mentioned, my confidence level in providing exact figures for maximum overshoot and settling time is low. Instead, I provided a qualitative analysis based on the damping ratio and typical system behaviors.

**Confidence Level: 30%**

**Calculation complexity**

**Claude Answer :**

Let's analyze the system step by step:

1. First, convert the system and control law into the standard form. The closed-loop transfer function with the PI controller is:

$$\frac{Y(s)}{R(s)} = \frac{2(K_p s + K_i)}{s^2 + (0.5 + 2K_p)s + 2K_i} \quad (253)$$

2. Comparing this to the standard second-order transfer function:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (254)$$

We can deduce that:

$$2\zeta\omega_n = 0.5 + 2K_p \quad (255)$$

$$\omega_n^2 = 2K_i \quad (256)$$

3. Given  $K_p = 1$  and  $\zeta = 0.7$ , we can solve for  $\omega_n$ :

$$2 \cdot 0.7 \cdot \omega_n = 0.5 + 2 \cdot 1 \quad (257)$$

$$1.4\omega_n = 2.5 \quad (258)$$

$$\omega_n = \frac{2.5}{1.4} \approx 1.786 \quad (259)$$

4. Now, we solve for  $K_i$ :

$$\omega_n^2 = 2K_i \quad (260)$$

$$(1.786)^2 \approx 2K_i \quad (261)$$

$$K_i \approx \frac{(1.786)^2}{2} \approx 1.595 \quad (262)$$

5. For a second-order system with damping ratio  $\zeta = 0.7$ , the estimated maximum overshoot (in percentage) is:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \cdot 100\% = e^{-\frac{0.7\pi}{\sqrt{1-0.7^2}}} \cdot 100\% \approx 4.6\% \quad (263)$$

6. The settling time (to within 5% of the final value) for a second-order system is approximately:

$$t_s \approx \frac{3}{\zeta\omega_n} = \frac{3}{0.7 \cdot 1.786} \approx 2.4 \text{ time units} \quad (264)$$

Therefore, with  $K_p = 1$  and  $K_i \approx 1.595$  (which gives a damping ratio of  $\zeta = 0.7$ ), the estimated maximum overshoot is about 4.6% and the 5% settling time is approximately 2.4 time units, neglecting the effect of the zero introduced by the PI controller.

**Confidence Level: 95%**

**Correct**