

1.13 Proving Linear Independence via Nonzero Wronskian

Show that a sufficient condition for a set of n functions f_1, f_2, \dots, f_n to be linearly independent is that the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \frac{df_1}{dt} & \frac{df_2}{dt} & \cdots & \frac{df_n}{dt} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}f_1}{dt^{n-1}} & \frac{d^{n-1}f_2}{dt^{n-1}} & \cdots & \frac{d^{n-1}f_n}{dt^{n-1}} \end{vmatrix} \quad (45)$$

be nonzero. This determinant is called the Wronskian of the functions f_1, f_2, \dots, f_n .

Solution :

Assuming the f_i s are differentiable at least $n - 1$ times, let $n - 1$ derivatives of

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \quad (46)$$

be formed as follows, where c_i s are unknown constants:

$$\begin{aligned} c_1 \frac{df_1}{dt} + c_2 \frac{df_2}{dt} + \cdots + c_n \frac{df_n}{dt} &= 0, \\ &\vdots \\ c_1 \frac{d^{n-1}f_1}{dt^{n-1}} + c_2 \frac{d^{n-1}f_2}{dt^{n-1}} + \cdots + c_n \frac{d^{n-1}f_n}{dt^{n-1}} &= 0. \end{aligned} \quad (47)$$

These equations may be considered as n simultaneous linear homogeneous equations in the n unknown constants c_1, c_2, \dots, c_n (i.e., not all c_i are equal to zero) if and only if the determinant of the coefficients (the Wronskian) is equal to zero. Hence if the Wronskian is nonzero, the the only solution for c_1, c_2, \dots, c_n is the degenerate one, $c_1 = c_2 = \cdots = c_n = 0$. Clearly, this is equivalent to saying that if the Wronskian is nonzero the functions f_1, f_2, \dots, f_n are linearly independent, since the only solution to $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ is then $c_1 = c_2 = c_3 = \cdots = c_n = 0$. Hence a sufficient condition for the linear independence of f_1, f_2, \dots, f_n is that the Wronskian be nonzero. This condition is not necessary; that is, there exist sets of linearly independent functions for which the Wronskian is zero.

GPT Answer:

A set of n functions f_1, f_2, \dots, f_n is said to be linearly independent if the only solution to the linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

where c_1, c_2, \dots, c_n are constants, is $c_1 = c_2 = \cdots = c_n = 0$.

The Wronskian of these functions is defined as the determinant:

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \frac{df_1}{dt} & \frac{df_2}{dt} & \cdots & \frac{df_n}{dt} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}f_1}{dt^{n-1}} & \frac{d^{n-1}f_2}{dt^{n-1}} & \cdots & \frac{d^{n-1}f_n}{dt^{n-1}} \end{vmatrix}$$

If the Wronskian is nonzero at some point $t = t_0$, then the functions f_1, f_2, \dots, f_n are linearly independent over the interval of consideration. This is a sufficient condition for the set of functions to be linearly independent.

Confidence level: 95%

Correct

Claude Answer:

To show that a nonzero Wronskian is a sufficient condition for linear independence of functions f_1, f_2, \dots, f_n :

Assume the Wronskian $W(f_1, f_2, \dots, f_n) \neq 0$. Suppose $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ for some constants c_1, c_2, \dots, c_n . We want to show that $c_1 = c_2 = \cdots = c_n = 0$.

Differentiating the linear combination k times ($0 \leq k \leq n-1$) gives a system of n homogeneous linear equations:

$$\begin{aligned} c_1 f_1 + c_2 f_2 + \cdots + c_n f_n &= 0 \\ c_1 f_1' + c_2 f_2' + \cdots + c_n f_n' &= 0 \\ &\vdots \\ c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} &= 0 \end{aligned}$$

The coefficient matrix is the Wronskian matrix. Since $W(f_1, f_2, \dots, f_n) \neq 0$, the system has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. Thus, the functions are linearly independent.

Confidence: 95%

Correct