

## 1.15 Laplace Transform

Show that the Laplace transform of the derivative  $\frac{df}{dt}$  of a function  $f(t)$  defined as below:

$$\mathcal{L} \left[ \frac{df}{dt} \right] = \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \int_{\epsilon}^T \frac{df}{dt} e^{-st} dt \quad (49)$$

is given by  $\mathcal{L} \left[ \frac{df}{dt} \right] = sF(s) - f(0^+)$ , where  $F(s) = \mathcal{L}[f(t)]$ .

**Solution :**

Integrating by parts.

$$\lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \int_{\epsilon}^T \frac{df}{dt} e^{-st} dt = \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \left[ f(t)e^{-st} \Big|_{\epsilon}^T + s \int_{\epsilon}^T f(t)e^{-st} dt \right] = -f(0^+) + sF(s) \quad (50)$$

where  $\lim_{\epsilon \rightarrow 0} f(\epsilon) = f(0^+)$ .

**GPT Answer :**

To show that the Laplace transform of the derivative  $\frac{df}{dt}$  is given by  $\mathcal{L} \left[ \frac{df}{dt} \right] = sF(s) - f(0^+)$ , we start by taking the definition of the Laplace transform of the derivative  $\frac{df}{dt}$ , integrate by parts, and then take the appropriate limits.

Given:

$$\mathcal{L} \left[ \frac{df}{dt} \right] = \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \int_{\epsilon}^T \frac{df}{dt} e^{-st} dt \quad (51)$$

Integrate by parts where  $u = e^{-st}$  and  $dv = \frac{df}{dt} dt$ , which gives  $du = -se^{-st} dt$  and  $v = f(t)$ .

Integration by parts formula:  $\int u dv = uv - \int v du$

Applying this formula:

$$\begin{aligned} \mathcal{L} \left[ \frac{df}{dt} \right] &= \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \left( f(t)e^{-st} \Big|_{\epsilon}^T + \int_{\epsilon}^T f(t) \cdot se^{-st} dt \right) \\ &= \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \left( f(T)e^{-sT} - f(\epsilon)e^{-s\epsilon} + s \int_{\epsilon}^T f(t)e^{-st} dt \right) \end{aligned}$$

As  $T \rightarrow +\infty$ ,  $f(T)e^{-sT} \rightarrow 0$  for stable  $f(t)$  and  $s > 0$ . Also, as  $\epsilon \downarrow 0$ ,  $f(\epsilon) \rightarrow f(0^+)$ .

Therefore, we have:

$$\mathcal{L} \left[ \frac{df}{dt} \right] = -f(0^+) + s\mathcal{L}[f(t)] \quad (52)$$

Since  $F(s) = \mathcal{L}[f(t)]$ , the final result is:

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^+) \quad (53)$$

**Confidence level: 95%**

**Correct**

**Claude Answer :**

$$\begin{aligned} \mathcal{L}\left\{\frac{df}{dt}\right\} &= \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \int_{\epsilon}^T \frac{df}{dt} e^{-st} dt \\ &= \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \left[ f(t) e^{-st} \right]_{\epsilon}^T - \int_{\epsilon}^T f(t) \frac{d}{dt} e^{-st} dt \\ &= \lim_{\substack{T \rightarrow +\infty \\ \epsilon \downarrow 0}} \left[ f(T) e^{-sT} - f(\epsilon) e^{-s\epsilon} + s \int_{\epsilon}^T f(t) e^{-st} dt \right] \\ &= -f(0^+) + s \lim_{T \rightarrow +\infty} \int_0^T f(t) e^{-st} dt \\ &= -f(0^+) + sF(s) \end{aligned}$$

Confidence Score: 100%

**Correct**