3.2 Free Response

Show that any free response $y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$ satisfies $\sum_{i=0}^{n} a_i(\frac{d' y}{dt'}) = 0$

Solution :

By the definition of a fundamental set, $y_k(t), k = 1, 2, \cdots, n$, satisfies $\sum_{i=0}^{n} a_i(\frac{d' y_k}{dt'}) = 0$. Substituting $\sum_{k=1}^{n} c_k y_k(t)$ into this differential equation yields

$$\sum_{i=0}^{n} a_i \frac{d'}{dt'} \left[\sum_{k=1}^{n} c_k y_k(t) \right] = \sum_{i=0}^{n} \sum_{k=1}^{n} a_i \frac{d'}{dt'} (c_k y_k(t)) = \sum_{k=1}^{n} c_k \left[\sum_{i=0}^{n} a_i \frac{d' y_k(t)}{dt'} \right] = 0 \tag{132}$$

The last equality is obtained because the term in the brackets is zero for all k.

GPT Answer:

To show that any free response $y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$ satisfies $\sum_{i=0}^{n} a_i \left(\frac{d^i y}{dt^i}\right) = 0$, we start by understanding that $y_a(t)$ is a linear combination of n solutions $y_k(t)$ of the homogeneous differential equation $\sum_{i=0}^{n} a_i \left(\frac{d^i y}{dt^i}\right) = 0$. Here, c_k are constants, and a_i are coefficients of the differential equation.

Since each $y_k(t)$ is a solution to the differential equation, we have:

$$\sum_{i=0}^{n} a_i \left(\frac{d^i y_k}{dt^i}\right) = 0, \quad \text{for all } k = 1, 2, \ldots, n$$

Now, considering the linear combination $y_a(t)$:

$$y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$$

Taking the i-th derivative of $y_a(t)$ with respect to t, we get:

$$\frac{d^i y_a}{dt^i} = \frac{d^i}{dt^i} \left(\sum_{k=1}^{n} c_k y_k(t) \right) = \sum_{k=1}^{n} c_k \frac{d^i y_k}{dt^i}$$

Then, applying the differential operator $\sum_{i=0}^{n} a_i \left(\frac{d^i}{dt^i}\right)$ to $y_a(t)$, we find:

$$\sum_{i=0}^{n} a_i \left(\frac{d^i y_a}{dt^i}\right) = \sum_{i=0}^{n} a_i \left(\sum_{k=1}^{n} c_k \frac{d^i y_k}{dt^i} \right)$$

Since each $y_k(t)$ satisfies the homogeneous differential equation, the inner sum results in 0 for each i, leading to:

$$\sum_{i=0}^{n} a_i \left(\frac{d^i y_a}{dt^i}\right) = 0$$

This shows that $y_a(t)$, being a linear combination of solutions $y_k(t)$, also satisfies the homogeneous differential equation.

Confidence Level: 95%

Correct
Claude Answer:
To show that any free response $y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$ satisfies $\sum_{i=0}^{n} a_i \left(\frac{d^i y}{dt^i}\right) = 0$, let's start by substituting $y_a(t)$ into the equation:

$$\sum_{i=0}^{n} a_i \left(\frac{d^i y_a(t)}{dt^i}\right) = \sum_{i=0}^{n} a_i \left(\frac{d^i \left(\sum_{k=1}^{n} c_k y_k(t)\right)}{dt^i}\right)$$

Using the linearity property of differentiation, we can bring the derivative inside the summation:

$$\sum_{i=0}^{n} a_i \left(\frac{d^i \left(\sum_{k=1}^{n} c_k y_k(t)\right)}{dt^i}\right) = \sum_{i=0}^{n} a_i \left(\sum_{k=1}^{n} c_k \frac{d^i y_k(t)}{dt^i}\right)$$

Now, we can rearrange the summations:

$$\sum_{i=0}^{n} a_i \left(\sum_{k=1}^{n} c_k \frac{d^i y_k(t)}{dt^i}\right) = \sum_{k=1}^{n} c_k \left(\sum_{i=0}^{n} a_i \frac{d^i y_k(t)}{dt^i}\right)$$

Since each $y_k(t)$ is a solution to the homogeneous linear differential equation, we know that:

$$\sum_{i=0}^{n} a_i \frac{d^i y_k(t)}{dt^i} = 0 \text{ for each } k = 1, 2, \ldots, n$$

Substituting this back into the previous equation:

$$\sum_{k=1}^{n} c_k \left(\sum_{i=0}^{n} a_i \frac{d^i y_k(t)}{dt^i}\right) = \sum_{k=1}^{n} c_k(0) = 0$$

Therefore, we have shown that any free response $y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$ satisfies $\sum_{i=0}^{n} a_i \left(\frac{d^i y}{dt^i}\right) = 0$.

Confidence Level: 95%

Correct