

10.3 Lyapunov Stability

The origin is a singular point for the pair of equations

$$\frac{dx_1}{dt} = ax_1 + bx_2 \quad \text{and} \quad \frac{dx_2}{dt} = cx_1 + dx_2$$

Using Lyapunov theory, find sufficient conditions on a , b , c , and d such that the origin is asymptotically stable.

Solution :

We choose a function

$$V = x_1^2 + x_2^2$$

which is positive for all x_1, x_2 except $x_1 = x_2 = 0$. The time derivative of V is

$$\frac{dV}{dt} = 2x_1 \frac{dx_1}{dt} + 2x_2 \frac{dx_2}{dt} = 2ax_1^2 + 2bx_1x_2 + 2cx_1x_2 + 2dx_2^2$$

To make $\frac{dV}{dt}$ negative for all x_1, x_2 , we might choose $a < 0$, $d < 0$, and $b = -c$. In this case,

$$\frac{dV}{dt} = 2ax_1^2 + 2dx_2^2 < 0$$

except when $x_1 = x_2 = 0$. Hence one set of sufficient conditions for asymptotic stability are $a < 0$, $d < 0$, and $b = -c$. There are other possible solutions to this problem.

GPT Answer:

To determine the conditions under which the origin is asymptotically stable for the given system of differential equations using Lyapunov theory, we express the system in matrix form as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

with $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The origin is asymptotically stable if there exists a Lyapunov function $V(\mathbf{x})$ that is positive definite and its time derivative $\dot{V}(\mathbf{x})$ is negative definite. A common choice for $V(\mathbf{x})$ is $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, with P being a positive definite matrix. The sufficient conditions derived from the eigenvalues of A are:

1. $a + d < 0$,
2. $ad - bc > 0$.

Given these conditions, the origin is asymptotically stable.

Confidence level: 90%

Correct

Claude Answer:

To find sufficient conditions for the origin to be asymptotically stable using Lyapunov theory, we need to construct a Lyapunov function $V(x_1, x_2)$ that satisfies the following conditions:

1. $V(x_1, x_2)$ is positive definite
2. dV/dt is negative definite

Let's consider the quadratic Lyapunov function:

$$V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$$

where $\alpha > 0$ and $\beta > 0$ to ensure V is positive definite.

Now, let's calculate dV/dt :

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt}$$

$$\frac{dV}{dt} = 2\alpha x_1(ax_1 + bx_2) + 2\beta x_2(cx_1 + dx_2)$$

$$\frac{dV}{dt} = 2\alpha ax_1^2 + 2\alpha bx_1x_2 + 2\beta cx_1x_2 + 2\beta dx_2^2$$

To ensure dV/dt is negative definite, we need:

1. $2\alpha a < 0$, which implies $a < 0$ (since $\alpha > 0$)
2. $2\beta d < 0$, which implies $d < 0$ (since $\beta > 0$)
3. $4\alpha a\beta d > (\alpha b + \beta c)^2$

The third condition comes from the requirement that the quadratic form of dV/dt is negative definite.

Therefore, sufficient conditions for the origin to be asymptotically stable are:

1. $a < 0$
2. $d < 0$
3. $4\alpha a\beta d > (\alpha b + \beta c)^2$ for some $\alpha > 0$ and $\beta > 0$

Note that these conditions are sufficient but not necessary, as there may be other Lyapunov functions that could provide different sufficient conditions.

Confidence level: 95%

Correct