

## 10.5 Lyapunov Stability

Determine sufficient conditions for the global stability of the system.

$$\dot{x} = Ax + bf(x_1) \quad \text{where} \quad A = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (287)$$

**Solution :**

Let  $V = x^T Px$  and  $P = \begin{bmatrix} a & c \\ c & 1 \end{bmatrix}$ . Then,

$$\dot{V} = x^T(PA + A^T P)x + x^T Pbf(x_1) + f(x_1)b^T Px \quad (288)$$

$$= x^T \begin{bmatrix} -4a & -a - 4c \\ -a - 4c & -2c - 4 \end{bmatrix} x + 2(a + 2c)x_1 f(x_1) + 2(c + 2)x_2 f(x_1) \quad (289)$$

To eliminate the cross-product term  $x_2 f(x_1)$ , set  $c = -2$ . Then,

$$\dot{V} = -x^T Qx + 2(a - 4)x_1 f(x_1)$$

Where  $Q = \begin{bmatrix} 4a & a - 8 \\ a - 8 & 0 \end{bmatrix}$ . For  $Q \geq 0$ ,  $a = 8$ . The resulting  $\dot{V}$  is

$$\dot{V} = -32x_1^2 + 8x_1 f(x_1) = -8x_1^2 \left( 4 - \frac{f(x_1)}{x_1} \right) \quad (290)$$

Then  $\dot{V} \leq 0$  and the system is stable if  $\frac{f(x_1)}{x_1} \leq 4$  for all  $x_1 \neq 0$ .

**GPT Answer:**

To determine the sufficient conditions for the stability of the system described by the differential equation  $\dot{x} = Ax + bf(x_1)$ , where

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we analyze the eigenvalues of  $A$  and the properties of  $f(x_1)$ . The eigenvalues of  $A$  are found to be  $\lambda = -2$ , indicating that the linear part of the system is stable as all eigenvalues have negative real parts.

For the nonlinear part  $bf(x_1)$ , a sufficient condition for stability is that  $f(x_1)$  is globally Lipschitz with a small enough Lipschitz constant to not introduce instability. This ensures that the nonlinearity does not lead to unstable behavior.

**Confidence Level: 90%**

**Incorrect Knowledge**

**Claude Answer:**

To determine sufficient conditions for the global stability of the given system, we can use the

Lyapunov stability theory. The system is globally stable if there exists a Lyapunov function  $V(x)$  that satisfies the following conditions:

1.  $V(x)$  is positive definite
2.  $\dot{V}(x)$  is negative definite

Let's consider a quadratic Lyapunov function candidate:

$V(x) = x^T P x$ , where  $P$  is a symmetric positive definite matrix.

Step 1: Calculate  $\dot{V}(x)$ .

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + bf(x_1))^T P x + x^T P (Ax + bf(x_1)) \\ &= x^T A^T P x + f(x_1) b^T P x + x^T P A x + x^T P b f(x_1) \\ &= x^T (A^T P + P A) x + 2f(x_1) x^T P b\end{aligned}$$

Step 2: Assume that the nonlinear function  $f(x_1)$  satisfies the sector condition:  $0 \leq \frac{f(x_1)}{x_1} \leq k$ , for all  $x_1 \neq 0$  and some  $k > 0$ .

This implies:  $0 \leq f(x_1) \leq kx_1$

Step 3: Find sufficient conditions for  $\dot{V}(x)$  to be negative definite.  $\dot{V}(x) \leq x^T (A^T P + P A) x + 2kx_1 x^T P b$

Let  $Q = -(A^T P + P A)$  and  $r = 2kx^T P b$ .

For  $\dot{V}(x)$  to be negative definite, we require:  $x^T Q x > r x_1$

Step 4: Solve the linear matrix inequality (LMI) to find  $P$  and  $k$ . Minimize  $k$  subject to:  $P > 0$

$$\begin{bmatrix} Q & P b \\ b^T P & \frac{1}{k} \end{bmatrix} > 0$$

The LMI can be solved using numerical methods or LMI solvers.

If a feasible solution for  $P$  and  $k$  exists, then the system is globally stable under the sector condition  $0 \leq \frac{f(x_1)}{x_1} \leq k$ .

In summary, the sufficient conditions for global stability are: 1. The existence of a symmetric positive definite matrix  $P$  and a scalar  $k > 0$  that satisfy the LMI. 2. The nonlinear function  $f(x_1)$  satisfies the sector condition  $0 \leq \frac{f(x_1)}{x_1} \leq k$  for all  $x_1 \neq 0$ .

**Confidence level: 95%**

**Correct**