2.14 Characterisitic Equation

Prove that a continuous system is unstable if any coefficients of the characteristic equation are zero.

Solution :
The characteristic equation may be written in the form

$$(s - s_1)(s - s_2)(s - s_3)\cdots(s - s_n) = 0 \tag{128}$$

where $s - 1, s_2, \cdots, s_n$ are the roots of the equation. If this equation is multiplied out, n new equations an be obtained relating the roots and the coefficients of the characteristic equation in the usual form. Thus

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0 \tag{129}$$

and the relations are

$$\frac{a_{n-1}}{a_n} = -\sum_{i=1}^{n} s_i, \frac{a_{n-2}}{a_n} = \sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n} s_i s_j, \frac{a_{n-3}}{a_n} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\i\neq j\neq k}}^{n} s_i s_j s_k, \cdots, \frac{a_0}{a_n}(-1)^n s_1 s_2 \cdots s_n \tag{130}$$

The coefficients $a_{n-1}, a_{n-2}, \cdots, a_0$ all have the same sign as a_n and are nonzero if all the roots $s_1, s_2, \cdots, s_n$ have negative real parts. The only way any one of the coefficients can be zero is for one or more of the roots to have zero or positive real parts. In either case, the system would be unstable.

GPT Answer:
The stability of a continuous system can often be determined by analyzing its characteristic equation, which typically arises from the linear differential equation describing the system. The characteristic equation is generally given in the form of a polynomial:

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients of the polynomial and λ represents the eigenvalues (or roots) of the equation. The system is considered stable if all the real parts of its eigenvalues are negative, which ensures that the system's response decays over time.

If any of the coefficients a_i (where $i = 0, 1, \ldots, n$) of the characteristic equation are zero, it affects the distribution of the roots in the complex plane. Specifically, a zero coefficient can result in a root with a non-negative real part, which indicates a potential for the system response not to decay, or even to grow over time. This situation can arise because the characteristic polynomial's degree is effectively reduced, altering the nature of its roots.

However, the assertion that "a continuous system is unstable if any coefficients of the characteristic equation are zero" requires a nuanced understanding. A zero coefficient does not automatically guarantee instability; it can indicate a boundary condition where the system might be on the verge of instability or require further analysis to determine stability conclusively.

For a rigorous proof, consider the Routh-Hurwitz criterion, which provides a systematic method to determine the stability of a system by examining the signs and magnitudes of the array constructed from the coefficients of the characteristic equation. According to the Routh-Hurwitz criterion, if any coefficient is zero (or missing), it implies that the system may have roots with positive real parts or roots lying on the imaginary axis, both conditions leading to potential instability.

In summary, while a zero coefficient in the characteristic equation can be a strong indicator of potential instability, concluding absolute instability requires further analysis, such as applying the Routh-Hurwitz criterion or examining the specific system dynamics. Hence, our confidence in the statement "a continuous system is unstable if any coefficients of the characteristic equation are zero" as an absolute rule is moderate.

Confidence level: 75%

Correct

Claude Answer:

To prove that a continuous system is unstable if any coefficients of the characteristic equation are zero, we'll use the Routh-Hurwitz stability criterion. This criterion states that a continuous system is stable if and only if all the coefficients of its characteristic equation and all the elements in the first column of the Routh array are positive.

Given:

- A continuous system with a characteristic equation.
- At least one coefficient of the characteristic equation is zero.

Step 1: Consider a general n-th order characteristic equation:

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

Where:

- s is the complex frequency variable
- a_i are the coefficients of the characteristic equation ($i = 0, 1, \ldots, n-1$)

Step 2: Construct the Routh array using the coefficients of the characteristic equation. The first two rows of the Routh array are:

$$\begin{array}{c|cccc} s^n & 1 & a_{n-2} & a_{n-4} & \ldots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \end{array}$$

The subsequent rows are calculated using the coefficients from the previous two rows.

Step 3: If any coefficient of the characteristic equation is zero, it will appear in the Routh array.

Case 1: If $a_0 = 0$, the system has a pole at the origin ($s = 0$), which makes the system unstable.

Case 2: If any other coefficient $a_i = 0$ ($i \neq 0$), it will lead to a zero element in the first column of the Routh array. According to the Routh-Hurwitz stability criterion, if any element in the first column of the Routh array is zero or negative, the system is unstable.

Therefore, if any coefficient of the characteristic equation is zero, the continuous system is unstable, as it violates the Routh-Hurwitz stability criterion, which requires all coefficients and all elements in the first column of the Routh array to be positive for a stable system.
Confidence level: 95%

Correct