

12.2 Proportional Controller

A model for a DC motor is:

$$J\dot{y} + by = cV$$

where y is the angular velocity of the motor shaft (deg/sec), and V is the input voltage (Volts). The model parameters are: $J = 3 \text{ N m sec}^2/\text{deg}^2$, $b = 5 \text{ Nm sec/deg}$, and gain from input voltage to applied torque $c = 12 \text{ Nm/Volts}$. We plan to design a controller $K(s)$ for this system. The controller and the system are in standard unity feedback setting. Let $e = r - y$ where r is the reference signal. Let n represent the output noise i.e. if $n \neq 0$ then $e = r - (y + n)$. Design a proportional control law $K(s) = K_p$ so that the closed loop is stable and has gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec.

Solution :

Firstly, we can show that the requirement of gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec translates to $|L(j\omega)| = |K(j\omega)G(j\omega)| \geq 100$ for $\omega \leq 0.1 \frac{\text{rad}}{\text{sec}}$. The system transfer function is $G(s) = \frac{12}{3s+5}$. This has a corner frequency at $\omega = 1.6 \text{ rad/sec}$ and DC gain $G(0) = 2.4$. Thus $K_p = \frac{100}{2.4}$ will ensure the gain requirement is approximately satisfied. Additionally, we can verify the closed loop pole of the system with $K_p = \frac{100}{2.4}$ is in the LHP so the closed loop system is stable. Thus both the requirements are met.

GPT Answer :

To design a proportional control law $K(s) = K_p$ for the given system where the DC motor model is defined by the differential equation:

$$J\dot{y} + by = cV$$

with $J = 3 \text{ Nm sec}^2/\text{deg}^2$, $b = 5 \text{ Nm sec/deg}$, and $c = 12 \text{ Nm/Volts}$, and ensuring that the closed-loop system is stable and has a gain of less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec, we follow these steps:

1. ****Transfer Function of the Motor:**** First, we find the transfer function of the motor. From the given differential equation, taking the Laplace transform (assuming zero initial conditions) gives:

$$JsY(s) + bY(s) = cV(s)$$

$$Y(s)(Js + b) = cV(s)$$

The transfer function $G(s) = \frac{Y(s)}{V(s)}$ is:

$$G(s) = \frac{c}{Js + b}$$

Substituting the given values:

$$G(s) = \frac{12}{3s + 5}$$

2. **Closed-Loop Transfer Function:** For a proportional controller $K(s) = K_p$, the closed-loop transfer function from the reference signal r to the output y in a unity feedback system is given by:

$$T(s) = \frac{K_p G(s)}{1 + K_p G(s)}$$

$$T(s) = \frac{K_p \frac{12}{3s+5}}{1 + K_p \frac{12}{3s+5}}$$

$$T(s) = \frac{12K_p}{3s + 5 + 12K_p}$$

3. **Stability Criterion:** For stability, all poles of $T(s)$ must have negative real parts. The pole of $T(s)$ is given by the denominator $3s + 5 + 12K_p = 0$. For stability, $5 + 12K_p > 0$, which is always true for positive K_p .

4. **Gain Requirement:** To ensure the gain from r to e is less than 0.01 for frequencies below 0.1 rad/sec, we solve the inequality for K_p based on the condition:

$$|1 - T(0)| < 0.01$$

This yields $K_p > 41.25$. Thus, choosing a proportional gain K_p greater than 41.25 will ensure that the closed-loop system meets the gain requirement.

Confidence Level: 75%

Correct

Claude Answer:

To design a proportional controller $K(s) = K_p$ that meets the given specifications, we need to analyze the closed-loop system and determine the appropriate value for K_p .

Step 1: Convert the DC motor model to the s-domain using the Laplace transform.

$$Js^2Y(s) + bsY(s) = cV(s)$$

Step 2: Rearrange the equation to find the transfer function of the DC motor, $G(s)$.

$$G(s) = \frac{Y(s)}{V(s)} = \frac{c}{Js^2 + bs}$$

Substituting the given values:

$$G(s) = \frac{12}{3s^2 + 5s}$$

Step 3: Write the closed-loop transfer function from the reference $R(s)$ to the output $Y(s)$.

$$\frac{Y(s)}{R(s)} = \frac{K_p G(s)}{1 + K_p G(s)}$$

Step 4: Write the transfer function from the reference $R(s)$ to the error $E(s)$.

$$\frac{E(s)}{R(s)} = \frac{1}{1 + K_p G(s)}$$

Step 5: To ensure closed-loop stability, the denominator of the closed-loop transfer function should have all poles in the left-half plane. The characteristic equation is:

$$1 + K_p G(s) = 0$$

Substituting $G(s)$:

$$1 + K_p \frac{12}{3s^2 + 5s} = 0$$

The poles are given by:

$$s = \frac{-5 \pm \sqrt{25 - 48K_p}}{6}$$

For stability, $K_p > 0$.

Step 6: To satisfy the gain requirement, we need:

$$\left| \frac{E(j\omega)}{R(j\omega)} \right| < 0.01 \text{ for } \omega < 0.1 \text{ rad/sec}$$

At low frequencies, the magnitude of the transfer function can be approximated by its DC gain ($s = 0$):

$$\left| \frac{E(j\omega)}{R(j\omega)} \right| \approx \left| \frac{1}{1 + K_p G(0)} \right| = \left| \frac{1}{1 + K_p \frac{12}{5}} \right| < 0.01$$

Solving for K_p :

$$K_p > \frac{495}{12} \approx 41.25$$

Therefore, choosing any $K_p > 41.25$ will ensure closed-loop stability and satisfy the gain requirement for frequencies below 0.1 rad/sec.

Confidence level: 95%

Correct