

2.5 Stability of Equilibrium

Consider an ODE

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y + u \end{cases} \quad (102)$$

where u is the input. Here we are interested in the stability of the equilibrium solution $(0, 0)$. Consider a linear feedback of the form $u = ax + by$. Can the linearization about the equilibrium be made asymptotically stable for any choice of constants (a, b) ?

Solution :

Substituting the feedback into the system yields

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -y + ax + by. \end{aligned}$$

The linearization of the system about the equilibrium point $(0, 0)$ is represented by the Jacobian matrix

$$J = \begin{bmatrix} 0 & 0 \\ a & b - 1 \end{bmatrix}.$$

The stability of the equilibrium is determined by the eigenvalues of J , which are solutions to the characteristic equation

$$\det(J - \lambda I) = 0,$$

resulting in eigenvalues $\lambda_1 = 0$ and $\lambda_2 = b - 1$.

For asymptotic stability, all eigenvalues must have strictly negative real parts. However, $\lambda_1 = 0$ does not satisfy this criterion, indicating that the equilibrium $(0, 0)$ cannot be made asymptotically stable through any choice of constants (a, b) .

GPT Answer :

To analyze the stability of the equilibrium solution $(0, 0)$ for the given system of Ordinary Differential Equations (ODEs) with a linear feedback $u = ax + by$, we first substitute the feedback control into the system and then linearize the resulting system around the equilibrium point.

The original system is:

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y + u. \end{cases} \quad (103)$$

Substituting the feedback $u = ax + by$ into the second equation, we get:

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y + ax + by. \end{cases} \quad (104)$$

The Jacobian matrix J of the system at any point (x, y) is given by:

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ a & b - 1 \end{bmatrix}. \quad (105)$$

Evaluating J at the equilibrium point $(0, 0)$ gives:

$$J|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ a & b - 1 \end{bmatrix}. \quad (106)$$

The eigenvalues of this matrix are critical for determining the stability of the equilibrium. They are solutions to the characteristic equation $\det(J - \lambda I) = 0$, where I is the identity matrix. For $J|_{(0,0)}$, the characteristic equation is:

$$\det \left(\begin{bmatrix} -\lambda & 0 \\ a & b - 1 - \lambda \end{bmatrix} \right) = 0 \Rightarrow \lambda(\lambda - (b - 1)) = 0. \quad (107)$$

Therefore, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = b - 1$.

For the linear system to be asymptotically stable, all eigenvalues must have strictly negative real parts. However, one of the eigenvalues, λ_1 , is always zero regardless of the values of a and b . Therefore, the linearization about the equilibrium $(0, 0)$ cannot be made asymptotically stable for any choice of constants (a, b) .

Confidence Level: 95%

Correct

Claude Answer :

Let's approach this problem step by step:

1) First, let's write the system with the linear feedback $u = ax + by$ substituted:

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y + ax + by \end{cases} \quad (108)$$

2) Now, let's compute the Jacobian matrix at the equilibrium point $(0, 0)$:

$$\begin{aligned} J(0, 0) &= \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}_{(0,0)} \\ &= \begin{bmatrix} y & x \\ a & b - 1 \end{bmatrix}_{(0,0)} \\ &= \begin{bmatrix} 0 & 0 \\ a & b - 1 \end{bmatrix} \end{aligned}$$

3) For asymptotic stability, all eigenvalues of the Jacobian matrix at the equilibrium point should have negative real parts. The characteristic equation is:

$$\det(J(0,0) - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 \\ a & b-1-\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - b + 1) = 0$$

4) The eigenvalues are:

$$\lambda_1 = 0 \quad \lambda_2 = b - 1$$

5) For asymptotic stability, we need $b - 1 < 0$, or $b < 1$.

6) However, we also have an eigenvalue at zero, which means the system is not asymptotically stable regardless of the choice of a and b .

Therefore, the linearization about the equilibrium cannot be made asymptotically stable for any choice of constants (a, b) . The system will at best be marginally stable (if $b < 1$), but not asymptotically stable.

Confidence Level: 95%

Correct