

## 1.22 State-Space Representation

Imagine two cars driving on the same road in the same direction one behind the other and trying to go at the same speed. This situation can be described by the linear system

$$\begin{aligned}\dot{v}_1 &= -(v_1 - v_2) + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2\end{aligned}\tag{63}$$

where, for  $i = 1, 2$ ,  $v_i$  is the velocity of car  $i$  and  $f_i$  is an external force (wind, road conditions, etc.) acting on it. The meaning of the above equations is that each car accelerates/decelerates depending on whether it is going slower/faster than the other. Now, suppose that we want to rewrite the above system in the following new coordinates:  $\bar{v}_1 := v_1$  (velocity of car 1) and  $\bar{v}_2 := v_2 - v_1$  (relative velocity of the two cars).

- Write down the differential equations for  $\dot{\bar{v}}_1, \dot{\bar{v}}_2$  in terms of  $\bar{v}_1, \bar{v}_2$  and  $f_1, f_2$ .
- Write down the original system in state-space form  $\dot{v} = Av + Bf$ , the new system in state-space form  $\dot{\bar{v}} = \bar{A}\bar{v} + \bar{B}f$ , the coordinate transformation matrix  $T$  from  $(v_1, v_2)$  to  $(\bar{v}_1, \bar{v}_2)$ , and verify that this transformation matrix  $T$  leads to the same ODE as part (a) if applied to the system in Eq. (63).

**Solution :**

$$\begin{aligned}\dot{v}_1 &= -(v_1 - v_2) + f_1 = \bar{v}_2 + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2 = -\bar{v}_2 + f_2\end{aligned}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{v}} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_v + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_B \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_u$$

(a)

$$\begin{aligned}\dot{\bar{v}}_1 &= \dot{v}_1 = +\bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 &= \dot{v}_2 - \dot{v}_1 = -\bar{v}_2 + f_2 - (\bar{v}_2 + f_1) = -2\bar{v}_2 - f_1 + f_2 \\ \Rightarrow \begin{cases} \dot{\bar{v}}_1 &= \bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 &= -2\bar{v}_2 + f_2 - f_1 \end{cases}\end{aligned}$$

(b)

$$\begin{aligned}\bar{v}_1 &= v_1 \\ \bar{v}_2 &= v_2 - v_1\end{aligned} \Rightarrow \underbrace{\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}}_{\bar{v}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_v$$

$$\begin{aligned}
 \dot{v} &= T\dot{v} = TAv + TBu \\
 \dot{v} &= TAT^{-1}\bar{v} + TBu \\
 &= \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{T^{-1}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_B \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_u \\
 &= \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}}_{\bar{A}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\bar{B}} u \\
 \dot{v}_1 &= \bar{v}_2 + f_1 \\
 \dot{v}_2 &= -2\bar{v}_2 - f_1 + f_2
 \end{aligned}$$

**GPT Answer :**

Given the system:

$$\begin{aligned}
 \dot{v}_1 &= -(v_1 - v_2) + f_1 \\
 \dot{v}_2 &= -(v_2 - v_1) + f_2
 \end{aligned} \tag{64}$$

With the transformation  $\bar{v}_1 := v_1$  and  $\bar{v}_2 := v_2 - v_1$ , the differential equations in terms of  $\bar{v}_1, \bar{v}_2$  and  $f_1, f_2$  are:

$$\begin{aligned}
 \dot{\bar{v}}_1 &= -\bar{v}_2 + f_1 \\
 \dot{\bar{v}}_2 &= -2\bar{v}_2 + (f_2 - f_1)
 \end{aligned} \tag{65}$$

For the original system in state-space form:  $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . For the new system:  $\bar{A} = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}$ ,  $\bar{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ . The transformation matrix  $T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ .

**Confidence Level: 95%**

**Calculation error**

**Claude Answer :**

**(a)** The differential equations for  $\dot{\bar{v}}_1$  and  $\dot{\bar{v}}_2$  in terms of  $\bar{v}_1, \bar{v}_2$ , and  $f_1, f_2$  are:

$$\begin{aligned}
 \dot{\bar{v}}_1 &= \bar{v}_2 + f_1 \\
 \dot{\bar{v}}_2 &= -2\bar{v}_2 + f_2 - f_1
 \end{aligned}$$

**(b)** The original system in state-space form is:

$$\dot{v} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} f$$

The new system in state-space form is:

$$\dot{\bar{v}} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \bar{v} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} f$$

The coordinate transformation matrix  $T$  from  $(v_1, v_2)$  to  $(\bar{v}_1, \bar{v}_2)$  is:

$$T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Applying the transformation matrix  $T$  to the original system in state-space form yields the same ODE as in part (a):

$$\begin{aligned} \dot{\bar{v}} &= T\dot{v} \\ &= T(Av + Bf) \\ &= TAT^{-1}\bar{v} + TBf \\ &= \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \bar{v} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} f \end{aligned}$$

Confidence level: 95/100

**Correct**