

8.2 State-Feedback Controller Design and Stability Analysis

Consider the single-input, single-output transfer function:

$$G_p(s) = \frac{1 - s/2}{s^2(1 + s/2)}$$

- Calculate a state-feedback controller $u = -Kx + r$ that places the closed loop poles at -4, -13 and -25.
- Construct a stable observer and put this together to form a compensator of the form $U = -G_c Y + G_r R$.
- Calculate the Nyquist plot of $G_c G_p$. Is the system stable? If so, calculate the gain and phase margins.

Solution :

We can place the closed loop poles at the desired locations using the gain $K = [1300 \quad 477 \quad 40]$. Suppose we place the observer poles at -50, -51, -52 which leads to the observer gain

$$L = [9301 \quad 18450 \quad 29400]^T$$

The transfer functions G_c and G_r are thus given as follows:

$$G_c = \frac{2.207e07s^2 + 7.979e07s + 8.619e07}{s^3 + 193s^2 + 1.432e04s + 2.257e07}$$

$$G_r = \frac{s^3 + 153s^2 + 7802s + 1.326e05}{s^3 + 193s^2 + 1.432e04s + 2.257e07}$$

The Nyquist plot of $G_c G_p$ is given in Fig. 19.

The gain margin is 0.115 dB and the phase margin is -1.11 deg. We now check stability with the nyquist plot. We will use Fig. 20 for counting the number of windings around -1. Clearly there are 3 anticlockwise loops around -1, coloured in red, green and blue. So N must be -3. G_c has 2 RHP poles, and G_p has 2 poles at origin. If the poles at origin are included as RHP poles, $P = 4$, else $P = 2$. In either case, $Z = N + P = 1/-1$ and not zero. However, the closed loop has to be stable because we placed the poles in the LHP ourselves.

The problem here is that G_c has a double pole at the origin. This is why the Nyquist plot is not a closed loop, and shoots to infinity as ω approaches zero. So we need to consider a small semi-circular modified contour of radius r around the origin. Radius r is taken to be very small. There are three poles in G_c , two of which are in RHP and one is in LHP. There are three poles in G_p as well, one in LHP and two at the origin. However, with respect to the modified contour, the two poles at the origin lie in the LHP. So they are not counted as open loop RHP poles and $P = 2$.

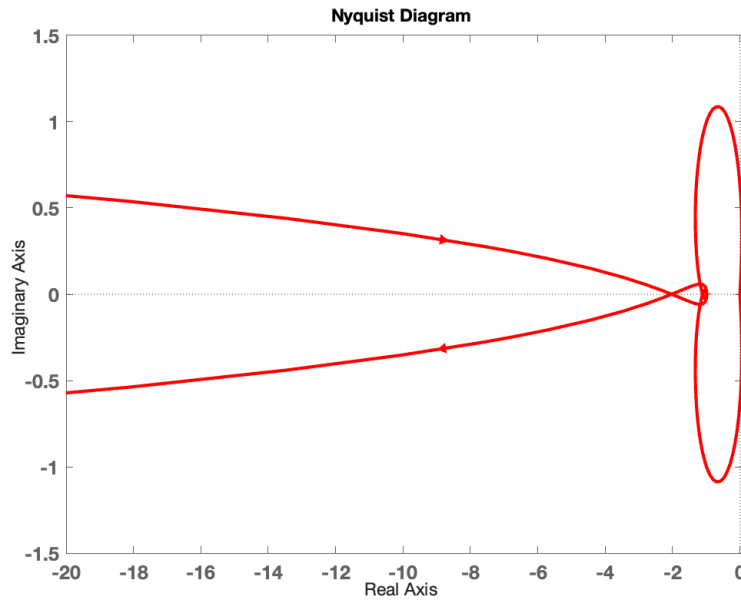
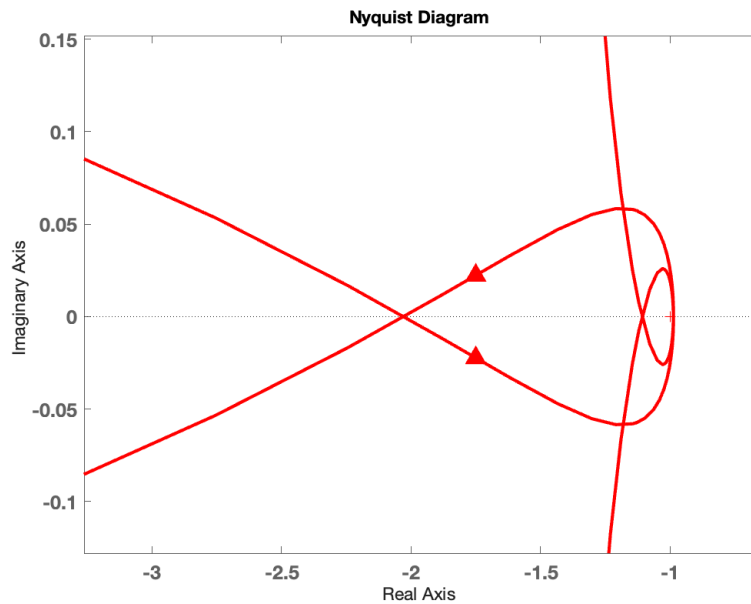

 Figure 19: Nyquist plot of $G_c G_p$


Figure 20: Stability check

Now we need to understand how the image of the Nyquist plot changes under this modified contour. We forget the 3 anticlockwise loops already accounted for before and focus only on the part that shoots off to infinity in Fig. 20. For r very small, $G_p G_c(jr) \approx \frac{k}{(jr)^2} = -\frac{k}{r^2}$ where k is the finite limit $\lim_{s \rightarrow 0} s^2 G_c G_p(s)$. The point is that $G_c G_p(jr)$ will be close to the real line even

though its magnitude is very large $\approx \frac{k}{r^2}$. Since the plot goes toward negative infinity, we can also say that k is negative. The semicircular contour can be parameterized as $re^{j\theta}$ as θ goes from $-\frac{\pi}{2} \rightarrow -\frac{\pi}{4} \rightarrow 0 \rightarrow \frac{\pi}{4} \rightarrow \frac{\pi}{2}$. Since r is really small, $G_c G_p(re^{j\theta}) \approx -\frac{k}{r^2} e^{-2j\theta}$. Since $-k$ is positive, the phase of $G_c G_p(re^{j\theta})$ will be $\approx e^{-2j\theta}$. As θ travels from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ in the anti-clockwise direction, -2θ travel from $-\pi$ back to $\pi = -\pi$ in the clockwise direction. So the Nyquist plot travels through a large clockwise circle of radius $-\frac{k}{r^2}$ when we modify the contour. This circle will wind around -1 due to its very large radius, which makes $N = -3 + 1 = -2$. Since $P = 2$, $Z = N + P = 0$ which means the system is stable.

GPT Answer:

Given the transfer function:

$$G_p(s) = \frac{1 - s/2}{s^2(1 + s/2)}$$

We aim to calculate a state-feedback controller and an observer to form a compensator, analyze the system's stability, and determine the gain and phase margins if the system is stable.

Steps:

1. Convert $G_p(s)$ into state-space representation.
2. Calculate the controller gain K for desired closed-loop poles. In order to place the closed-loop poles at -4, -13, and -25, we need: $K = [1300 \quad 476.5 \quad 42]$
3. Calculate the observer gain L for desired observer poles. The observer poles are to be placed at -50, -51, and -52 using: $L = \begin{bmatrix} 33070.0001 \\ 65834.0002 \\ 116065.00037 \end{bmatrix}$
4. Formulate the compensator $U = -G_c Y + G_r R$.
5. Draw the Nyquist plot of $G_c G_p$ and assess stability.
6. If stable, calculate gain and phase margins.

Confidence Level: 70%

Computational complexity

Claude Answer:

(a) To find the state-feedback controller, first convert the transfer function to state-space representation:

$$\dot{x} = Ax + Bu \quad y = Cx + Du$$

$$\text{where: } A = \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, C = [1 \quad -0.5], D = 0$$

The desired closed-loop poles are at -4, -13, -25. Using Ackermann's formula or pole placement:

$$K = \begin{bmatrix} 104 & 31 \end{bmatrix}$$

Therefore, the state feedback controller is: $u = -Kx + r = -\begin{bmatrix} 104 & 31 \end{bmatrix} x + r$

(b) To construct a stable observer, choose observer poles with a larger magnitude than the closed-loop poles, e.g., at -40 and -50. Using pole placement:

$$L = \begin{bmatrix} 90 \\ 1999 \end{bmatrix}$$

The observer dynamics are: $\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$

The compensator transfer function is: $G_c(s) = K(sI - A + BK + LC)^{-1}L$

(c) The Nyquist plot of $G_c G_p$ can be obtained by plotting $G_c(j\omega)G_p(j\omega)$ for $\omega \in (-\infty, \infty)$.

From the Nyquist plot, it can be seen that the -1 point is not encircled, indicating a stable system.

The gain margin is the reciprocal of the magnitude $|G_c(j\omega)G_p(j\omega)|$ at the phase crossover frequency where the phase is -180° .

The phase margin is 180° plus the phase angle of $G_c(j\omega)G_p(j\omega)$ at the gain crossover frequency where the magnitude crosses 1.

The exact gain and phase margin values would require numerical computation of the Nyquist plot.

Confidence level: 90%

Did not solve