

5.18 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \quad y(0) = 0$$

with a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. Under the constraint that $K_p \in [-0.2, 1]$, find K_p such that we minimize the steady-state error due to the unit step reference command. What is the time constant of the closed-loop system for this gain?

Solution :

For $K_p \in [-0.2, 1]$ it is clear that steady-state error to a unit-step reference input is maximized at the left end-point of the interval and minimized at the right end-point. To see why derive the transfer function:

$$\frac{Y(s)}{R(s)} = \frac{2K_p}{s + 2K_p + 0.5}$$

and we see using Final Value Theorem that for higher K_p the steady-state value approaches 1. Hence, we choose $K_p = 1$ to satisfy the constraint. The corresponding transfer function is given as follows:

$$H(s) = \frac{2}{s + 5/2} \implies \tau = \frac{2}{5}$$

GPT Answer :

The system is described by:

$$\dot{y} = -0.5y + 2u \quad y(0) = 0$$

with a proportional control law:

$$u(t) = K_p(r(t) - y(t))$$

where $r(t)$ is the reference command. Under the constraint that $K_p \in [-0.2, 1]$, the optimal K_p to minimize the steady-state error for a unit step input is found to be 1.

The closed-loop transfer function with $K_p = 1$ is:

$$T(s) = \frac{2}{s + 2.5}$$

indicating a time constant of the system as $\tau = 1/2.5$.

Confidence Level: 90%

Correct

Claude Answer :

Let's approach this problem step by step:

1) First, let's find the transfer function of the closed-loop system. The plant is given by:

$$G(s) = \frac{2}{s+0.5}$$

With the proportional control law, the closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{K_p G(s)}{1 + K_p G(s)} = \frac{2K_p}{s+0.5+2K_p}$$

2) For a unit step input, $R(s) = \frac{1}{s}$, the steady-state output is given by the Final Value Theorem:

$$y_{ss} = \lim_{s \rightarrow 0} s \cdot \frac{Y(s)}{R(s)} \cdot \frac{1}{s} = \lim_{s \rightarrow 0} \frac{2K_p}{s+0.5+2K_p} = \frac{2K_p}{0.5+2K_p}$$

3) The steady-state error is the difference between the reference input (which is 1 for a unit step) and the steady-state output:

$$e_{ss} = 1 - y_{ss} = 1 - \frac{2K_p}{0.5+2K_p} = \frac{0.5}{0.5+2K_p}$$

4) To minimize the steady-state error, we need to maximize K_p within the given constraint. The maximum allowed value for K_p is 1. Therefore, we choose:

$$K_p = 1$$

5) With $K_p = 1$, the closed-loop transfer function becomes:

$$\frac{Y(s)}{R(s)} = \frac{2}{s+2.5}$$

This is a first-order system with a time constant $\tau = \frac{1}{2.5} = 0.4$.

Therefore, to minimize the steady-state error due to a unit step reference command under the given constraint, we choose $K_p = 1$. The time constant of the resulting closed-loop system is 0.4.

Confidence Level: 95%

Correct