

3.8 Trajectory

Show that the phase plane trajectory of the solution of the differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

with initial conditions $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = 1$ is a circle of unit radius centered at the origin.

Solution :

Letting $x = x_1$ and $x_2 = \frac{dx_1}{dt}$, we obtain the pair of equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2, & x_1(0) &= 0 \\ \frac{dx_2}{dt} &= -x_1, & x_2(0) &= 1\end{aligned}$$

We eliminate time as the independent variable by writing

$$\frac{dx_1}{dx_2} = \frac{-x_2}{x_1}$$

or

$$x_1 dx_1 + x_2 dx_2 = 0$$

Integrating this equation for the given initial conditions, we obtain

$$x_1^2 + x_2^2 = 1$$

which is the equation of a circle of unit radius centered at the origin.

GPT Answer:

To solve this problem, we first recognize that the given differential equation is a simple harmonic oscillator equation, which is a second-order linear differential equation. The general solution to this differential equation can be expressed in terms of trigonometric functions. Given the equation:

$$\frac{d^2x}{dt^2} + x = 0$$

The characteristic equation of this differential equation is $r^2 + 1 = 0$, leading to the roots $r = \pm i$. Therefore, the general solution to the differential equation is:

$$x(t) = A \cos(t) + B \sin(t)$$

where A and B are constants determined by the initial conditions. Given the initial conditions $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = 1$, we can solve for A and B .

For $x(0) = 0$:

$$0 = A \cos(0) + B \sin(0)$$

$$0 = A$$

Thus, $A = 0$.

For $\left. \frac{dx}{dt} \right|_{t=0} = 1$: Taking the derivative of $x(t)$ gives:

$$\frac{dx}{dt} = -A \sin(t) + B \cos(t)$$

Substituting $t = 0$:

$$1 = -A \sin(0) + B \cos(0)$$

$$1 = B$$

Thus, $B = 1$.

So the specific solution to the differential equation given the initial conditions is:

$$x(t) = \sin(t)$$

$$\frac{dx}{dt} = \cos(t)$$

To show that the phase plane trajectory is a circle of unit radius centered at the origin, we can use the expressions for $x(t)$ and its derivative. The phase plane is described by plotting $\frac{dx}{dt}$ against x . Substituting the expressions for $x(t)$ and $\frac{dx}{dt}$ into the equation for a circle $x^2 + y^2 = 1$, we get:

$$x^2 + \left(\frac{dx}{dt} \right)^2 = \sin^2(t) + \cos^2(t)$$

Using the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$, we find:

$$x^2 + \left(\frac{dx}{dt} \right)^2 = 1$$

This equation represents a circle of unit radius centered at the origin in the phase plane, which confirms that the phase plane trajectory of the solution is indeed a circle of unit radius centered at the origin.

Confidence level: 100%

Correct

Claude Answer:

To show that the phase plane trajectory is a circle of unit radius centered at the origin, let's follow these steps:

1) First, let's solve the differential equation. The characteristic equation is:

$$r^2 + 1 = 0$$

which has roots $r = \pm i$. Therefore, the general solution is:

$$x(t) = c_1 \cos t + c_2 \sin t$$

2) Now, let's apply the initial conditions to find the specific solution.

$$x(0) = 0 \text{ gives:}$$

$$0 = c_1 \cos 0 + c_2 \sin 0 = c_1$$

$$\text{So, } c_1 = 0.$$

$$\left. \frac{dx}{dt} \right|_{t=0} = 1 \text{ gives:}$$

$$1 = -c_1 \sin 0 + c_2 \cos 0 = c_2$$

$$\text{So, } c_2 = 1.$$

Therefore, the specific solution is:

$$x(t) = \sin t$$

3) In the phase plane, we plot x against $\frac{dx}{dt}$. Let's call $\frac{dx}{dt} = y$. Then:

$$y = \frac{dx}{dt} = \cos t$$

4) Now, let's consider the equation of a circle with radius 1 centered at the origin:

$$x^2 + y^2 = 1$$

5) Substituting our expressions for x and y into this equation:

$$(\sin t)^2 + (\cos t)^2 = 1$$

This is the fundamental trigonometric identity and is always true.

6) Therefore, as t varies, the point $(x, y) = (\sin t, \cos t)$ traces out a circle of unit radius centered at the origin in the phase plane.

Thus, we have shown that the phase plane trajectory of the solution to the given differential equation with the given initial conditions is indeed a circle of unit radius centered at the origin.

Confidence level: 95%

Correct