

7.3 Minimizing Performance Index

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 + u \\ y &= 2x_1 + x_2\end{aligned}\tag{270}$$

and suppose that the control objective is to minimize the performance $\int_0^\infty [\rho y^2(t) + u^2(t)] dt$ $\rho > 0$.

- See why in the limit as $\rho \rightarrow 0$ (“expensive control” case), the optimal closed-loop poles become mirror images of the open-loop poles across the imaginary axis.
- See why in the limit as $\rho \rightarrow \infty$ (“cheap control” case), one optimal closed-loop pole cancels the open-loop zero and the other moves off to $-\infty$.

Solution :

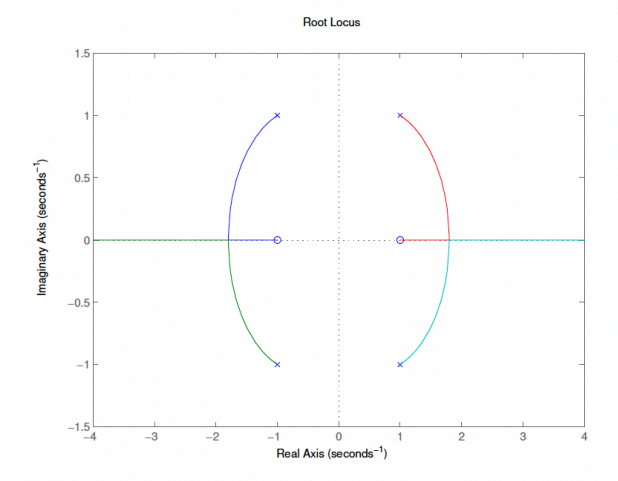


Figure 14: Root locus of the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 1 \end{bmatrix} x\end{aligned}$$

- open-loop poles: $1 \pm j$ from Fig. 14, we see that for every ρ , we assign the LHP pole for the closed loop system (due to symmetry vs imaginary axis) For $\rho = 0$, the CL-optimal poles

would be $-1 \pm j$ because the OL-poles are on RHP, it would be the mirror images of OL-poles across the imaginary axis (Note: if the OL-poles were on LHP, then CL-poles would be equal to OL poles in this case).

- (b) For $\rho \rightarrow -\infty$, one pole moves to zero, the zero is on LHP, so it cancels the pole, and the other one goes to $-\infty$.

GPT Answer :

The system under consideration and the performance index to be minimized can be represented as follows:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 + u \\ y &= 2x_1 + x_2\end{aligned}\tag{271}$$

With the objective to minimize:

$$\int_0^\infty [\rho y^2(t) + u^2(t)] dt \quad \rho > 0$$

The analysis involves understanding the root locus as a function of ρ and exploring the behavior of the system as $\rho \rightarrow 0$ (expensive control) and $\rho \rightarrow \infty$ (cheap control).

- (a) As $\rho \rightarrow 0$: The system prioritizes minimal control effort, leading to closed-loop poles that mirror the open-loop poles across the imaginary axis.
- (b) As $\rho \rightarrow \infty$: The system allows for higher control effort for minimizing output variance, resulting in one pole canceling the open-loop zero while the other moves towards $-\infty$.

Confidence Level: 70%

Correct

Claude Answer :

Let's approach this problem step by step:

- (a) To show the locations of the optimal closed-loop poles graphically as ρ varies, we need to draw the symmetric root locus. First, let's find the open-loop transfer function.

From the state-space representation, we can write: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = [2 \quad 1]$, $D = 0$

The open-loop transfer function is: $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$

$$G(s) = \frac{s+2}{s^2-2s-1} = \frac{s+2}{(s-1)^2-2}$$

The open-loop poles are at $s = 1 \pm \sqrt{2}$, and the zero is at $s = -2$.

The symmetric root locus starts from the open-loop poles and ends at the open-loop zeros (or at infinity if the number of poles is greater than the number of zeros). As ρ varies from 0 to ∞ , the optimal closed-loop poles move along the symmetric root locus.

(a) In the limit as $\rho \rightarrow 0$ ("expensive control" case), the control effort is penalized heavily, and the optimal controller tends to use minimal control effort. In this case, the optimal closed-loop poles become the mirror images of the open-loop poles about the imaginary axis. This is because the open-loop system is unstable, and the optimal controller tries to stabilize the system with minimal control effort.

(b) In the limit as $\rho \rightarrow \infty$ ("cheap control" case), the control effort is not penalized significantly, and the optimal controller can use large control efforts to minimize the output. In this case, one of the optimal closed-loop poles cancels the open-loop zero at $s = -2$, and the other pole moves towards negative infinity. This is because the optimal controller tries to make the closed-loop system as fast as possible while minimizing the output.

In summary, the symmetric root locus shows the locations of the optimal closed-loop poles as ρ varies. In the "expensive control" case, the poles become mirror images of the open-loop poles, while in the "cheap control" case, one pole cancels the open-loop zero, and the other moves towards negative infinity.

Confidence Level: 95%

Correct