

1.8 State-Space Representation of an ODE

Consider the triple integrator:

$$\frac{d^3y}{dt^3} = u \quad (35)$$

- (a) Obtain a state-space representation of the form $\dot{x} = Ax + Bu$ for this system.
- (b) Compute e^{At} and use it to obtain the solution with $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ and $u(t) = 0$

Solution :

- (a) To obtain a state-space representation of the form $\dot{x} = Ax + Bu$ for this system, we define:

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{y}, \\ x_3 &= \ddot{y}. \end{aligned}$$

Thus, the state-space model is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- (b) To compute e^{At} and obtain the solution with initial conditions $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ and $u(t) = 0$, we find:

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Given $x(0) = [1 \ 1 \ 1]^T$, the solution $x(t)$ is:

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 + t + \frac{t^2}{2} \\ 1 + t \\ 1 \end{bmatrix} \quad (36)$$

GPT Answer :

Given the system:

$$\frac{d^3y}{dt^3} = u, \quad (37)$$

we define the state variables as $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = \ddot{y}$. The state-space representation becomes:

$$\dot{x} = Ax + Bu, \quad (38)$$

with $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

To compute e^{At} and find the solution for $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ and $u(t) = 0$, we have:

$$e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad (39)$$

$$x(t) = \begin{pmatrix} 1 + t + \frac{t^2}{2} \\ 1 + t \\ 1 \end{pmatrix}. \quad (40)$$

Hence, $y(t) = 1 + t + \frac{t^2}{2}$.

Confidence level: 95%

Correct

Claude Answer :

(a) Let $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = \ddot{y}$. Then a state-space representation is:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(b) The matrix exponential e^{At} is:

$$e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Using the matrix exponential, the solution with the given initial conditions and $u(t) = 0$ is:

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + t + \frac{1}{2}t^2 \\ 1 + t \\ 1 \end{bmatrix}$$

Confidence level: 95/100

Correct