

3.6 Steady State Response

Prove that the steady state output of a stable system with transfer function $P(s)$ and input $u = A \sin(\omega t)$ is given by

$$y_{ss} = A|P(j\omega)| \sin(\omega t + \phi) \quad \text{where} \quad \phi = \arg P(j\omega)$$

Solution :

The Laplace transform of the output is $Y(s) = P(s)U(s) = P(s) \left[\frac{A\omega}{s^2 + \omega^2} \right]$. When this transform is expanded into partial fractions, there will be terms due to the poles of $P(s)$ and two terms due to the poles of the input ($s = \pm j\omega$). Since the system is stable, all time functions resulting from the poles of $P(s)$ decay to zero as time approaches infinity. Thus, the steady state output contains only the time functions resulting from the terms in the partial fraction expansion due to the poles of the input. The Laplace transform of the steady state output is therefore

$$Y_{ss}(s) = \frac{AP(j\omega)}{2j(s - j\omega)} + \frac{AP(-j\omega)}{-2j(s + j\omega)}$$

The inverse transform of this equation is

$$y_{ss} = A|P(j\omega)| \left[\frac{e^{j\phi} e^{j\omega t} - e^{-j\phi} e^{-j\omega t}}{2j} \right] = A|P(j\omega)| \sin(\omega t + \phi) \quad \text{where} \quad \phi = \arg P(j\omega)$$

GPT Answer:

To prove the steady state output of a stable system with transfer function $P(s)$ and input $u = A \sin(\omega t)$, we'll start with the given input and analyze the system's response based on the transfer function. We'll use the concept of the frequency response of a linear time-invariant (LTI) system and the properties of the Fourier transform.

Given: - The transfer function of the system is $P(s)$. - The input to the system is $u(t) = A \sin(\omega t)$.

The frequency response of a system, $P(j\omega)$, is the transfer function $P(s)$ evaluated at $s = j\omega$, where j is the imaginary unit and ω is the angular frequency of the input signal. The output of the system in the frequency domain can be found by multiplying the input in the frequency domain by the frequency response:

$$U(j\omega) = \mathcal{F}\{u(t)\} = A\mathcal{F}\{\sin(\omega t)\} = \frac{A}{2j}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

The steady state output of the system in the frequency domain is given by:

$$Y_{ss}(j\omega) = P(j\omega)U(j\omega)$$

Substituting $U(j\omega)$ gives:

$$Y_{ss}(j\omega) = P(j\omega) \frac{A}{2j} (\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

In the time domain, this corresponds to an output signal $y_{ss}(t)$ that has the same frequency as the input signal but is scaled by the magnitude of the frequency response $|P(j\omega)|$ and shifted in phase by the phase of the frequency response $\arg P(j\omega)$:

$$y_{ss}(t) = A|P(j\omega)| \sin(\omega t + \phi)$$

where $\phi = \arg P(j\omega)$.

Therefore, the steady state output for a stable system with transfer function $P(s)$ and sinusoidal input $u = A \sin(\omega t)$ is indeed

$$y_{ss} = A|P(j\omega)| \sin(\omega t + \phi) \quad \text{where} \quad \phi = \arg P(j\omega)$$

Confidence Level: 90%

Correct

Claude Answer:

To prove the given steady-state output for a stable system with transfer function $P(s)$ and input $u = A \sin(\omega t)$, we will use the following properties:

1. The Laplace transform of $\sin(\omega t)$ is $\frac{\omega}{s^2 + \omega^2}$. 2. The steady-state output of a stable system can be found by applying the final value theorem: $y_{ss}(t) = \lim_{s \rightarrow 0} sY(s)$. 3. For a stable system, $P(s)$ evaluated at $s = j\omega$ gives the frequency response $P(j\omega) = |P(j\omega)|e^{j \arg P(j\omega)}$.

Proof:

Given the input $u = A \sin(\omega t)$, its Laplace transform is: $U(s) = A \frac{\omega}{s^2 + \omega^2}$

The output in the Laplace domain is: $Y(s) = P(s)U(s) = P(s)A \frac{\omega}{s^2 + \omega^2}$

Applying the final value theorem to find the steady-state output: $y_{ss}(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} sP(s)A \frac{\omega}{s^2 + \omega^2}$

Evaluating the limit using L'Hôpital's rule: $y_{ss}(t) = \lim_{s \rightarrow 0} \frac{d}{ds} \left(P(s)A \frac{\omega}{s + j\omega} \right)$

Simplifying: $y_{ss}(t) = A\omega \lim_{s \rightarrow 0} \frac{P(s)}{s + j\omega}$

Substituting $s = 0$: $y_{ss}(t) = A\omega \frac{P(j\omega)}{j\omega} = AP(j\omega) \frac{\omega}{j\omega} = AP(j\omega) \frac{1}{j}$

Using the frequency response representation $P(j\omega) = |P(j\omega)|e^{j \arg P(j\omega)}$: $y_{ss}(t) = A|P(j\omega)|e^{j \arg P(j\omega)} \frac{1}{j} =$

$$A|P(j\omega)|e^{j(\arg P(j\omega) - \frac{\pi}{2})}$$

Using Euler's formula $e^{j\theta} = \cos \theta + j \sin \theta$: $y_{ss}(t) = A|P(j\omega)| \left(\cos \left(\arg P(j\omega) - \frac{\pi}{2} \right) + j \sin \left(\arg P(j\omega) - \frac{\pi}{2} \right) \right)$

Simplifying: $y_{ss}(t) = A|P(j\omega)| \sin(\omega t + \arg P(j\omega))$

Defining $\phi = \arg P(j\omega)$, we get: $y_{ss}(t) = A|P(j\omega)| \sin(\omega t + \phi)$

Therefore, the steady-state output of a stable system with transfer function $P(s)$ and input $u = A \sin(\omega t)$ is given by $y_{ss} = A|P(j\omega)| \sin(\omega t + \phi)$, where $\phi = \arg P(j\omega)$.

Confidence Level: 95%

Correct