

3.3 Free Response

Find the free response of the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 4y = u \quad (133)$$

with initial conditions $y(0) = 1$, $(dy/dt)|_{t=0} = 0$ and $(d^2y/dt^2)|_{t=0} = -1$

Solution :

Fundamental set for this equation is e^{-3t} , $e^{-t} \cos t$, $e^{-t} \sin t$. Hence the free response can be written as

$$y_a(t) = c_1 e^{-2t} + c_2 e^{-t} \cos t + c_3 e^{-t} \sin t \quad (134)$$

The initial conditions provide the following set of algebraic equations for c_1, c_2, c_3 :

$$y_a(0) = c_1 + c_2 = 1 \quad \frac{dy_a}{dt}|_{t=0} = -2c_1 - c_2 + c_3 = 0 \quad \frac{d^2y_a}{dt^2}|_{t=0} = 4c_1 - 2c_3 = -1 \quad (135)$$

from which $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{3}{2}$. Therefore the free response is

$$y_a(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t} \cos t + \frac{3}{2}e^{-t} \sin t \quad (136)$$

GPT Answer:

The free response of the differential equation is given by:

$$y_h(t) = \frac{1}{2}e^{-2t} + e^{-t} \left(\frac{1}{2} \cos(t) + \frac{3}{2} \sin(t) \right)$$

Confidence level: 95%

Correct

Claude Answer:

Given:

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 4y = u \quad (137)$$

Initial conditions: $y(0) = 1$, $\frac{dy}{dt}|_{t=0} = 0$, and $\frac{d^2y}{dt^2}|_{t=0} = -1$

Step 1: Find the characteristic equation by setting $u = 0$ and assuming a solution of the form $y = e^{rt}$.

$$r^3 + 4r^2 + 6r + 4 = 0 \quad (138)$$

Step 2: Solve the characteristic equation.

$$(r + 2)(r^2 + 2r + 2) = 0 \quad (139)$$

$$(r + 2)(r + 1)^2 = 0 \quad (140)$$

The roots are $r_1 = -2$ and $r_2 = r_3 = -1$ (a double root).

Step 3: Write the general solution using the roots.

$$y(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 t e^{-t} \quad (141)$$

Step 4: Apply the initial conditions to find the values of c_1 , c_2 , and c_3 .

$$y(0) = c_1 + c_2 = 1 \quad (142)$$

$$\left. \frac{dy}{dt} \right|_{t=0} = -2c_1 - c_2 + c_3 = 0 \quad (143)$$

$$\left. \frac{d^2y}{dt^2} \right|_{t=0} = 4c_1 + c_2 - 2c_3 = -1 \quad (144)$$

Solving the system of equations, we get $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, and $c_3 = -\frac{1}{2}$.

Therefore, the free response of the differential equation with the given initial conditions is:

$$y(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t} \quad (145)$$

Confidence level: 99%

Calculation Error