

10.7 Controllability

Consider the following system:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ b \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

What value of b leads to the loss of controllability in the system?

Solution :

System's transfer function is given by:

$$G(s) = C(sI - A)^{-1}B = \frac{s + 2 + b(s - 1)}{(s + 2)(s - 1)}$$

We can see that setting $b = 0$ leads to pole-zero cancellation and loss of controllability in the system. Alternatively, it can also be checked by:

$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & 1 \\ b & -2b \end{pmatrix}$$

and for $b = 0$, $\text{rank}(\mathcal{C}(A, B)) = 1 < 2$.

GPT Answer :

To analyze the controllability of the given system, we can use the controllability matrix. The system is described by:

$$\dot{x} = Ax + Bu \quad y = Cx$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The controllability matrix, \mathcal{C} , is given by:

$$\mathcal{C} = [B, AB, A^2B, \dots, A^{n-1}B]$$

where n is the number of states. For our 2-state system ($n = 2$), the controllability matrix simplifies to:

$$\mathcal{C} = [B, AB].$$

Let's calculate AB and then construct the controllability matrix \mathcal{C} .

$$AB = A \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -2b \end{bmatrix}.$$

Thus,

$$\mathcal{C} = \left[\begin{bmatrix} 1 \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ -2b \end{bmatrix} \right] = \begin{bmatrix} 1 & 1 \\ b & -2b \end{bmatrix}.$$

A system is controllable if and only if its controllability matrix has full rank. In this case, \mathcal{C} must have rank 2, since our system has 2 states.

The rank of \mathcal{C} is less than 2 if its columns are linearly dependent. The columns of \mathcal{C} are linearly dependent if there exists a scalar λ such that:

$$\begin{bmatrix} 1 \\ b \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2b \end{bmatrix}.$$

From the first component, $\lambda = 1$. Applying this to the second component gives:

$$b = -2b,$$

which leads to $b = 0$ as the condition for linear dependence of the columns of \mathcal{C} .

Therefore, the value of b that leads to the loss of controllability in the system is $b = 0$.

Confidence Level: 95%

Correct

Claude Answer:

To determine the value of b that leads to a loss of controllability, we need to check the rank of the controllability matrix.

For a linear time-invariant system: $\dot{x} = Ax + Bu$ $y = Cx$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$, the system is controllable if and only if the controllability matrix $\mathcal{C} = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$ has full row rank, i.e., rank n .

In this case, we have:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad C = [1 \ 1]$$

The controllability matrix is:

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ b & -2b \end{bmatrix}$$

For the system to be controllable, the rank of \mathcal{C} should be equal to $n = 2$. This means that the determinant of \mathcal{C} should be non-zero.

$$\det(\mathcal{C}) = (-2b) - (1 \cdot b) = -3b$$

The determinant is zero when $b = 0$. Therefore, the system loses controllability when $b = 0$.

Confidence level: 95%

Correct