

## 1.24 Characterization of Second-Order Systems

Consider the following mass-spring system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where  $m$  is the mass,  $k$  is the spring constant and  $\rho$  is the friction coefficient. Find the values of  $c_1$  and  $c_2$  to guarantee that the transfer function of the resulting system has the form of the standard 2nd-order system. Write down the expressions for the parameters  $\zeta$  and  $\omega_n$  in terms of  $k$ ,  $\rho$ , and  $m$ .

### Solution:

The given system can be transformed into its transfer function as follows:

$$C(sI - A)^{-1}B = \frac{\frac{c_2}{m}s + \frac{c_1}{m}}{s^2 + \frac{\rho}{m}s + \frac{k}{m}}$$

Comparison with the transfer function of a standard 2nd-order system yields:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\frac{c_2}{m}s + \frac{c_1}{m}}{s^2 + \frac{\rho}{m}s + \frac{k}{m}}$$

which leads to the following result:

$$\frac{c_2}{m} = 0, \frac{c_1}{m} = \frac{k}{m} \implies c_1 = k, c_2 = 0$$

Additionally, we get the following expressions for  $\zeta$  and  $\omega_n$ :

$$2\zeta\omega_n = \frac{\rho}{m}, \omega_n = \frac{k}{m} \implies \omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{\rho}{2\sqrt{km}}$$

### GPT Answer :

To guarantee that the transfer function of the given mass-spring system has the form of the standard 2nd-order system, we must first express the system in terms of its transfer function and then manipulate the coefficients to match the standard form.

The given mass-spring system can be written in state-space form as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \text{with} \quad y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$

The transfer function  $H(s)$  of a system defined by  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$ , without direct feedthrough (assuming  $\mathbf{D} = 0$ ), is given by:

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

Substituting  $\mathbf{A}$  and  $\mathbf{B}$  into this equation, we get:

$$H(s) = [c_1 \quad c_2] \left( s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$H(s) = [c_1 \quad c_2] \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{\rho}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

For the system to have the form of a standard 2nd-order system, the output must be proportional to the displacement  $x_1$ , which implies choosing  $c_1 = 1$  and  $c_2 = 0$ . This choice simplifies the transfer function to a form where the coefficients of the denominator match the standard 2nd-order system form,  $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ .

After some algebra, the transfer function simplifies to:

$$H(s) = \frac{\frac{1}{m}}{s^2 + \frac{\rho}{m}s + \frac{k}{m}}$$

Comparing this with the standard form, we get:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{\rho}{2\sqrt{km}}$$

Thus, the values of  $c_1$  and  $c_2$  to guarantee the transfer function has the standard 2nd-order system form are  $c_1 = 1$  and  $c_2 = 0$ , and the expressions for the parameters  $\zeta$  and  $\omega_n$  in terms of  $k$ ,  $\rho$ , and  $m$  are as derived above.

**Confidence Level: 100%**

**Calculation Error**

**Claude Answer**

Given: The standard 2nd-order system transfer function is:  $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Comparing the coefficients of the derived transfer function with the standard 2nd-order system:  
 $c_1 + c_2 s = \omega_n^2$

This means:  $c_1 = \omega_n^2$   $c_2 = 0$

Comparing the denominators:  $ms^2 + \rho s + k = s^2 + 2\zeta\omega_n s + \omega_n^2$

Equating the coefficients:  $m = 1$   $\rho = 2\zeta\omega_n$   $k = \omega_n^2$

Expressing  $\zeta$  and  $\omega_n$  in terms of  $k$ ,  $\rho$ , and  $m$ :  $\omega_n = \sqrt{k}$  (since  $m = 1$ )  $\zeta = \frac{\rho}{2\omega_n} = \frac{\rho}{2\sqrt{k}}$

Therefore, to guarantee that the transfer function of the given system has the form of the standard 2nd-order system, the values of  $c_1$  and  $c_2$  should be:  $c_1 = k$   $c_2 = 0$

The parameters  $\zeta$  and  $\omega_n$  can be expressed in terms of  $k$ ,  $\rho$ , and  $m$  as:  $\omega_n = \sqrt{k}$   $\zeta = \frac{\rho}{2\sqrt{k}}$

**Correct**