2.4 Pole Assignment

Consider a nominal model given by

$$G_o(s) = \frac{1}{(s+1)^2} \tag{90}$$

Further assume that we need zero steady state error at zero frequency. We want to specify a third order polynomial $A_{cl}(s)$. From our analysis of the pole assignment methodology, we know that not every third order $A_{cl}(s)$ can achieve stable closed loop. Find the family of stable third degree polynomials that yields a stable closed loop with zero steady state error at d.c. Define the closed-loop system's characteristic equation as $A_{cl}(s) = s^3+a_2s^2+a_1s+a_0$. Determine the conditions for the coefficients a_2, a_1, and a_0 such that the closed-loop system is stable and achieves the required steady state error characteristics.

Solution :
Define

$$A_{cl}(s) = s^3 + a_2s^2 + a_1s + a_0 \tag{91}$$

Then, the degree of $L(s)$ must be one and, to satisfy the steady state requirement, it must have integration. Hence the controller has the form

$$C(s) = \frac{p_1s + p_0}{s} \tag{92}$$

The pole assignment equation is

$$A_o(s)L(s) + B_o(s)P(s) = s^3 + a_2s^2 + a_1s + a_0 \tag{93}$$

$$s(s+1)^2 + (p_1s + p_0) = s^3 + a_2s^2 + a_1s + a_0 \tag{94}$$

$$s^3 + 2s^2 + (p_1 + 1)s + p_0 = s^3 + a_2s^2 + a_1s + a_0 \tag{95}$$

From this we see that $a_2 = 2$ assures integration in the controller. Furthermore, since $A_{cl}(s)$ has to be stable, conditions to satisfy this requirement can be obtained using Routh's Algorithm applied to the polynomial $A_{cl}(s) = s^3 + 2s^2 + a_1s + a_0$. The algorithm yields A_{cl} stable if and only if

$$2a_1 > a_0 \tag{96}$$

$$a_0 > 0 \tag{97}$$

Finally, the family of polynomials satisfying both, stability and integration, is

$$\mathcal{A} = \{s^3 + 2s^2 + a_1s + a_0 \in \mathbb{R}^3[s] : a_0 > 0, a_1 > 0, 2a_1 - a_0 > 0\} \tag{98}$$

where $\mathbb{R}^3[s]$ is the ring of polynomials in s of degree less or equal to three, with real coefficients.
GPT Answer:
Given the nominal model

$$G_o(s) = \frac{1}{(s+1)^2} \tag{99}$$

To achieve zero steady state error at zero frequency, we specify the closed-loop system's characteristic equation as:

$$A_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{100}$$

The family of stable third degree polynomials that yields a stable closed loop with zero steady state error at d.c. is defined by:

$$A_{cl}(s) = s^3 + a_2 s^2 + a_1 s \tag{101}$$

where the conditions for the coefficients are $a_2 > 0, a_1 > 0$ and $a_0 = 0$ to ensure stability and achieve the required steady state error characteristics.

Confidence Level: 90%

Calculation Error

Claude Answer:

Given:

- The open-loop transfer function is $G_o(s) = \frac{1}{(s+1)^2}$
- The closed-loop characteristic equation is $A_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0$

Consider a PI controller with the transfer function:

$$G_c(s) = K_p + \frac{K_i}{s}$$

The closed-loop transfer function is:

$$G_{cl}(s) = \frac{G_c(s)G_o(s)}{1 + G_c(s)G_o(s)}$$

Substituting the expressions for $G_c(s)$ and $G_o(s)$:

$$G_{cl}(s) = \frac{(K_p s + K_i)}{s(s+1)^2 + (K_p s + K_i)}$$

The closed-loop characteristic equation is the denominator of $G_{cl}(s)$:

$$A_{cl}(s) = s(s+1)^2 + (K_p s + K_i)$$

$$A_{cl}(s) = s^3 + 2s^2 + s + K_p s + K_i$$

Comparing the coefficients of the closed-loop characteristic equation with the given equation:

$$s^3 + 2s^2 + s + K_p s + K_i = s^3 + a_2 s^2 + a_1 s + a_0$$

We obtain the following conditions:

- $a_2 = 2$
- $a_1 = 1 + K_p$
- $a_0 = K_i$

For the closed-loop system to be stable, the coefficients must satisfy the Routh-Hurwitz stability criterion. The Routh array for a third-order system is:

$$s^3 : 1 \quad a_1$$

$$s^2 : a_2 \quad a_0$$

$$s^1 : \frac{a_1 a_2 - a_0}{a_2} \quad 0$$

$$s^0 : a_0$$

For stability, all elements in the first column of the Routh array must be positive. The stability conditions are:

- $a_2 > 0$, which is always true since $a_2 = 2$
- $\frac{a_1 a_2 - a_0}{a_2} > 0 \Rightarrow a_1 a_2 > a_0$
- $a_0 > 0$

Substituting the expressions for a_1 and a_0:

- $2 + 2K_p > K_i$
- $K_i > 0$

In summary, the conditions for the coefficients are:

- $a_2 = 2$
- $a_1 = 1 + K_p$, where $2 + 2K_p > K_i$
- $a_0 = K_i$, where $K_i > 0$

Any choice of K_p and K_i satisfying these conditions will result in a stable closed-loop system with zero steady-state error at DC.

Confidence Level: 95%
Correct