Listed below are the comprehensive topics from which we have extracted questions, covering various aspects of control theory, to compile our dataset.

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1 Differential Equations, Laplace Transform and Preliminaries

1.1 Differential Equations

Determine the transfer function of a linear time invariant (LTI) system given the following information:

1. The system has relative degree 3.
2. It has 3 poles, of which 2 are at -2 and -4.
3. The impulse response resembles a step response for a stable linear system with a steady state value of 0.25.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $G(s) = \frac{4}{(s+2)^2(s+4)}$

(b) $G(s) = \frac{2(s+1)}{s(s+1)(s+2)(s+4)}$

(c) $G(s) = \frac{2}{s(s+2)(s+4)}$

(d) $G(s) = \frac{2}{(s+1)(s+2)(s+4)}$

Solution :
Correct answer is (c).
Given a linear time-invariant (LTI) system with specific characteristics, we aim to determine its transfer function. The details and solution steps are as follows:

System Characteristics

- The system has a relative degree 3 with 3 poles, hence it has no finite zeros.

- It has 3 poles, hence it takes the form:

$$G(s) = \frac{K}{A(s)(s+2)(s+4)} \tag{1}$$

- Since the impulse response resembles a step response with a steady state value, we conclude the system must contain a pole at zero. Therefore, the transfer function is of the form:

$$G(s) = \frac{K}{s(s+2)(s+4)} \tag{2}$$

Determination of K

Using the final value theorem to determine K, we have:

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{sK}{s(s+2)(s+4)} \tag{3}$$

$$= \frac{K}{8} \tag{4}$$

Given the steady state value of 0.25, i.e., $\lim_{s\to 0} sG(s) = 0.25$, we find $K = 2$. Therefore, the transfer function of the system is:

$$G(s) = \frac{2}{s(s+2)(s+4)} \tag{5}$$

1.2 Initial Value Theorem

Consider a system with the transfer function given by

$$H(s) = \frac{2(s+1)}{s^2+4s+9} \tag{6}$$

and denote by $g(t)$ the unit step response with zero initial conditions. Compute the initial slope of $g(t)$, i.e., the slope at $t = 0+$.
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

- (a) 0
- (b) 1
- (c) 2
- (d) 9

Solution :
Correct answer is (c).
We first note that the derivative of $g(t)$ is the response to the derivative of the unit step, i.e., it is equal to the response to a Dirac delta. Hence, we have

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \mathcal{L}\{h(t)\} = H(s) \tag{7}$$

Therefore, the answer to the question translates into computing $h(0+)$. This can be computed via the Initial Value Theorem. This yields

$$\dot{g}(0+) = h(0+) = \lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{2s(s+1)}{s^2+4s+9} = 2 \tag{8}$$

1.3 Undershoot

Consider a system with a transfer function given by

$$H(s) = \frac{2}{s+1} + \frac{\alpha}{s+2} \tag{9}$$

where α is a real number. Is there a range of real values for α such that the system's unit step response exhibits undershoot?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $2 < \alpha < 4$

(b) $-4 < \alpha < -2$

(c) $-2 < \alpha$

(d) $-4 < \alpha < 4$

Solution :
Correct answer is (b).
We first note that if $\alpha > 0$, the unit step response is always positive, thus no undershoot occurs.

To have undershoot in the unit step response, it is sufficient that a linear system have a real zero in the open Right Half Plane (RHP) – a Non-Minimum Phase (NMP) zero. To investigate that possibility, we add the two fractions in $H(s)$ to yield:

$$H(s) = \frac{2s + 4 + \alpha s + \alpha}{(s+1)(s+2)} = (4+\alpha)\frac{\frac{2+\alpha}{4+\alpha}s + 1}{(s+1)(s+2)} \tag{10}$$

Then, we have a NMP zero if and only if $-4 < \alpha < -2$.

1.4 Superposition Principle

Consider a linear system with transfer function $G(s) = \frac{4}{s+2}$. Let u denote the input and y denote the output. The response of $G(s)$ with $u(t) = 2$ for $t \geq 0$, $u(t) = 0$ for $t < 0$, and zero initial conditions is $y(t) = 4(1 - e^{-2t})$.

(a) What is the response y_A from zero initial conditions if $u_A(t) = -1$ for $t \geq 0$? Provide your answer as an explicit function of $y(t)$.

(b) Analyze the response $y_B(t)$ from zero initial conditions for a piecewise input:

$$u_B(t) = \begin{cases} 2 & 0 \leq t < 10, \\ 4 & t \geq 10. \end{cases}$$

Express $y_B(t)$ as a mathematical expression in terms of $y(t)$, using the principle of superposition and time-invariance.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $y_A = +\frac{1}{2}y$ b) $y_B = y(t) + 2y(t - 10)$

(b) a) $y_A = -2y$ b) $y_B = y(t) + y(t - 10)$

(c) a) $y_A = -\frac{1}{2}y$ b) $y_B = y(t) - y(t - 10)$

(d) a) $y_A = -\frac{1}{2}y$ b) $y_B = y(t) + y(t - 10)$

Solution :
Correct answer is (d).

(a) Note that $u_A = -\frac{1}{2}u$. By the principle of superposition (scaling), the response due to u_A is

$$y_A = -\frac{1}{2}y = -2(1 - e^{-2t}).$$

(b) Note that $u_B(t) = u(t) + u(t-10)$ where $u(t-10)$ is the original input shifted by 10 sec. By time-invariance, the response due to $u(t-10)$ is given by $y(t-10)$. By the principle of superposition (addition), the response due to u_B is given by $y_B(t) = y(t) + y(t-10)$. This plot is shown below.

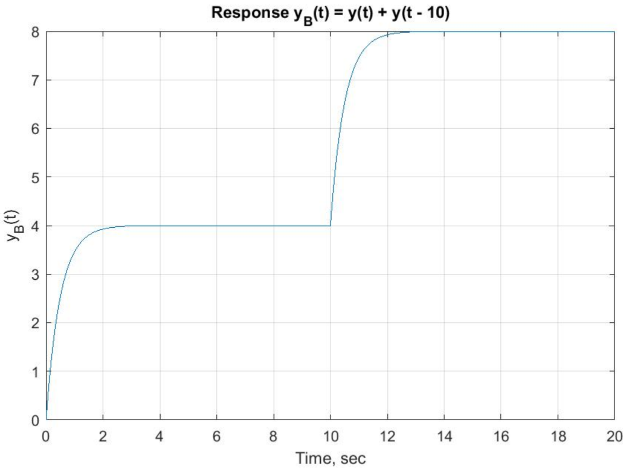

Figure 1: The system response $y_B(t)$ over time.

1.5 Ordinary Differential Equation Representation of Transfer Functions

Consider two systems $G_1(s) = \frac{-3}{s+11}$ and $G_2(s) = \frac{2}{s+9}$.

(a) What is the ODE that represents the serial interconnection $H(s) = G_2(s)G_1(s)$?

(b) What is the ODE that represents the parallel interconnection $H(s) = G_1(s) + G_2(s)$?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = -6u(t)$ b) $\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = -\dot{u}(t) - 5u(t)$

(b) a) $\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = 6u(t)$ b) $\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = \dot{u}(t)5u(t)$

(c) a) $\ddot{y}(t) - 20\dot{y}(t) - 99y(t) = -6u(t)$ b) $\ddot{y}(t) - 20\dot{y}(t) - 99y(t) = -\dot{u}(t) - 5u(t)$

(d) a) $\ddot{y}(t) + 20\dot{y}(t) - 99y(t) = -6u(t)$ b) $\ddot{y}(t) + 20\dot{y}(t) - 99y(t) = -\dot{u}(t) - 5u(t)$

Solution :
Correct answer is (a).

(a) Multiplication of the transfer functions yields:

$$H(s) = \frac{2}{s+9} \times \left(\frac{-3}{s+11}\right) = \frac{-6}{s^2 + 20s + 99} \tag{11}$$

The corresponding ODE from input u to output y is:

$$\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = -6u(t) \tag{12}$$

(b) Addition of the transfer functions yields:

$$H(s) = \frac{-3}{s+11} + \frac{2}{s+9} = \frac{-3(s+9) + 2(s+11)}{(s+11)(s+9)} = \frac{-s-5}{s^2 + 20s + 99} \tag{13}$$

The corresponding ODE from input u to output y is:

$$\ddot{y}(t) + 20\dot{y}(t) + 99y(t) = -\dot{u}(t) - 5u(t) \tag{14}$$

1.6 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u; \quad y(0) = 0 \tag{15}$$

Consider a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. Consider a unit step command:

$$r(t) = \begin{cases} 0 & \text{for } t < 0 \text{ sec} \\ 1 & \text{for } t \geq 0 \text{ sec} \end{cases} \quad (2)$$

For what gains K_p is $|u(t)| \leq 1$ for all time?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $-0.25 < K_p$

(b) $-0.25 < K_p < 0$

(c) $-0.2 \leq K_p$

(d) $-0.2 \leq K_p \leq 1$

Solution :
Correct answer is (d).
First, note that we must ensure that the closed-loop is stable otherwise the control will grow unbounded. The closed-loop is modeled by:

$$\dot{y}(t) + (0.5 + 2K_p)y(t) = 2K_p r(t) \tag{16}$$

Thus, $K_p > -0.25$ is required for stability. There are two cases to consider: (i) $K_p \geq 0$ and (ii) $-0.25 < K_p < 0$.

First consider the case $K_p \geq 0$. In this case, the output starts at $y(0) = 0$ and converges exponentially to the final value $\bar{y} = \frac{2K_p}{0.5+2K_p}$. The error $e(t) = r(t) - y(t)$ thus starts at $e(0) = 1$ and converges to the following final value:

$$\bar{e} = 1 - \frac{2K_p}{0.5 + 2K_p} = \frac{0.5}{0.5 + 2K_p} \leq 1 \tag{17}$$

In other words, e achieves its largest value at $t = 0$ and decays to a smaller value. Thus the control effort $u(t) = K_p e(t)$ achieves its largest value at $t = 0$. To ensure $|u(t)| \leq 1$ it is sufficient to require $u(0) = K_p e(0) = K_p \leq 1$.

The alternative case $-0.25 < K_p < 0$ actually causes the error to grow from $e(0) = 1$ to its final value. In this case, the controller is acting to degrade tracking, which would not be typical. The hint given in the problem statement is incorrect for this case. It is fine if you only considered positive gains. However, to analyze this case you can use the expression above for the final value $\bar{e}$. Use this expression to show that $K_p \geq -0.2$ is required to ensure $|u(t)| \leq 1$.

Combining these results, the input satisfies $|u(t)| \leq 1$ for all time if the gain is selected in the range $-0.2 \leq K_p \leq 1$.

1.7 Equilibrium

Consider the scalar ODE

$$\dot{x} = x^2 - 1 \tag{18}$$

(a) Find all the equilibria

(b) Compute the linearization of the ODE about each equilibrium and determine the stability.

Solution :

(a) The equilibria occur where $\dot{x} = 0$. Solving the equation $x^2 - 1 = 0$, we find the equilibrium points are $x = 1$ and $x = -1$.

(b) The linearization involves computing the derivative of $f(x) = x^2 - 1$ with respect to x, which is $f'(x) = 2x$. Evaluating this derivative at each equilibrium point:

- For $x = 1$, $f'(1) = 2$. Since $f'(1) > 0$, the equilibrium at $x = 1$ is unstable.
- For $x = -1$, $f'(-1) = -2$. Since $f'(-1) < 0$, the equilibrium at $x = -1$ is stable.

Thus, the equilibrium at $x = 1$ is unstable, while the equilibrium at $x = -1$ is stable.

1.8 State-Space Representation of an ODE

Consider the triple integrator:

$$\frac{d^3y}{dt^3} = u \tag{19}$$

(a) Obtain a state-space representation of the form $\dot{x} = Ax + Bu$ for this system.

(b) Compute e^{At} and use it to obtain the solution with $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ and $u(t) = 0$

Solution :

(a) To obtain a state-space representation of the form $\dot{x} = Ax + Bu$ for this system, we define:

$$x_1 = y,$$
$$x_2 = \dot{y},$$
$$x_3 = \ddot{y}.$$

Thus, the state-space model is given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) To compute e^{At} and obtain the solution with initial conditions $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ and $u(t) = 0$, we find:

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Given $x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, the solution $x(t)$ is:

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 + t + \frac{t^2}{2} \\ 1 + t \\ 1 \end{bmatrix} \tag{20}$$

1.9 Convolution Integral

Let $f_1(t) = 3e^{-3t}u_s(t)$ and $f_2(t) = te^{-2t}u_s(t)$. Compute the following convolution:

$$g(t) = f_1(t) * f_2(t).$$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $g(t) = 3(te^{-3t} + te^{-2t} - e^{-2t})$

(b) $g(t) = 3(e^{-3t} + te^{-2t} - e^{-2t})$

(c) $g(t) = 3e^{-3t} + te^{-2t} - 3e^{-2t}$

(d) $g(t) = 3(e^{-3t} + 3te^{-2t} - e^{-2t})$

Solution :
Correct answer is (b).
Calculating the Laplace transform of $f_1(t)$ and $f_2(t)$, we have:

$$F_1(s) = \frac{3}{s+3}, \quad F_2(s) = \frac{1}{(s+2)^2} \tag{21}$$

It is known that $G(s) = F_1(s).F_2(s)$. Using partial fraction, we have

$$G(s) = \frac{3}{(s+3)(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3} \tag{22}$$

Solving for A, B and C, we have

$$A = -3, \quad B = 3, \quad C = 3 \tag{23}$$

we have:

$$g(t) = \mathcal{L}^{-1}(G(s)) = 3(e^{-3t} + te^{-2t} - e^{-2t}) \tag{24}$$

1.10 Classification of Differential Equations

Classify the following differential equations according to whether they are ordinary or partial. Indicate the dependent and independent variables.

(a) $\frac{dx}{dt} + \frac{dy}{dt} + x + y = 0 \quad x = x(t) \quad y = y(t)$

(b) $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + x + y = 0 \quad f = f(x, y)$

(c) $\frac{d}{dt}\left[\frac{\partial f}{\partial x}\right] = 0 \quad f = x^2 + \frac{dx}{dt}$

(d) $\frac{df}{dx} = x \quad f = y^2(x) + \frac{dy}{dx}$

Solution :

(a) Ordinary; independent variable t; dependent variables x and y.

(b) Partial; independent variables x and y; dependent variable f.

(c) Since $\frac{\partial f}{\partial x} = 2x$, then $\frac{d}{dt}\left[\frac{\partial f}{\partial x}\right] = 2\frac{dx}{dt} = 0$, which is an ordinary differential equation independent variable t; dependent variable x

(d) $\frac{df}{dx} = 2y\frac{dy}{dx} + \frac{d^2y}{dx^2} = x$, which is an ordinary differential equation; independent variable x; dependent variable y.

1.11 Time-Invariance vs. Time-Variance in ODEs

Classify the following linear differential equations according to whether they are time-variable or time-invariant. Indicate any time-variable terms.

(a) $\frac{d^2y}{dt^2} + 2y = 0$

(b) $\frac{d}{dt}(t^2y) = 0$

(c) $\left(\frac{1}{t+1}\right)\frac{d^2y}{dt^2} + \left(\frac{1}{t+1}\right)y = 0$

(d) $\frac{d^2y}{dt^2} + (\cos t)y = 0$

You may assume $t \geq 0$ in these differential equations.

Solution :

(a) Time-invariant.

(b) $\frac{d}{dt}(t^2y) = 2ty + t^2\left(\frac{dy}{dt}\right) = 0$. Dividing through by t, $t\left(\frac{dy}{dt} + 2y\right) = 0$ which is time-variable. The time-variable term is $t\left(\frac{dy}{dt}\right)$

(c) Multiplying through by $t+1$, we obtain $\frac{d^2y}{dt^2} + y = 0$ which is time-invariant.

(d) Time-variable. The time-variable term is $(\cos t)y$

1.12 Casualty

Two systems are defined by the relationships between their input and outputs as follows:

system 1: The input is $u(t)$ and at the same instant of time the output is $y(t) = u(t+T), T > 0$

system 2: The input is $u(t)$ and at the same instant of time the output is $y(t) = u(t-T), T > 0$
Are either of these systems causal?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) Causal b) Non-causal

(b) a) Non-causal b) Causal

(c) a) Non-causal b) Non-causal

(d) a) Causal b) Causal

Solution :
Correct answer is (b).
In system 1, the output depends only on the input T seconds in the future. Thus it is not causal. An operation of this type is called prediction.

In system 2 the output depends only on the input T seconds in the past. Thus it is causal. An operation of this type is called a time delay.

1.13 Proving Linear Independence via Nonzero Wronskian

Show that a sufficient condition for a set of n functions $f_1, f_2, \cdots, f_n$ to be linearly independent is that the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \frac{df_1}{dt} & \frac{df_2}{dt} & \cdots & \frac{df_n}{dt} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{n-1}f_1}{dt^{n-1}} & \frac{d^{n-1}f_2}{dt^{n-1}} & \cdots & \frac{d^{n-1}f_n}{dt^{n-1}} \end{vmatrix} \tag{25}$$

be nonzero. This determinant is called the Wronskian of the functions $f_1, f_2, \cdots, f_n$.

Solution :

Assuming the f_is are differentiable at least $n-1$ times, let $n-1$ derivatives of

$$c_1f_1 + c_2f_2 + \cdots c_nf_n = 0 \tag{26}$$

be formed as follows, where c_is are unknown constants:

$$\begin{aligned} c_1\frac{df_1}{dt} + c_2\frac{df_2}{dt} + \cdots + c_n\frac{df_n}{dt} &= 0, \\ &\vdots \\ c_1\frac{d^{n-1}f_1}{dt^{n-1}} + c_2\frac{d^{n-1}f_2}{dt^{n-1}} + \cdots + c_n\frac{d^{n-1}f_n}{dt^{n-1}} &= 0. \end{aligned} \tag{27}$$

These equations may be considered as n simultaneous linear homogeneous equations in the n unknown constants $c_1, c_2, \cdots, C_n$(i.e., not all c_i are equal to zero) if and only if the determinant of the coefficients (the Wronskian) is equal to zero. Hence if the Wronskian is nonzero, the the only solution for $c_1, c_2, \cdots, C_n$ is the degenerate one, $c_1 = c_2 = \cdots = c_n = 0$. Clearly, this it equivalent to saying that if the Wronskian is nonzero the functions $f_1, f_2, \cdots, f_n$ are linearly independent, since the only solution to $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ is then $c_1 = c_2 = c_3 = \cdots = c_n = 0$. Hence a sufficient condition for the linear independence of $f_1, f_2, \cdots, d_n$ is that the Wronskian be nonzero. This condition is not necessary; that is , there exist sets of linearly independent functions for which the Wronskian is zero.

1.14 Linear Independence

Show that the function $1, t, t^2$ are linearly independent.

Solution :

The Wronskian of these three functions is

$$\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} \tag{28}$$

Since the Wronskian is nonzero, the functions are linearly independent.

1.15 Laplace Transform

Show that the Laplace transform of the derivative $\frac{df}{dt}$ of a function $f(t)$ defined as below:

$$\mathcal{L}\left[\frac{df}{dt}\right] = \lim_{\substack{T\to+\infty \\ \epsilon\downarrow 0}} \int_{\epsilon}^{T} \frac{df}{dt} e^{-st} dt \tag{29}$$

is given by $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^+)$, where $F(s) = \mathcal{L}[f(t)]$.

Solution :
Integrating by parts.

$$\lim_{\substack{T\to+\infty \\ \epsilon\to 0}} \int_{\epsilon}^{T} \frac{df}{dt} e^{-st} dt = \lim_{\substack{T\to+\infty \\ \epsilon\to 0}} \left[f(t)e^{-st}|_{\epsilon}^{T} + s\int_{\epsilon}^{T} f(t)e^{-st}dt \right] = -f(0^+) + sF(s) \tag{30}$$

where $\lim_{\epsilon\to 0} f(\epsilon) = f(0^+)$.

1.16 Characterization of Second-Order Systems

Determine

(a) the undamped natural frequency ω_n

(b) the damping ratio ζ

(c) the constant time $\tau = \frac{1}{\zeta\omega_n}$

(d) the damped natural frequency ω_d

(e) characteristic equation for the second-order system given by

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 9y = 9u \tag{31}$$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $\omega_n = 3$ rad/sec b) $\zeta = \frac{5}{3}$ c) $\tau = \frac{1}{5}$ d) $\omega_d = 1.66$ rad/sec e) $s^2 + 5s + 9 = 0$

(b) a) $\omega_n = 9$ rad/sec b) $\zeta = \frac{5}{18}$ c) $\tau = \frac{9}{5}$ d) $\omega_d = 8.65$ rad/sec e) $s^2 + 5s + 9 = 0$

(c) a) $\omega_n = 3$ rad/sec b) $\zeta = \frac{5}{6}$ c) $\tau = \frac{2}{5}$ d) $\omega_d = 1.66$ rad/sec e) $s^2 + 5s + 9 = 0$

(d) a) $\omega_n = 9$ rad/sec b) $\zeta = \frac{5}{9}$ c) $\tau = \frac{9}{10}$ d) $\omega_d = 7.48$ rad/sec e) $s^2 + 5s + 9 = 0$

Solution :
Correct answer is (c).

(a) $\omega_n^2 = 9$ or $\omega_n = 3$ rad/sec

(b) $2\zeta\omega_n = 5$ or $\zeta = \frac{5}{2\omega_n} = \frac{5}{6}$

(c) $\tau = \frac{1}{\zeta\omega_n} = \frac{2}{5}$ sec

(d) $\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.66$ rad/sec

(e) $s^2 + 5s + 9 = 0$

1.17 Transfer Function

What is the transfer function of a system whose input and output are related by the following differential equation?

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} = 2y = u + \frac{du}{dt} \tag{32}$$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $T(s) = \frac{s(s+1)}{s^2+3s+2}$

(b) $T(s) = \frac{(s+1)^2}{s^2+3s+2}$

(c) $T(s) = \frac{s+1}{s(s^2+3s+2)}$

(d) $T(s) = \frac{s+1}{s^2+3s+2}$

Solution :
Correct answer is (d).
Taking the Laplace transform of this equation, ignoring terms due to initial conditions, we obtain

$$s^2Y(s) + 3sY(s) + 2Y(s) = U(s) + sU(s) \tag{33}$$

This equation can be writtten as

$$Y(s) = \left[\frac{s+1}{s^2+3s+2}U(s)\right] \tag{34}$$

The transfer function of this system is therefore given by

$$P(s) = \frac{s+1}{s^2+3s+2} \tag{35}$$

1.18 Time-Delayed ODEs: Transfer Function Approach

A particular system containing a time delay has the differential equation $\frac{dy(t)}{dt} + y(t) = u(t-T)$. Find the transfer function of this system.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $T(s) = \frac{e^{-sT}}{(s+1)^2}$

(b) $T(s) = \frac{se^{-sT}}{s+1}$

(c) $T(s) = \frac{e^{-sT}}{s+1}$

(d) $T(s) = \frac{s-T}{s+1}$

Solution :

Correct answer is (c).

The Laplace transform of the differential equation ignoring terms due to initial conditions, is $sY(s)+Y(s) = e^{-sT}U(s)$. The system transfer function would be

$$P(s) = \frac{Y(s)}{U(s)} = \frac{e^{-sT}}{s+1} \tag{36}$$

1.19 State-Space Representation of ODEs

Show the equation $\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt})$ can be equivalently described by a pair of first-order differential equations.

Solution :
We define a set of new variables: $x_1 = x$ and $x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$.

$$\frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} = \frac{dx_2}{dt} = f(x, \frac{dx}{dt}) = f(x_1, \frac{dx_1}{dt}) = f(x_1, x_2) \tag{37}$$

The two desired equations are therefore:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= f(x_1, x_2) \end{aligned}$$

1.20 Characterization of Second-Order Systems

For each of the system below, determine the followings:

- What is the natural frequency and damping ratio?
- Is the system under. over, or critically damped?
- For the unit step input, find the final value, settling time (5%) and overshoot (if underdamped)

(a) $G_A(s) = \frac{20}{s^2+2s+10}$

(b) $G_B(s) = \frac{20}{s^2+11s+10}$

Solution :

(a)

- $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \rightarrow \omega_n = \sqrt{10}/rad/sec, \quad \zeta = \frac{2}{2\omega_n} = \frac{1}{\sqrt{10}} \approx 0.316$
- $\zeta < 1 \quad \rightarrow$ underdamped
- $y_f = \frac{20*1}{10} = 2 \quad t_s = \frac{3}{\zeta\omega_n} = 3 \quad Mp = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.35$

(b)

- $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \rightarrow \omega_n = \sqrt{10}/rad/sec, \quad \zeta = \frac{11}{2\omega_n} = \frac{1}{\sqrt{10}} \approx 1.74$
- $\zeta > 1 \quad \rightarrow$ overdamped
- $y_f = \frac{20*1}{10} = 2 \quad \tau_1 = 1, \tau_2 = 10 \quad t_s = 3 * \min\{\tau_1, \tau_2\} = 3$

1.21 Dominant Pole Approximation

For each system:

- Construct a first-order or second-order approximation from the dominant pole
- Do you expect the dominant pole approximation to be accurate?

(a) $G_A(s) = \frac{5000}{(s+2)(s+20)(s^2+20s+500)}$

(b) $G_B(s) = \frac{2}{(s+1)(0.5s+1)^2(s/3+1)}$

(c) $G_C(s) = \frac{15}{(s+1)^2(s+10)}$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $\hat{G}(s) = \frac{0.5}{s+2}$, Accurate b) $\hat{G}(s) = \frac{2}{s+1}$, Not that accurate c) $\hat{G}(s) = \frac{1.5}{(s+1)^2}$, Accurate

(b) a) $\hat{G}(s) = \frac{0.5}{s+2}$, Accurate b) $\hat{G}(s) = \frac{2}{s+1}$, Accurate c) $\hat{G}(s) = \frac{1.5}{(s+1)^2}$, Accurate

(c) a) $\hat{G}(s) = \frac{0.5}{s+2}$, Accurate b) $\hat{G}(s) = \frac{1}{s+1}$, Accurate c) $\hat{G}(s) = \frac{1.5}{(s+1)^2}$, Accurate

(d) a) $\hat{G}(s) = \frac{0.5}{s+2}$, Accurate b) $\hat{G}(s) = \frac{1}{s+1}$, Accurate c) $\hat{G}(s) = \frac{1.5}{(s+1)^2}$, Not that accurate

Solution :
Correct answer is (a).

(a)

- Poles are $s_1 = -2, s_2 = -20, s_{3,4} = -10 \pm 20j$. The dominant pole is s_1.
  DC Gain: $G(0) = \frac{5000}{2\times20\times500} = \frac{1}{4}$
  First-order approximation: $\hat{G}(s) = \frac{0.5}{s+2}$

- Accurate, since $s_1 = -2$ is much slower than s_2 and $s_{3,4}$

(b)

- Poles are $s_1 = -1, s_{2,3} = -2, s_4 = -3$. The dominant pole is s_1.
  DC Gain: $G(0) = \frac{24}{1 \times 2^2 \times 3} = 2$
  First-order approximation: $\hat{G}(s) = \frac{2}{s+1}$
- Not that accurate, since all poles are of similar time scale.

(c)

- Poles are $s_{1,2} = -1, s_3 = -10$. The dominant poles are $s_{1,2}$.
  DC Gain: $G(0) = \frac{15}{1^2 \times 10} = 1.5$
  Second-order approximation: $\hat{G}(s) = \frac{1.5}{(s+1)^2}$
- Accurate, since $s_{1,2}$ are much slower than s_3

1.22 State-Space Representation

Imagine two cars driving on the same road in the same direction one behind the other and trying to go at the same speed. This situation can be described by the linear system

$$\begin{aligned} \dot{v}_1 &= -(v_1 - v_2) + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2 \end{aligned} \tag{38}$$

where, for $i = 1, 2$, v_i is the velocity of car i and f_i is an external force (wind, road conditions, etc.) acting on it. The meaning of the above equations is that each car accelerates/decelerates depending on whether it is going slower/faster than the other. Now, suppose that we want to rewrite the above system in the following new coordinates: $\bar{v}_1 := v_1$ (velocity of car 1) and $\bar{v}_2 := v_2 - v_1$ (relative velocity of the two cars).

(a) Write down the differential equations for $\dot{\bar{v}}_1, \dot{\bar{v}}_2$ in terms of $\bar{v}_1, \bar{v}_2$ and f_1, f_2.

(b) Write down the original system in state-space form $\dot{v} = Av + Bf$, the new system in state-space form $\dot{\bar{v}} = \bar{A}\bar{v} + \bar{B}f$, the coordinate transformation matrix T from (v_1, v_2) to $(\bar{v}_1, \bar{v}_2)$, and verify that this transformation matrix T leads to the same ODE as part (a) if applied to the system in Eq. (38).

Solution :

$$\begin{aligned} \dot{v}_1 &= -(v_1 - v_2) + f_1 = \bar{v}_2 + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2 = -\bar{v}_2 + f_2 \end{aligned}$$

$$\Longrightarrow \underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{v}} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_{u}$$

(a)

$$\begin{aligned} \dot{\bar{v}}_1 &= \dot{v}_1 = +\bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 &= \dot{v}_2 - \dot{v}_1 = -\bar{v}_2 + f_2 - (\bar{v}_2 + f_1) = -2\bar{v}_2 - f_1 + f_2 \end{aligned}$$

$$\Longrightarrow \begin{cases} \dot{\bar{v}}_1 = \bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 = -2\bar{v}_2 + f_2 - f_1 \end{cases}$$

(b)

$$\begin{aligned} \bar{v}_1 &= v_1 \\ \bar{v}_2 &= v_2 - v_1 \end{aligned} \Longrightarrow \underbrace{\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}}_{\bar{v}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{v}$$

$$
\begin{aligned}
\dot{\bar{v}} &= T\dot{v} = TAv + TBu \\
\dot{\bar{v}} &= TAT^{-1}\bar{v} + TBu \\
&= \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{T^{-1}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_{u} \\
&= \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}}_{\bar{A}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\bar{B}} u
\end{aligned}
$$

$$
\begin{aligned}
\dot{\bar{v}}_1 &= \bar{v}_2 + f_1 \\
\dot{\bar{v}}_2 &= -2\bar{v}_2 - f_1 + f_2
\end{aligned}
$$

1.23 Final Value Theorem

Consider the following transfer functions

$$G_1(s) = \frac{2}{s^2 - 2s + 4} \qquad G_2(s) = \frac{2s - 3}{s^2 + 4s + 1}$$

Use the final value theorem to compute the DC gain for $G_1(s)$ and $G_2(s)$. In each case, explain whether the final value theorem gives the right answer and why.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) DC Gain for $G_1(s)$ is $\frac{1}{2}$ and is applicable, DC Gain for $G_2(s)$ is -3 and is applicable

(b) DC Gain for $G_1(s)$ is $\frac{1}{2}$ and is not applicable, DC Gain for $G_2(s)$ is -3 and is applicable

(c) DC Gain for $G_1(s)$ is $\frac{1}{2}$ and is not applicable, DC Gain for $G_2(s)$ is -3 and is not applicable

(d) DC Gain for $G_1(s)$ is $\frac{1}{2}$ and is applicable, DC Gain for $G_2(s)$ is -3 and is not applicable

Solution :
Correct answer is (b).
DC gain for $G_1(s) = G_1(s)|_{s=0} = \frac{1}{2}$
DC gain for $G_2(s) = G_2(s)|_{s=0} = -3$
The final value theorem computes the correct DC gain for $G_2(s)$ but not for $G_1(s)$ because the poles of $G_1(s)$ are not in LHP, hence Final Value Theorem does not apply.

1.24 Characterization of Second-Order Systems

Consider the following mass-spring system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}1 & 1\\ -\frac{k}{m} & -\frac{\rho}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u \qquad y = \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

where m is the mass, k is the spring constant and ρ is the friction coefficient. Find the values of c_1 and c_2 to guarantee that the transfer function of the resulting system has the form of the standard 2nd-order system. Write down the expressions for the parameters ζ and ω_n in terms of k, ρ, and m. Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{\rho}{2\sqrt{km}}$

(b) $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{1}{2\sqrt{km\rho}}$

(c) $\omega_n = \sqrt{\frac{1}{m}}$ and $\zeta = \frac{\rho}{2\sqrt{km}}$

(d) $\omega_n = \sqrt{\frac{1}{m}}$ and $\zeta = \frac{1}{2\sqrt{km\rho}}$

Solution:
Correct answer is (a).
The given system can be transformed into its transfer function as follows:

$$C(sI - A)^{-1}B = \frac{\frac{c_2}{m}s + \frac{c_1}{m}}{s^2 + \frac{\rho}{m}s + \frac{k}{m}}$$

Comparison with the transfer function of a standard 2nd-order system yields:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{\frac{c_2}{m}s + \frac{c_1}{m}}{s^2 + \frac{\rho}{m}s + \frac{k}{m}}$$

which leads to the following result:

$$\frac{c_2}{m} = 0, \frac{c_1}{m} = \frac{k}{m} \qquad \Longrightarrow \qquad c_1 = k, c_2 = 0$$

Additionally, we get the following expressions for ζ and ω_n:

$$2\zeta\omega_n = \frac{\rho}{m}, \omega_n = \frac{k}{m} \quad \implies \quad \omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{\rho}{2\sqrt{km}}$$

1.25 Time-Domain Specs

Consider the transfer function

$$H(s) = \frac{16}{s^2 + 4s + 16}$$

Suppose that you are given the following time-domain specs: $t_r \leq 0.6$, $t_s \leq 1.6$ (defined by the time system's response reach and stay within 5% of steady state). Does the given system satisfy these specs? Recall that $t_r = \frac{1.8}{\omega_n}$ and $t_s = \frac{3}{\zeta\omega_n}$.

Solution:

Recall that $t_r = \frac{1.8}{\omega_n}$ and $t_s = \frac{3}{\sigma}$. Therefore by the given specs $t_r \leq 0.6$ and $t_s \leq 1.6$ we can conclude that:

$$\omega_n \geq 3 \text{ and } \sigma \geq 1.875$$

The admissible pole locations are shown in the figure below, where the blue shaded region is due to rise time constraint and red shaded region is due to the settling time constraint.

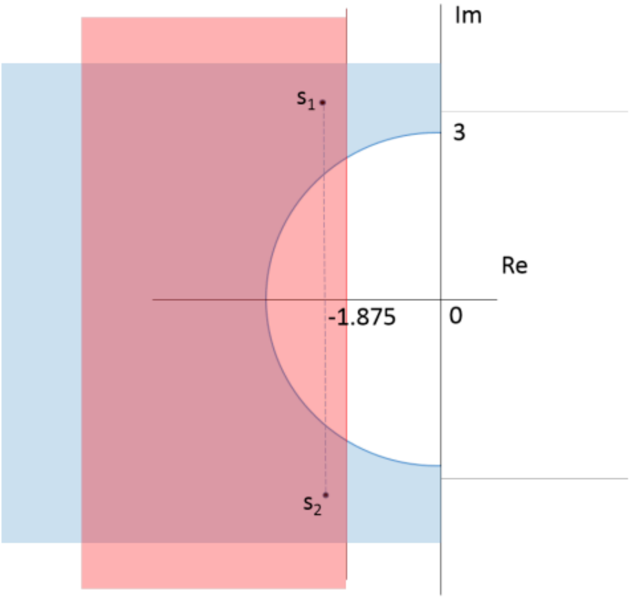

Figure 2: Locations of the Poles of $H(s)$

The poles of $H(s)$ are the roots of $s^2 + 4s + 16$, which can be computed to be $s_{1,2} = -2 \pm 2\sqrt{3}$. They are plotted in the complex plane and found to be in both of shaded regions. Hence the given system satisfy these specs.

1.26 Time-Domain Specs

Consider the transfer function

$$H(s) = \frac{16}{s^2 + 4s + 16}$$

Suppose that you are given the following time-domain specs: $t_r \leq 0.6$, $t_s \leq 1.6$ (defined by the time system's response reach and stay within 5% of steady state). Suppose additionally, we have overshoot: $M_p \leq \frac{1}{e^2}$. Does the given system satisfy these specs?

Solution:

Recall that $t_r = \frac{1.8}{\omega_n}$, $t_s = \frac{3}{\sigma}$ and $M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\pi \arctan\theta}$. Therefore by the given specs $t_r \leq 0.6$, $t_s \leq 1.6$ and $M_p \leq \frac{1}{e^2}$, we can conclude that:

$$\omega_n \geq 3, \text{ and } \sigma \geq 1.875, \text{ and } \theta \geq 42^\circ$$

The admissible region due to overshoot constraint is plotted as the yellow shaded area in the figure given below. Since the poles of the system are not in the yellow region, the given system does not satisfy the specs.

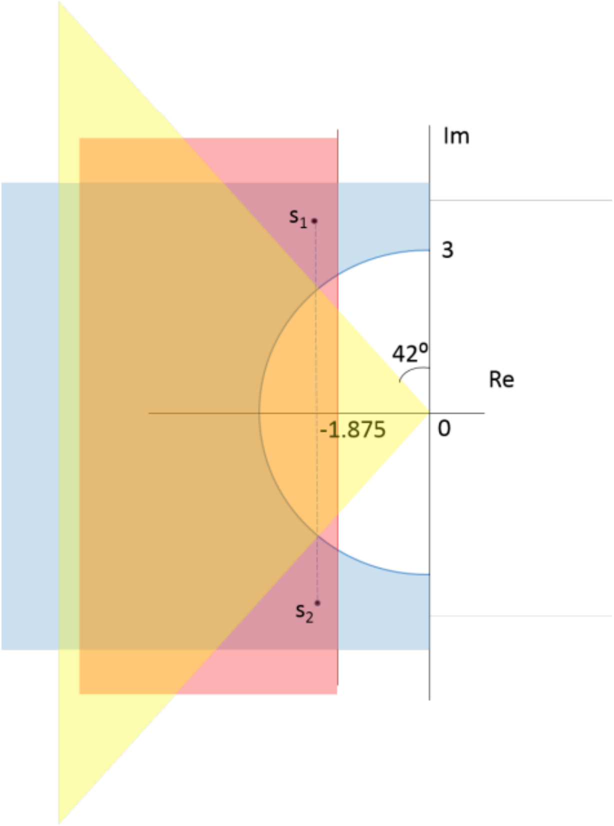

Figure 3: Admissible Region

1.27 Second-Order Systems

Consider a second-order system with peak time $t_p = 0.5$ sec. and 5% settling time $t_s = 1.5$ sec. Determine the poles of this system.
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $s = -1 \pm \pi$

(b) $s = -2 \pm 2\pi$

(c) $s = 2 \pm 2\pi$

(d) $s = -2 \pm \pi$

Solution:
Correct answer is (b).
The poles of this system are given by $s = -\sigma \pm j\omega_d$ where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Given information $t_p = 0.5$ sec and 5% settling time $t_s = 1.5$ sec leads to $\omega_d = 2\pi$ and $\sigma = 2$. Therefore the poles of the system are $s = -2 \pm 2\pi$

1.28 Time-Domain Specs

Consider a second-order system with peak time $t_p = 0.5$ sec. and 5% settling time $t_s = 1.5$ sec. Does this system satisfy rising time $t_r \leq 0.18$ sec?

Solution:

Recall that $t_r = \frac{1.8}{\omega_n}$. Therefore this system must have $\omega_n \geq 10$ in order to satisfy the rise time requirement.

We know that $\omega_n^2 = \sigma^2 + \omega_d^2$. Given that $t_p = 0.5$ sec and 5% settling time $t_s = 1.5$ sec, we can compute the corresponding $\omega_d = 2\pi$ and $\sigma = 2$.

Therefore this system's $\omega_n^2 = 2^2 + (2\pi)^2 = 68$. Since $\omega_n^2 < 100$, therefore this system can't satisfy $t_r \leq 0.18$ sec.

2 Stability

2.1 Stability Analysis

In a feedback control loop, the open-loop transfer function $G(s)$ and the controller $C(s)$ are given by

$$G(s) = \frac{s-2}{(s-1)(s+4)}, \quad C(s) = K\frac{s+1}{s} \tag{39}$$

Determine $K \in \mathbb{R}$, if exists, such that the control loop is stable.

Select the correct option from the following choices:

(a) -1

(b) 0

(c) 1

(d) No such value exits

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (d).
The closed-loop characteristic polynomial is given by the numerator of $1 + G(s)C(s)$, i.e.,

$$p(s) = s^3 + (3+K)s^2 - (K+4)s - 2K \tag{40}$$

On applying the Routh criterion, we observe that $p(s)$ has all its roots in the open Left Half Plane (LHP) if and only if the following three conditions are simultaneously satisfied:

$$3 + K > 0 \tag{41}$$

$$K^2 + 5K + 12 < 0 \tag{42}$$

$$-2K > 0 \tag{43}$$

The first and third conditions require that $-3 < K < 0$. On the other hand, the second condition can be re-written as $(K + 2.5)^2 + 5.75 < 0$. This form allows us to see that the condition cannot be met by any real value of K. In summary, there is no real value for K which stabilizes the closed loop.

2.2 Stability Analysis using PI Controller

Consider a plant with a nominal transfer function given by

$$G(s) = \frac{1}{(s-1)^2} \tag{44}$$

Prove that this system cannot be stabilized with a PI controller.

Solution :
We consider the general form of a PI controller as

$$C(s) = \frac{as+b}{s} \tag{45}$$

Thus, the closed-loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = \text{numerator of } \{1 + G(s)C(s)\} = s^3 - 2s^2 + (1+a)s + b \tag{46}$$

Hence, there are no values for a and b such that $A_{cl}(s)$ is strictly Hurwitz. This originates from the fact that one of the coefficients of the polynomial $A_{cl}(s)$ is negative no matter what values we choose for a and b.

2.3 Designing Minimal Complexity Stabilizing Controller

Consider a continuous time system with a transfer function $G_o(s)$ defined as

$$G_o(s) = \frac{2}{(s-1)(s+2)} = \frac{2}{s^2+s-2}. \tag{47}$$

Your task is to design a controller of minimal complexity that stabilizes the plant, ensures zero steady state error for step disturbances, and achieves a closed-loop response where natural modes decay at a rate no slower than e^{-3t}.

Solution :
We begin by recalling that we can choose an arbitrary set of closed loop natural frequencies if the closed loop characteristic polynomial, $A_{cl}(s)$, has degree at least equal to $2n-1$ (the degree of $A_o(s)$, the plant nominal model denominator). However, since we want to force zero steady state error at d.c., an additional degree-of-freedom is needed. We are aiming for a minimum complexity controller; hence, we choose the degree of $A_{cl}(s)$ equal to 4.

In addition, the roots of $A_{cl}(s)$ should be to the left of $s=-3$ to ensure that the response time specification is met. Say we choose

$$A_{cl}(s) = (s^2+6s+9)(s+4)(s+5) = s^4+15s^3+83s^2+201s+180 \tag{48}$$

We can now solve the Diophantine equation

$$(s-1)(s+2)\underbrace{s(s+l_0)}_{L(s)}+2\underbrace{(p_2s^2+p_1s+p_0)}_{P(s)} = s^4+15s^3+83s^2+201s+180 \tag{49}$$

where $A_o(s)$ is the plant nominal model denominator, $B_o(s)$ is the plant numerator, $L(s)$ represents the controller's pole polynomial, and $P(s)$ represents the controller's zero polynomial. We finally obtain

$$P(s) = 35.5s^2+114.5s+90 \tag{50}$$

$$L(s) = s^2+14s \tag{51}$$

$$C(s) = \frac{35.5s^2+114.5s+90}{s^2+14s} \tag{52}$$

2.4 Pole Assignment

Consider a nominal model given by

$$G_o(s) = \frac{1}{(s+1)^2} \tag{53}$$

Further assume that we need zero steady state error at zero frequency. We want to specify a third order polynomial $A_{cl}(s)$. From our analysis of the pole assignment methodology, we know that not every third order $A_{cl}(s)$ can achieve stable closed loop. Find the family of stable third degree polynomials that yields a stable closed loop with zero steady state error at d.c. Define the closed-loop system's characteristic equation as $A_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0$. Determine the values for the coefficients a_2, a_1, and a_0 such that the closed-loop system is stable and achieves the required steady state error characteristics.

Select the correct option from the following choices:

(a) $a_0 = 1, a_1 = -1, a_2 = 2$

(b) $a_0 = -1, a_1 = -1, a_2 = 2$

(c) $a_0 = 1, a_1 = 1, a_2 = 2$

(d) $a_0 = 3, a_1 = 1, a_2 = 2$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (c).
Define

$$A_{cl}(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{54}$$

Then, the degree of $L(s)$ must be one and, to satisfy the steady state requirement, it must have integration. Hence the controller has the form

$$C(s) = \frac{p_1 s + p_0}{s} \tag{55}$$

The pole assignment equation is

$$A_o(s)L(s) + B_o(s)P(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{56}$$

$$s(s+1)^2 + (p_1 s + p_0) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{57}$$

$$s^3 + 2s^2 + (p_1 + 1)s + p_0 = s^3 + a_2 s^2 + a_1 s + a_0 \tag{58}$$

From this we see that $a_2 = 2$ assures integration in the controller. Furthermore, since $A_{cl}(s)$ has to be stable, conditions to satisfy this requirement can be obtained using Routh's Algorithm applied to the polynomial $A_{cl}(s) = s^3 + 2s^2 + a_1 s + a_0$. The algorithm yields A_{cl} stable if and only if

$$2a_1 > a_0 \tag{59}$$

$$a_0 > 0 \tag{60}$$

Finally, the family of polynomials satisfying both, stability and integration, is

$$\mathcal{A} = \{s^3 + 2s^2 + a_1 s + a_0 \in \mathbb{R}^3[s] : a_0 > 0, a_1 > 0, 2a_1 - a_0 > 0\} \tag{61}$$

where $\mathbb{R}^3[s]$ is the ring of polynomials in s of degree less or equal to three, with real coefficients.

2.5 Stability of Equilibrium

Consider an ODE

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = -y + u \end{cases} \tag{62}$$

where u is the input. Here we are interested in the stability of the equilibrium solution $(0, 0)$. Consider a linear feedback of the form $u = ax + by$. Can the linearization about the equilibrium be made asymptotically stable for any choice of constants (a, b)?

Solution :
Substituting the feedback into the system yields

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -y + ax + by. \end{aligned}$$

The linearization of the system about the equilibrium point $(0, 0)$ is represented by the Jacobian matrix

$$J = \begin{bmatrix} 0 & 0 \\ a & b-1 \end{bmatrix}.$$

The stability of the equilibrium is determined by the eigenvalues of J, which are solutions to the characteristic equation

$$\text{det}(J - \lambda I) = 0,$$

resulting in eigenvalues $\lambda_1 = 0$ and $\lambda_2 = b - 1$.

For asymptotic stability, all eigenvalues must have strictly negative real parts. However, $\lambda_1 = 0$ does not satisfy this criterion, indicating that the equilibrium $(0, 0)$ cannot be made asymptotically stable through any choice of constants (a, b).

2.6 Routh-Hurwitz Stability

A system has a characteristic equation given by

$$s^3 + Ks^2 + (K+1)s + 6 = 0. \tag{63}$$

Determine the range of K for a stable system. You can use Routh-Hurwitz Stability Criteria.

Select the correct option from the following choices:

(a) $K > 2$
(b) $K < -3$
(c) $K = 0$
(d) $K < 0$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).
The Routh array is structured as follows:

$$\begin{array}{l} s^3 : 1 \quad K+1 \\ s^2 : K \quad 6 \\ s^1 : b \quad 0 \\ s^0 : 6 \end{array}$$

where $b = \frac{K(K+1)-6}{K}$ for the s^1 row. For stability, we require $K > 0$ and $b > 0$. Therefore, using the condition that $b > 0$, we obtain

$$K^2 + K - 6 > 0, \tag{64}$$

and solving for K yields $K > 2$ and $K < -3$. We select $K > 2$, since we also have the condition that $K > 0$.

2.7 Routh-Hurwitz Stability

A system has the second-order characteristic equation

$$s^2 + as + b = 0, \tag{65}$$

where a and b are constant parameters. Determine the necessary and sufficient conditions for the system to be stable. Is it possible to determine the stability of a second-order system just by inspecting the coefficients of the characteristic equation?

Solution :
The characteristic equation is $s^2 + as + b = 0$, so, the Routh array is

$$\begin{array}{ll} s^2 : 1 & b \\ s^1 : a & 0 \\ s^0 : b & \end{array}$$

The system is stable if and only if $a > 0$ and $b > 0$. For a second-order system, a necessary and sufficient condition for stability is that all the coefficients have the same sign.

2.8 Assessing Stability

A system has poles at -1 and -5 and zeros at 1 and -2. Is this system stable?

Solution :
The system is stable since the poles are the roots of the system characteristic equation which have negative real parts. The fact that the system has a zero with a postive real part does not affect its stability.

2.9 Routh-Hurwitz Stability

Determine if the following characteristic equation has any roots with positive real parts:

$$s^4 + s^3 - s - 1 = 0 \tag{66}$$

Solution :
Note that the coefficient of the s^2 term is zero. The Routh table for this equation is

$$\begin{array}{r|lll}
s^4 & 1 & 0 & -1 \\
s^3 & 1 & -1 & 0 \\
s^2 & 1 & -1 & \\
s^1 & 0 & 0 & \\
\text{new } s^1 & 2 & 0 & \\
s^0 & 0 & &
\end{array}$$

The presence of the zeros in the s^1 row indicates that the characteristic equation has two roots which satisfy the auxiliary equation formed from the s^2 row as follows: $s^2 - 1 = 0$. The roots of the equation are $= 1$ and -1. The new s^1 row was formed using the coefficients from the derivative of the auxiliary equation: $2s - 0 = 0$. Since there is one change of sign, the characteristic equation has one root with a positive real part, the one at $+1$ determine from the auxiliary equation.

2.10 Routh-Hurwitz Stability

The characteristic equation of a given system is

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0 \tag{67}$$

What restrictions must be placed upon the parameter K in order to ensure that the system is stable?

Select the correct option from the following choices:

(a) $K < 0$

(b) $K < 10$

(c) $0 < K < 10$

(d) $5 < K < 10$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (c).
The Routh table for this system is

$$\begin{array}{c|ccc} s^4 & 1 & 11 & K \\ s^3 & 6 & 6 & 0 \\ s^2 & 10 & K & 0 \\ s^1 & \frac{60-6K}{10} & 0 & \\ s^0 & K & & \end{array}$$

For the system to be stable. $60 - 6K > 0$. or $K < 10$, and $K > 0$. Thus $0 < K < 10$.

2.11 Routh-Hurwitz Stability

Construct a Routh table and determine the number of roots with positive real parts for the equation

$$2s^3 + 4s^2 + 4s + 12 = 0 \tag{68}$$

Select the correct option from the following choices:

(a) 0

(b) 1

(c) 2

(d) 3

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

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”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

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Solution :
Correct answer is (c).
The Routh table for this equation is given below. Here the s^2 row was divided by 4 before the s^1 row was computed. The s^1 row was then divided by 2 before the s^0 was computed.

$$\begin{array}{c|cc} s^3 & 2 & 4 \\ s^2 & 1 & 3 \\ s^1 & -1 & 0 \\ s^0 & 3 & \end{array}$$

Since there are two changes of sing in the first column of the Routh table, the equation above has two roots with positive real parts.

2.12 Routh-Hurwitz Stability

Use Hurwitz stability and show that for what range of values of K is the system with the following characteristic equation stable?

$$s^2 + Ks + 2K - 1 = 0 \tag{69}$$

Select the correct option from the following choices:

(a) $K > 0.5$

(b) $K > 1$

(c) $K < 0$

(d) $K < -1$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).
The Hurwitz determinants for this system are

$$\Delta_2 = \begin{vmatrix} K & 0 \\ 1 & 2K-1 \end{vmatrix} = 2K^2 - K = K(2K-1) \quad \Delta_1 = K \tag{70}$$

In order for these determinants to be positive, it is necessary that $K > 0$ and $2K - 1 > 0$. Thus the system is stable if $K > \frac{1}{2}$.

2.13 Routh-Hurwitz Stability

Determine the Hurwitz conditions for stability of the following general fourth-order characteristic equation, assuming a_4 is positive.

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \tag{71}$$

Solution :

The Hurwitz determinants are:

$$\begin{aligned} \Delta_1 &= a_3, \\ \Delta_2 &= a_3 a_2 - a_4 a_1, \\ \Delta_3 &= \begin{vmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix}, \\ \Delta_4 &= \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \end{vmatrix}. \end{aligned}$$

For the system to be stable:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \Delta_3 > 0, \quad \Delta_4 > 0$$

2.14 Characterisitic Equation

Prove that a continuous system is unstable if any coefficients of the characteristic equation are zero.

Solution :

The characteristic equation may be written in the form

$$(s - s_1)(s - s_2)(s - s_3)\cdots(s - s_n) = 0 \tag{72}$$

where $s - 1, s_2, \cdots, s_n$ are the roots of the equation. If this equation is multiplied out, n new equations an be obtained relating the roots and the coefficients of the characteristic equation in the usual form. Thus

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0 \tag{73}$$

and the relations are

$$\frac{a_{n-1}}{a_n} = -\sum_{i=1}^{n} s_i,\ \frac{a_{n-2}}{a_n} = \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} s_i s_j,\ \frac{a_{n-3}}{a_n} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ i\neq j\neq k}}^{n} s_i s_j s_k, \cdots, \frac{a_0}{a_n}(-1)^n s_1 s_2 \cdots s_n \tag{74}$$

The coefficients $a_{n-1}, a_{n-2}, \cdots, a_0$ all have the same sign as a_n and are nonzero if all the roots $s_1, s_2, \cdots, s_n$ have negative real parts. The only way any one of the coefficients can be zero is for one or more of the roots to have zero or positive real parts. In either case, the system would be unstable.

2.15 Assessing Stability (Discrete-time system)

What is the transfer function of a discrete system with a gain factor of 2, zeros at 0.2 and -0.5, and poles at 0.5, 0.6, and -0.4? Is it stable?

Select the correct option from the following choices:

(a)
$$P(z) = \frac{(z-0.2)(z+0.5)}{2(z-0.5)(z-0.6)(z+0.4)}$$
; the system is stable

(b)
$$P(z) = \frac{2(z-0.2)(z+0.5)}{(z-0.5)(z-0.6)(z+0.4)}$$
; the system is stable

(c)
$$P(z) = \frac{2(z-0.2)(z+0.5)}{(z-0.5)(z-0.6)(z+0.4)}$$
; the system is not stable

(d)
$$P(z) = \frac{(z-0.2)(z+0.5)}{(z-0.5)(z-0.6)(z+0.4)}$$
; the system is stable

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (b).
The transfer function is
$$P(z) = \frac{2(z-0.2)(z+0.5)}{(z-0.5)(z-0.6)(z+0.4)}$$
Since all the system poles are inside the unit circle, the system is stable.

2.16 Identifying Singular Points in ODEs

Find the singular points of the pair of equations

$$\frac{dx_1}{dt} = \sin x_2 \quad \text{and} \quad \frac{dx_2}{dt} = -x_1 + x_2$$

Select the correct option from the following choices:

(a)

$$x_1 = n\pi, \quad x_2 = n\pi,$$

where $n = 0, 1, 2, \ldots$

(b)

$$x_1 = 2n\pi, \quad x_2 = -n\pi,$$

where $n = 0, 1, 2, \ldots$

(c)

$$x_1 = n\pi, \quad x_2 = 2n\pi,$$

where $n = 0, 1, 2, \ldots$

(d)

$$x_1 = -2n\pi, \quad x_2 = n\pi,$$

where $n = 0, 1, 2, \ldots$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]", "Reason": "[Your explanation]"

‘‘‘

Solution :
Correct answer is (a).
Singular points are found by setting $\frac{dx_1}{dt} = 0$ and $\frac{dx_2}{dt} = 0$. The first equation is satisfied by

$$x_2 = n\pi, \quad n = 0, 1, 2, \ldots$$

The second is satisfied by $x_1 = x_2$. Hence the singular points are defined by

$$x_1 = n\pi, \quad x_2 = n\pi,$$

where $n = 0, 1, 2, \ldots$

2.17 Stability and Time Constant (Sub-Problem 1)

For the systems $G(s) = \frac{5}{(s^2+4s+13)(s-5)}$:

- What are the poles? Is the system stable?
- What is the time constant associated with each pole? Which time constant is the slowest? (Provide a time constant regardless of stability.)

Select the correct option from the following choices:

(a) Poles: $s_{1,2} = -2 \pm 3j$, $s_3 = +5$; system is unstable
Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

(b) Poles: $s_{1,2} = -2 \pm 3j$, $s_3 = -5$; system is stable
Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

(c) Poles: $s_{1,2} = -2 \pm 3j$, $s_3 = +5$; system is stable
Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

(d) Poles: $s_{1,2} = -2 \pm 3j$, $s_3 = +5$; system is unstable
Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, τ_3 is slowest time constant

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).

- **Poles:** $s_{1,2} = -2 \pm 3j$, $s_3 = +5$ system is unstable
  Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

Stability and Time Constant (Sub-Problem 2)

For the systems $G(s) = \frac{s-2}{s+7}$:

- What are the poles? Is the system stable?
- What is the time constant associated with each pole?

Select the correct option from the following choices:

(a) $s = -7$ system is unstable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(b) $s = -7$ system is stable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(c) $s = 7$ system is unstable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(d) $s = 7$ system is stable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (b).

- **Poles:** $s_{1,2} = -2 \pm 3j, s_3 = +5$ system is unstable
  Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

Stability and Time Constant (Sub-Problem 3)

For the systems $G(s) = \frac{s+2}{s-7}$:

- What are the poles? Is the system stable?
- What is the time constant associated with each pole?

Select the correct option from the following choices:

(a) $s = -7$ system is unstable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(b) $s = -7$ system is stable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(c) $s = 7$ system is unstable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

(d) $s = 7$ system is stable
Time constant: $\tau = \frac{1}{|Re(s)|} = \frac{1}{7}$ sec

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (c).

- **Poles:** $s_{1,2} = -2 \pm 3j, s_3 = +5$ system is unstable
  Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

Stability and Time Constant (Sub-Problem 4)

For the systems $G(s) = \frac{-9}{s^2+2s-8}$:

- What are the poles? Is the system stable?
- What is the time constant associated with each pole?

Select the correct option from the following choices:

(a) **Poles:** $s_1 = -4, s_2 = 2$ system is unstable
Time constant: $\tau_1 = \frac{1}{|Re(s)|} = \frac{1}{4}$ sec, $\tau_2 = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, τ_2 is slowest time constant

(b) **Poles:** $s_1 = -4, s_2 = 2$ system is stable
Time constant: $\tau_1 = \frac{1}{|Re(s)|} = \frac{1}{4}$ sec, $\tau_2 = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, τ_2 is slowest time constant

(c) **Poles:** $s_1 = -4, s_2 = -2$ system is stable
Time constant: $\tau_1 = \frac{1}{|Re(s)|} = \frac{1}{4}$ sec, $\tau_2 = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, τ_2 is slowest time constant

(d) **Poles:** $s_1 = 4, s_2 = 2$ system is unstable
Time constant: $\tau_1 = \frac{1}{|Re(s)|} = \frac{1}{4}$ sec, $\tau_2 = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, τ_2 is slowest time constant

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).

- **Poles:** $s_{1,2} = -2 \pm 3j, s_3 = +5$ system is unstable
  Time constant: $\tau_{1,2} = \frac{1}{|Re(s)|} = \frac{1}{2}$ sec, $\tau_3 = \frac{1}{|Re(s)|} = \frac{1}{5}$ sec, $\tau_{1,2}$ is slowest time constant

2.18 Observer

Consider the single-input, single-output transfer function:

$$Y(s) = \frac{s+1}{s^2+2s+2}U(s)$$

(a) Find a second-order state-space model that represents this transfer function.

(b) For this state-space model, calculate a state-feedback controller $u = -Kx + r$ that places the closed-loop poles at -4 and -25.

(c) Construct an observer to estimate x based on the known inputs u and observations y (put the observer poles at $\{-50, -51\}$).

(d) With the controller and observer from the previous problems in place, calculate k_r such that $u = -K\hat{x} + k_r r$ yields a closed-loop system Y/R with unity gain.

Solution :

(a) Recall that for a transfer function:

$$\frac{Q(s)}{P(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

the controllable canonical realization is:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T$$
$$C = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1}b_0 & b_{n-2} - a_{n-2}b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix}$$
$$D = [b_0]$$

Here $b_0 = 0$ and therefore we have that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(b) If we want to place the closed loop poles at -4 and -25 , then the new denominator would be $(s+4)(s+25) = s^2 + 29s + 100$. Therefore, $a_0 + k_1 = 100 \Longrightarrow k_1 = 100 - 2 = 98$ and $a_1 + k_2 = 29 \Longrightarrow k_2 = 29 - 2 = 27$. Hence, $K = \begin{bmatrix} 98 & 27 \end{bmatrix} \Longrightarrow u = -\begin{bmatrix} 98 & 27 \end{bmatrix} x + r$.

(c) For an observer gain L, the observer poles are the eigenvalues of $A \quad LC$, which coincide with the eigenvalues of $A^T - C^T L^T$. Consequently to compute the observer gain in MATLAB, we apply the place command for $\left(A^T, C^T\right)$. It is a rule of thumb, to pick observer poles to be 2-5 times further than the controller poles. Suppose we want to place the observer poles at $\{-50, -51\}$. By using the MATLAB command

$$L = \text{place}\left(A', C', [-50, -51]\right)$$

where $\left(A^T, C^T\right)$ are as above we get $L = \begin{bmatrix} -2449 & 2548 \end{bmatrix}^T$.

(d) To obtain k_r, recall that $\hat{x}(t) \equiv 0$ if $\hat{x}(0) = 0$, so we can ignore the observer. The closed loop system transfer function disregards initial conditions. With full state feedback, the closed loop transfer function is

$$\frac{Y(s)}{R(s)} = C[sI - (A - BK)]^{-1} B k_r$$

To set the DC gain to unity we need

$$1 = C[sI - (A - BK)]^{-1} B k_r \implies k_r = 100$$

2.19 Discrete-Time Implementation

Consider the following plant:

$$\dot{y} + 5y = 3u$$

Suppose a PI controller is used with P gain K_p and I gain K_i. Assume the closed-loop system is stable. Then what sampling time would you recommend for a discrete-time implementation? Write out Δt as a function of K_p and K_i. Use $u(t) = k_p e(t) + K_i \int_0^t e(\tau)d\tau$ with $e(t) = r(t) - y(t)$.

Solution :

Since we are using PI control, $u(t) = k_p e(t) + K_i \int_0^t e(\tau)d\tau$ with $e(t) = r(t) - y(t)$. The closed loop system can be expressed as:

$$\ddot{y} + \left(5 + 3K_p\right)\dot{y} + 3K_i y = 3K_p \dot{r} + 3K_i r.$$

Performing Laplace transform on both sides, we have the closed-loop transfer function

$$T(s) = \frac{3K_p s + 3K_i}{s^2 + \left(5 + 3K_p\right)s + 3K_i}.$$

The poles of the closed-loop system is

$$s_{1,2} = \frac{-\left(5 + 3K_p\right) \pm \sqrt{\left(5 + 3K_p\right)^2 - 12K_i}}{2}.$$

Now we need to consider the following two scenarios: (i) If $\left(5 + 3K_p\right)^2 - 12K_i \geq 0, s_1$ and s_2 are both real, the corresponding time constant is $\tau_1 = \frac{2}{\left|-(5+3K_p)+\sqrt{(5+3K_p)^2-12K_i}\right|}, \tau_2 = \frac{2}{(5+3K_p)+\sqrt{(5+3K_p)^2-12K_i}}$.

For τ_1, we have

$$\Delta t_1 = \frac{1}{5\mid\left(5 + 3K_p\right) - \sqrt{\left(5 + 3K_p\right)^2 - 12K_i}},$$

For τ_2, we have

$$\Delta t_2 = \frac{1}{5\left(5 + 3K_p\right) + 5\sqrt{\left(5 + 3K_p\right)^2 - 12K_i}}.$$

Since $\Delta t_2 \leq \Delta t_1$, we can choose $\Delta t = \Delta t_2$. (ii) If $\left(5 + 3K_p\right)^2 - 12K_i \geq 0, s_1$ and s_2 are a complex conjugate, which has same real part and hence same time constant $\tau = \frac{2}{5+3K_p}$. In this case, we should design $\Delta t = \frac{1}{10}\frac{2}{5+3K_p} = \frac{1}{25+15K_p}$.

3 Time Response of Dynamical Systems

3.1 Steady-State Response

Consider a second-order linear system described by the following ordinary differential equation (ODE):

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = \cos(3t) + 2\sin(4t), \quad x(0) = 0, \quad \dot{x}(0) = 0,$$

where a, b, and c are positive real numbers. The system is known to have a steady-state response of the form:

$$x_{ss}(t) = A\cos(3t - \phi_1) + B\sin(4t - \phi_2),$$

where A, B, ϕ_1, and ϕ_2 are constants.

Determine the values of a, b, and c given that $A = \frac{1}{3\sqrt{2}}$, $B = \frac{1}{2\sqrt{2}}$, $\phi_1 = \frac{\pi}{4}$, and $\phi_2 = \frac{3\pi}{4}$.

Select the correct option from the following choices:

(a) $b = 1$, $c = 12$, and $a = 1$

(b) $b = 12$, $c = 1$, and $a = 1$

(c) $b = 1$, $c = 12$, and $a = -1$

(d) $b = 1$, $c = 1$, and $a = 12$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).

Transfer function of the system is

$$G(s) = \frac{1}{as^2 + bs + c} \tag{75}$$

and

$$\begin{aligned} M(\omega) &= \frac{1}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}}, \\ \phi(\omega) &= -\tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right) \end{aligned}$$

The steady-state response $x_{ss}(t)$ is given by:

$$\begin{aligned} x_{ss}(t) &= 1 \times M(3) \times \cos(3t + \phi(3)) + 2 \times M(4) \times \sin(4t + \phi(4)) \\ &= 1 \times \frac{1}{\sqrt{(c - 9a)^2 + (3b)^2}} \times \cos\left(3t - \tan^{-1}\left(\frac{3b}{c - 9a}\right)\right) \\ &\quad + 2 \times \frac{1}{\sqrt{(c - 16a)^2 + (4b)^2}} \times \sin\left(4t - \tan^{-1}\left(\frac{4b}{c - 16a}\right)\right) \end{aligned}$$

Given the following conditions:

$$\begin{aligned} (c - 9a)^2 + (3b)^2 &= 18 \\ (c - 16a)^2 + (4b)^2 &= 32 \\ \tan^{-1}\left(\frac{3b}{c - 9a}\right) &= \frac{\pi}{4} \Rightarrow c - 9a = 3b \\ \tan^{-1}\left(\frac{4b}{c - 16a}\right) &= \frac{3\pi}{4} \Rightarrow c - 16a = -4b \end{aligned}$$

Solving these equations, we find that $b = 1$, $c = 12$, and $a = 1$.

3.2 Free Response

Show that any free response $y_a(t) = \sum_{k=1}^{n} c_k y_k(t)$ satisfies $\sum_{i=0}^{n} a_i(\frac{d'y}{dt'}) = 0$

Solution :

By the definition of a fundamental set, $y_k(t), k = 1, 2, \cdots, n$, satisfies $\sum_{i=0}^{n} a_i(\frac{d'y_k}{dt'}) = 0$. Substituting $\sum_{k=1}^{n} c_k y_k(t)$ into this differential equation yields

$$\sum_{i=0}^{n} a_i \frac{d'}{dt'} \left[\sum_{k=1}^{n} c_k y_k(t) \right] = \sum_{i=0}^{n} \sum_{k=1}^{n} a_i \frac{d'}{dt'} (c_k y_k(t)) = \sum_{k=1}^{n} c_k \left[\sum_{i=0}^{n} a_i \frac{d' y_k(t)}{dt'} \right] = 0 \qquad (76)$$

The last equality is obtained because the term in the brackets is zero for all k.

3.3 Free Response

Find the free response of the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 4y = u \tag{77}$$

with initial conditions $y(0) = 1$, $(dy/dt)|_{t=0} = 0$ and $(d^2y/dt^2)|_{t=0} = -1$ Select the correct option from the following choices:

(a) $y_a(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}\cos t + \frac{3}{2}e^{-t}\sin t$

(b) $y_a(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t$

(c) $y_a(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{t}\cos t + \frac{3}{2}e^{-4t}\sin t$

(d) $y_a(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t}\cos t + \frac{3}{2}e^{-t}\sin t$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :

Correct answer is (b).

Fundamental set for this equation is $e^{-3t}, e^{-t}\cos t, e^{-t}\sin t$. Hence the free response can be written as

$$y_a(t) = c_1e^{-2t} + c_2e^{-t}\cos t + c_3e^{-t}\sin t \tag{78}$$

The initial conditions provide the following set of algebraic equations for c_1, c_2, c_3:

$$y_a(0) = c_1 + c_2 = 1 \quad \frac{dy_a}{dt}|_{t=0} = -2c_1 - c_2 + c_3 = 0 \quad \frac{d^2y_a}{dt^2}|_{t=0} = 4c_1 - 2c_3 = -1 \tag{79}$$

fro which $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}, c_3 = \frac{3}{2}$. Therefor the free response is

$$y_a(t) = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t \tag{80}$$

3.4 Inputs for Unbounded Outputs in Systems

Determine a bounded input which will produce an unbounded output from an integrator.

Select the correct option from the following choices:

(a) For $t \geq 0$, $u = t$ will produce $y = t^2$, which is unbounded.

(b) For $t \geq 0$, $u = 1$ will produce $y = t^2$, which is unbounded.

(c) For $t \geq 0$, $u = 1$ will produce $y = t$, which is unbounded.

(d) For $t \geq 0$, $u = t$ will produce $y = t$, which is unbounded.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).
The input $u = 1$ will produce the output $y = t$, which is unbounded.

3.5 Step Response

What is the unit step response of a continuous system whose transfer function has a zero at -1, a pole at -2, and a gain factor of 2? Select the correct option from the following choices:

(a) $y(t) = 1 + e^{-t}, \quad t \geq 0$

(b) $y(t) = 2 + e^{-2t}, \quad t \geq 0$

(c) $y(t) = 1 + e^{2t}, \quad t \geq 0$

(d) $y(t) = 1 + e^{-2t}, \quad t \geq 0$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (d).
The Laplace transform of the output is given by $Y(s) = P(s)U(s)$. Here

$$P(s) = \frac{2(s+1)}{s+2} \quad U(s) = \frac{1}{s} \quad Y(s) = \frac{2(s+1)}{s(s+2)} = \frac{1}{s} + \frac{1}{s+2} \tag{81}$$

Evaluating the inverse transform of the partial fraction expansion of $Y(s)$ gives $y(t) = 1 + e^{-2t}$

3.6 Steady State Response

Prove that the steady state output of a stable system with transfer function $P(s)$ and input $u = A\sin(\omega t)$ is given by

$$y_{ss} = A|P(j\omega)|\sin(\omega t + \phi) \quad \text{where} \quad \phi = \arg P(j\omega)$$

Solution :

The Laplace transform of the output is $Y(s) = P(s)U(s) = P(s)\left[\frac{A\omega}{s^2+\omega^2}\right]$. When this transform is expanded into partial fractions, there will be terms due to the poles of $P(s)$ and two terms due to the poles of the input $(s = \pm j\omega)$. Since the system is stable, all time functions resulting from the poles of $P(s)$ decay to zero as time approaches infinity. Thus, the steady state output contains only the time functions resulting from the terms in the partial fraction expansion due to the poles of the input. The Laplace transform of the steady state output is therefore

$$Y_{ss}(s) = \frac{AP(j\omega)}{2j(s - j\omega)} + \frac{AP(-j\omega)}{-2j(s + j\omega)}$$

The inverse transform of this equation is

$$y_{ss} = A|P(j\omega)|\left[\frac{e^{j\phi}e^{j\omega t} - e^{-j\phi}e^{-j\omega t}}{2j}\right] = A|P(j\omega)|\sin(\omega t + \phi) \quad \text{where} \quad \phi = \arg P(j\omega)$$

3.7 Unit Step Response

The output in response to a unit step function input for a particular continuous control system is

$$c(t) = 1 - e^{-t}.$$

What is the delay time T_d? Select the correct option from the following choices:

(a) $-\ln 1.5$

(b) $-\ln 0.5$

(c) $\ln 0.5$

(d) $\ln 1.5$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (b).
The output is given as a function of time. The final value of the output is $\lim_{t\to\infty} c(t) = 1$. Hence T_d (at 50% of the final value) is the solution of

$$0.5 = 1 - e^{-T_d},$$

and is equal to $\log_e(2)$, or 0.693.

3.8 Trajectory

Show that the phase plane trajectory of the solution of the differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

with initial conditions $x(0) = 0$ and $\frac{dx}{dt}\Big|_{t=0} = 1$ is a circle of unit radius centered at the origin.

Solution :
Letting $x = x_1$ and $x_2 = \frac{dx_1}{dt}$, we obtain the pair of equations:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \quad x_1(0) = 0 \\ \frac{dx_2}{dt} &= -x_1, \quad x_2(0) = 1 \end{aligned}$$

We eliminate time as the independent variable by writing

$$\frac{dx_1}{dx_2} = \frac{-x_2}{x_1}$$

or

$$x_1 dx_1 + x_2 dx_2 = 0$$

Integrating this equation for the given initial conditions, we obtain

$$x_1^2 + x_2^2 = 1$$

which is the equation of a circle of unit radius centered at the origin.

3.9 Time Response of ODEs

For the given system $G(s) = \frac{5}{s+7}$:

- What are the poles, zeros, and DC gain?
- What is the general form of the free response?

If any poles are complex then express the free/forced response in its "real" form.
Select the correct option from the following choices:

(a) **Poles:** $s = -7$, **Zeros:** No zeros , **DC Gain:** $\frac{5}{7}$, **Free response:** $y(t) = Ce^{-7t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(b) **Poles:** $s = 7$, **Zeros:** No zeros , **DC Gain:** $\frac{5}{7}$, **Free response:** $y(t) = Ce^{7t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(c) **Poles:** $s = -7$, **Zeros:** $s = 5$, **DC Gain:** $\frac{5}{7}$, **Free response:** $y(t) = 5 + Ce^{-7t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(d) **Poles:** $s = -7$, **Zeros:** $s = -5$, **DC Gain:** $\frac{7}{5}$, **Free response:** $y(t) = -5 + Ce^{-7t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

Solution :
Correct answer is (a).
Poles: $s = -7$
Zeros: No zeros
DC Gain: $G(0) = \frac{5}{7}$
Free response: $y(t) = Ce^{-7t}$
Forced response: $y_p(t)$ depends on $u(t)$

3.10 Time Response of ODEs

For the given system $G(s) = \frac{4s-6}{s^2+2s-3}$:

- What are the poles, zeros, and DC gain?
- What is the general form of the free response?
- What is the general form of the forced response?

If any poles are complex then express the free/forced response in its "real" form. Select the correct option from the following choices:

(a) **Poles:** $s_1 = -3, s_2 = 1$, **Zeros:** $s = \frac{6}{4}$, **DC Gain:** $\frac{6}{3}$, **Free response:** $y(t) = C_1e^{-3t} + C_2e^{t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(b) **Poles:** $s_1 = 3, s_2 = 1$, **Zeros:** $s = \frac{6}{4}$, **DC Gain:** $\frac{6}{3}$, **Free response:** $y(t) = C_1e^{3t} + C_2e^{t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(c) **Poles:** $s_1 = 3, s_2 = -1$, **Zeros:** $s = \frac{6}{4}$, **DC Gain:** $\frac{6}{3}$, **Free response:** $y(t) = C_1e^{-3t} + C_2e^{-t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

(d) **Poles:** $s_1 = -3, s_2 = 1$, **Zeros:** $s = \frac{6}{4}$, **DC Gain:** $\frac{6}{3}$, **Free response:** $y(t) = C_1e^{3t} + C_2e^{-t}$, **Forced response:** $y_p(t)$ depends on $u(t)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

"‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

Solution :
Correct answer is (a).
Poles: $s_1 = -3, s_2 = 1$
Zeros: $s = \frac{6}{4}$
DC Gain: $G(0) = \frac{-6}{-3}$
Free response: $y(t) = C_1e^{-3t} + C_2e^{t}$
Forced response: $y_p(t)$ depends on $u(t)$

3.11 Time Response of ODEs

For the given system $G(s) = \frac{5}{s^2-2s+5}$:

- What are the poles, zeros, and DC gain?
- What is the general form of the free response?
- What is the general form of the forced response?

If any poles are complex then express the free/forced response in its “real” form. Select the correct option from the following choices:

(a) **Poles:** $s_{1,2} = -1 \pm 2j$, **Zeros:** No zeros , **DC Gain:** -1 , **Free response:**$y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{t}\sin(2t)$, **Forced response:** $y_p(t)$ depends on $u(t)$

(b) **Poles:** $s_{1,2} = 1 \pm 2j$, **Zeros:** No zeros , **DC Gain:** 1 , **Free response:**$y(t) = C_1 e^{t}\cos(2t) + C_2 e^{t}\sin(2t)$, **Forced response:** $y_p(t)$ depends on $u(t)$

(c) **Poles:** $s_{1,2} = 1 \pm 2j$, **Zeros:** No zeros , **DC Gain:** 1 , **Free response:**$y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{5t}\sin(2t)$, **Forced response:** $y_p(t)$ depends on $u(t)$

(d) **Poles:** $s_{1,2} = 1 \pm 2j$, **Zeros:** No zeros , **DC Gain:** -1 , **Free response:**$y(t) = C_1 e^{t}\cos(2t + 1) + C_2 e^{-t}\sin(2t)$, **Forced response:** $y_p(t)$ depends on $u(t)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :

Correct answer is (b).
Poles: $s_{1,2} = 1 \pm 2j$
Zeros: No zeros
DC Gain: $G(0) = \frac{5}{5}$
Free response: $y(t) = C_1 e^{t}\cos(2t) + C_2 e^{t}\sin(2t)$
Forced response: $y_p(t)$ depends on $u(t)$

3.12 Sketching Time Response

(a) Roughly sketch the response for the following:

$$\dot{y}(t) + 2y(t) = 4u(t) \quad y(0) = 0, \quad u(t) = 3 \text{ for all } t \geq 0 \tag{82}$$

(b) Roughly sketch the response for the following:

$$\dot{y}(t) - 3y(t) = 2u(t) \quad y(0) = 0, \quad u(t) = 1 \text{ for all } t \geq 0 \tag{83}$$

Solution :

(a)

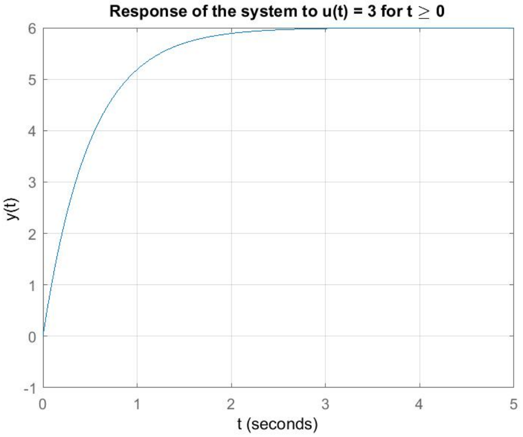


Figure 4: The system response over time.

(b)

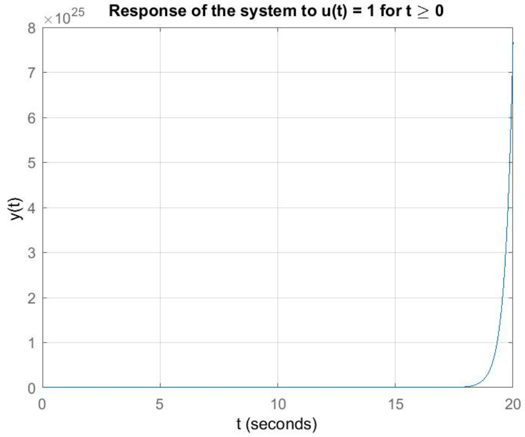


Figure 5: The system response over time.

3.13 Unit Step Response

Four systems and four unit step responses are given below. Match each system to its unit step response.

(a) $G_A(s) = \frac{-2s+10}{s^2+2s+5}$

(b) $G_B(s) = \frac{-10s+10}{s^2+2s+5}$

(c) $G_c(s) = \frac{2s+10}{s^2+2s+5}$

(d) $G_D(s) = \frac{10s+10}{s^2+2s+5}$

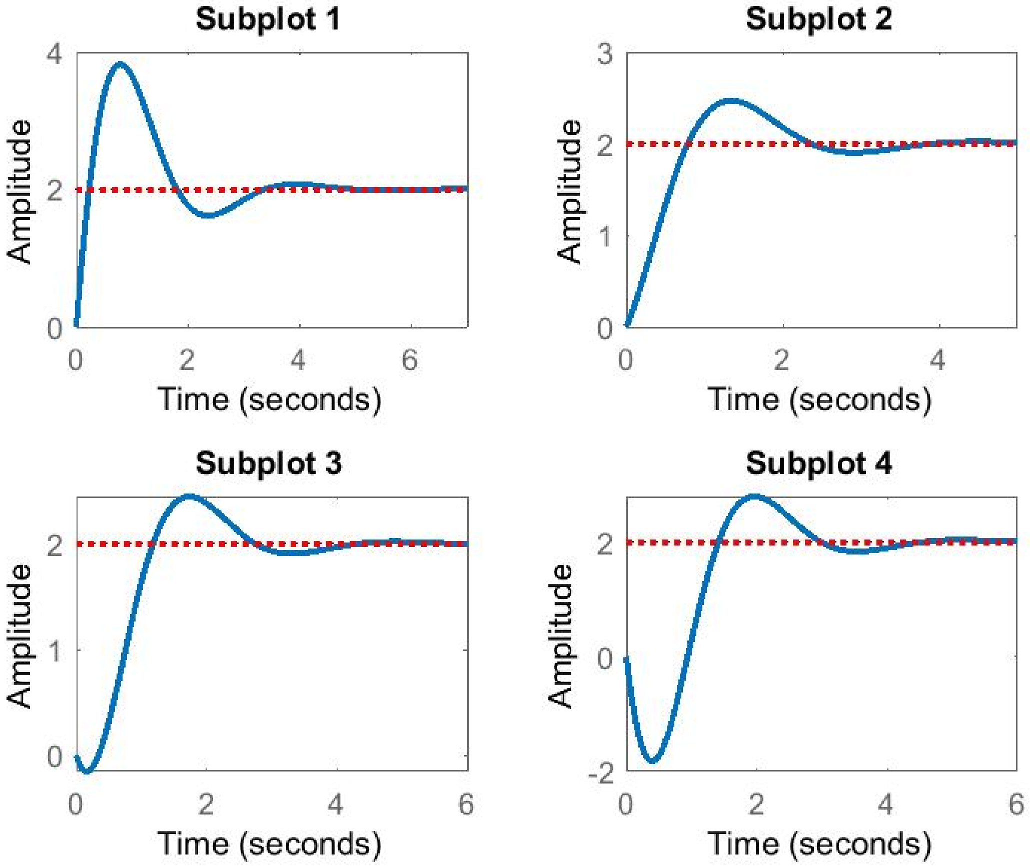


Figure 6: Step responses

Select the correct option from the following choices:

(a) **Subplot1**: $G_B(s)$, **Subplot2**: $G_C(s)$, **Subplot3**: $G_A(s)$, **Subplot4**: $G_D(s)$

(b) **Subplot1**: $G_D(s)$, **Subplot2**: $G_A(s)$, **Subplot3**: $G_C(s)$, **Subplot4**: $G_B(s)$

(c) **Subplot1**: $G_A(s)$, **Subplot2**: $G_B(s)$, **Subplot3**: $G_C(s)$, **Subplot4**: $G_D(s)$

(d) **Subplot1**: $G_D(s)$, **Subplot2**: $G_C(s)$, **Subplot3**: $G_A(s)$, **Subplot4**: $G_B(s)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (d).
Subplot1: $G_D(s)$
Subplot2: $G_C(s)$
Subplot3: $G_A(s)$
Subplot4: $G_B(s)$

3.14 Steady State Response

For the following first-order systems with the sinusoidal input $u(t) = 5\sin(4t + 0.1)$

- What is the magnitude and phase of $G(j\omega)$?
- Is the steady-state response bounded? If yes, what is it?

(a) $-2\dot{y}(t) - y(t) = 3u(t)$

(b) $-2\dot{y}(t) + y(t) = 3u(t)$

Select the correct option from the following choices:

(a) For (a) magnitude is 0.37, and Phase is 1.7, and solution is bounded with $y(t) = 1.85\sin(2t+0.8)$. For (b) magnitude is 0.37, and Phase is 1.44, and solution is not bounded.

(b) For (a) magnitude is 0.37, and Phase is 1.7, and solution is not bounded. For (b) magnitude is 0.37, and Phase is 1.44, and solution is bounded with $y(t) = 1.85\sin(4t + 1.8)$.

(c) For (a) magnitude is 0.37, and Phase is 1.7, and solution is bounded with $y(t) = 1.85\sin(4t+1.8)$. For (b) magnitude is 0.37, and Phase is 1.44, and solution is not bounded.

(d) For (a) magnitude is 0.37, and Phase is 1.44, and solution is bounded with $y(t) = 1.85\sin(4t+1.8)$. For (b) magnitude is 0.37, and Phase is 1.7, and solution is not bounded.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).

(a) $G(s) = \frac{3}{-2s-1} \rightarrow s_1 = \frac{-1}{2} \rightarrow$ system is stable
$\omega = 4 \rightarrow |G(4j)| = 0.37 \quad \text{and} \quad \angle G(j\omega) = 1.7$

- Yes, $y(t) = 1.85\sin(4t + 1.8)$

(b) $G(s) = \frac{3}{-2s+1} \rightarrow s_1 = \frac{1}{2} \rightarrow$ system is unstable
$\omega = 4 \rightarrow |G(4j)| = 0.37$ and $\angle G(j\omega) = 1.44$

- No, response is not bounded.

3.15 Steady State Response

The figure shows the output $y(t)$ generated by a linear system $G(s)$ with input $u(t) = A_0 \cos(\omega_0 t)$.

(1) What are the values of A_0 and ω_0 for the input signal $u(t)$?

(2) What is the magnitude $|G(j\omega_0)|$?

(3) What is the phase $\angle G(j\omega_0)$ in degrees?

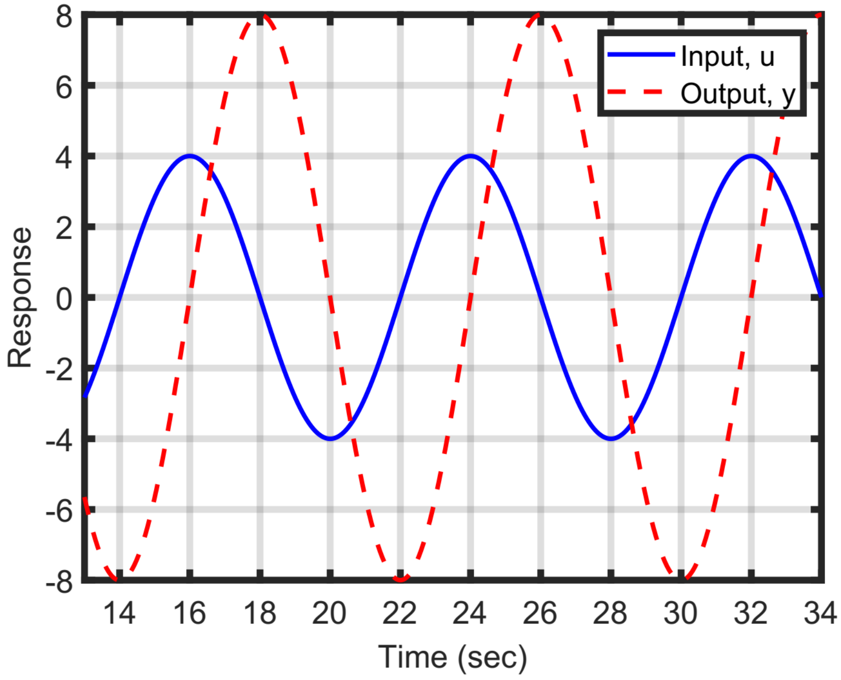


Figure 7: System response

Select the correct option from the following choices:

(a) $A_0 = 4$ and $\omega_0 = \frac{\pi}{4}$ rad/sec. $|G(j\omega_0)| = 2$ and $\angle G(j\omega_0) = -\frac{\pi}{2}$.

(b) $A_0 = 8$ and $\omega_0 = -\frac{\pi}{2}$ rad/sec. $|G(j\omega_0)| = 2$ and $\angle G(j\omega_0) = \frac{\pi}{4}$.

(c) $A_0 = 4$ and $\omega_0 = -\frac{\pi}{4}$ rad/sec. $|G(j\omega_0)| = 2$ and $\angle G(j\omega_0) = \frac{\pi}{2}$.

(d) $A_0 = 4$ and $\omega_0 = \frac{\pi}{4}$ rad/sec. $|G(j\omega_0)| = 8$ and $\angle G(j\omega_0) = -\frac{\pi}{2}$.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).

(1) $A_0 = 4$ and $\omega_0 = \frac{\pi}{4}$ rad/sec

(2) $|G(j\omega_0)| = 2$

(3) $\angle G(j\omega_0) = -\frac{\pi}{2}$

3.16 Zeros or Poles

Let $y(t)$ be the response of a stable, LTI system $G(s)$ due to a sinusoidal input $u(t) = \sin(2t)$. If $y(t)$ converges back to zero in steady-state, then what can we say about the zeros or poles of $G(s)$?

Select the correct option from the following choices:

(a) $G(s)$ has a pole at j

(b) $G(s)$ has a zero at $2j$

(c) $G(s)$ has a pole at $2j$

(d) $G(s)$ has a zero at j

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (b).

$$y(t) = |G(2j)|sin(2t + \angle G(2j)) = 0 \rightarrow |G(2j)| = 0$$

Which we can conclude that $G(s)$ has a zero at $2j$.

3.17 Steady State Response

Consider a first-order LTI system $G(s) = \frac{b_0}{s+a_0}$ where a_0 and b_0 are non-zero numbers. If the phase of $G(j\omega)$ at $\omega = 0$ is expressed in degrees, what values can it have?

Select the correct option from the following choices:

(a) The phase of $G(j\omega)$ at $\omega = 0$ is -90 degrees.

(b) The phase of $G(j\omega)$ at $\omega = 0$ is 90 degrees.

(c) The phase of $G(j\omega)$ at $\omega = 0$ can be either 0 degrees or ± 180 degrees.

(d) The phase of $G(j\omega)$ at $\omega = 0$ is 120 degrees.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).
It can only be 0 degrees or 180 degrees.

3.18 Steady-State Response

Answer the following questions.

(1) Consider the following system $G(s)$ and sinusoidal input:

$$\begin{aligned} &-3\dot{y}(t) - 2y(t) = 7u(t) \\ &u(t) = 6\cos(t+4) \end{aligned}$$

What is the magnitude and phase of $G(1j)$? Is the steady-state output bounded? If yes, what is it?

(2) Consider the following system $G(s)$ and sinusoidal input:

$$\begin{aligned} &\ddot{y}(t) + 0.1\dot{y}(t) + 4y(t) = \dot{u}(t) + 2u(t) \\ &u(t) = -\cos(2t) \end{aligned}$$

What is the magnitude and phase of $G(2j)$? Is the steady-state output bounded? If yes, what is it?

Select the correct option from the following choices:

(a) For (1) $|G(j)| = \frac{7}{\sqrt{13}},\ \angle G(j) = 123.69^\circ$ and $y_{ss}(t) \approx 11.65\cos(t + 6.16)$.
For (2) $|G(2j)| = 10\sqrt{2},\ \angle G(2j) = \frac{\pi}{4}$ and $y_{ss}(t) = -10\sqrt{2}\cos\left(2t + \frac{\pi}{4}\right)$

(b) For (1) $|G(j)| = \frac{7}{13},\ \angle G(j) = 123.69^\circ$ and $y_{ss}(t) \approx 11.65\cos(2t + 6.16)$.
For (2) $|G(2j)| = 10\sqrt{2},\ \angle G(2j) = \frac{-\pi}{4}$ and $y_{ss}(t) = -10\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$

(c) For (1) $|G(j)| = \frac{7}{\sqrt{13}},\ \angle G(j) = 123.69^\circ$ and $y_{ss}(t) \approx 11.65\cos(t + 6.16)$.
For (2) $|G(2j)| = 14,\ \angle G(2j) = \frac{-\pi}{4}$ and $y_{ss}(t) = -14\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$

(d) For (1) $|G(j)| = \frac{7}{\sqrt{13}},\ \angle G(j) = 123.69^\circ$ and $y_{ss}(t) \approx 11.65\cos(t + 6.16)$.
For (2) $|G(2j)| = 10\sqrt{2},\ \angle G(2j) = \frac{-\pi}{4}$ and $y_{ss}(t) = -10\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :

Correct answer is (d).

For a sinusoidal input $u(t) = A\cos(\omega t)$ an LTI system G with zero initial conditions, the system output can be expressed as follows:

$$y(t) = A|G(j\omega)|\cos(\omega t + \angle G(j\omega))$$

(1) We derive the transfer function as $G(s) = \frac{-7}{3s+2}$. This gives $|G(j)| = \frac{7\sqrt{13}}{13}$ and $\angle G(j) = \arctan\left(\frac{3}{-2}\right) = \pi - \arctan(3/2) = 123.69°$. The steady state output is bounded and can be readily obtained:

$$y_{ss}(t) = 6|G(j)|\cos(t + 4 + \angle G(j)) \approx 11.65\cos(t + 6.16)$$

(2) Here the transfer function is given by

$$G(s) = \frac{s+2}{s^2 + 0.1s + 4}$$

and so $|G(2j)| = 10\sqrt{2}$ and $\angle G(2j) = \frac{-\pi}{4}$. Again, the steady state output is bounded and given by:

$$y_{ss}(t) = -10\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$$

3.19 Unbounded Response

Consider the transfer function

$$G(s) = \frac{1}{s^2 + \omega_n^2} \qquad \omega_n > 0$$

and the input $u(t) = sin(\omega t), \omega > 0$. What value of ω in $u(t)$ leads to an unbounded response and why it does so?
Select the correct option from the following choices:

(a) Response becomes unbounded when $\omega \to \omega_n^2$, indicating an unstable pole.

(b) Response becomes unbounded when $\omega \to \omega_n$, indicating a pole-zero cancellation.

(c) Response becomes unbounded when $\omega \to \omega_n$, indicating a resonance condition.

(d) Response becomes unbounded when $\omega^2 \to \omega_n$, indicating a resonance condition.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).

$$G(s) = \frac{1}{s^2 + \omega_n^2} \implies G(j\omega) = \frac{1}{(j\omega)^2 + \omega_n^2} = \frac{1}{\omega_n^2 - \omega^2}$$

$\implies |G(j\omega)| = \left|\frac{1}{\omega_n^2 - \omega^2}\right| \to \infty$ when $\omega \to \omega_n$.

3.20 Step Response

Consider the following transfer function:

$$G(s) = \frac{20e^{-s}}{0.5s + 1}$$

Plot the step response of this system. Identify anything unusual or different in the response and report the reason for this behaviour. Select the correct option from the following choices:

(a) Over damped response as a result of e^{-s} term in the transfer function.

(b) Undershoot as a result of e^{-s} term in the transfer function.

(c) Overshoot as a result of e^{-s} term in the transfer function.

(d) Undershoot as a result of e^{s} term in the transfer function.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

Solution:
Correct answer is (b).
The step response of the given system can be interpreted as the step response of the transfer function $G(s) = \frac{20}{0.5s+1}$, followed by the application of the effect of e^{-s}, which shifts the response by 1 second.

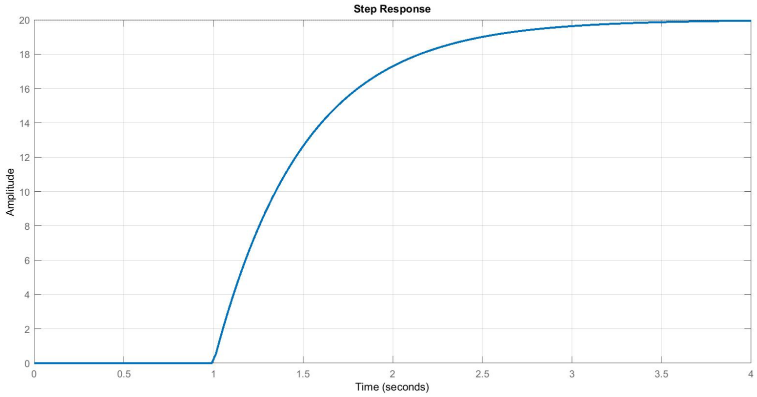


Figure 8: Step response of G(s)

3.21 Step Response

Consider the transfer function:

$$G(s) = \frac{1}{s^2 + s + 1}$$

Which of the following is the corresponding step response? Explain your choice and why you rejected the other two possibilities.

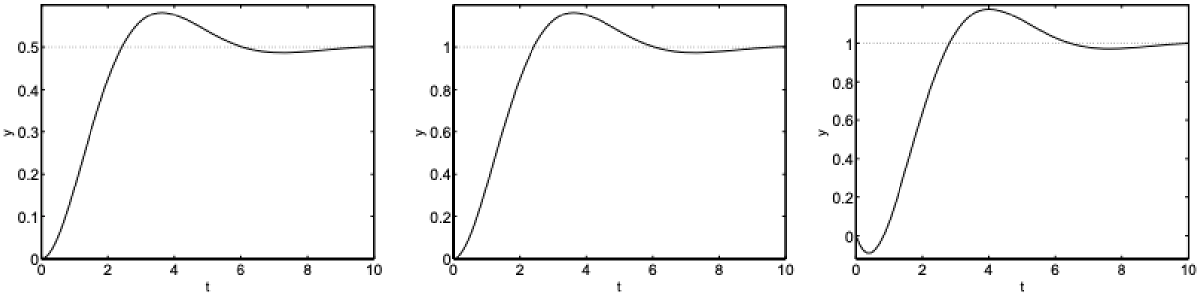


Figure 9: Sample step response plots

Select the correct option from the following choices:

(a) Second plot

(b) First plot

(c) Third plot

(d) Second and third plot

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution:
Correct answer is (a).
The DC gain of H(s) equals 1, so the left plot is immediately rejected because it corresponds to DC gain of 0.5. The right plot show the correct DC gain, but it has an initial dip which indicates the presence of a RHP zero, while H(s) has no zeros. The middle plot is consistent with a stable second-order transfer function with DC gain of 1 and with no zeros, so it is the correct one.

4 Block Diagrams

4.1 Design Pre-Compensators

Consider a feedback system in the below figure with $G(s) = \frac{2}{s+2}$ and $K(s) = \frac{3s+12.5}{s}$.

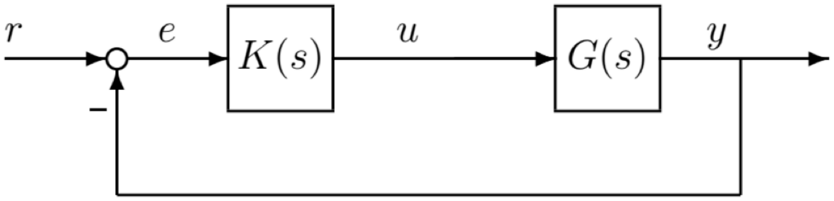


Figure 10: Standard Feedback System

(a) What is the transfer function $T(s)$ from r to y?

(b) Design a low pass $F(s)$ to filter the reference command.

Solution :

(a) $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{6s+25}{s^2+8s+25}$.

(b) Ensuring that $F(0) = 1$, we set $F(s) = \frac{25}{6s+25}$. The resulting closed-loop from $r \to y$ is now $T(s)F(s) = \frac{25}{s^2+8s+25}$.

4.2 Feedback System Design for Error Specifications

Consider the general feedback system in the below figure.

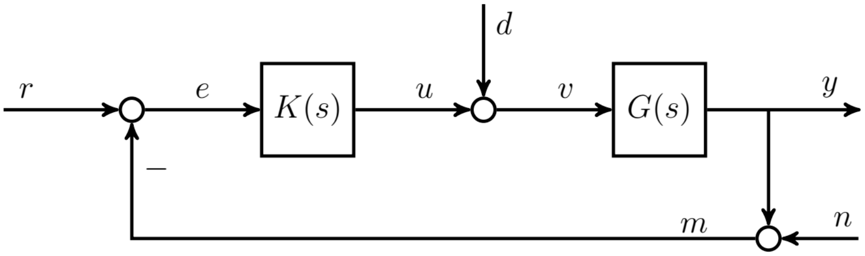


Figure 11: Standard Feedback System

(a) The closed-loop should have zero steady-state error for step reference commands. Translate this into a requirement on the sensitivity $S(s)$. What does this imply about the $L(s) = G(s)K(s)$?

(b) The closed-loop should have an error ≤ 0.1 for reference commands $r(t) = 2\sin(\omega t)$ with $\omega \leq 10$ rad/sec. Translate this into a requirement on the sensitivity $S(s)$.

(c) The closed-loop should have an error ≤ 0.05 for noise $n(t) = 10\sin(\omega t)$ with $\omega \geq 1000$ rad/sec. Translate this into a requirement on the complementary sensitivity $T(s)$.

Solution :

(a) Since $S(s) = \frac{1}{1+L(s)}$ and we desire that $S(0) = 0$, then we must have $L(0) = +\infty$. $L(s)$ must have a pole at $s = 0$.

(b) $e(t) \to 2|S(j\omega)|\sin(\omega t + \measuredangle S(j\omega)) \leq 0.1$ implies that $|S(j\omega)| \leq 0.05$ for $\omega \leq 10$.

(c) $y(t) \to -10|T(j\omega)|\sin(\omega t + \measuredangle T(j\omega)) \leq 0.05$ implies that $|T(j\omega)| \leq 0.05/10 = 0.005$ for $\omega \geq 1000$.

4.3 Overshoot Constraints on Proportional Gains

For the feedback system shown below determine the range of proportional gains K so that the overshoot of the closed-loop system (in response to the unit step reference input) is no more than 10%.

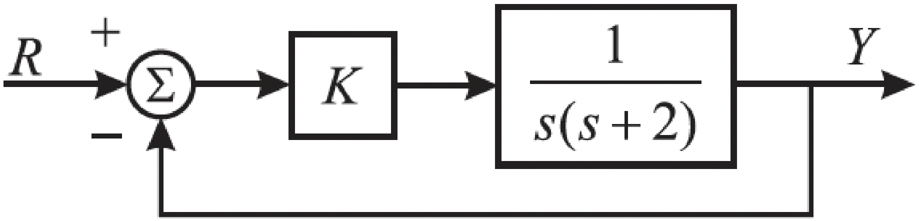


Figure 12: Standard Feedback System

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

(a) $0 \leq K \leq 5$

(b) $0 \leq K \leq 2.86$

(c) $-2.86 \leq K \leq 2.86$

(d) $-5 \leq K \leq 5$

Solution :
Correct answer is (b).
The closed loop transfer function from R to Y is given by:

$$T(s) = \frac{K}{s^2 + 2s + K}$$

$\implies \omega_n^2 = K, \quad \zeta = \frac{1}{\sqrt{K}}$ or $K = \frac{1}{\zeta^2}$.
$M_p \leq 10\% \implies \zeta \geq 0.59$. Therefore, $0 \leq K \leq 2.86$

4.4 Tracking a Ramp Reference

The plant transfer function $G_p(s) = \frac{1}{s^2}$ shown below describes the angle of a pendulum without damping.

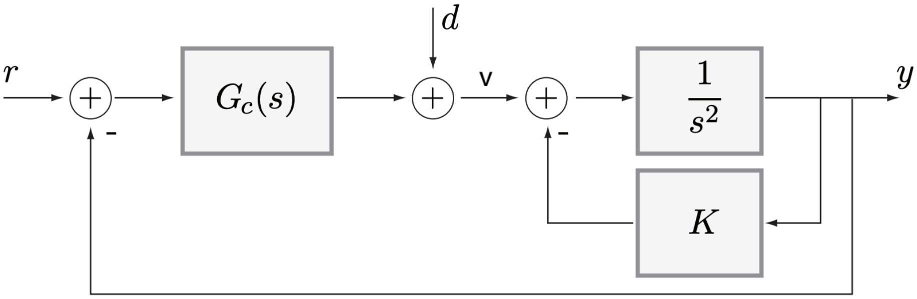


Figure 13: Representation of an Undamped Pendulum Dynamics

(a) What conditions must G_c satisfy to ensure that the system can track a ramp reference input with finite steady-state error?

(b) Find a compensator that satisfies the conditions of (a). What class of disturbances d(t) will the system reject perfectly?

Solution :

(a) The plant transfer function G_p is given by:

$$G_p(s) = \frac{Y}{V} = \frac{1}{s^2 + K}$$

Let $G = G_c(s)G_p(s)$ then with $e = r - y$, we have $\frac{E}{R} = \frac{1}{1+G}$.
Since $r(t) = t \implies R(s) = \frac{1}{s^2}$, the steady state error is given by:

$$e_\infty = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{s(1 + G(s))}$$

Therefore, $G_c(s)$ must have an integrator controller and we can choose a PID controller.

(b) We can try a PID controller with $G_c(s) = K_p + \frac{K_i}{s} + K_d s$. The closed-loop transfer function $\frac{Y}{d}$ is given by:

$$\frac{Y}{d} = \frac{G_p}{1 + G} = \frac{s}{s^3 + K_d s^2 + (K + K_p)s + K_i}$$

In order to place all the poles at -1, we set $K_i = 1$, $K_p = 3 - K$ and $K_d = 3$. This PID compensator will perfectly reject DC disturbances.

4.5 Stability Analysis

Consider two systems in a negative feedback configuration shown below:

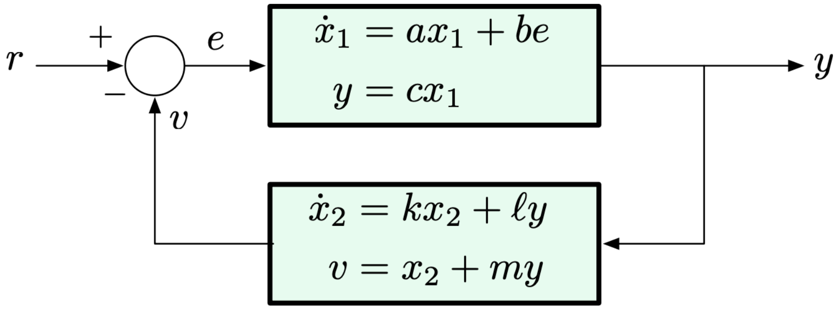


Figure 14: Negative Feedback Configuration

Write down the conditions that must be satisfied by the system parameters a, b, c, k, l, m for this transfer function to be stable (i.e., for all poles to have negative real parts).
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a)

$$\begin{cases} bcm + a + k > 0 \\ ak + bcl - bckm > 0 \end{cases}$$

(b)

$$\begin{cases} bcm - a - k > 0 \\ ak + bcl - bckm > 0 \end{cases}$$

(c)

$$\begin{cases} bcm - a - k > 0 \\ ak + bcl + bckm > 0 \end{cases}$$

(d)

$$\begin{cases} bcm + a + k > 0 \\ ak + bcl + bckm > 0 \end{cases}$$

Solution :

Correct answer is (b).

Notice that the denominator of the transfer function is a monic second order polynomial in the form $p(s) = s^2 + a_1 s + a_2$

In order for the system to be stable, we need all roots of (5) to be in the OLHP. When the determinant of (5) is non-negative, that is, $a_1^2 - 4a_2 \geq 0$, we need both roots to be negative:

$$\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} < 0 \Rightarrow a_1 > 0, 0 < a_2 \leq \frac{a_1^2}{4}$$

When $a_1^2 - 4a_2 < 0$, we need the real part of the roots to be negative:

$$\frac{-a_1}{2} < 0 \Rightarrow a_1 > 0, a_2 > \frac{a_1^2}{4}$$

Hence it can be summarized that we need both $a_1 > 0, a_2 > 0$ in order for the system to be stable. This condition can be directly obtained by using Routh-Hurwitz criterion. Substitute a_1, a_2 with the system parameters given, all we need is:

$$\begin{cases} bcm - a - k > 0 \\ ak + bcl - bckm > 0 \end{cases}$$

5 Control System Design

5.1 PI Control Design

Consider a plant with a nominal model given by

$$G(s) = \frac{1}{s+2} \tag{84}$$

Compute the parameters K_p and K_i of a PI controller so that the natural modes of the closed loop response decay as fast as e^{-5t}.
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $K_p > 8, K_i < 25$

(b) $K_p = 8, K_i < 25$

(c) $K_p < 8, K_i \geq 25$

(d) $K_p = 8, K_i \geq 25$

Solution :
Correct answer is (d).
A PI controller has a transfer function given by

$$C(s) = \frac{K_p s + K_i}{s}; \tag{85}$$

The closed-loop characteristic polynomial, $A_{cl}(s)$, is derived as

$$A_{cl}(s) = \text{numerator of } \{1 + G(s)C(s)\} = s^2 + (2 + K_p)s + K_i. \tag{86}$$

To achieve a closed-loop transient response that decays as fast as e^{-5t}, the controller must generate a pair of complex conjugate poles with real parts equal to -5. This requirement determines the values of K_p and K_i, with K_p needing to be 8 and K_i needing to be greater than or equal to 25. Therefore, the appropriate PI controller parameters are $K_p = 8$ and $K_i \geq 25$.

5.2 Pole Assignment

A plant with nominal model

$$G_o(s) = \frac{1}{(s+1)^2} \tag{87}$$

is in a feedback loop under control with a PI controller having transfer function

$$C(s) = \frac{K_p s + K_i}{s} \tag{88}$$

Determine whether this controller configuration can be used to achieve full pole placement.

Solution :
The plant is of second-order, and the controller introduces an additional third pole. However, the controller only have two gains, corresponding to two numerator coefficients, which generally precludes it from being used to place all three poles effectively.

5.3 Minimal-Order Control Design

Consider a nominal model for a plant given by:

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{1}{(s+1)(s-2)} \tag{89}$$

The closed-loop characteristic polynomial is defined as:

$$A_{cl}(s) = (s+2)(s+3)(s+4)(s+5) \tag{90}$$

Design a controller $C(s)$ that satisfies the following conditions:
- Results in the given closed-loop characteristic polynomial.
- Has the minimum order numerator to ensure system stability.
- Express both the numerator and the denominator of $C(s)$ as expanded polynomials.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $C(s) = \frac{88s^2+184s+120}{s^2+15s}$

(b) $C(s) = \frac{272s+296}{s^2+15s+88}$

(c) $C(s) = \frac{40s^2+232s+216}{s^2+15s+48}$

(d) $C(s) = \frac{48s^2+224s+200}{s^2+15s+40}$

Solution :
Correct answer is (b).
The degree of $A_{cl}(s)$ is four and the degree of $A_o(s)$ is two. Hence $L(s)$ should have a degree equal to two. Then

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s^2 + \lambda_1 s + \lambda_0} \tag{91}$$

The corresponding pole assignment equation becomes

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4)(s+5) \tag{92}$$

$$(s+1)(s-2)(s^2+\lambda_1 s+\lambda_0)+(p_2s^2+p_1s+p_0)=(s+2)(s+3)(s+4)(s+5) \tag{93}$$

$$s^4+(\lambda_1-1)s^3+(p_2-\lambda_1+\lambda_0-2)s^2+(p_1-\lambda_0-2\lambda_1)s+(p_0-2\lambda_0)=s^4+14s^3+71s^2+154s+120 \tag{94}$$

This polynomial identity leads to the equations

$$\lambda_1-1=14 \tag{95}$$

$$p_2-\lambda_1+\lambda_0-2=71 \tag{96}$$

$$p_1-\lambda_0-2\lambda_1=154 \tag{97}$$

$$p_0-2\lambda_0=120 \tag{98}$$

The solution of these equations is

$$\lambda_1=15 \tag{99}$$

$$p_2=73+\lambda_1-\lambda_0=88-\lambda_0 \tag{100}$$

$$p_1=154+2\lambda_1+\lambda_0=184+\lambda_0 \tag{101}$$

$$p_0=120+2\lambda_0 \tag{102}$$

We thus observe that there is an infinite number of solutions. Every choice of λ_0 leads to a different stabilizing controller, for example, to force integration in the controller we can choose $\lambda_0=0$

$$C(s)=\frac{88s^2+184s+120}{s^2+15s} \tag{103}$$

In MATLAB we can solve this problem with the command line

```
>> [L,P]=paq([1 -1 -2],1,[1 14 71 154 120])
```

This leads to

$$C(s)=\frac{272s+296}{s^2+15s+88} \tag{104}$$

This is equivalent to choosing $\lambda_0=88$. The algorithm in `paq.m` has been designed to yield a minimum degree $P(s)$.

5.4 Pole Placement

Consider a nominal model given by:

$$G_o(s) = \frac{3s+1}{(s+2)(s-3)} \tag{105}$$

The goal is to design a control law that tracks a constant reference and cancels the pole at $s = -2$ in $G_o(s)$. Design a suitable controller. If pole placement is required for the characteristic equation, start with a pole at $s = -2$ and for any additional poles required, increase their magnitude sequentially (e.g., for three required poles, assign them at -2, -3, and -4; and so forth).
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $C(s) = \frac{(s+2)(-44s+300)}{s(5s+207)}$

(b) $C(s) = \frac{(s+2)}{s(5s+207)}$

(c) $C(s) = \frac{(-44s+300)}{(5s+207)}$

(d) $C(s) = \frac{(-44s+300)}{s(5s+207)}$

Solution :
Correct answer is (a).
The minimum degree of $A_{cl}(s)$ is four since we need to force integration in the controller, then

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s(s+\lambda_1)} \tag{106}$$

Since $(s+2)$ is cancelled if and only if $(s+2)$ is a factor of $A_{cl}(s)$, we choose

$$A_{cl}(s) = (s+2)(s+3)(s+4)(s+5) \tag{107}$$

The pole assignment equation then becomes

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4)(s+5) \tag{108}$$

$$(s+2)(s-3)(s+\lambda_1)s + (3s+1)(p_2 s^2 + p_1 s + p_0) = (s+2)(s+3)(s+4)(s+5) \tag{109}$$

In this polynomial identity we note that $(s+2)$ has to be a factor of $P(s)$ (a plant pole can only be cancelled by a controller zero). Thus we define

$$(s+2)(\tilde{p}_1 s + \tilde{p}_0) = p_2 s^2 + p_1 s + p_0 \tag{110}$$

The pole assignment equation then simplifies as follows:

$$(s-3)(s+\lambda_1)s + (3s+1)(\tilde{p}_1 s + \tilde{p}_0) = (s+3)(s+4)(s+5) \tag{111}$$

$$s^3 + (3\tilde{p}_1 + \lambda_1 - 3)s^2 + (\tilde{p}_1 + 3\tilde{p}_0 - 3\lambda_1)s + \tilde{p}_0 = s^3 + 12s^2 + 47s + 60 \tag{112}$$

This leads to equations

$$3\tilde{p}_1 + \lambda_1 - 3 = 12 \tag{113}$$

$$\tilde{p}_1 + 3\tilde{p}_0 - 3\lambda_1 = 47 \tag{114}$$

$$\tilde{p}_0 = 60 \tag{115}$$

The solution is $\tilde{p}_0 = 60$, $\tilde{p}_1 = -\frac{44}{5}$ and $\lambda_1 = \frac{207}{5}$. Finally the controller is

$$C(s) = \frac{(s+2)(-44s+300)}{s(5s+207)} \tag{116}$$

5.5 Proportional Controller

Answer True or False to the following questions. Also justify your answers.

(a) Let $y(t)$ be the response of an LTI system $G(s)$ with zero initial conditions due to the input $u(t)$. The response of $G(s)$ with nonzero initial conditions and input $2 * u(t)$ will always be equal to $2 * y(t)$.

(b) An LTI system $G(s)$ with a zero in the right half plane will always have an unbounded step response.

(c) Let $G(s)$ be an LTI system with one real pole that has a time constant five times larger than all the other poles. A dominant pole approximation for $G(s)$ will be reasonably accurate.

(d) Let $G(s)$ be a second-order LTI system and $K(s)$ be a PID controller designed to place all closed-loop poles in the left half plane. If the derivative term is implemented using an approximate derivative then the resulting closed-loop will be fourth order.

(e) Consider a control system with the plant $G(s) = \frac{5}{s^2+s}$ and a proportional controller with gain k_p. It is possible to choose k_p so that the closed-loop is stable and has zero steady-state error due to a unit step reference command.

Solution :

(a) False

(b) False

(c) True

(d) True

(e) True

5.6 PI Controller

Consider a feedback system with the plant and controller:

$$G(s) = \frac{2}{s+2}, \qquad K(s) = \frac{s+0.5}{s}.$$

What type of controller has the transfer function $K(s)$?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) PI Controller with $K_p = 0.5$ and $K_i = 1$

(b) P Controller with $K_p = 1$

(c) PI Controller with $K_p = 1$ and $K_i = 0.5$

(d) P Controller with $K_p = 0.5$

Solution :
$K(s)$ is a PI controller with $K_p = 1$ and $K_i = 0.5$.

5.7 Proportional Controller

Consider the following plant:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 20u(t) + 10d(t) \tag{117}$$

(a) What is the model from inputs (r, d) to output y if we use an open-loop controller $u(t) = K_{ol}r(t)$?

(b) Can the gain K_{ol} be selected so that the control system is overdamped from r to y?

(c) Select K_{ol} so that $y(t) \to \bar{r}$ when $r(t) = \bar{r}$ and $d(t) = 0$

(d) What is the impact of the disturbance $d(t)$ when $r(t) = 2$ and $d(t) = 1$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 20K_{ol}r(t) + 10d(t)$ b) Yes c) $K_{ol} = \frac{1}{10}$ d) Steady-state error

(b) a) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 20K_{ol}r(t) + 10d(t)$ b) No c) $K_{ol} = \frac{1}{2}$ d) Steady-state error

(c) a) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 20K_{ol}r(t) + 10d(t)$ b) Yes c) $K_{ol} = \frac{1}{2}$ d) Increasing settling time

(d) a) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 30d(t)$ b) No c) $K_{ol} = 0$ d) Steady-state error

Solution :
Correct answer is (b).

(a) $\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 20K_{ol}r(t) + 10d(t)$

(b) No, it is underdamped for any choice of K_{ol}.

(c) Set $\ddot{\bar{y}} = \dot{\bar{y}} = 0 \to 10\bar{y} = 20K_{ol}\bar{r}$, thus $K_{ol} = \frac{1}{2}$

(d) It causes error in steady state solution.

5.8 Proportional Controller

Consider the following plant:

$$2\dot{y}(t) + 3y(t) = -4u(t) + d(t) \tag{118}$$

(a) What is the model from inputs (r, d) to output y if we use an proportional controller $u(t) = K_p(r(t) - y(t))$?

(b) Select K_p so that the steady-state error $\bar{e} = \bar{r} - \bar{y}$ is less than 0.1 when $r(t) = \bar{r} = 2$ and $d(t) = \bar{d} = 1$.

(c) What is the time constant of the closed-loop?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $2\dot{y}(t) + (3 + 4K_p)y(t) = +4K_p r(t) + d(t)$ b) $K_p > 11.75$ c) $\tau = \frac{2}{3+4K_p}$

(b) a) $2\dot{y}(t) + (3 + 4K_p)y(t) = -4K_p r(t) + d(t)$ b) $K_p < 11.75$ c) $\tau = \frac{2}{3+4K_p}$

(c) a) $2\dot{y}(t) + (3 - 4K_p)y(t) = -4K_p r(t) + d(t)$ b) $K_p > -11.75$ c) $\tau = \frac{2}{3-4K_p}$

(d) a) $2\dot{y}(t) + (3 - 4K_p)y(t) = -4K_p r(t) + d(t)$ b) $K_p < -11.75$ c) $\tau = \frac{2}{3-4K_p}$

Solution :
Correct answer is (d).

(a) $2\dot{y}(t) + (3 - 4K_p)y(t) = -4K_p r(t) + d(t)$

(b) The closed-loop is stable if and only if $K_p < \frac{3}{4}$. Also, in steady-state we have:

$$\bar{e} = \bar{r} - \bar{y} = \left(1 + \frac{4K_p}{3 - 4K_p}\right)\bar{r} - \frac{1}{3 - 4K_p}\bar{d} \tag{119}$$

$$= \left(1 + \frac{4K_p}{3 - 4K_p}\right)2 - \frac{1}{3 - 4K_p}1 \tag{120}$$

$$= 2 + \frac{8K_p - 1}{3 - 4K_p} < 0.1 \tag{121}$$

Thus,

$$K_p < -11.75 \tag{122}$$

(c) $\tau = \frac{2}{3-4K_p}$

5.9 PI Controller

Consider the following plant:

$$2\dot{y}(t) + 6y(t) = 8u(t) \tag{123}$$

with a PI controller in the following form:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau \tag{124}$$

where $e(t) = r(t) - y(t)$

(a) What is the ODE model for the closed-loop from $r(t)$ to $y(t)$?

(b) Choose (K_p, K_i) so that the closed-loop :

- Stable
- Settling time $t_s = \frac{3}{\zeta\omega_n}$ is 3
- damping ratio $\zeta = 0.5$

(c) How will rising time, overshoot and settling time change if K_i is increased further?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $2\ddot{y}(t) + (6 + 8K_p)\dot{y}(t) + 8K_i y(t) = 8K_p\dot{r}(t) + 8K_i r(t)$ b) $K_p = -\frac{1}{4} \quad K_i = 1$ c) 1. rising time ↓ 2. overshoot ↓ 3. settling time ↑

(b) a) $2\ddot{y}(t) + (6 + 8K_p)\dot{y}(t) + 8K_i y(t) = 8K_p\dot{r}(t) + 8K_i r(t)$ b) $K_p = \frac{1}{4} \quad K_i = 1$ c) 1. rising time ↑ 2. overshoot ↓ 3. settling time ↑

(c) a) $2\ddot{y}(t) + (6 + 8K_p)\dot{y}(t) + 8K_i y(t) = 8K_i r(t)$ b) $K_p = \frac{1}{4} \quad K_i = \frac{1}{2}$ c) 1. rising time ↓ 2. overshoot ↓ 3. settling time ↓

(d) a) $2\ddot{y}(t) + (6 + 8K_p)\dot{y}(t) + 8K_i y(t) = 8K_p\dot{r}(t) + 8K_i r(t)$ b) $K_p = -\frac{1}{4} \quad K_i = 1$ c) 1. rising time ↑ 2. overshoot ↓ 3. settling time, does not change

Solution :
Correct answer is (d).

(a) $2\ddot{y}(t) + (6 + 8K_p)\dot{y}(t) + 8K_i y(t) = 8K_p\dot{r}(t) + 8K_i r(t)$

(b) $\zeta\omega_n = 1$ and $\zeta = 0.5 \rightarrow \omega_n = 2$
Thus,

$$K_p = -\frac{1}{4} \quad K_i = 1 \tag{125}$$

(c) $2\zeta\omega_n$ remains unchanged and ω_n increases. Increasing K_i makes the following change:

- rising time ↑
- overshoot ↓
- settling time, does not change

5.10 PD Controller

Consider the following plant:

$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = u(t) \tag{126}$$

with a PD controller in the following form:

$$u(t) = K_p(r(t) - y(t)) - K_d\dot{y}(t) \tag{127}$$

(a) What is the ODE model for the closd-loop from $r(t)$ to $y(t)$?

(b) Choose (K_p, K_d) so that the closed-loop is:

- stable
- has (ω_n, ζ) = (2rad/sec,0.5)

(c) What is the steady-state error if r is a unit step reference?

(d) Would you increase or decrease K_p to reduce the steady-state error?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

(a) a) $\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t)$ b) $K_p = 3$ and $K_d = 4$ c) $\bar{e} = \bar{r} - \bar{y} = \frac{1}{4}$ d) Increasing K_p

(b) a) $\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t)$ b) $K_p = 3$ and $K_d = 4$ c) $\bar{e} = \bar{r} - \bar{y} = \frac{1}{4}$ d) Decreasing K_p

(c) a) $\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t)$ b) $K_p = 4$ and $K_d = 3$ c) $\bar{e} = \bar{r} - \bar{y} = \frac{1}{4}$ d) Increasing K_p

(d) a) $\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t)$ b) $K_p = \frac{1}{3}$ and $K_d = \frac{1}{4}$ c) $\bar{e} = \bar{r} - \bar{y} = 1$ d) Decreasing K_p

Solution :
Correct answer is (a).

(a) $\ddot{y}(t) + (K_d - 2)\dot{y}(t) + (K_p + 1)y(t) = K_p r(t)$

(b) $K_p = 3$ and $K_d = 4$

(c) $4\bar{y} = 3\bar{r} = 3 \rightarrow \bar{y} = \frac{3}{4}$
Thus,

$$\bar{e} = \bar{r} - \bar{y} = \frac{1}{4} \tag{128}$$

(d) $\bar{y} = \frac{K_p}{K_p+1}\bar{r}$. Thus, increasing K_p reduce steady-state error.

5.11 PID Controller

Consider the plant with the following transfer function:

$$G(s) = \frac{505}{s^3 + 21s^2 + 121s + 101} \tag{129}$$

The poles of this system are at $s = -1, -10 \pm j$.

(a) What is the dominant pole approximation $G_a(s)$ for this plant?

(b) Would you recommend using a PI, PD, or PID controller?

(c) Choose the controller gains so that the closed-loop with $G(s)$ has poles repeated at $s = -1$

Solution :

(a) Set $G_a(s) = \frac{b_0}{s+1}$. Since $G(0) = \frac{505}{101}$, we have $b_0 = 5$. Thus,

$$G_a(s) = \frac{5}{s+1} \tag{130}$$

The time-domain approximation of $G_a(s)$ is:

$$\dot{y}(t) + y(t) = 5u(t) \tag{131}$$

(b) PI controller

(c) Desired characteristic equation:

$$(s+1)^2 = 0$$

Using the following PI controller:

$$u(t) = K_p(r(t) - y(t)) + K_i \int (r(t) - y(t))$$

we have:

$$\ddot{y}(t) + (1 + 5K_p)\dot{y}(t) + 5K_i y(t) = 5K_p \dot{r}(t) + 5K_i r(t)$$

Thus: $K_p = 0.2$ and $K_i = 0.2$

5.12 PI Controller

Consider the plant with the following transfer function:

$$G(s) = \frac{505}{s^3 + 21s^2 + 121s + 101} \tag{132}$$

The poles of this system are at $s = -1, -10 \pm j$.

(a) What is the dominant pole approximation $G_a(s)$ for this plant?

(b) Choose the controller gains so that the closed-loop with $G(s)$ has poles repeated at $s = -2$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $G_a(s) = \frac{505}{s+1}$ b) $K_p = 0.6, K_i = 0.8$

(b) a) $G_a(s) = \frac{505}{s+1}$ b) $K_p = 0.6, K_i = 8$

(c) a) $G_a(s) = \frac{5}{s+1}$ b) $K_p = 6, K_i = 0.8$

(d) a) $G_a(s) = \frac{5}{s+1}$ b) $K_p = 0.6, K_i = 0.8$

Solution :
Correct answer is (d).

(a) Set $G_a(s) = \frac{b_0}{s+1}$. Since $G(0) = \frac{505}{101}$, we have $b_0 = 5$. Thus,

$$G_a(s) = \frac{5}{s+1} \tag{133}$$

The time-domain approximation of $G_a(s)$ is:

$$\dot{y}(t) + y(t) = 5u(t) \tag{134}$$

(b) Desired characteristic equation:

$$(s+2)^2 = 0$$

Using the following PI controller:

$$u(t) = K_p(r(t) - y(t)) + K_i \int (r(t) - y(t))$$

we have:

$$\ddot{y}(t) + (1 + 5K_p)\dot{y}(t) + 5K_i y(t) = 5K_p \dot{r}(t) + 5K_i r(t)$$

Thus: $K_p = 0.6$ and $K_i = 0.8$

5.13 PID Controller

Consider the plant with the following transfer function:

$$G(s) = \frac{20}{s^2 - 6s + 10} \tag{135}$$

(a) What is the closed-loop ODE from reference r to output y if we use the following PID controller?

$$u(t) = K_p e(t) + K_i \int e(t) + K_d \dot{e}(t)$$

where $e(t) = r(t) - y(t)$

(b) Choose the controller gains so that the closed-loop has poles repeated at $s = -3$. Hint: $(s+3)^3 = s^3 + 9s^2 + 27s + 27$

(c) What is the impact of the implementing the derivative term $K_d \dot{e}(t)$ as versus the rate feedback from $-K_d \dot{y}(t)$?

Solution :

(a) $G(s)$ in the time-domain can be written as

$$\ddot{y}(t) - 6\dot{y}(t) + 10y(t) = 20u(t)$$

using the PID controller $u(t)$, we have:

$$\dddot{y}(t) + (20K_d - 6)\ddot{y}(t) + (10 + 20K_p)\dot{y}(t) + 20K_i y(t) = 20K_d \ddot{r}(t) + 20K_p \dot{r}(t) + 20K_i r(t)$$

(b) we have:

$$K_p = \frac{17}{20}, K_i = \frac{27}{20}, K_d = \frac{15}{20}$$

(c) $K_d \dot{e}(t)$ causes an extra term in the numerator of the transfer function (extra zero). If $\dot{r}(t)$ changes fast, this causes overshoot/oscillatory.

5.14 PI Controller

Consider the following plant and PI controller:

$$2\dot{y}(t) + 6y(t) = 8u(t) \qquad u(t) = 2.5e(t) + 9\int_0^t e(\tau)d\tau$$

(a) What sampling time Δt would you recommend for a discrete-time implementation?

(b) The value of $u(t)$ at $t = \Delta t$ is:

$$u(\Delta t) = 2.5e(\Delta t) + 9\int_0^{\Delta t} e(\tau)d\tau$$

Approximate $u_1 := u(\Delta t)$ in terms $e_0 := e(0)$ and $e_1 := e(\Delta t)$.

Solution :

(a)

$$2\ddot{y}(t) + 26\dot{y}(t) + 72y(t) = 20\dot{r}(t) + 72r(t)$$

Thus,

$$T_{r\to y}(s) = \frac{20s + 72}{2s^2 + 26s + 72}$$

The poles of the transfer function are $s_1 = -4$ and $s_2 = -9$. The corresponding time constants are $\tau_1 = \frac{1}{4}$ and $\tau = \frac{1}{9}$. One good approximation for Δt would be

$$\Delta t = \frac{1/9}{10} \approx \frac{1}{100} = 10\text{msec}$$

(b) Using this approximation $\int_0^{\Delta t} e(\tau)d\tau \approx \frac{1}{2}(e_0 + e_1)\Delta t$, we have:

$$u_1 \approx 2.5e_1 + 9(\frac{1}{2}(e_0 + e_1)\Delta t)$$

5.15 Proportional Controller

Consider the transfer function

$$T(s) = \frac{b_0 K_p}{s^2 + a_1 s + (a_0 + b_0 K_p)} \tag{136}$$

where $b_0 > 0$ and the closed-loop is critically damped. What happens to the settling time if K_p is increased? K_p is decreased?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) Increasing the settling time b) Decreasing the settling time

(b) a) Settling time remains unchanged b) Increasing the settling time

(c) a) Increasing the settling time b) Increasing the settling time

(d) a) Settling time remains unchanged b) Decreasing the settling time

Solution :
Correct answer is (b).
We have $a_1 = 2\zeta\omega_n$ and $a_0 + b_0 K_p = 2\omega_n^2$.

- Increasing K_p does not change the real part and thus settling time t_s remain unchanged.
- Decreasing K_p brings one of the poles closer to the origin, thereby increasing the settling time t_s.

5.16 PI Controller

Consider a car whose longitudinal motion is modeled by the following ODE:

$$2085\dot{v}(t) + 23.2v(t) = 40u(t) + 108.4 - F_{grav}(t)$$

The input is the throttle u and the output is the velocity v. The gravitational force F_{grav} is a disturbance. Let $e(t) = v_{des} - v(t)$ denote the tracking error between the desired velocity $v_{des} = 29$ m/s and actual velocity v(t). Consider a PI controller of the following form:

$$u(t) = \bar{u} + K_p e(t) + K_i \int_0^t e(\tau)d\tau$$

where $\bar{u} = 14.11$ is the open-loop input to maintain v_{des} when on at road $\theta = 0°$. Choose the PI gains so that the cruise control system is stable and rejects disturbances due to changing road slopes within ≈ 10 sec. The closed-loop should also be over or critically damped as oscillations are uncomfortable for the driver.

Solution :

This particular problem is of interest as it is one of the first design problems encountered by undergraduate control engineer. Moreover, it requires reasoning about the impact of the control gains on the closed-loop poles and the connection to transient response properties. There is more than one design that satisfies the given objectives. One approach is to note that the closed-loop characteristic equation is:

$$2085s^2 + (23.2 + 40K_p)s + 40K_i = 0$$

We can place the closed-loop poles in the LHP to have any desired damping ratio ζ and natural frequency ω_n by choosing the controller gains to satisfy $\omega_n^2 = \frac{40K_i}{2085}$ and $2\zeta\omega_n = \frac{23.2+40K_p}{2085}$. Select $\zeta = 1$ to obtain two critically damped closed-loop poles placed at $s = -\omega_n$. The time constant for these poles is $\tau = \frac{1}{\omega_n}$ sec and, for critically damped poles, the 5% settling time is approximately $4.75\tau = \frac{4.75}{\omega_n}$. Thus we select $\omega_n = \frac{4.75}{10} = 0.475$ to obtain a settling time near 10sec. Finally, use $\zeta = 1$ and $\omega_n = 0.475$ to solve for the PI gains using the relations above. This yields $K_p = 48.94$ and $K_i = 11.76$. The corresponding transfer function from F_{grav} to v is

$$T_{Fgrav \to v}(s) = \frac{-s}{2085s^2 + 1981s + 470.4}$$

This has critically damped poles at $s = -0.475$ as expected. A step change increase in gravitational force (due to a step increase in the road slope) will cause the velocity to initially drop from the desired value. However, the PI controller rejects the disturbance in $\approx 10sec$ with the velocity converging to the desired value with a nice overdamped response.

5.17 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \qquad y(0) = 0$$

with a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. Assume a unit step for $r(t)$. For what gains K_p is $|u(t)| \leq 1$ for all time?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $-1 \leq K_p \leq 0.2$

(b) $0.2 \leq K_p \leq 1$

(c) $-0.2 \leq K_p \leq 1$

(d) $-1 \leq K_p \leq 1$

Solution :
Correct answer is (c).
Note that the magnitude $|u(t)|$ of the control law is directly proportional to the difference between the state value and the reference value. i.e. $|r(t) - y(t)|$ via the gain K_p. Notice that the largest value of $|u(t)|$ will occur at t = 0 which gives us

$$\max_t |u(t)| = |K_p||r(0) - y(0)| = |K_p||1 - 0| = |K_p|$$

Therefore a preliminary condition for $|u(t)| \leq 1$ for all $t \in \mathbb{R}_+$ is that $|u \leq 1|$. However, note that we can write:

$$\dot{y} = -0.5y + 2u = -y(0.5 + 2K_p) + 2K_p r$$

and so when $K_p = 0$ we get a stable autonomous system with eigenvalue 0.5. On the other hand, for any constant reference signal $r(t) \equiv$const we have that $K_p < -0.25$ will result in an unstable system. But note, that by solving for y(t) we see that further we need,

$$|u(t)| \leq \left| K_p \left(1 - \frac{2K_p}{0.5 + 2K_p} \right) \right| \leq 1$$

Therefore, the acceptable range is $K_p \in [-0.2, 1]$.

5.18 Proportional Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \qquad y(0) = 0$$

with a proportional control law $u(t) = K_p(r(t) - y(t))$ where $r(t)$ is the reference command. Under the constraint that $K_p \in [-0.2, 1]$, find K_p such that we minimizes the steady-state error due to the unit step reference command. What is the time constant of the closed-loop system for this gain?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $K_p = -0.2, \tau = 10$

(b) $K_p = 0, \tau = 2$

(c) $K_p = 1, \tau = \frac{5}{2}$

(d) $K_p = 1, \tau = \frac{2}{5}$

Solution :
Correct answer is (d).
For $K_p \in [-0.2, 1]$ it is clear that steady-state error to a unit-step reference input is maximized at the left end-point of the interval and minimized at the right end-point. To see why derive the transfer function:

$$\frac{Y(s)}{R(s)} = \frac{2K_p}{s + 2K_p + 0.5}$$

and we see using Final Value Theorem that for higher K_p the steady-state value approaches 1. Hence, we choose $K_p = 1$ to satisfy the constraint. The corresponding transfer function is given as follows:

$$H(s) = \frac{2}{s + 5/2} \qquad \Longrightarrow \tau = \frac{2}{5}$$

5.19 PI Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \qquad y(0) = 0$$

with a proportional-integral (PI) control law:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

with $e(t) = r(t) - y(t)$ is the tracking error. Combining the system model and PI controller leads to the closed-loop system in the following form:

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{r} + b_0 r$$

How do the damping ratio and natural frequency depend on K_p and K_i? What is the steady state error if r is a unit step?
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $\omega_n = \sqrt{2K_i}, \zeta = \frac{4K_p+1}{4\sqrt{2K_i}}$, with zero steady-state error

(b) $\omega_n = \sqrt{2K_p}, \zeta = \frac{4K_p-1}{4\sqrt{2K_i}}$, with zero steady-state error

(c) $\omega_n = \sqrt{2K_p}, \zeta = \frac{4\sqrt{2K_i}}{4K_p+1}$, with constant steady-state error

(d) $\omega_n = \sqrt{2K_i}, \zeta = \frac{4K_p-1}{4\sqrt{2K_i}}$, with constant steady-state error

Solution :
Correct answer is (a).
The second order system in the above ODE form gives us:

$$a_0 = 2K_p + 1 = 2 \qquad a_1 = 2K_i \qquad b_0 = 2K_i \qquad b_1 = 2K_p$$

The characteristic polynomial then is:

$$s^2 + s(2K_p + \frac{1}{2}) + 2K_i \implies \omega_n = \sqrt{2K_i} \quad \text{and} \quad \zeta = \frac{4K_p + 1}{4\sqrt{2K_i}}$$

For a unit-step reference signal this results in zero steady state error.

5.20 PI Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \qquad y(0) = 0$$

with a proportional-integral (PI) control law: $u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau$ with $e(t) = r(t) - y(t)$. Let $K_p = 1$ and choose K_i to obtain a damping ratio of $\zeta = 0.7$. For these PI gains, what are the estimated maximum overshoot and 5% settling time (neglecting the integral effect of the zero)? Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $M_p = 6.4\%, T_s = 2.4s$

(b) $M_p = 4.6\%, T_s = 4.2s$

(c) $M_p = 4.6\%, T_s = 2.4s$

(d) $M_p = 6.4\%, T_s = 4.2s$

Solution :
Correct answer is (c).
The characteristic polynomial for the system with PI controller is given by:

$$s^2 + s(2K_p + \frac{1}{2}) + 2K_i \implies \omega_n = \sqrt{2K_i} \quad \text{and} \quad \zeta = \frac{4K_p + 1}{4\sqrt{2K_i}}$$

Plugging in $K_p = 1$ and $\zeta = 0.7$ into the last equation above yields that $K_i = \frac{625}{392}$ and $\omega_n = 1.7857$. We know,

$$M_p = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \approx 4.6\%$$

and for 5% settling time,

$$T_s = \frac{-ln(5/100)}{\zeta\omega_n} = \frac{-ln(.05)}{0.7 \times 1.7857} = 2.4s$$

5.21 PD Controller

Consider the following first order system:

$$\ddot{y} + 2\dot{y} + y = u \qquad y(0) = 0$$

with a PD controller in the form $u(t) = K_p(r(t) - y(t)) + K_d\dot{y}(t)$. Choose (K_p, K_d) so that the closed loop system is stable and has $(\omega_n, \zeta) = (2, 0.25)$.
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $K_p = 3, K_d = 1$

(b) $K_p = 1, K_d = 3$

(c) $K_p = 2, K_d = 2$

(d) $K_p = 2, K_d = 3$

Solution:
Correct answer is (a).
We can derive the transfer function as follows:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{K_p}{s^2 + (2 - K_d)s + (K_p + 1)}$$
$$= \left(\frac{K_p}{K_p + 1}\right) \frac{K_p + 1}{s^2 + (2 - K_d)s + (K_p + 1)}$$

So, $\omega_n^2 = K_p + 1$ and $2\zeta\omega_n = (2 - K_d)$.
This means $K_p = \omega_n^2 - 1 = 3$ and $K_d = 2 - 2 \times 0.25 \times 2 = 1$.

5.22 Proportional Controller

Consider the system

$$G(s) = \frac{s+1}{s^2+2s+3}$$

is in standard feedback configuration using a controller K. Is it possible to make the settling time t_s arbitrarily small by choosing the gain K large enough for the closed-loop system?

Solution :

It is worth noting that the closed loop poles for the given system are:

$$s = \frac{-(K+2) \pm \sqrt{K^2-8}}{2}$$

As $K \to \infty$, both of the poles for the closed loop equation are real with one of them going to $-\infty$ and at the same time the other root goes to -1. Thus, the rate limiting step is going to be -1. This limits our ability to make t_s arbitrarily small. Therefore, it's not possible to make t_s arbitrarily small by choosing large enough gain K.

5.23 Dynamic Output Feedback Controller

Consider the following system:

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u \qquad y = x_2$$

Design a dynamic output feedback controller such that the poles of the closed-loop system are at -10, $-10 \pm 5j$.

Solution:

We can obtain a dynamic output feedback controller by combining state feedback controller with a full state observer design. The state-space model for the resulting controller will be:

$$\begin{aligned} \dot{\hat{x}} &= (A - LC - BK)\hat{x} + Ly \\ u &= -K\hat{x} \end{aligned} \tag{137}$$

where in order to place the poles of closed-loop system at -10, $-10 \pm 5j$ we need to use $K = \begin{pmatrix} 325 & -1251 & 735 \end{pmatrix}$ and $L = \begin{pmatrix} 26769 & 59 & 28032 \end{pmatrix}^T$ to place the observer poles at -20, $-20 \pm 2j$.

5.24 Controller Design

Consider a feedback control system under unity feedback setting with a controller $K(s) = K$ and the system $G(s)$ given as follows:

$$G(s) = \frac{1}{s^2(s+1)}$$

Suppose we would like to place a closed loop pole at $\bar{s} = -1 + j$. What is the phase of $K(\bar{s})$? Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $\angle K(\bar{s}) = -\pi$

(b) $\angle K(\bar{s}) = -\frac{\pi}{2}$

(c) $\angle K(\bar{s}) = \frac{\pi}{2}$

(d) $\angle K(\bar{s}) = \pi$

Solution:
Correct answer is (d).
The controller needs to contribute π rad since $\angle G(\bar{s}) + \angle K(\bar{s}) \equiv \pi$ which leads to:

$$\angle K(\bar{s}) = \pi - \left(0 - 2\left(\frac{3\pi}{4}\right) - \frac{\pi}{2}\right) = \pi + 2\pi = \pi$$

6 Bode Analysis

6.1 Stability Analysis of a Closed-Loop System Using Bode Plot

Consider a standard closed-loop system with the loop transfer function $L(s)$ with the Bode plot below. Assume the closed-loop is stable with the loop $L(s)$:

$$L(s) = \frac{-4s + 72}{0.39s^2 + 8.02s + 18}$$

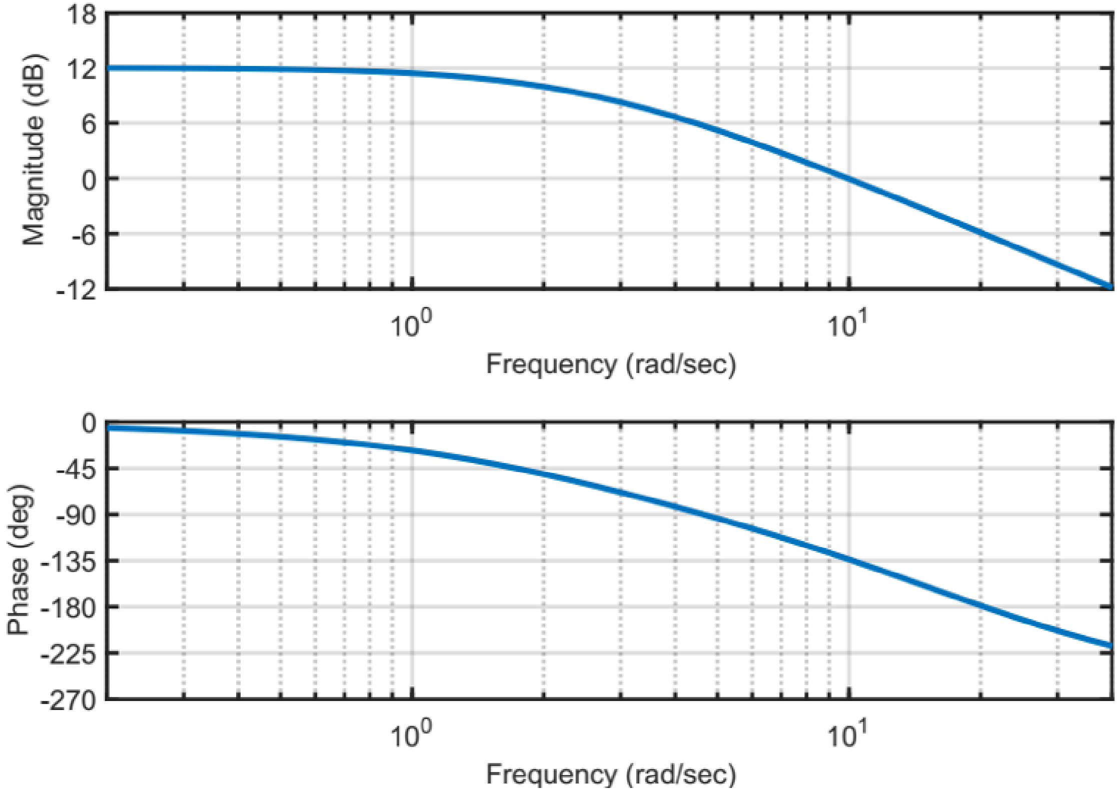


Figure 15: Bode Plot

(a) What is the phase crossover frequency, ω_0?

(b) What is the gain margin, g_0, of the closed-loop?

(c) Is the closed-loop stable if the open-loop transfer function is $1.5L(s)$?

Solution :

(a) $\omega_0 = 20\text{rad}/\sec$.

(b) $g_0 = 2$.

(c) Yes. Because $1.5 < g_0 = 2$.

6.2 Gain Crossover and Phase Margin Analysis

Consider a standard closed-loop system with the loop transfer function $L(s)$ with the Bode plot below. Assume the closed-loop is stable with the loop $L(s)$:

$$L(s) = \frac{-4s + 72}{0.39s^2 + 8.02s + 18}$$

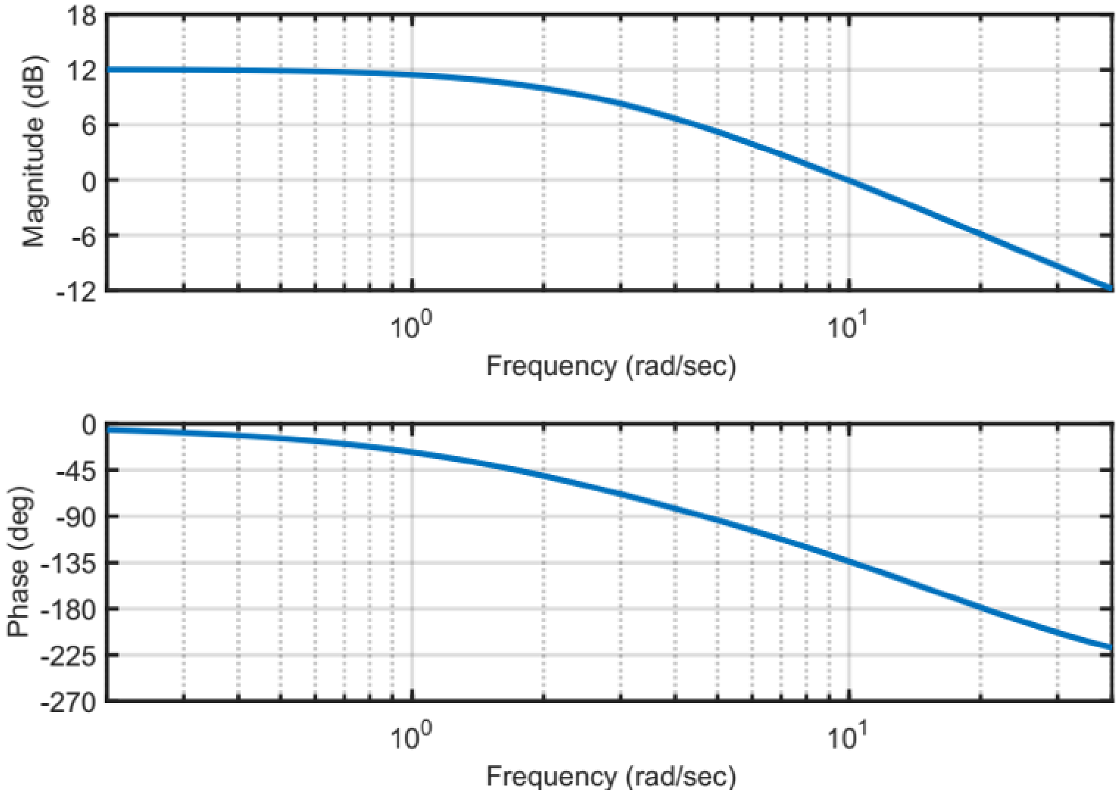


Figure 16: Bode Plot

(a) What is the gain crossover frequency, ω_0?

(b) What is the phase margin, θ_0, of the closed-loop?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $\omega_0 = 1\,\mathrm{rad/\,sec}, \theta_0 = 150^\circ$

(b) $\omega_0 = 2\,\mathrm{rad/\,sec}, \theta_0 = 135^\circ$

(c) $\omega_0 = 10\,\mathrm{rad/\,sec}, \theta_0 = 45^\circ$

(d) $\omega_0 = 20\,\mathrm{rad/\,sec}, \theta_0 = 0^\circ$

Solution :
Correct answer is (c).

(a) $\omega_0 = 10\,\mathrm{rad/\,sec}$.

(b) $\theta_0 = 45^\circ$.

6.3 Bode Plot Analysis and Steady-State Response of a Linear System

A linear system $G(s)$ with input u and output y has the bode plot shown below:

(a) What is $|G(10j)|$ in dB and actual units?

(b) What is $\angle G(10j)$ om degs and radians?

(c) What is the output response $y(t)$ in steady-state for the input $u(t) = 2\cos(10t)$?

(d) What is the steady-state value of $y(t)$ if the input is a unit step $u(t) = 1$ for all $t \geq 0$?

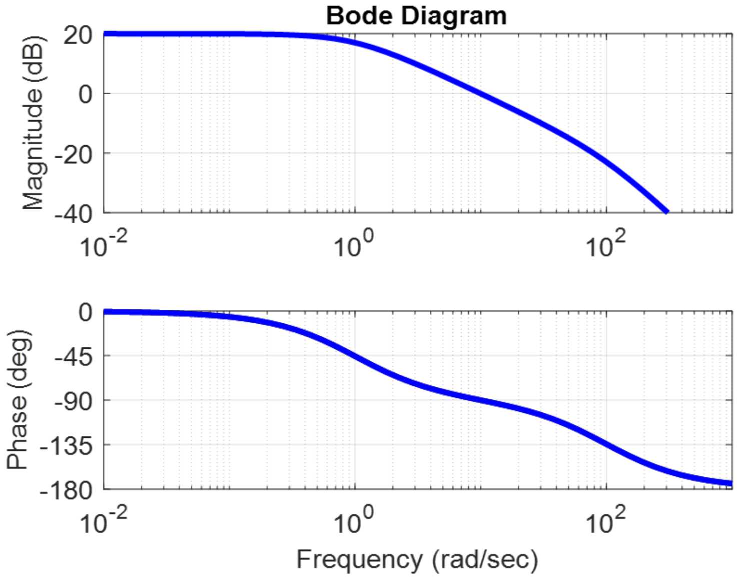


Figure 17: Bode Plot

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $|G(j10)| = -20dB = -10$ b) $\angle G(j10) = -180\text{deg} = -\pi\text{rad}$ c) $y(t) = -20\cos(10t-\pi)$ d) 10

(b) a) $|G(j10)| = 0dB = 1$ b) $\angle G(j10) = -90\text{deg} = -\frac{\pi}{2}\text{rad}$ c) $y(t) = 2\cos(10t - \frac{\pi}{2})$ d) 10

(c) a) $|G(j10)| = 0dB = 1$ b) $\angle G(j10) = -90\text{deg} = -\frac{\pi}{2}\text{rad}$ c) $y(t) = 2\cos(10t - \frac{\pi}{2})$ d) $\cos(10t)$

(d) a) $|G(j10)| = -20dB = -10$ b) $\angle G(j10) = -90\text{deg} = -\frac{\pi}{2}\text{rad}$ c) $y(t) = -20\cos(10t - \frac{\pi}{2})$ d) 10

Solution :
Correct answer is (b).

(a) $|G(j10)| = 0dB = 1$

(b) $\angle G(j10) = -90\text{deg} = -\frac{\pi}{2}\text{rad}$

(c) $y(t) = 2|G(j10)|\cos(10t + \angle G(j10)) = 2\cos(10t - \frac{\pi}{2})$

(d) $u(t) = \cos(\omega t) \quad \text{with} \quad \omega = 0$

$$|G(j0)| = 20dB = 10 \quad \text{and} \quad \angle G(j0) = 0\text{deg} = 0\text{rad}$$

Thus, we have:

$$y(t) = |G(j0)|\cos(\omega t + \angle G(j0)) = 10$$

6.4 Closed-Loop Stability Analysis with Bode Plot

Consider a standard closed-loop system with the loop transfer function $L(s)$ with Bode plot below. Assume the closed-loop is stable with the loop $L(s)$.

$$L(s) = \frac{-4s + 72}{0.39s^2 + 8.02s + 18} \tag{138}$$

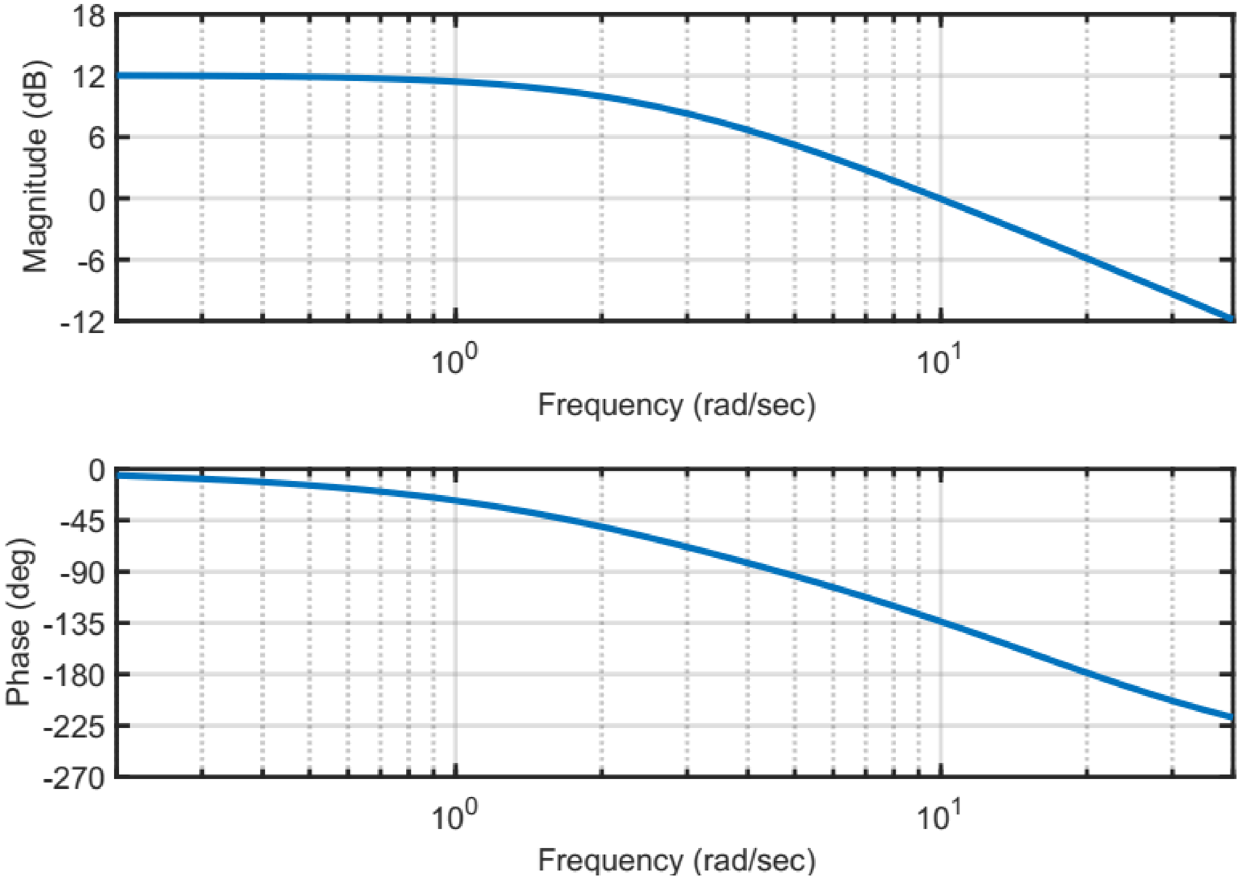


Figure 18: Bode Plot

(a) What is the gain cross-over frequency ω_0 in rads/sec?

(b) What is the phase margin, θ_0, of the closed-loop in degrees?

(c) What is the delay margin, τ_0, of the closed-loop in seconds?

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $\omega_0 = 1$ rad/sec b) $\theta_0 = 150$ deg c) $\tau_0 = 5\pi/6$ secs

(b) a) $\omega_0 = 2$ rad/sec b) $\theta_0 = 135$ deg c) $\tau_0 = 3\pi/8$ secs

(c) a) $\omega_0 = 10$ rad/sec b) $\theta_0 = 45$ deg c) $\tau_0 = \pi/40$ secs

(d) a) $\omega_0 = 20$ rad/sec b) $\theta_0 = 0$ deg c) $\tau_0 = 0$ secs

Solution :
Correct answer is (c).

(a) The gain cross-over frequency is $\omega_0 = 10$ rad/sec

(b) The phase margin is $\theta_0 = 45$ deg.

(c) The delay margin is $\tau_0 = \pi/40 \approx 0.08$ secs.

6.5 Deriving Transfer Function from Bode Plot for First-Order System

The system $G(s)$ is first-order. Determine the transfer function for this system from the Bode plot.

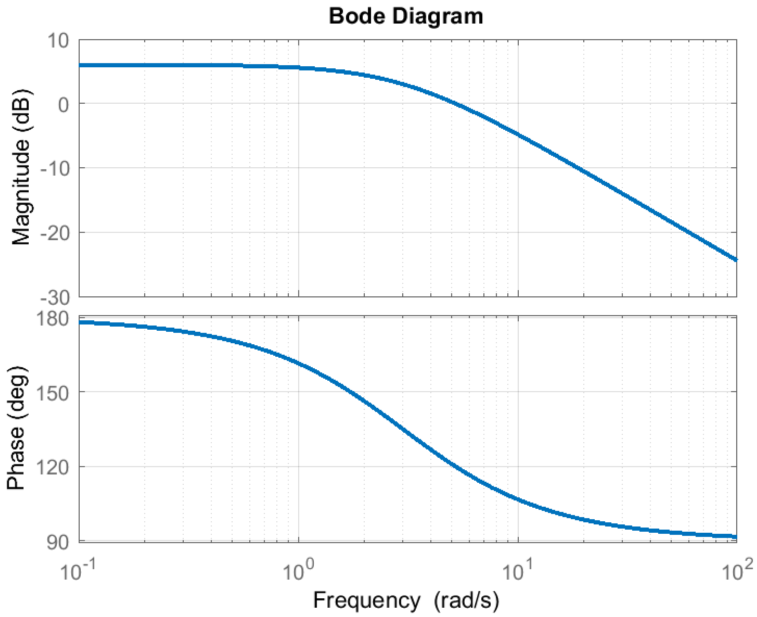


Figure 19: Bode Plot

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $G(s) = -\frac{6}{(s+3)^2}$

(b) $G(s) = -\frac{18}{(s+3)^2}$

(c) $G(s) = -\frac{18}{(s+3)}$

(d) $G(s) = -\frac{6}{(s+3)}$

Solution :

Correct answer is (d).

$$G(s) = -\frac{6}{s+3}$$

6.6 Disk Margin

Consider the following loop transfer function:

$$L(s) = \frac{4s+1}{s^3+4s-2s+0.5} \tag{139}$$

Below is the Bode magnitude plot of the closed-loop sensitivity $S(s)$.

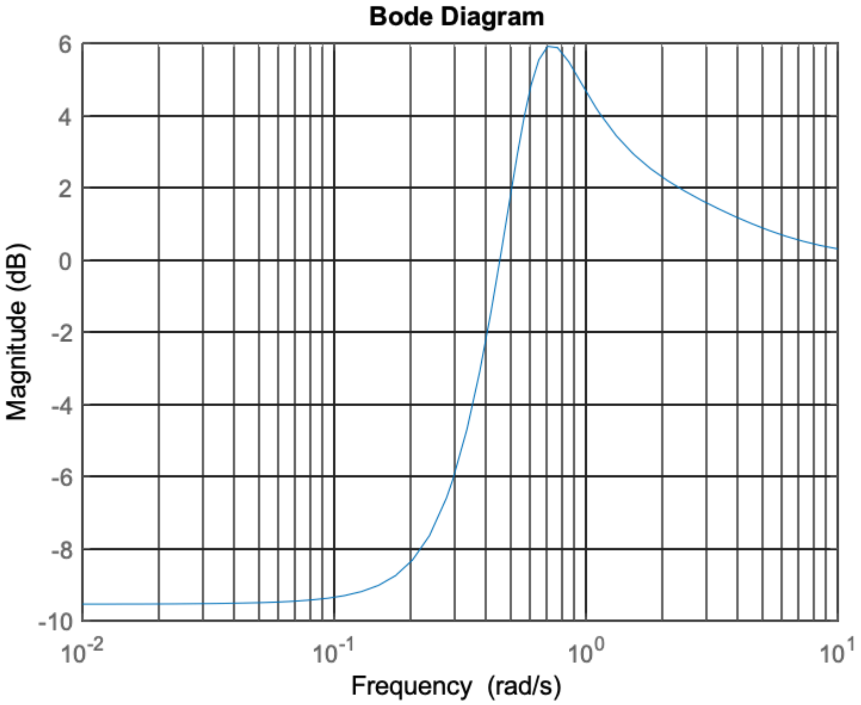


Figure 20: Bode Plot

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $d_{\text{min}} \approx 1/2$

(b) $d_{\text{min}} \approx 2$

(c) $d_{\text{min}} \approx 1$

(d) $d_{\text{min}} \approx 4$

Solution :

Correct answer is (a).

Since $d_{\text{min}} = \min_{\omega}|1 + L(j\omega)|$ and $S(j\omega) = \frac{1}{1+L(j\omega)}$, the minimizing frequency will also maximize closed-loop sensitivity $\max_{\omega}|S(j\omega)| = 6dB \approx 2$ in absolute units. Therefore the disk margin is $d_{\text{min}} \approx 1/2$.

6.7 Matching Transfer Functions with Their Corresponding Bode Plots

Let $G(s) = \frac{1}{s+1}$. Name each component $K_i(s)$ below and match with the corresponding Bode plot for the loop $L(s) = G(s)K_i(s)$.

(i) $K_1(s) = 2$ (ii) $K_2(s) = \frac{s+0.1}{s}$ (iii) $K_3(s) = \frac{10}{s+10}$ (iv) $K_4(s) = \frac{2s+10}{s+20}$

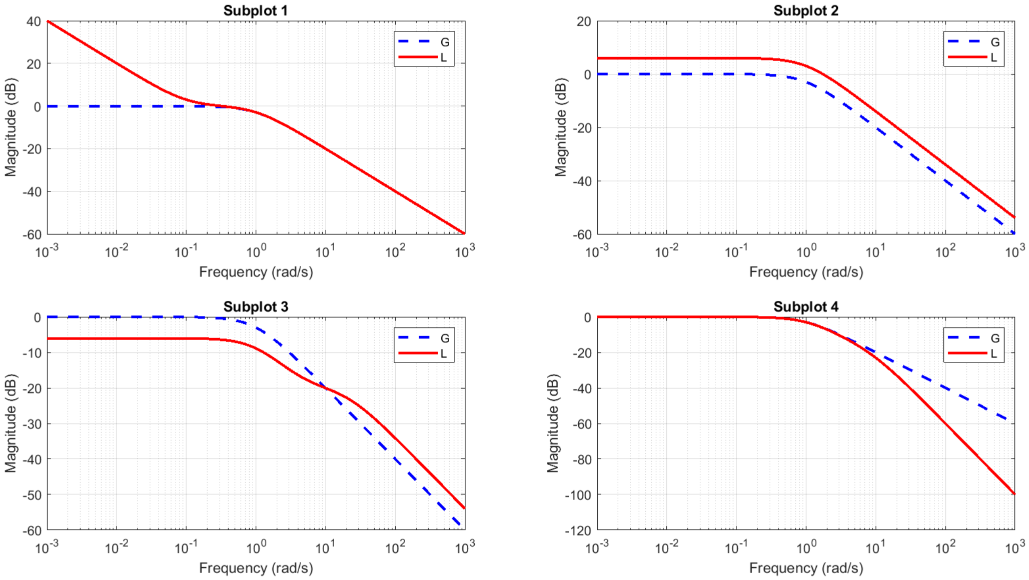


Figure 21: Bode Plots for Loop Transfer Functions $L(s)$

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) (i) Subplot 1 (ii) Subplot 2 (iii) Subplot 4 (iv) Subplot 3

(b) (i) Subplot 3 (ii) Subplot 2 (iii) Subplot 4 (iv) Subplot 3

(c) (i) Subplot 2 (ii) Subplot 1 (iii) Subplot 4 (iv) Subplot 3

(d) (i) Subplot 4 (ii) Subplot 1 (iii) Subplot 2 (iv) Subplot 3

Solution :

Correct answer is (c).

(i) Subplot 2 (ii) Subplot 1 (iii) Subplot 4 (iv) Subplot 3

6.8 Design of a Proportional Controller Using Bode Plots

Consider a feedback system where the plant $G(s)$ is stable and has the Bode magnitude plot shown below. We want to design a controller so that: i) the closed-loop is stable, ii) the system has a loop cross-over frequency near 50 rad/sec, and iii) the closed-loop can track $r(t) = \sin(0.1t)$ with less than 1% error.

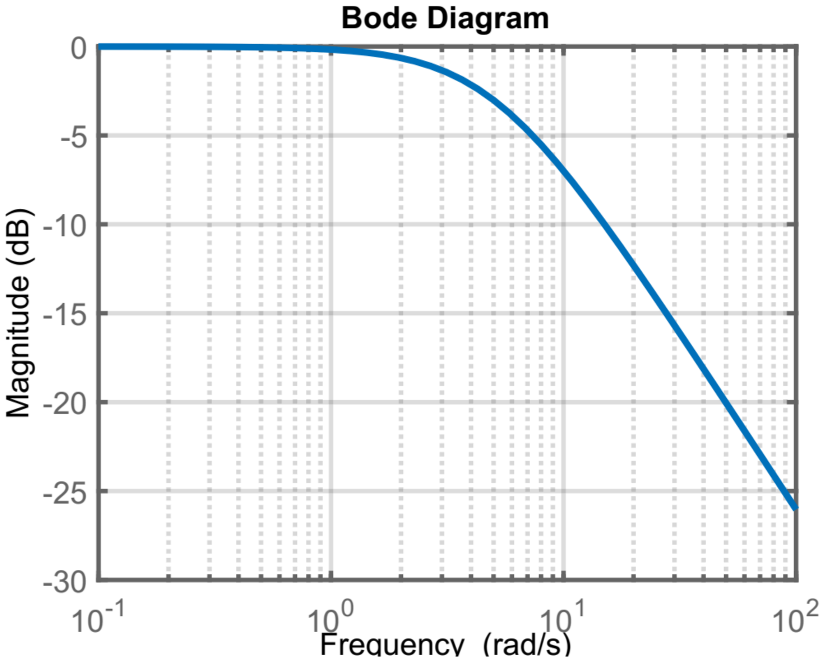


Figure 22: Bode Plot

(a) Using the Bode diagram, choose a gain K_p so that $K_pG(s)$ has the desired cross-over frequency of 50 rad/sec.

(b) Convert the requirement (iii) into a requirement on the closed-loop transfer function $L(s) = G(s)K(s)$.

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $K_p \approx 10$ b) $100 \leq |L(j\omega)|$

(b) a) $K_p \approx 10$ b) $|L(j\omega)| \leq 100$

(c) a) $K_p \approx 1$ b) $100 \leq |L(j\omega)|$

(d) a) $K_p \approx 1$ b) $|L(j\omega)| \leq 100$

Solution :
Correct answer is (a).

(a) $|K_pG(j50)| = 1$ implies that $K_p = 1/|G(j50)| \approx 1/0.1 = 10$

(b) With $r(t) = \sin(0.1t)$, and $S(jw) = \frac{1}{1+L(j\omega)}$ the response from $r \rightarrow e$, then we must have $|S(j\omega)| \sin(0.1t + \measuredangle S(j\omega)) \leq 0.01$. Therefore $|S(j\omega)| = |\frac{1}{1+L(j\omega)}| \leq 0.01$ and so $100 \leq |1 + L(j\omega)| \approx |L(j\omega)|$.

6.9 Design of a Proportional Controller Using Bode Plots

Consider a feedback system where the plant $G(s)$ is stable and has the Bode magnitude plot shown below. We want to design a controller so that: i) the closed-loop is stable, ii) the system has a loop cross-over frequency near 1 rad/sec, and iii) the gain from the noise n to the output y is ≤ 0.001 for $\omega \geq 100$ rad/sec.

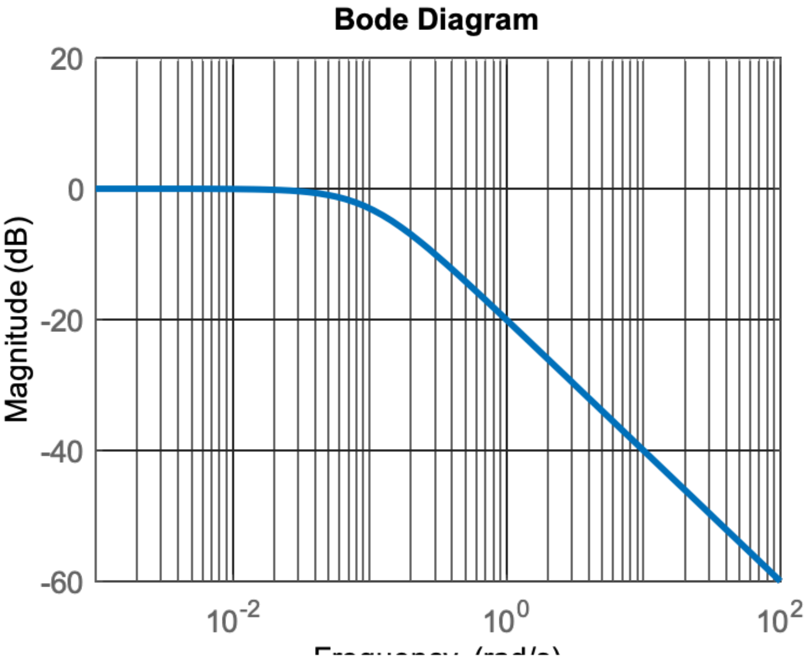


Figure 23: Bode Plot

(a) Using the Bode diagram, choose a gain K_p so that $K_pG(s)$ has the desired cross-over frequency of 1 rad/sec.

(b) Convert the requirement (iii) into a requirement on the closed-loop transfer function $L(s) = G(s)K(s)$. The transfer function from noise n to y is $T_{n\rightarrow y} = -\frac{L(j\omega)}{1+L(j\omega)}$

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $K_p \approx 1$ b) $|T(j\omega)| \geq 0.001$ for all $\omega \geq 100$

(b) a) $K_p \approx 10$ b) $|T(j\omega)| \geq 0.001$ for all $\omega \geq 100$

(c) a) $K_p \approx 1$ b) $|T(j\omega)| \leq 0.001$ for all $\omega \geq 100$

(d) a) $K_p \approx 10$ b) $|T(j\omega)| \leq 0.001$ for all $\omega \geq 100$

Solution :
Correct answer is (d).

(a) $|K_pG(j1)| = 1$ implies that $K_p = 1/|G(j1)| \approx 1/0.1 = 10$.

(b) With $n(t) = \sin(\omega t)$, and $-T(jw) = -\frac{L(j\omega)}{1+L(j\omega)}$ the response from $n \to y$, then we must have $-|T(j\omega)| \sin(\omega t + \measuredangle T(j\omega)) \leq 0.001$ for all $\omega \geq 100$. Therefore we should have $|T(j\omega)| = |\frac{L(j\omega)}{1+L(j\omega)}| \leq 0.001$ for all $\omega \geq 100$.

6.10 Estimating Steady-State Output from Bode Plot Analysis

The Bode plot for the system with transfer function

$$\frac{X(s)}{G(s)} = \frac{s^2 + 0.002s + 10000}{(s + 0.01)(s + 1000)} \tag{140}$$

is shown in the figure below. Estimate the steady-state output $x_{ss}(t)$ corresponding to the input $f(t) = 0.0001 \sin 0.0001t + \sin 100t + \sin 10000t$.

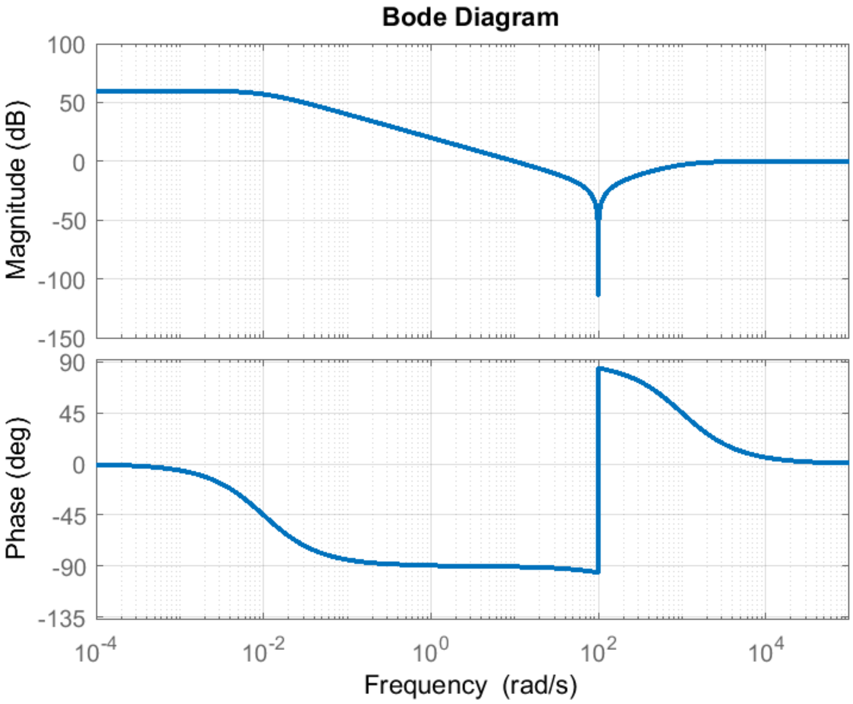


Figure 24: Bode Plot

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $x(t) = 0.1\sin(10^{-4}t) + \sin(100t) + \sin(10^{4}t)$

(b) $x(t) = \sin(10^{-4}t) + 10\sin(10^4 t)$

(c) $x(t) = 0.1\sin(10^{-4}t) + 10\sin(100t) + 10\sin(10^4 t)$

(d) $x(t) = 0.1\sin(10^{-4}t) + \sin(10^4 t)$

Solution :

Correct answer is (d).

We gave three periodic terms, so we consult the Bode plot at each frequency to determine the gain and the phase shift of each.

- The frequency $\omega = 10^{-4}$ is not included in the Bode plot, but the plot is level at 10^{-3} so we assume the same gain of approximately $60dB$. Likewise, the phase shift is not shown, but appears to be very close to zero. Since $60dB = 20log(|G|) \rightarrow |G| = 10^3$, we have an output component of $0.1\sin(10^{-4}t)$/
- At $\omega = 10^2$, the gain is $< -150dB$. This gain effectiverly attenuates this signal to zero.
- At $\omega = 10^4$, the gain is approximately $0dB$, which is a gain of 1. The phase shift is also effectively 0, so this gives a component of $\sin(10^4 t)$.

Therefore, our output is approximately: $x(t) = 0.1\sin(10^{-4}t) + \sin(10^4 t)$.

6.11 Matching Step Responses with Their Corresponding Bode Plots

Match the Bode plots

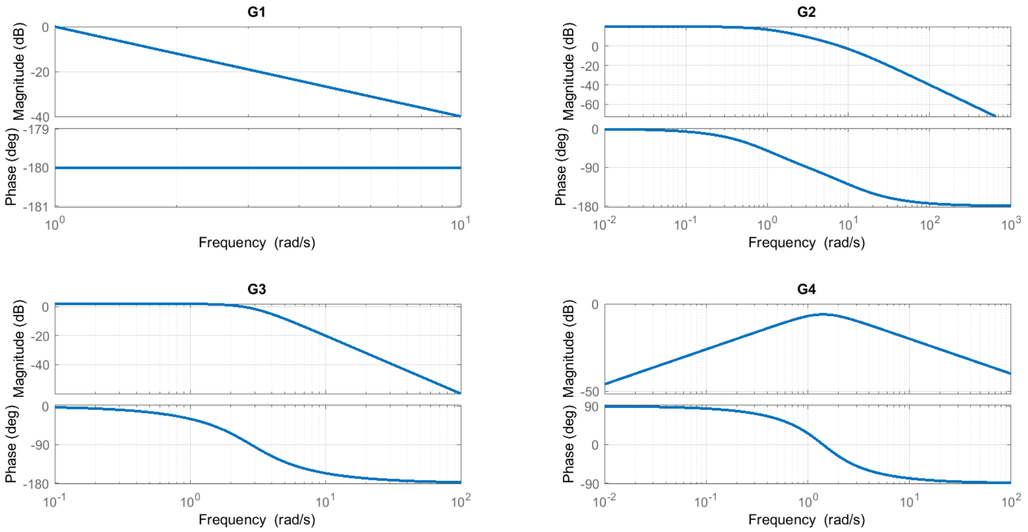


Figure 25: Bode Plots

with the step responses.

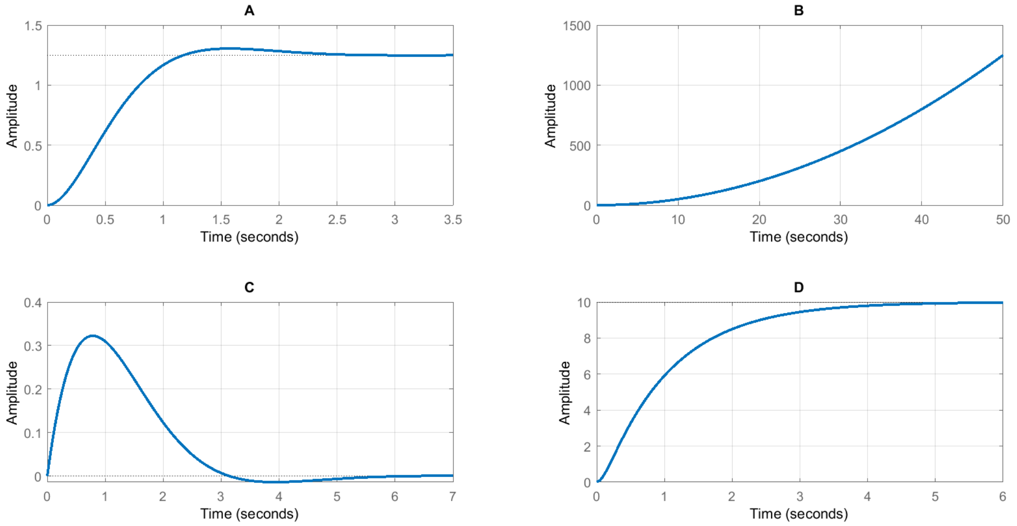


Figure 26: Step Responses

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

"‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

(a) $G_1 - D, G_2 - B, G_3 - A, G_4 - C$

(b) $G_1 - B, G_2 - D, G_3 - C, G_4 - A$

(c) $G_1 - C, G_2 - A, G_3 - D, G_4 - B$

(d) $G_1 - B, G_2 - D, G_3 - A, G_4 - C$

Solution :
Correct answer is (d).
$G_1 - B, G_2 - D, G_3 - A, G_4 - C$

6.12 Lag Controller Design

Design a lag controller that provides PM of at least $60°$ and steady-state tracking of constant references within 10% for the following system

$$G(s) = \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

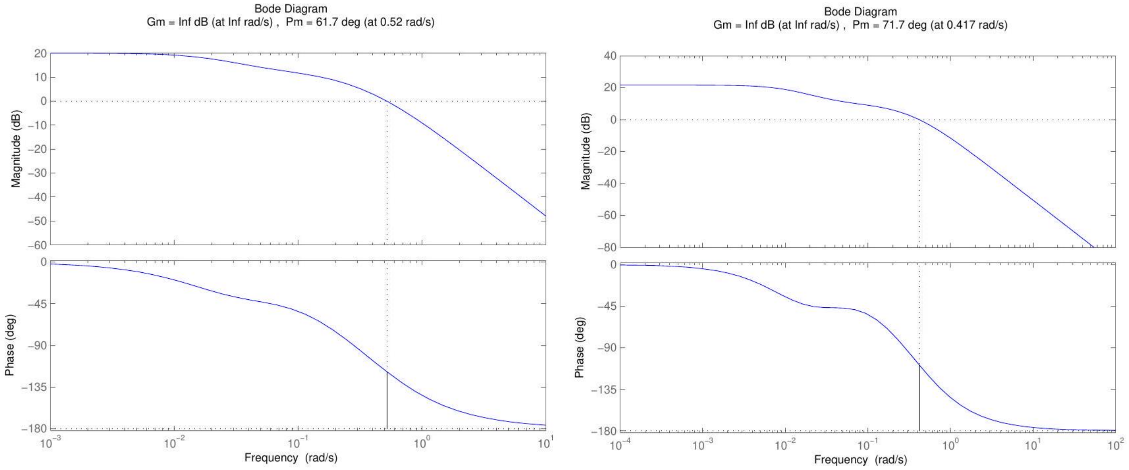
Compute the PM and steady-state tracking error for the following controller to verify that the specs are met:

$$KD(s) = 0.4\frac{s+0.05}{s+0.02}$$

Solution :

In Fig. 27(a), we can see that PM $= 61.7\text{deg}, \omega_c = 0.52$, and GM $= \infty$ for the system with controller $KD(s) = 0.4\frac{s+0.05}{s+0.02}$. For steady-state error for constant input:

$$e(\infty) = \left.\frac{1}{1+KD(s)G(s)}\right|_{s=0} = \frac{1}{1+0.4\times 2.5\times 10} = \frac{1}{11} < 0.1$$



(a) Bode plot using given controller

(b) Bode plot using modified controller

Figure 27: Bode plots

6.13 Closed-Loop Poles

Consider the transfer function

$$G(s) = \frac{1}{(s-1)(s^2+2s+5)}$$

Assume standard feedback configuration with K as the controller. Using the Bode plots of KG(s) explain how you can confirm the presence of $j\omega$-axis closed-loop poles if $K = 5$ or $K = 8$. You can use the following bode plot to answer the question.

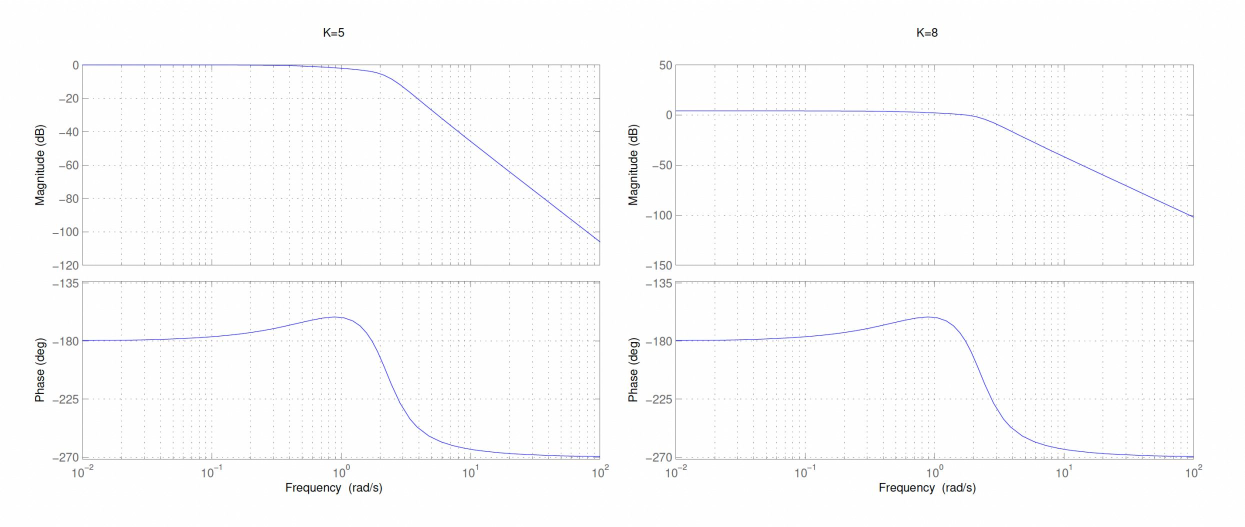


Figure 28: Bode Plots

Solution :
According to the bode plots given below for K = 5 and K = 8, we can see that both GM and PM are zero which confirm the presence of $j\omega$-axis closed-loop poles.

6.14 Phase Margin

Consider the transfer function

$$G(s) = \frac{1}{(s-1)(s^2+2s+5)}$$

Assume standard feedback configuration with K as the controller. What is the largest possible phase margin that can be achieved for this system? Determine the gain K for which it is achieved. You can use the following bode plot to answer the question.

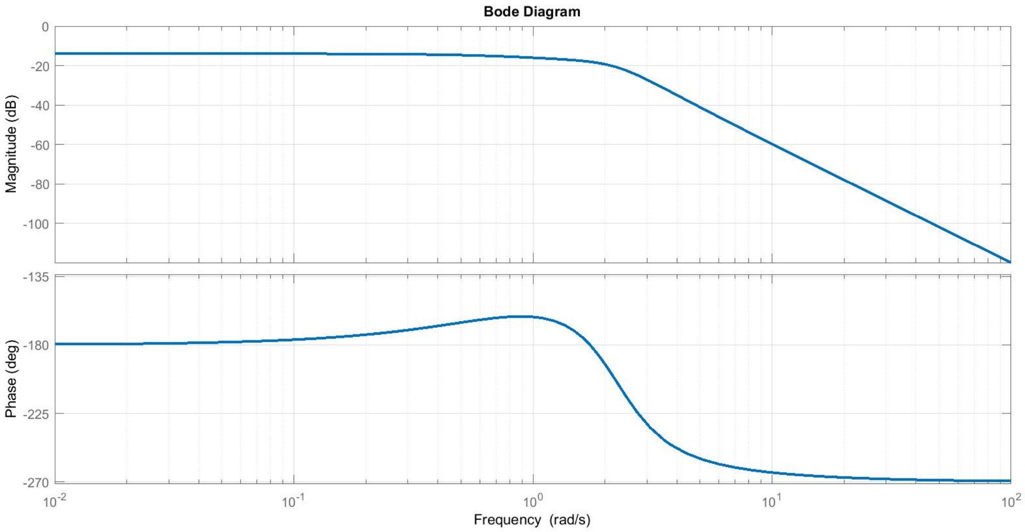


Figure 29: Bode Plot

Compute the disk margin from the Bode magnitude plot of $S(s)$ Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $K \approx 6.3$

(b) $K \approx 0.63$

(c) $K \approx 63$

(d) $K \approx 630$

Solution :

Correct answer is (b).

According to Bode plot given below, the maximum phase (and PM in this case) achieved on $\omega \approx 1$ (rad/sec), so we need to choose K in such a way that this is also equal to ω_c. The result would be therefore be $K \approx 6.3$.

6.15 Phase Margin

Consider the system

$$G(s) = \frac{1}{(s-1)(s+1)}$$

(a) Design a PD controller that achieves phase margin $PM \approx 90°$ and closed-loop bandwidth $\omega_{BW} \approx 10$. Verify that the specs are met.

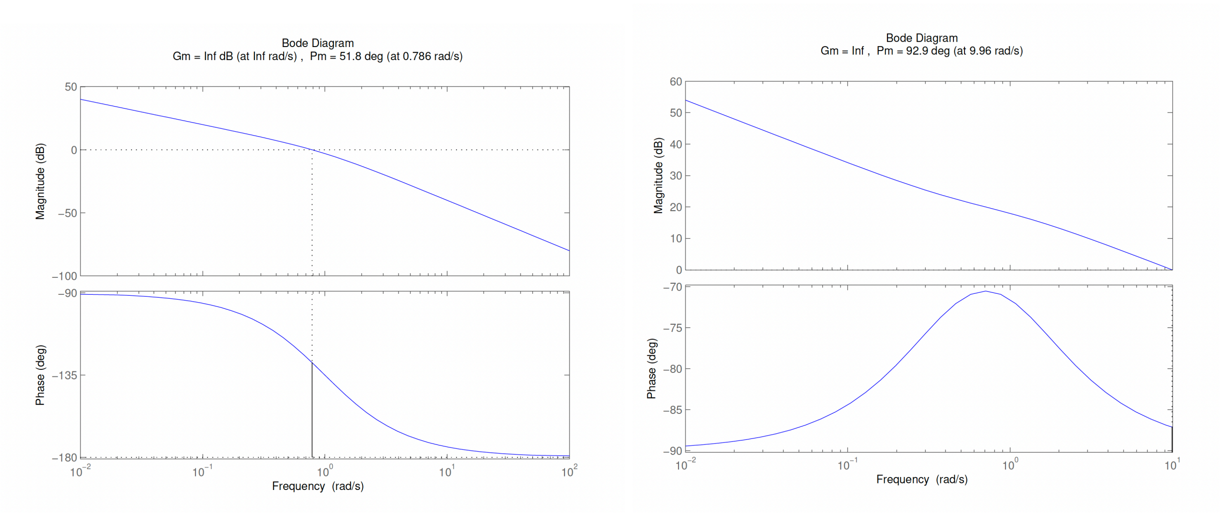
(b) Can you modify the above design to get $\omega_{BW} \approx 1$, while maintaining $PM \approx 90°$? Explain how or why not.

Solution :

(a) The bode plot of G(s) given below shows that we have a phase margin of $PM \approx 52°$ (but small ω_c). We want our PD controller to increase ω_c as well as PM.

$D(s) = K(\tau s + 1)$, we choose $\frac{1}{\tau} \ll 10$ to make sure the gain is high enough at $\omega_c = 10$. Also, we choose $\frac{1}{\tau} < 1$ to make sure that magnitude slope at $\omega_c = 10$ is -1.

Let $\tau = 2$ and $K \left| \frac{2j\omega_c + 1}{j\omega_c(j\omega_c + 1)} \right|_{\omega_c = 10} = 1 \implies K \approx 5 \implies D(s) = 5(2s+1)$



(a) Bode plot of system

(b) Bode plot of closed loop system

Figure 30: Bode Plots

(b) Achieving $\omega_{BW} = 1$ and $PM \approx 90°$ is impossible unless we cancel the pole at $s = -1$ i.e. $D(s) = s + 1$. Because there is a break point at $\omega = 1$ so we can't maintain slope = -1 on that point. Therefore, we cannot make $\omega_{BW} = 1$ and $PM \approx 90°$ unless we take D(s) = s + 1.

7 Root-Locus Design

7.1 Root Locus Sketches of Transfer Functions

Sketch the root loci for the following $L(s)$ by hand. Recall that the root locus plots how the solutions of $1+KL(s)=0$ vary as K goes from 0 to $+\infty$.

$$(a) \quad L(s)=\frac{1}{s^2+2s+20} \qquad (b) \quad L(s)=\frac{s-3}{s^2+2s+20}$$
$$(c) \quad L(s)=\frac{(s+1)(s+2)}{s\left(s^2+4\right)\left(s^2+5\right)} \qquad (d) \quad L(s)=\frac{s+3}{s^5+1}$$

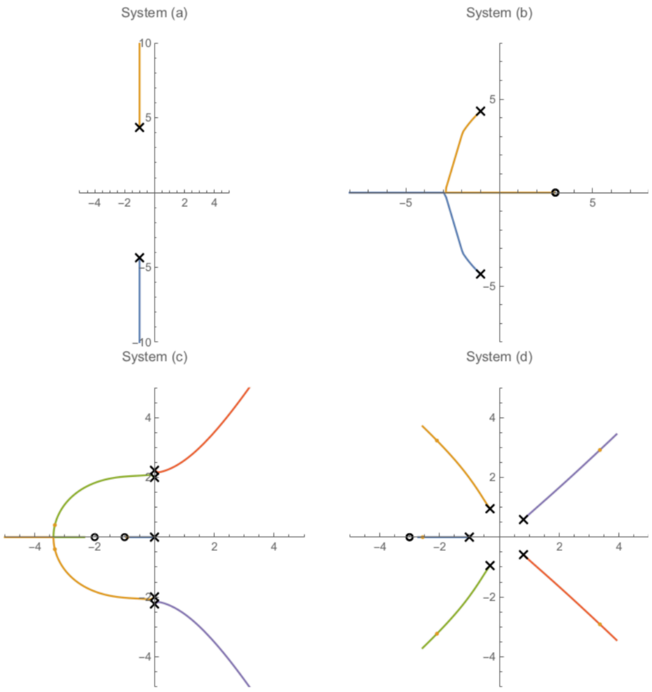


Figure 1: The root-locus plots for the given systems

Figure 31: Root Locus for Systems

Solution :
The application of Rules for the given systems results in the following table:

Rule / System System	System (a)	System (b)	System (c)	System (d)
No. of branches	2	2	5	5
Branch starts	$-1 \pm \sqrt{19}j$	$-1 \pm \sqrt{19}j$	$0, \pm 2j, \pm\sqrt{5}j$	-1,$\pm\sqrt{2}/2, \pm\sqrt{2}j/2$
Branch ends	∞	$3, \pm\infty$	$-1, -2, \infty$	$-3, \pm\infty$
Real RL	None	$(\infty, 3)$	$(-\infty, -2) \cup (-1, 0)$	$(-3, -1)$
Exit angles	$90°, 270°$	$180°$	$\frac{2\pi k}{3} \quad k = 1, 2, 3$	$\frac{\pi k}{4} \quad k = 1, 3, 5, 7$
$j\omega$- crossing	None	$\omega = 0$	None	None

Table 1: Root Locus rules for systems in Problem 7.1

The required plots are shown in Fig. 31.

7.2 Closed-Loop Stability

Suppose the closed loop transfer function is given by:

$$\frac{KL(s)}{1+KL(s)}$$

where K is some constant control gain.

(1) If L(s) has 3 LHP poles and 1 LHP zero, is the closed-loop system stable for very large values of $K > 0$?

(2) If L(s) has 2 LHP poles, 1 RHP poles, and 3 LHP zeros, is the closed-loop system stable for very large values of $K > 0$?

(3) If L(s) has 5 LHP poles, 4 LHP zeros, and 1 RHP zeroes, is the closed-loop system stable for very large values of $K > 0$?

Select the correct option from the following choices:

(a) (1) No, (2) Yes and (3) No

(b) (1) No, (2) Yes and (3) Yes

(c) (1) Yes, (2) Yes and (3) No

(d) (1) Yes, (2) Yes and (3) Yes

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).

(1) Although, all poles and zeros are in the LHP, it does not necessary mean that all values of gain, K, will keep the system stable. Suppose that the transfer function is strictly proper,

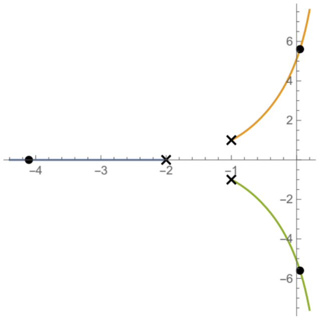


Figure 32: All poles and zeros in LHP and still high gain can make a system unstable.

then the asymptotes should be taken into consideration. If there are $j\omega$-crossings, then some asymptotes may escape into RHP with high gain, causing the system to be unstable (see Fig. 32). For example, consider a system with a zero at $s = -5$ and poles at $s = -2, s = -1 \pm j$. Although, it has 3 LHP poles and 1 LHP zero, the root locus plot of this system shows that there are $j\omega$-crossings and the system becomes unstable with high gain.

(2) Root locus branches start at the poles and end at the zeros. If the number of poles is greater than the number of zeros, then some branch(es) will go to infinity. In this case, there is one unstable pole (RHP pole). Since there are equal numbers of poles and zeros, all branches will end at the zeros and because all the zeros are in the LHP, a large enough K can stabilize the closed-loop system.

(3) Similar to the previous problem, all branches will start at the poles and end at the zeros. This case also has equal number of both. However, one of the zeros is in the RHP. Therefore, a large value of K can cause the closed-loop system to be unstable.

7.3 Minimizing Performance Index

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_2 + u \\ y &= 2x_1 + x_2 \end{aligned} \tag{141}$$

and suppose that the control objective is to minimize the performance $\int_0^\infty [\rho y^2(t) + u^2(t)]dt \quad \rho > 0$.

(a) See why in the limit as $\rho \to 0$ ("expensive control" case), the optimal closed-loop poles become mirror images of the open-loop poles across the imaginary axis.

(b) See why in the limit as $\rho \to \infty$("cheap control" case), one optimal closed-loop pole cancels the open-loop zero and the other moves off to $-\infty$.

Solution :

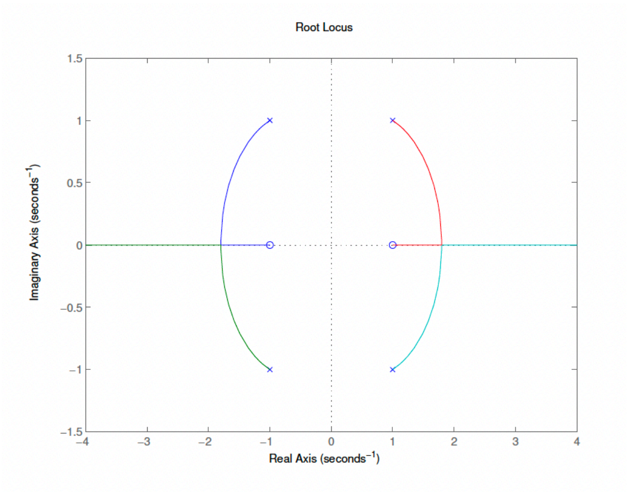


Figure 33: Root Locus of the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

(a) open-loop poles: $1 \pm j$ from Fig. 33, we see that for every ρ, we assign the LHP pole for the closed loop system (due to symmetry vs imaginary axis) For $\rho = 0$, the CL-optimal poles would be $-1 \pm j$ because the OL-poles are on RHP, it would be the mirror images of OL-poles across the imaginary axis (Note: if the OL-poles were on LHP, then CL-poles would be equal to OL poles in this case).

(b) For $\rho \rightarrow -\infty$, one pole moves to zero, the zero is on LHP, so it cancels the pole, and the other one goes to $-\infty$.

7.4 PI Controller

Consider the approximate model of a DC motor is given as:

$$G(s) = \frac{.5}{(.01s+1)(.1s+1)}$$

Design a PI controller such that it meets the following specifications:

$$\zeta \geq 0.6, \quad e_{ss}\big|_{\text{step}} \leq 0.01$$

. Select the correct option from the following choices:

(a) $K(s) = \frac{10}{s}$

(b) $K(s) = \frac{10(s+10)}{s}$

(c) $K(s) = 10(s+10)$

(d) $K(s) = 10$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

"‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

Solution :

Correct answer is (b).

We choose a PI controller, $K(s) = \frac{K(s+z_c)}{s}$, with unity-gain feedback $H(s) = 1$. The loop gain is given as:

$$K(s)G(s) = \frac{.5K\,(s+z_c)}{s(.01s+1)(.1s+1)}$$

The position error constant is $K_p = \infty$. Hence $e_{ss}\big|_{\text{step}} = 0$. The controller zero can be selected to cancel one of the plant poles; hence $z_c = 10$. Next, from the Root Locus plot given in Fig. 34, we may choose, e.g., $K = 10$. The PI controller design is given as:

$$K(s) = \frac{10(s+10)}{s}$$

The closed-loop system transfer function is thus given by $T(s) = \frac{5000}{s^2+100s+5000}$.

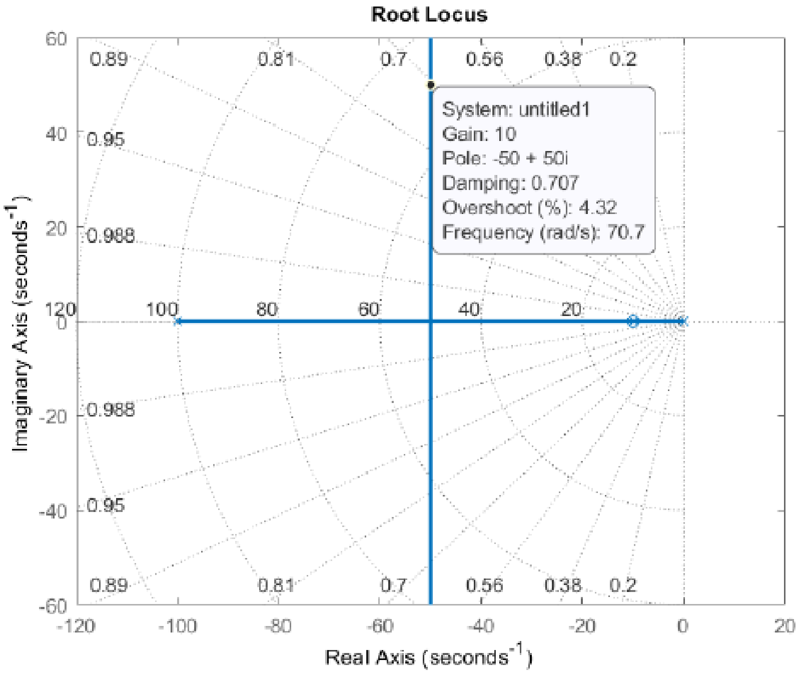


Figure 34: Root Locus Plot for the PI Controller

7.5 Phase-Lag Controller

Consider the following system:

$$G(s) = \frac{10}{s(s+2)(s+5)}$$

Design a Phase-Lag controller such that it meets the following design specifications:

$$\zeta \geq 0.6, \quad e_{ss}|_{\text{ramp}} \leq 0.1$$

Select the correct option from the following choices:

(a) $K(s) = \frac{s+0.02}{s+0.0012}$

(b) $K(s) = \frac{0.02}{s+0.0012}$

(c) $K(s) = \frac{s+0.02}{s}$

(d) $K(s) = \frac{s+0.02}{s+0.012}$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

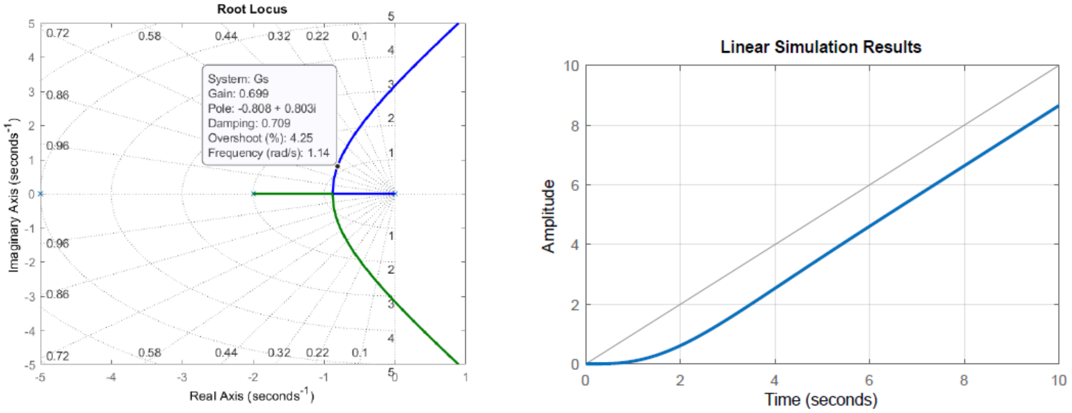
”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).
Since transient response improvement is not aimed, we may use a static controller to satisfy the damping ratio requirement. Let $K = 0.7$; then, the closed-loop system has dominant roots at: $s = -0.8 \pm j0.8 (\zeta = 0.7)$. The characteristic polynomial is factored as:

$$\Delta(s) = (s+5.38)\left(s^2 + 1.62s + 1.3\right)$$

Next, the velocity error constant is evaluated as: $K_v = 0.7$. In order to raise $K_v > 10$, we consider a phase-lag controller: $K(s) = \frac{s+0.02}{s+0.0012}$, where $\angle K(s_1) = -0.3°$. The compensated system has: $K_v = 11.67$; hence $e_{ss}|_{\text{ramp}} \leq 0.1$. The ramp response of the closed-loop system is plotted to confirm the results.



(a) Root locus plot for phase-lag design

(b) Closed-loop system response

Figure 35: Root Locus and Response of the Systems

With the addition of the phase-lag controller, the closed-loop transfer function is given as:

$$T(s) = \frac{7(s+0.02)}{(s+0.0202)(s+5.38)\left(s^2+1.61s+1.29\right)}$$

The closed-loop root at $s = -0.0202$ is almost canceled by the presence of the zero at $s = -0.02$. Hence, the slow mode $e^{-0.0202t}$ has a very small coefficient and only minimally affects the system response. The dominant closed-loop roots are also minimally affected.

7.6 Overshoot

Consider the system

$$G(s) = \frac{s+2}{(s+1)^2+1}$$

The positive root locus for $G(s)$ is given in Fig. 36.Suppose you want to find a feedback gain that is a positive integer $(K = 1, 2, 3, \ldots)$ such that the closed-loop system has overshoot M_p as small as possible and settling time ts is as fast possible. Explain what value of K you would pick?
Select the correct option from the following choices:

(a) K= 4

(b) K= 5

(c) K= 6

(d) K= 3

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (b).

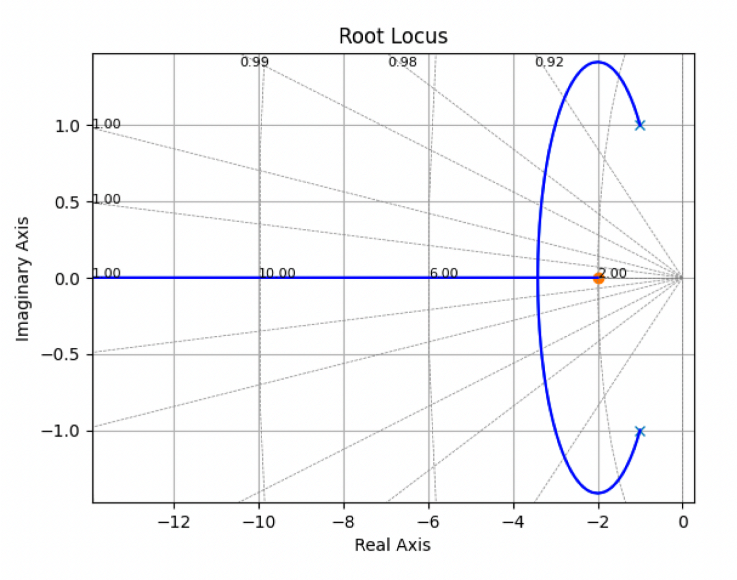


Figure 36: Root Locus of the System

Referring to the root locus in Fig. 36. The best location for closed-loop poles is at $-2-\sqrt{2}$, the point of multiple roots, because negative real roots means no overshoot. For large K, one pole moves right towards -2, which is not good for settling time. Thus, closed-loop characteristic equation is $(s+2+\sqrt{2})^2$ and it also equals to $s^2+(K+2)s+2(K+1)$. We finally get $K=2+2+\sqrt{2}$ and the nearest integer is K = 5.

7.7 Stability Analysis

Consider a feedback control system under unity feedback setting with a controller $K(s) = K$ and the system $G(s)$ given as follows:

$$G(s) = \frac{s(s+10)(s+12)}{(s-1)(s+11)(s+13)(s+14)}$$

Is there a non-negative value of K for which the closed loop system is stable? Explain why or why not?

Select the correct option from the following choices:

(a) Yes because for a high enough non-negative value of K all poles become stable.

(b) Yes because for any non-negative value of K all root locus branches lie in the stable region.

(c) No because for any non-negative value of K one branch of root locus lies in the unstable region.

(d) No because for any non-negative value of K at-least two system poles remain unstable.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).
No, there is no non-negative value of K for which the closed loop system is stable. Check the root locus given below. The root locus branch between the RHP pole and the imaginary axis zero keeps the system unstable.

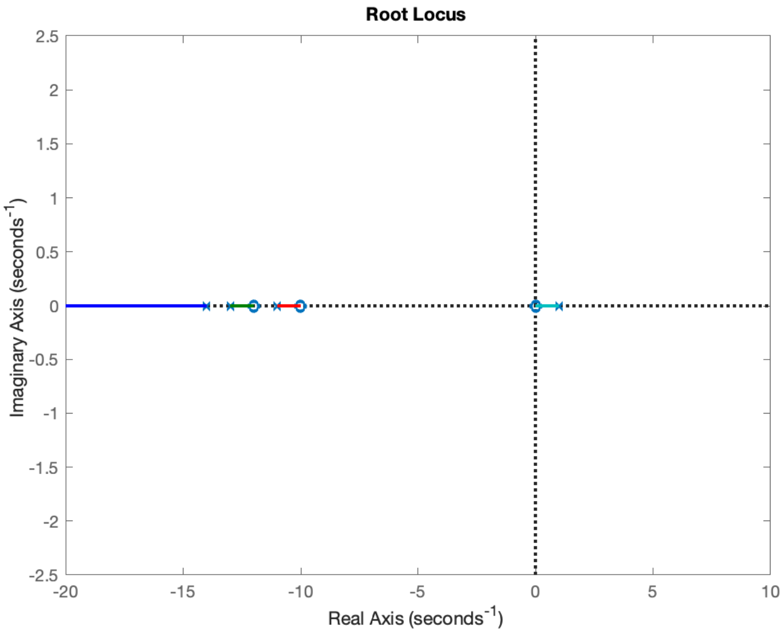
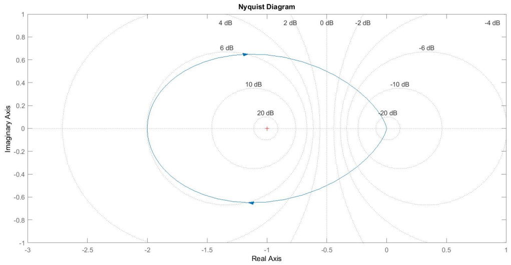


Figure 37: Root Locus of G(s)

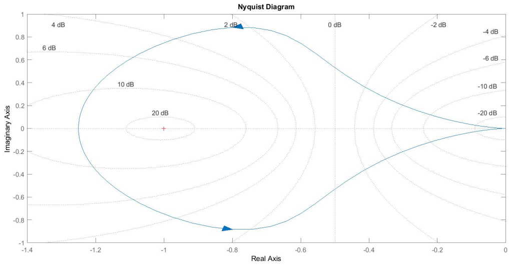
8 Nyquist Design

8.1 Nyquist Stability Analysis

Consider a standard closed-loop system characterized by the loop transfer function $L(s)$. The stability of the closed-loop system can be predicted by applying the Nyquist stability criterion, which involves analyzing the Nyquist plots of $L(s)$. Below are the Nyquist plots for two different loop transfer functions, labeled as (a) and (b). Use the Nyquist stability theorem to determine the number of poles in the right-half plane (RHP) for the closed-loop system based on these plots. Then, classify each system as either stable or unstable.



(a) Nyquist Plot for $L(s)$ in scenario (a)



(b) Nyquist Plot for $L(s)$ in scenario (b)

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) Stable, $P_{CL} = 0$ b) Unstable, $P_{CL} = 2$

(b) a) Unstable, $P_{CL} = 1$ b) Stable, $P_{CL} = 0$

(c) a) Unstable, $P_{CL} = 3$ b) Stable, $P_{CL} = 0$

(d) a) Unstable, $P_{CL} = 2$ b) Stable, $P_{CL} = 0$

Solution :
Correct answer is (d).

(a) Unstable.
Steps: $P_{CL} = P_{OL} - N_{CCW} = 1 - (-1) = +2$.

(b) Stable.
Steps: $P_{CL} = P_{OL} - N_{CCW} = 1 - 1 = 0$.

8.2 State-Feedback Controller Design and Stability Analysis

Consider the single-input, single-output transfer function:

$$G_p(s) = \frac{1 - s/2}{s^2(1 + s/2)}$$

(a) Calculate a state-feedback controller $u = -Kx + r$ that places the closed loop poles at -4, -13 and -25.

(b) Construct a stable observer and put this together to form a compensator of the form $U = -G_cY + G_rR$.

(c) Calculate the Nyquist plot of G_cG_p. Is the system stable? If so, calculate the gain and phase margins.

Solution :
We can place the closed loop poles at the desired locations using the gain $K = \begin{bmatrix}1300 & 477 & 40\end{bmatrix}$. Suppose we place the observer poles at -50, -51, -52 which leads to the observer gain

$$L = \begin{bmatrix}9301 & 18450 & 29400\end{bmatrix}^T$$

The transfer functions G_c and G_r are thus given as follows:

$$\begin{aligned} G_c &= \frac{2.207e07s^2 + 7.979e07s + 8.619e07}{s^3 + 193s^2 + 1.432e04s + 2.257e07} \\ G_r &= \frac{s^3 + 153s^2 + 7802s + 1.326e05}{s^3 + 193s^2 + 1.432e04s + 2.257e07} \end{aligned}$$

The Nyquist plot of G_cG_p is given in Fig. 38.

The gain margin is 0.115 dB and the phase margin is -1.11 deg. We now check stability with the nyquist plot. We will use Fig. 39 for counting the number of windings around -1. Clearly there are 3 anticlockwise loops around -1, coloured in red, green and blue. So N must be -3. Gc has 2 RHP poles, and G_p has 2 poles at origin. If the poles at origin are included as RHP poles, P = 4, else P = 2. In either case, $Z = N + P = 1/-1$ and not zero. However, the closed loop has to be stable because we placed the poles in the LHP ourselves.

The problem here is that G_c has a double pole at the origin. This is why the Nyquist plot is not a closed loop, and shoots to infinity as ω approaches zero. So we need to consider a small semi-circular modified contour of radius r around the origin. Radius r is taken to be very small. There are three poles in G_c, two of which are in RHP and one is in LHP. There are three poles in G_p as well, one in LHP and two at the origin. However, with respect to the modified contour, the two poles at the origin lie in the LHP. So they are not counted as open loop RHP poles and P = 2.

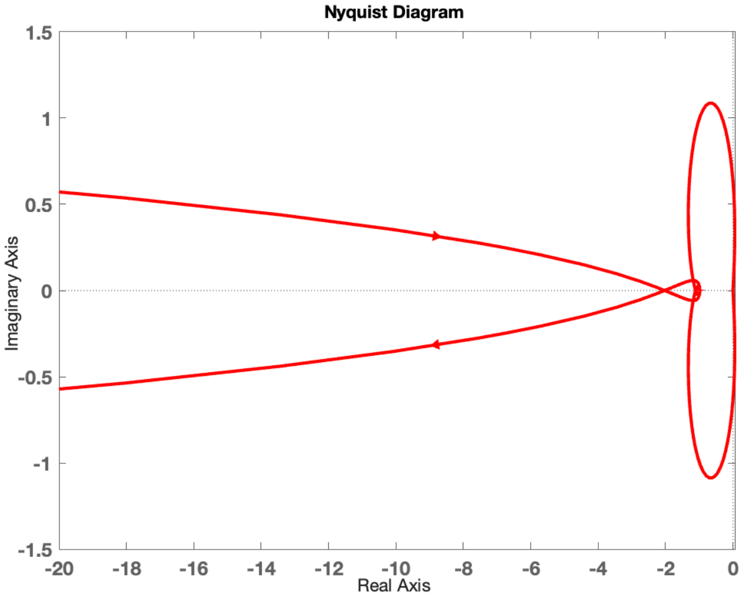


Figure 38: Nyquist Plot of $G_c G_p$

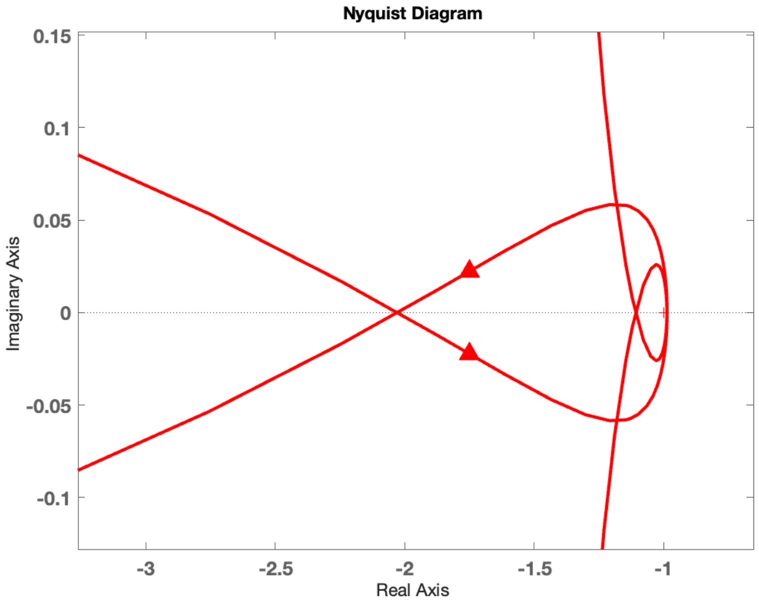


Figure 39: Stability check

Now we need to understand how the image of the Nyquist plot changes under this modified contour. We forget the 3 anticlockwise loops already accounted for before and focus only on the part that shoots off to infinity in Fig. 39. For r very small, $G_p G_c\,(jr) \approx \frac{k}{(jr)^2} = -\frac{k}{r^2}$ where k is the finite limit $\lim s \rightarrow 0 s^2 G_c G_p(s)$. The point is that $G_c G_p(jr)$ will be close to the real line even though its magnitude is very large $\approx \frac{k}{r^2}$. Since the plot goes toward negative infinity, we can also say that k is negative. The semicircular contour can be parameterized as $re^{j\theta}$ as θ goes from $-\frac{\pi}{2} \to -\frac{\pi}{4} \to 0 \to \frac{\pi}{4} \to \frac{\pi}{2}$. Since r is really small, $G_cG_p(re^{j\theta}) \approx -\frac{k}{r^2}e^{-2j\theta}$. Since $-k$ is positive, the phase of $G_cG_p(re^{j\theta})$ will be $\approx e^{-2j\theta}$. As θ travels from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ in the anti-clockwise direction, -2θ travel from $-\pi$ back to $\pi = -\pi$ in the clockwise direction. So the Nyquist plot travels through a large clockwise circle of radius $-\frac{k}{r^2}$ when we modify the contour. This circle will wind around -1 due to its very large radius, which makes $N = -3 + 1 = -2$. Since $P = 2$, $Z = N + P = 0$ which means the system is stable.

8.3 Stabilizing Controllers

Consider the following:

$$G(s) = \frac{1}{(s+2)(s2+2s+5)}$$

Suppose this is in unity feedback with a constant gain controller K. In other words, we have a negative feedback loop where the forward gain is KG(s) and the loop gain is also KG(s).

(a) Determine what values of K stabilize the closed-loop system using the Routh-Hurwitz stability criterion.

(b) Using the Nyquist plot, determine what values of K stabilize the closed-loop system. Does this match your answer from the Routh-Hurwitz criterion?

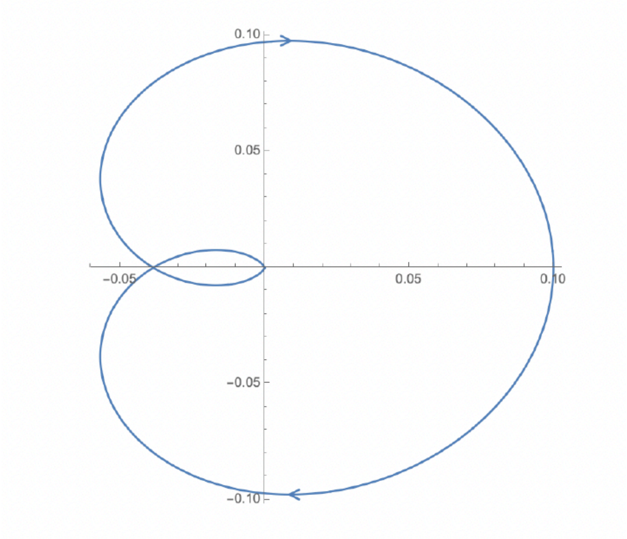


Figure 40: Nyquist plot of the system

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) $-26 < K < 26$

(b) $-26 < K < 10$

(c) $-10 < K < 10$

(d) $-10 < K < 26$

Solution :

Correct answer is (d).

(a) The characteristic equation is: $s^3 + 4s^2 + 9s + 10 + K$ and the Routh table reads as follows and it yields that $-10 < K < 26$.

$$\begin{array}{lcc}
s^3 & 1 & 9 \\
s^2 & 4 & \text{10+K} \\
s^1 & -\frac{1}{4}(K-26) & 0 \\
s^0 & \text{K+10} &
\end{array}$$

(b) Fig. 40 shows the Nyquist plot for this system. Let N be the number of encirclements of $\frac{-1}{K}$ and P be the number of open loop poles. We have that $N = Z - P$ where Z is the number of closed loop poles. In this particular case P = 0 yielding N = Z. This gives that

$$-\frac{1}{K} > 10 \qquad -\frac{1}{K} < 0.0384$$

which is the same condition as derived using Routh-Hurwitz criterion.

8.4 Gain/Phase Margin

Use the Nyquist plot of KG(s) to calculate the gain margin and phase margin for the following system $G(s)$ and controller $K(s)$:

$$G(s) = \frac{1}{(s+1)(s+2)(s+4)} \qquad K = 10$$

Assume the system and the controller are in standard unity feedback setting.

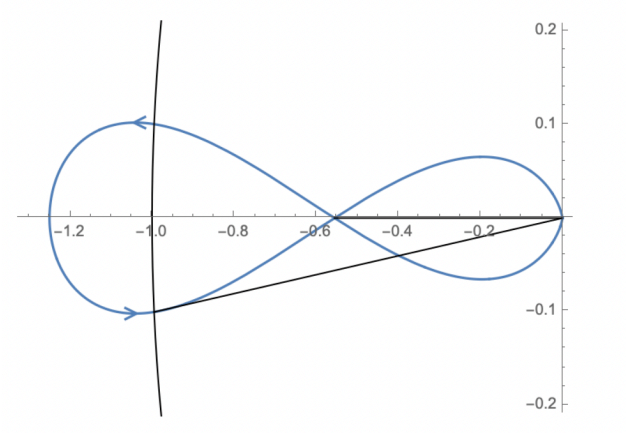


Figure 41: Nyquist Plot of the System

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

- (a) $PM \approx -5.74$ degrees
- (b) $PM \approx -57.4$ degrees
- (c) $PM \approx 57.4$ degrees
- (d) $PM \approx 5.74$ degrees

Solution :
Correct answer is (d).

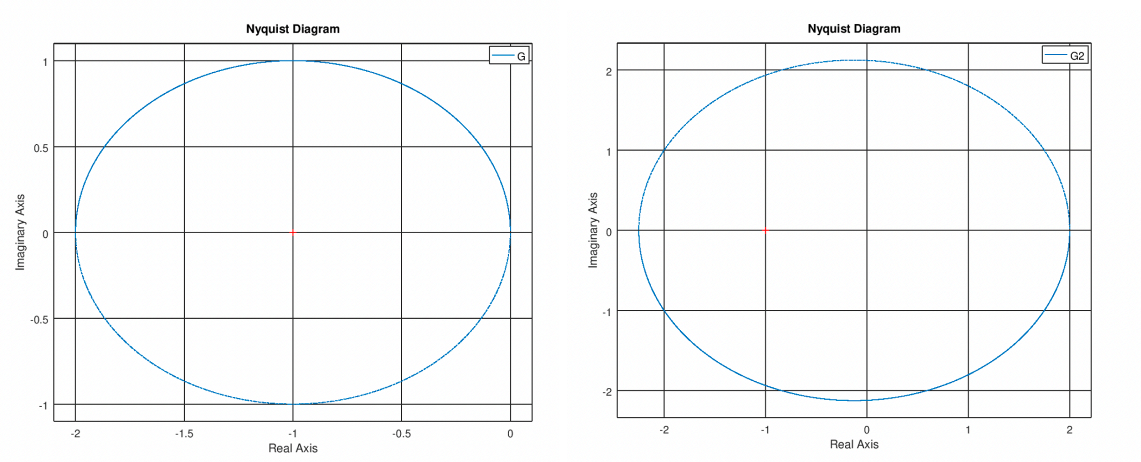
Nyquist plot of KG(s) is shown in Fig. 41. From the Nyquist plot, two points (-0.55, -1.25) intersect the negative real axis. The corresponding gain margins are $\frac{1}{0.55} \approx 1.8$ and $\frac{1}{1.25} \approx 0.8$. The GM of the system is $[0.8, 1.8]$.

To obtain PM from the Nyquist plot, draw a unit circle and mark the points where unit circle intersects the Nyquist plot. Draw a line from the origin to one of the points. The angle formed between that line and the $180°$ axis is the PM. The Nyquist plot here intersects the unit circle at two symmetric points, one of which is $\approx (-0.995, -0.1)$ and so the PM is arctan $\left(\frac{-0.1}{-0.995}\right) = 5.74$ degrees.

8.5 Closed-Loop Poles of the Feedback

Consider the system $G(s)$ and the controller $K(s)$ are in standard unity feedback setting. Define $L(s) := G(s)K(s)$. Use the Nyquist plot for the following loop transfer functions and apply the Nyquist stability theorem to predict the number of closed-loop poles of the feedback system in the right-half plane.

$$(a) \quad L(s) = \frac{-8}{s+4} \qquad\qquad (b) \quad L(s) = \frac{2s+9}{s-4}$$



(a) Nyquist plot of $L(s)$ in (a)

(b) Nyquist plot of $L(s)$ in (b)

Figure 42: Nyquist Plot for the Systems

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $P_{CL} = 0$ b) $P_{CL} = 2$

(b) a) $P_{CL} = 1$ b) $P_{CL} = 0$

(c) a) $P_{CL} = 2$ b) $P_{CL} = 0$

(d) a) $P_{CL} = 0$ b) $P_{CL} = 1$

Solution :

Correct answer is (b).

The Nyquist plot for the two $L(s)$ is given in Fig. 42.The Nyquist stability theorem says that if the number of RHP poles of the closed loop transfer function $\frac{L(s)}{1+L(s)}$ is Z, the number of RHP poles of the open loop transfer function L(s) is P, and N is the number of clockwise rotations of the Nyquist plot around -1, then Z = N + P. Therefore,

(a) N = 1; P = 0 so Z = 1.

(b) N = -1; P = 1 so Z = 0.

9 Gain/Phase Margins

9.1 Gain/Phase Margins

Determine the gain and phase margin for the system in which $GH(j\omega) = \frac{1}{(j\omega+1)^3}$

Select the correct option from the following choices:

(a) Gain margin is 8 and Phase margin is π.

(b) Gain margin is 8 and Phase margin is $-\pi$.

(c) Gain margin is 1 and Phase margin is 2π.

(d) Gain margin is 4 and Phase margin is π.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).
Writing $GH(j\omega)$ in polar form, we have

- The magnitude $|GH(j\omega)| = \frac{1}{\sqrt{\omega^2+1}^3}$
- The phase $\angle GH(j\omega) = -3\tan^{-1}(\omega)$

Then $-3\tan^{-1}\omega_\pi = -\pi, \omega_\pi = \tan(\pi/3) = \sqrt{3}$. Hence, gain margin is

$$\frac{1}{|GH(j\omega_\pi)|} = 8$$

Also,

$$|GH(j\omega_1)| = \frac{1}{(\omega_1^2+1)^{3/2}} = 1$$

happens only at $\omega_1 = 0$, therefore

$$\phi_{PM} = 180^\circ + (-3\tan^{-1} 0) = 180^\circ = \pi \quad \text{radians}$$

9.2 Bandwidth

Determine the bandwidth for the system with transfer function $(C/R)(s) = 1/(s+1)$.

Select the correct option from the following choices:

(a) 2 rad/s

(b) 1 rad/s

(c) $\sqrt{2}$ rad/s

(d) $\frac{1}{\sqrt{2}}$ rad/s

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (b).
We have

$$\left|\frac{C}{R}(j\omega)\right| = \frac{1}{\sqrt{\omega^2+1}} \tag{142}$$

The bandwidth is the frequency at which this magnitude drops to $\frac{1}{\sqrt{2}}$ of its maximum value. Since $|(C/R)(j\omega)|$ is a strictly decreasing function of positive frequency, maximum happens at $\omega = 0$. Setting the magnitude equal to $\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{0+1}}$ and solving for ω yields:

$$\frac{1}{\sqrt{\omega^2+1}} = \frac{1}{\sqrt{2}}$$

Solving this equation:

$$\begin{aligned}\sqrt{\omega^2+1} &= \sqrt{2}\\ \omega^2+1 &= 2\\ \omega^2 &= 1\\ \omega &= 1\end{aligned}$$

Therefore, the bandwidth of the system is $\omega = 1$ rad/s.

9.3 Resonance Peak

Determine the resonance peak M_p and the resonant frequency ω_p for the system whose transfer function is $\frac{C}{R}(s) = \frac{5}{s^2+2s+5}$.

Select the correct option from the following choices:

(a) $M_p = \frac{1}{4}$ **and** $\omega_p = \sqrt{3}$

(b) $M_p = \frac{5}{4}$ **and** $\omega_p = -\sqrt{3}$

(c) $M_p = \frac{5}{4}$ **and** $\omega_p = \sqrt{3}$

(d) $M_p = \frac{1}{4}$ **and** $\omega_p = \sqrt{5}$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (c).
The magnitude $|C/R(j\omega)|$ is given by:

$$\left|\frac{C}{R}(j\omega)\right| = \left|\frac{5}{-\omega^2+2j\omega+5}\right| = \frac{5}{\sqrt{\omega^4-6\omega^2+25}}$$

Setting the derivative of $\left|\frac{C}{R}(j\omega)\right|$ equal to zero, we get $\omega_p = \pm\sqrt{3}$. Therefore,

$$M_P = \max_{\omega}\left|\frac{C}{R}(j\omega)\right| = \left|\frac{C}{R}(j\sqrt{3})\right| = \frac{5}{4} \tag{143}$$

9.4 Gain/Phase Margins

Calculate the gain and phase margin for $GH = \frac{432}{s(s^2+13s+115)}$.

Select the correct option from the following choices:

(a) Gain margin is 3.46 and Phase margin is $-60°$.

(b) Gain margin is 4.4 and Phase margin is $90°$.

(c) Gain margin is 4 and Phase margin is $30°$.

(d) Gain margin is 3.46 and Phase margin is $64°$.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

“‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :

Correct answer is (d).

Given the open-loop transfer function $GH(s) = \frac{432}{s(s^2+13s+115)}$, we can write $GH(j\omega)$ in polar form

- The magnitude $|GH(j\omega)| = \frac{432}{\sqrt{(115\omega-\omega^3)^2+(-13\omega^2)^2}}$
- The phase $\angle GH(j\omega) = -\tan^{-1}(\frac{115\omega-\omega^3}{-13\omega^2})$

Then $-\tan^{-1}(\frac{115\omega_\pi-\omega_\pi^3}{-13\omega_\pi^2}) = -\pi, \omega_\pi = \sqrt{115}$. Hence, gain margin is

$$\frac{1}{|GH(j\omega_\pi)|} = \frac{13 \times 115}{432} = 3.46$$

Also,

$$|GH(j\omega_1)| = \frac{432}{\sqrt{(115\omega_1 - \omega_1^3)^2 + (-13\omega_1^2)^2}} = 1$$

happens only at $\omega_1 = 3.86$, therefore

$$\phi_{PM} = 180° - \tan^{-1}\left(\frac{115\omega_1 - \omega_1^3}{-13\omega_1^2}\right) = 64°$$

9.5 Gain Margin

True or False:

(a) The gain margin is the amount of allowable variation in the phase of the plant before the closed loop becomes unstable.

(b) As a rule of thumb (not being strict), the closed-loop should remain stable for gain variations in the range $[0.5, 2](= \pm 6dB)$.

Select the correct option from the following choices:

(a) For (a) False and for (b) True

(b) For (a) True and for (b) False

(c) For (a) True and for (b) True

(d) For (a) False and for (b) False

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :
Correct answer is (a).

(a) False. (should be the allowable variation in the gain of the plant)

(b) True.

9.6 Maximum Phase Lead

Show that the maximum phase lead of the lead compensator

$$P_{Lead}(j\omega) = \frac{(a/b)(1 + j\omega/a)}{1 + j\omega/b} \tag{144}$$

occurs at $\omega_m = \sqrt{ab}$ and find the maximum phase. What magnitude is produced at the maximum phase lead?

Solution :
The phase angle of the lead compensator is $\phi = \arg P_{Lead}(j\omega) = \tan^{-1}\omega/a - \tan^{-1}\omega/b$. Then

$$\frac{d\phi}{d\omega} = \frac{1}{a[1 + (\omega/a)^2]} - \frac{1}{b[1 + (\omega/b)^2]} \tag{145}$$

Setting $d\phi/d\omega$ equal to zero yields $\omega^2 = ab$. Thus the maximum phase lead occurs at $\omega_m = \sqrt{ab}$. Hence $\phi_{\text{max}} = \tan^{-1}\sqrt{b/a} - \tan^{1}\sqrt{a/b}$. But since $\tan^{-1}\sqrt{b/a} = \pi/2 - \tan^{-1}\sqrt{a/b}$, we have $\phi_{\text{max}} = (90 - 2\tan^{-1}\sqrt{a/b})$ degrees.

The magnitude factor is given by

$$|P_{Lead}(j\sqrt{ab})| = |\frac{(a/b)(1 + j\sqrt{b/a})}{1 + j\sqrt{a/b}}| = \sqrt{\frac{a}{b}}$$

9.7 Gain/Phase Margin

Suppose we have the open-loop transfer function $G(s) = \frac{K}{s(s+20)(s+85)}$, and we put it through unity feedback, i.e. the closed loop transfer function is $\frac{G(s)}{1+G(s)}$.

(a) Set the gain K so that the magnitude is 1 (0 dB) at $\omega = 1$. What K achieves this?

(b) We wish to achieve 15% overshoot in the transient response for a step input. What phase margin is required to achieve this?

(c) What frequency on the Bode phase diagram yields this phase margin?

(d) Find the adjusted gain necessary to produce the required phase margin.

Select the correct option from the following choices:

(a) K=1702, Desired phase margin is $-53°$, $\omega = 11.2$rad/s and adjusted gain $K^* = 12010$

(b) K=1702, Desired phase margin is $53°$, $\omega = 11.2$rad/s and adjusted gain $K^* = 22010$

(c) K=1802, Desired phase margin is $53°$, $\omega =$ 12rad/s and adjusted gain $K^* = 22010$

(d) K=1702, Desired phase margin is $50°$, $\omega =$ 12rad/s and adjusted gain $K^* = 32010$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, "(a)" is correct, not "a":

‘‘‘

"Choice": "[ONLY ONE choice from (a), (b), (c), (d)]",

"Reason": "[Your explanation]"

Solution :
Correct answer is (b).

(a) For 0 db at $\omega = 1$, we have that,

$$|G(j1)| = \left|\frac{K}{j(j+20)(j+85)}\right| = 1$$

so, $K = |j(j+20)(j+85)| = \sqrt{2897626} = 1702.24$

(b) For 15% overshoot, we have that,

$$\zeta = \frac{-\ln(0.15)}{\sqrt{\pi^2 + \ln^2(0.15)}} = 0.5169$$

using this damping ratio, we can obtain the desired phase margin

$$\Phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right) = 53.17^\circ$$

(c) The phase which yields this phase margin is $\phi = -180 + \Phi_M = -127$. The Bode plot shows the phase is -127 when $\omega = 11.2\text{rad/s}$.

(d) We achieve the desired phase margin by shifting the magnitude at $\omega_{\text{PM}} = 11.2$ to 0 dB. This can be done by changing the gain:

$$\begin{aligned} 0 \text{ dB} &= 20 \log \left|K^* G\left(j\omega_{\text{PM}}\right)\right| \\ 1 &= \left|K^* G\left(j\omega_{\text{PM}}\right)\right| \end{aligned}$$

Therefore $K^* = 1/\left|G\left(j\omega_{\text{PM}}\right)\right| = 12.93$ and the adjusted gain is $K^* = \sqrt{2897626} \times 12.93 \approx 22010$.

9.8 Phase Margin

Show that for the transfer function $KG(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s}$, the phase margin is independent of ω_n and is given as $\tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}\right)$.

Solution :

Given,

$$KG(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s}$$

To calculate the phase margin, we first find the gain-crossover-frequency (ω_c) :

$$\begin{aligned} |KG(j\omega)||_{\omega=\omega_c} = 1 &\Rightarrow \frac{\omega_n^2}{|-\omega_c^2 + 2j\omega_n\omega_c\zeta|} = 1 \\ &\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_c^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega_n^4 &= \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 \\ \Leftrightarrow \omega_n^4 + \left(2\zeta^2\omega_n^2\right)^2 &= \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 + \left(2\zeta^2\omega_n^2\right)^2 \\ \Leftrightarrow \omega_n^4\left(1+4\zeta^4\right) &= \left(\omega_c^2 + 2\zeta^2\omega_n^2\right)^2 \end{aligned}$$

Thus, excluding the negative root (why?),

$$\begin{aligned} \omega_c^2 &= -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4+1} \\ &= \left(\sqrt{4\zeta^4+1} - 2\zeta^2\right)\omega_n^2 \end{aligned}$$

So we get,

$$KG(j\omega) = \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c} \quad \text{and} \quad \angle KG(j\omega) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega_c}{-\omega_c^2}\right)$$

which is

$$\tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)$$

Note that $\theta = \tan^{-1} x \Longleftrightarrow \pi + \theta = \tan^{-1}(x)$.

If we ask for formulas, Claude-3 gives $PM = \pi - \text{arctan}\left(\frac{2\zeta}{\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}}\right) = 180^\circ - \tan^{-1}\left(\frac{2\zeta(\zeta+\sqrt{\zeta^2+1})}{(\zeta+\sqrt{\zeta^2+1})^2-2\zeta(\zeta+\sqrt{\zeta^2+1})}\right)$, which is close but stil wrong.

9.9 Comparing the Contributions of Poles and Zeros in Magnitude and Phase Plots

Consider the transfer function

$$G(s) = \frac{1}{(s-1)(s+1)}$$

The transfer function $G(j\omega)$ in this problem has an unstable real pole of the form $(j\omega\tau - 1)^{-1}$ and a stable real pole of the form $(j\omega\tau + 1)^{-1}$. Compare the role of both poles and explain their contribution both to the magnitude and to the phase plot.

Select the correct option from the following choices:

(a) Magnitude for stable pole is $\frac{1}{\sqrt{(\omega\tau)^2+1}}$ and for unstable pole $-\frac{1}{\sqrt{(\omega\tau)^2+1}}$. For stable pole phase changes from $0°$ to $-90°$ while for unstable pole phase changes from $-180°$ to $-90°$.

(b) Magnitude for stable pole is $\frac{1}{\sqrt{(\omega\tau)^2+1}}$ and for unstable pole $-\frac{1}{\sqrt{(\omega\tau)^2+1}}$. For stable pole phase changes from $0°$ to $-90°$ while for unstable pole phase changes from $-180°$ to $-90°$.

(c) Both poles have same magnitude $\frac{1}{\sqrt{(\omega\tau)^2+1}}$. For stable pole phase changes from $0°$ to $-90°$ while for unstable pole phase changes from $-180°$ to $-90°$.

(d) Both poles have same magnitude $\frac{1}{\sqrt{(\omega\tau)^2+1}}$. For stable pole phase changes from $-180°$ to $-90°$ while for unstable pole phase changes from $0°$ to $-90°$.

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

Solution :

Correct answer is (c).

$(j\omega\tau - 1)^{-1}$ and $(j\omega\tau + 1)^{-1}$ have the same magnitude $\frac{1}{\sqrt{(\omega\tau)^2+1}}$.

Before the break point $\omega = \frac{1}{\tau}$, $(j\omega\tau + 1)^{-1} \approx 1$. After the break point, $(j\omega\tau + 1)^{-1} \approx (j\omega\tau)^{-1}$. Therefore, its phase changes from $0°$ to $-90°$.

The phase of unstable real pole is trickier. Before the break point $\omega = \frac{1}{\tau}$, $(j\omega\tau - 1)^{-1} \approx -1$. After the break point, $(j\omega\tau + 1)^{-1} \approx (j\omega\tau)^{-1}$. Therefore, its phase changes from -180° to -90°.

10 Advanced Topics (Lyapunov Stability, Controllability and Observability)

10.1 Lyapunov Stability

Consider the differential equation

$$\ddot{y}(t) + 2\delta\dot{y}(t) + y(t) = 0 \tag{146}$$

with the initial condition $y(0) = 1; \dot{y}(0) = 0$. Evaluate

$$\eta = \int_0^\infty y^2(t)dt \tag{147}$$

(i) First express the system in its state-space form.

(ii) Write η as a quadratic form $x_0^T P x_0$ where x_0 is the initial condition

(iii) Derive and solve the Lyapunov equation for P

Determine which of the following are correct answer to $i) - iii)$?

(a) (i):

$$A = \begin{bmatrix} 0 & -1 \\ -1 & -2\delta \end{bmatrix} \tag{148}$$

(ii):

$$\eta = x_0^T \underbrace{\int_0^\infty e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{At} dt}_{P} \, x_0 \tag{149}$$

(iii):

$$P = \begin{bmatrix} \delta + \frac{1}{4\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4\delta}. \end{bmatrix} \tag{150}$$

(b) (i):

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \tag{151}$$

(ii):

$$\eta = x_0^T \underbrace{\int_0^\infty e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{At} dt}_{P} \, x_0 \tag{152}$$

(iii):

$$P = \begin{bmatrix} \delta + \frac{1}{2\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2\delta}. \end{bmatrix} \tag{153}$$

(c) (i):

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \tag{154}$$

(ii):

$$\eta = x_0^T \underbrace{\int_0^{\infty} e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{At} dt\, x_0}_{P} \tag{155}$$

(iii):

$$P = \begin{bmatrix} \delta + \frac{1}{4\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4\delta}. \end{bmatrix} \tag{156}$$

(d) (i):

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \tag{157}$$

(ii):

$$\eta = x_0^T \underbrace{\int_0^{\infty} e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{At} dt\, x_0}_{P} \tag{158}$$

(iii):

$$P = \begin{bmatrix} \delta + \frac{1}{4\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4\delta}. \end{bmatrix} \tag{159}$$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”: “‘ ”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]” “‘

Solution :

Correct answer is (d).

To express this in state-space form, we introduce the state vector $x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$, leading to

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix}}_{A} x(t) \tag{160}$$

Writing $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} x_0$, we can write η as follows

$$\eta = \int_0^{\infty} x_0^T e^{A^T t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} dt = x_0^T \underbrace{\int_0^{\infty} e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{At} dt\, x_0}_{P} \tag{161}$$

Assuming P has the form

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix},$$

we plug into the Lyapunov equation to get

$$\begin{bmatrix} 0 & -1 \\ 1 & -2\delta \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving this system of equations leads to

$$P = \begin{bmatrix} \delta + \frac{1}{4\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4\delta}. \end{bmatrix} \tag{162}$$

Therefore, $\eta = \delta + \frac{1}{4\delta}$ and the minimum happens at $\frac{d\eta}{d\delta} = 0$, which is $\delta = \frac{1}{2}$.

10.2 Controllability

Consider the second-order system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} x(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} u(t)$$

and the output equation

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t).$$

The parameters a, b, c_1, and c_2 are unknown a priori.

For which of the following choices for a, b, c_1 an c_2 below ensure that the system is completely controllable?

(a) $(a, b, c_1, c_2) = (-4, 3, 1, 1)$

(b) $(a, b, c_1, c_2) = (10, 3, 0, 10)$

(c) $(a, b, c_1, c_2) = (3, -7/2, 1, 2)$

(d) $(a, b, c_1, c_2) = (10, 3, 0, 0)$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”: “‘ ”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]” “‘

Solution :
Correct answer is (b).

The controllability matrix is

$$P_c = \begin{bmatrix} c_1 & c_2 \\ c_2 & -ac_1 - bc_2 \end{bmatrix}$$

and $\det(P_c) = c_2^2 + [bc_1]c_2 + ac_1^2$. For controllability, we require $\det(P_c) \neq 0$, hence

$$c_2^2 + [bc_1]c_2 + ac_1^2 \neq 0$$

implies

$$\frac{c_2}{c_1} \neq -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - a}$$

where $\left(\frac{b}{2}\right)^2 - a \geq 0$. For real-valued c_1 and c_2, if $\left(\frac{b}{2}\right)^2 - a < 0$, all real values of c_1 and c_2 are valid. Valid values of the constants are $c_1 = 0$, $c_2 = 10$, $a = 10$, and $b = 3$. Valid values of the constants are $c_1 = 0$, $c_2 = 10$, $a = 10$, and $b = 3$. The step response is shown below:

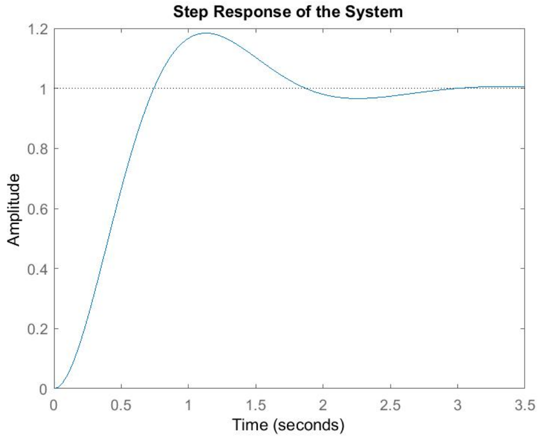


Figure 43: The System Response Over Time.

10.3 Lyapunov Stability

The origin is a singular point for the pair of equations

$$\frac{dx_1}{dt} = ax_1 + bx_2 \quad \text{and} \quad \frac{dx_2}{dt} = cx_1 + dx_2$$

Using Lyapunov theory, we want to find sufficient conditions on a, b, c, and d such that the origin is asymptotically stable. Which of the following choices of Lyapunov functions and parameters is a valid suffiicient condition?

(a) $V = x_1^2 + x_2^2$, and $a > 0, d > 0$ and $b = -c$

(b) $V = x_1^2 + x_2^2$, and $a < 0, d < 0$ and $b = -c$

(c) $V = x_1^2$, and $a < 0, b = 0$

(d) $V = x_2^2$, and $c = 0, d < 0$

Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”: “‘ ”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”, ”Reason”: ”[Your explanation]” “‘

Solution :
Correct answer is (b).
Using the Lyapunov function

$$V = x_1^2 + x_2^2$$

which is positive for all x_1, x_2 except $x_1 = x_2 = 0$. The time derivative of V is

$$\frac{dV}{dt} = 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt} = 2ax_1^2 + 2bx_1x_2 + 2cx_1x_2 + 2dx_2^2$$

To make $\frac{dV}{dt}$ negative for all x_1, x_2, we might choose $a < 0$, $d < 0$, and $b = -c$. In this case,

$$\frac{dV}{dt} = 2ax_1^2 + 2dx_2^2 < 0$$

except when $x_1 = x_2 = 0$. Hence one set of sufficient conditions for asymptotic stability are $a < 0$, $d < 0$, and $b = -c$. There are other possible solutions to this problem.

10.4 Lyapunov Stability

Consider the nonlinear discrete-time system described by

$$x_1(k+1) = x_1(k) - f[x_1(k)]$$

and candidate Lyapunov function $V = x_1^2$. Which of the following conditions are valid sufficient conditions for proving asymptotic stability of the origin using Lyapunov's direct method?

(a) $x_1 f(x_1) \leq 0$ and $\frac{f(x_1)}{x_1} \leq 2$ for all $x_1 \neq 0$

(b) $x_1 f(x_1) \geq 1$ and $\frac{f(x_1)}{x_1} \leq 2$ for all $x_1 \neq 0$

(c) $x_1 f(x_1) \geq 0$ and $\frac{f(x_1)}{x_1} \geq 2$ for all $x_1 \neq 0$

(d) $x_1 f(x_1) \geq 0$ and $\frac{f(x_1)}{x_1} \leq 2$ for all $x_1 \neq 0$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (d).
Let $V[x(k)] = [x_1(k)]^2$, which is greater than 0 for all $x \neq 0$. Then,

$$\Delta V = x_1^2(k+1) - x_1^2(k) = (x_1(k) - f[x_1(k)])^2 - x_1^2(k) \tag{163}$$
$$= x_1(k) f[x_1(k)] \left(\frac{f[x_1(k)]}{x_1} - 2 \right) \tag{164}$$

Therefore sufficient conditions for $\Delta V \leq 0$ and thus stability of the system are

$$x_1 f(x_1) \geq 0$$

and

$$\frac{f(x_1)}{x_1} \leq 2 \quad \text{for all} \quad x_1$$

10.5 Lyapunov Stability

Determine sufficient conditions for the global stability of the system.

$$\dot{x} = Ax + bf(x_1) \quad \text{where} \quad A = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{165}$$

Given a quadratic Lyapunv function $V = x^\top Px$ and $P = \begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}$ positive definite, which of the following are valid sufficient conditions for

(a) $\frac{f(x_1)}{x_1} \leq 5$ for all $x_1 \neq 0$

(b) $f(x_1) \geq 0$ for all $x_1 \neq 0$

(c) $\frac{f(x_1)}{x_1} \leq 4$ for all $x_1 \neq 0$

(d) $\frac{f(x_1)}{x_1} \geq 0$ for all $x_1 \neq 0$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

"‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :

Correct answer is (c).

Let $V = x^T Px$ and $P = \begin{bmatrix} a & c \\ c & 1 \end{bmatrix}$. Then,

$$\begin{aligned} \dot{V} &= x^T(PA + A^T P)x + x^T Pbf(x_1) + f(x_1)b^T Px \quad (166) \\ &= x^T \begin{bmatrix} -4a & -a-4c \\ -a-4c & -2c-4 \end{bmatrix} x + 2(a+2c)x_1 f(x_1) + 2(c+2)x_2 f(x_1) \quad (167) \end{aligned}$$

To eliminate the cross-product term $x_2 f(x_1)$, set $c = -2$. Then,

$$\dot{V} = -x^T Qx + 2(a-4)x_1 f(x_1)$$

Where $Q = \begin{bmatrix} 4a & a-8 \\ a-8 & 0 \end{bmatrix}$. For $Q \geq 0$, $a = 8$. The resulting $\dot{V}$ is

$$\dot{V} = -32x_1^2 + 8x_1 f(x_1) = -8x_1^2 \left(4 - \frac{f(x_1)}{x_1}\right) \tag{168}$$

Then $\dot{V} \leq 0$ and the system is stable if $\frac{f(x_1)}{x_1} \leq 4$ for all $x_1 \neq 0$.

10.6 Linearization

Consider the following nonlinear system:

$$\dot{x}(t) = f(x,u) = -4x(t) - 2u(t) + u(t)^3.$$

(i) Find an equilibrium $(\bar{x}, \bar{u})$ with $\bar{u} = 2$.

(ii) Linearize the dynamics around $(\bar{x}, \bar{u})$. Find the value of a_0 and b_0 of the linearized model in the form:

$$\dot{\delta}_x(t) + a_0\delta_x(t) = b_0\delta_u(t),$$
where $\delta_x(t) := x(t) - \bar{x}$ and $\delta_u(t) := u(t) - \bar{u}$.

Which of following choices corresponds to the correct solution?

(a) (i) $\bar{x} = 1$, (ii) $(a_0, b_0) = (4, 10)$

(b) (i) $\bar{x} = 0$, (ii) $(a_0, b_0) = (4, 10)$

(c) (i) $\bar{x} = 1$, (ii) $(a_0, b_0) = (0, 6)$

(d) (i) $\bar{x} = 0$, (iii) $(a_0, b_0) = (0, 6)$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).

(a) $\bar{x} = 1$.
Steps: Solve $-4\bar{x} - 2\bar{u} + \bar{u}^3 = 0$ with $\bar{u} = 2$ gives $\bar{x} = 1$.

(b) $(a_0, b_0) = (4, 10)$.
Steps: $\delta_x = x - \bar{x} = x - 1$, $\delta_u = u - \bar{u} = u - 2$. Then we have

$$\dot{\delta}_x(t) \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\delta_x + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\delta_u = -4\delta_x + 10\delta_u.$$

10.7 Controllability

Consider the following system:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ b \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

What value of b leads to the loss of controllability in the system?

(a) $b = 1$

(b) $b = 0$

(c) $b = -1$

(d) $b = 10$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (b).
System’s transfer function is given by:

$$G(s) = C(sI - A)^{-1}B = \frac{s + 2 + b(s - 1)}{(s + 2)(s - 1)}$$

We can see that setting $b = 0$ leads to pole-zero cancellation and loss of controllability in the system. Alternatively, it can also be checked by:

$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & 1 \\ b & -2b \end{pmatrix}$$

and for $b = 0$, $\text{rank}(\mathcal{C}(A, B)) = 1 < 2$.

11 System Sensitivity Measures

11.1 Sensitivity Transfer Function

The closed-loop should have zero steady-state error for step reference commands.

(i) What is the the value of the sensitivity transfer function $S(s)$ at $s = 0$?

(ii) What is the the value of the loop transfer function $|L(0)|$ at $s = 0$?

Which of the following give the correct answer for (i) and (ii)?

(a) $S(0) = 0$, $|L(0)| = +\infty$

(b) $S(0) = \infty$, $|L(0)| = +\infty$

(c) $S(0) = 0$, $|L(0)| = 0$

(d) $S(0) = \infty$, $|L(0)| = -\infty$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (a).

(a) $S(0) = 0$.
Since the closed-loop system have zero steady-state error: $e(t) \to 0$ as $t \to \infty$. We must have $S(0) = 0$.

(b) $|L(0)| = +\infty$.
Since $S(s) = \dfrac{1}{1 + L(s)}$, $S(0) = 0$ implies that $|L(0)| = +\infty$.

11.2 Sensitivity Transfer Function

The closed-loop should have an error $e(t) \leq 0.1$ for reference commands $r(t) = 2\sin(\omega t)$ with $\omega \leq 10\,\text{rad/sec}$. This means that $|S(j\omega)| \leq A$ for $\omega \leq B\,\text{rad/sec}$. What is the value of A and B?

Which of the following gives the correct values for A and B?

(a) $(A, B) = (0.10, 10)$

(b) $(A, B) = (0.05, 10)$

(c) $(A, B) = (0.01, 5)$

(d) $(A, B) = (0.05, 5)$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (b).
$(A, B) = (0.05, 10)$.
Steps:
We have $e(t) = 2|S(j\omega)|\sin(\omega t + \measuredangle S(j\omega))$.
Since $e(t) \leq 0.1$ for $\omega \leq 10\,\text{rad/sec}$, we must have:

$$|S(j\omega)| \leq \frac{0.1}{2} = 0.05$$

for $\omega \leq 10\,\text{rad/sec}$.

11.3 Closed-Loop Transfer Function

The closed-loop should have an output $y(t) \leq 0.05$ for reference commands $n(t) = 10\sin(\omega t)$ with $\omega \geq 1000\,\text{rad/sec}$. This means that $|T(j\omega)| \leq A$ for $\omega \geq B\,\text{rad/sec}$. What is the value of A and B?

Which of the following gives the correct values for A and B?

(a) $(A, B) = (0.5, 10)$

(b) $(A, B) = (0.05, 100)$

(c) $(A, B) = (0.005, 1000)$

(d) $(A, B) = (0.05, 5)$

Please output your choice by just choosing ONLY ONE option from (a), (b), (c), (d) and a short explanation below in a JSON format by filling in the placeholders in []:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

‘‘‘

Solution :
Correct answer is (c).
$(A, B) = (0.005, 1000)$.
Steps:
We have $y(t) = -10|T(j\omega)|\sin(\omega t + \measuredangle T(j\omega))$.
Since $y(t) \leq 0.1$ for $\omega \geq 1000\,\text{rad/sec}$, we must have:

$$|T(j\omega)| \leq \frac{0.05}{10} = 0.005$$

for $\omega \geq 1000\,\text{rad/sec}$.

12 Loop-Shaping

12.1 Loop Gain

We plan to design a controller $K(s)$ for a system with transfer function given by $G(s)$. The controller and the system are in standard unity feedback setting. Let $e = r - y$ where r and y representing the reference and the output signals respectively. Let n represent the output noise i.e. if $n \neq 0$ then $e = r - (y + n)$. The specifications are to design a controller so that the closed loop is stable and has:

(a) gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec.

(b) gain less than 0.04 from input n to output y for frequencies above 200 rad/sec.

Translate the above specifications into requirements on the loop gain $|L(j\omega)|$ where loop gain is given by $L(s) = K(s)G(s)$.
Please output your choice by just choosing *ONLY ONE* option from (a), (b), (c), or (d), and provide a short explanation below in JSON format by filling in the placeholders in []. Your answer should not include any further explanation, and remember to use parentheses; for example, ”(a)” is correct, not ”a”:

‘‘‘

”Choice”: ”[ONLY ONE choice from (a), (b), (c), (d)]”,

”Reason”: ”[Your explanation]”

(a) a) $|L(j\omega)| \geq 99$ for $\omega \leq 0.1\frac{rad}{sec}$ b) $|L(j\omega)| \leq 0.04$ for $\omega \geq 200\frac{rad}{sec}$

(b) a) $|L(j\omega)| \leq 99$ for $\omega \leq 0.1\frac{rad}{sec}$ b) $|L(j\omega)| \geq 0.04$ for $\omega \geq 200\frac{rad}{sec}$

(c) a) $|L(j\omega)| \leq 0.04$ for $\omega \leq 0.1\frac{rad}{sec}$ b) $|L(j\omega)| \geq 99$ for $\omega \geq 200\frac{rad}{sec}$

(d) a) $|L(j\omega)| \geq 0.04$ for $\omega \leq 0.1\frac{rad}{sec}$ b) $|L(j\omega)| \leq 99$ for $\omega \geq 200\frac{rad}{sec}$

Solution :
Correct answer is (a).
The transfer function from r to e is $S(s) = \frac{1}{1+L(s)}$. The steady error due to a unit step reference input $r(t) = \bar{r}$ is given by $e = S(0)\bar{r}$. Thus specification (a) is equivalent to $|S(j\omega)| \leq 0.01$ for $\omega \leq 0.1\frac{rad}{sec}$. This is approximately equivalent to $|L(j\omega)| \geq 99$ for $\omega \leq 0.1\frac{rad}{sec}$.

The transfer function from n to y is $-T(s) = \frac{-L(s)}{1+L(s)}$. Specification (b) corresponds to $|T(j\omega)| \leq 0.04$ for $\omega \geq 200\frac{rad}{sec}$ and this is approximately equivalent to $|L(j\omega)| \leq 0.04$ for $\omega \geq 200\frac{rad}{sec}$.

12.2 Proportional Controller

A model for a DC motor is:

$$J\dot{y} + by = cV$$

where y is the angular velocity of the motor shaft (deg/sec), and V is the input voltage (Volts). The model parameters are: $J = 3$ N m sec2/deg2 , $b = 5$ Nm sec/deg, and gain from input voltage to applied torque $c = 12$ Nm/Volts. We plan to design a controller $K(s)$ for this system. The controller and the system are in standard unity feedback setting. Let $e = r - y$ where r is the reference signal. Let n represent the output noise i.e. if $n \neq 0$ then $e = r - (y + n)$. Design a a proportional control law $K(s) = K_p$ so that the closed loop is stable and has gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec.

Solution :

Firstly, we can show that the requirement of gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec translates to $|L(j\omega)| = |K(j\omega)G(j\omega)| \geq 100$ for $\omega \leq 0.1\frac{rad}{sec}$. The system transfer function is $G(s) = \frac{12}{3s+5}$. This has a corner frequency at $\omega = 1.6$ rad/sec and DC gain $G(0) = 2.4$. Thus $K_p = \frac{100}{2.4}$ will ensure the gain requirement is approximately satisfied. Additionally, we can verify the closed loop pole of the system with $K_p = \frac{100}{2.4}$ is in the LHP so the closed loop system is stable. Thus both the requirements are met.

12.3 Proportional Controller

We plan to design a controller $K(s) = K_p$ for a system with transfer function $G(s)$ given by:

$$G(s) = \frac{12}{3s + 5}$$

The controller and the system are in standard unity feedback setting. Let $e = r - y$ where r and y representing the reference and the output signals respectively. Let n represent the output noise i.e. if $n \neq 0$ then $e = r - (y + n)$. The specification is to design a controller so that the closed loop is stable and has gain less than 0.04 from input n to output y for frequencies above 200 rad/sec. What is the cross-over frequency.

Solution :
Firstly, we can show that the requirement of gain less than 0.04 from input n to output y for frequencies greater then 200 rad/sec translates to $|L(j\omega)| = |K(j\omega)G(j\omega)| \leq 0.04$ for $\omega \geq 200\frac{rad}{sec}$. Thus $K_p \leq \frac{0.04}{|G(j200)|} \approx 2$ will ensure this requirement. Additionally, we can verify the closed loop pole of the system with $\frac{-5}{12} < K_p$ is in the LHP so the closed loop system is stable. Thus choosing $K_p = 2$ meet the both requirements.

The cross-over frequency is approximately 7.82 rad/sec.

12.4 Design a Stable Controller

Given a system's transfer function $G(s) = \frac{12}{3s+5}$, use loop-shaping procedure to design a controller $K(s)$ such that the closed loop system is stable and it satisfies:

(a) has a loop crossover frequency near 20 rad/sec.

(b) gain less than 0.01 from reference r to error e for frequencies below 0.1 rad/sec.

(c) gain less than 0.04 from input n to output y for frequencies above 200 rad/sec.

Verify that the closed-loop is stable with your control design.

Solution :
We can use a proportional gain to set the loop cross-over frequency (requirement (a)). Then use an integral boost and roll-off to satisfy requirements (b) and (c) respectively. This may require some iteration to get values that meet all design specifications.

First we choose the proportional gain so that the loop gain has crossover $|L(j\omega_c)| = |K(j\omega_c)G(j\omega_c)| = 1$ at $\omega_c = 20$ rad/sec. This can be done with the gain $K_1 = \frac{1}{|G(j\omega_c)|} \approx 5.02$. This will give the first loop-shape $L_1(s) = G(s)K_1$. The first loop shape has DC gain $L_1(s) = K_1G(s)$. Thus we need to increase the gain by a factor of $\frac{100}{12} = 8.4$ to ensure that we satisfy requirement (b). We'll use an integral boost $K_2(s) = \frac{s+\bar{\omega}_2}{s}$ where $\bar{\omega}_2$ is the frequency below which the gain starts to increase. With some iteration the choice $\bar{\omega}_2 = 5$rad/sec seem to easily exceed the low frequency requirement. Notice that $\bar{\omega}_2 < \omega_c$. For stability and robustness reasons we need the slope of $|L|$ to be shallow near ω_c. Thus we need to choose our low-frequency boost to start sufficiently below ω_c that it doesn't have a significant impact on the slope at ω_c. After this stage our controller is $K_1K_2(s)$ and our loop-shape is $L_2(s) = K_1K_2(s)G(s)$. Finally, we need to modify the loop shape to satisfy the design requirement iv. We can choose a roll-off to decrease the high frequency gain $K_3(s) = \frac{\bar{\omega}_3}{s+\bar{\omega}_3}$. Again, we want to choose $\bar{\omega}_3 > \omega_c$ so that the roll-off has negligible effect on the slope of $|L|$ near cross-over. However, we also want to choose $\bar{\omega}_3$ small enough that we decrease the high frequency gain enough to satisfy $|L(j200)| \leq 0.0385$. After some trial and error $\bar{\omega}_3 = 80$ rad/sec ensures that the requirements are satisfied. Our final controller is $K(s) = K_1K_2(s)K_3(s)$ and our final loop-shape is $L_2(s) = K(s)G(s)$. The Bode plot and step responses are shown below. The step response has a small steady state error and good noise rejection. The response also has a small overshoot.

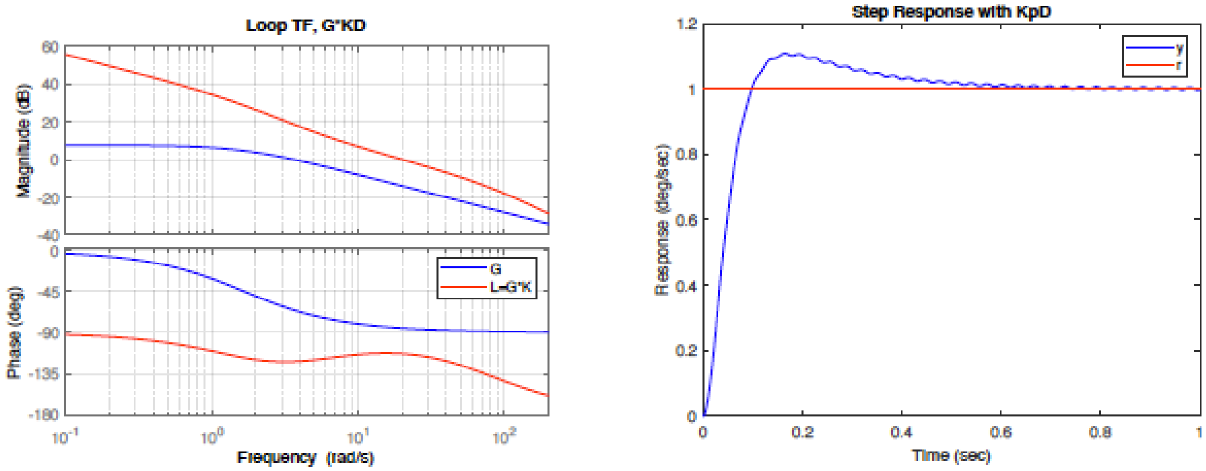


Figure 44: Bode Plot and step response for closed loop system.