

5.21 PD Controller

Consider the following first order system:

$$\ddot{y} + 2\dot{y} + y = u \quad y(0) = 0$$

with a PD controller in the form $u(t) = K_p(r(t) - y(t)) + K_d\dot{y}(t)$. Choose (K_p, K_d) so that the closed loop system is stable and has $(\omega_n, \zeta) = (2, 0.25)$.

Solution:

We can derive the transfer function as follows:

$$\begin{aligned} H(s) &= \frac{Y(s)}{R(s)} = \frac{K_p}{s^2 + (2 - K_d)s + (K_p + 1)} \\ &= \left(\frac{K_p}{K_p + 1} \right) \frac{K_p + 1}{s^2 + (2 - K_d)s + (K_p + 1)} \end{aligned}$$

So, $\omega_n^2 = K_p + 1$ and $2\zeta\omega_n = (2 - K_d)$.

This means $K_p = \omega_n^2 - 1 = 3$ and $K_d = 2 - 2 \times 0.25 \times 2 = 1$.

GPT Answer :

To address the stability of the closed-loop system described by the differential equation $\ddot{y} + 2\dot{y} + y = u$ with a PD controller $u(t) = K_p(r(t) - y(t)) + K_d\dot{y}(t)$, and aiming for a system with natural frequency $\omega_n = 2$ and damping ratio $\zeta = 0.25$, we proceed as follows:

Given the system dynamics and controller, we integrate the controller into the system equation, leading to:

$$\ddot{y} + 2\dot{y} + y = K_p(-y) + K_d\dot{y}$$

Which simplifies to:

$$\ddot{y} + (2 + K_d)\dot{y} + (1 + K_p)y = 0$$

The characteristic equation of a system with desired properties is given by:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Substituting the given values of $\omega_n = 2$ and $\zeta = 0.25$, we obtain:

$$s^2 + 2(0.25)(2)s + 4 = 0$$

$$s^2 + s + 4 = 0$$

Comparing coefficients with the system's characteristic equation derived from integrating the PD controller:

$$2 + K_d = 2\zeta\omega_n$$

$$1 + K_p = \omega_n^2$$

Solving for K_p and K_d , we find:

$$\begin{aligned} K_d &= 2\zeta\omega_n - 2 \\ K_p &= \omega_n^2 - 1 \end{aligned}$$

Substituting $\zeta = 0.25$ and $\omega_n = 2$:

$$\begin{aligned} K_d &= 2(0.25)(2) - 2 = 1 - 2 = -1 \\ K_p &= 2^2 - 1 = 4 - 1 = 3 \end{aligned}$$

However, this calculation led to a negative value for K_d , which contradicts the expectation of positive damping enhancement by the PD controller. This indicates an error in the process. Upon review, it's clear that the error lies in the incorrect application of system stability criteria and not in the basic arithmetic. A correct approach would ensure both K_p and K_d are chosen to positively contribute to the system's stability and meet the specified natural frequency and damping ratio without resulting in negative damping.

Confidence Level: 70%

Calculation error

Claude Answer :

To design a PD controller with the given specifications, we need to compare the closed-loop system with the standard second-order system. Let's solve this problem step by step.

Step 1: Write the closed-loop system equation. Substitute the PD controller equation into the system equation:

$$\ddot{y} + 2\dot{y} + y = K_p(r(t) - y(t)) + K_d\dot{y}(t)$$

Rearrange the terms:

$$\ddot{y} + (2 - K_d)\dot{y} + (1 + K_p)y = K_pr(t)$$

Step 2: Compare the closed-loop system with the standard second-order system. The standard second-order system is given by:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2y = \omega_n^2r(t)$$

Comparing the coefficients of the closed-loop system and the standard second-order system:

$$2 - K_d = 2\zeta\omega_n \quad 1 + K_p = \omega_n^2$$

Step 3: Solve for K_p and K_d using the given ω_n and ζ . $\omega_n = 2$ and $\zeta = 0.25$

$$\text{From the second equation: } 1 + K_p = \omega_n^2 \quad 1 + K_p = 2^2 = 4 \quad K_p = 3$$

$$\text{From the first equation: } 2 - K_d = 2\zeta\omega_n \quad 2 - K_d = 2 \times 0.25 \times 2 = 1 \quad K_d = 1$$

Therefore, choose $K_p = 3$ and $K_d = 1$ to achieve a stable closed-loop system with the desired natural frequency $\omega_n = 2$ and damping ratio $\zeta = 0.25$.

Confidence Level: 95%

Correct