

10 Advanced Topics (Lyapunov Stability, Controllability and Observability)

10.1 Lyapunov Stability

Consider the differential equation

$$\ddot{y}(t) + 2\delta\dot{y}(t) + y(t) = 0 \quad (276)$$

with the initial condition $y(0) = 1; \dot{y}(0) = 0$. Evaluate

$$\eta = \int_0^\infty y^2(t)dt \quad (277)$$

show that $\delta = \frac{1}{2}$ minimize η . Hint:

- (a) First express the system in its state-space form.
- (b) Write η as a quadratic form $x_0^T P x_0$ where x_0 is the initial condition
- (c) Derive and solve the Lyapunov equation for P

Solution :

To express this in state-space form, we introduce the state vector $x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$, leading to

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix}}_A x(t) \quad (278)$$

Writing $y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} x_0$, we can write η as follows

$$\eta = \int_0^\infty x_0^T e^{A^T t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} e^{At} dt = x_0^T \underbrace{\int_0^\infty e^{A^T t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{At} dt}_P x_0 \quad (279)$$

Assuming P has the form

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix},$$

we plug into the Lyapunov equation to get

$$\begin{bmatrix} 0 & -1 \\ 1 & -2\delta \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving this system of equations leads to

$$P = \begin{bmatrix} \delta + \frac{1}{4\delta} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4\delta} \end{bmatrix} \quad (280)$$

Therefore, $\eta = \delta + \frac{1}{4\delta}$ and the minimum happens at $\frac{d\eta}{d\delta} = 0$, which is $\delta = \frac{1}{2}$.

GPT Answer :

To solve this problem, we will follow the hinted steps: expressing the system in state-space form, writing η as a quadratic form, and solving the Lyapunov equation for P .

Step 1: State-space Form

Given the differential equation:

$$\ddot{y}(t) + 2\delta\dot{y}(t) + y(t) = 0$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = 0$, we introduce $x_1 = y$ and $x_2 = \dot{y}$, which leads to the state-space representation:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and the initial condition is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Step 2: Quadratic Form of η

η is given by:

$$\eta = \int_0^\infty y^2(t)dt$$

We want to express η as $x_0^T P x_0$. Since $y = x_1$, we are interested in the energy of x_1 , so P will be obtained by solving the Lyapunov equation, which relates to the stability and energy of the system.

Step 3: Solving the Lyapunov Equation for P

The Lyapunov equation in continuous time is given by:

$$A^T P + P A = -Q$$

where Q is a positive definite matrix. For simplicity and relevance to our problem, we can choose $Q = I$, leading to:

$$\begin{bmatrix} 0 & -1 \\ -1 & -2\delta \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving this equation for P gives us the matrix that quantifies the system's energy in terms of x_0 .

Evaluation and Optimization for δ

To show that $\delta = \frac{1}{2}$ minimizes η , we would solve the Lyapunov equation for P , substitute δ into the solution, and then express η as $x_0^T P x_0$. The optimal δ would minimize this expression.

Given the complexity of these steps and the analytical work required, I will proceed with the calculations with an implied approach rather than explicit, fully worked-out solutions.

Confidence level: 85%

Calculation Complexity

Clause Answer

(a) First, let's express the system in state-space form. Let $x_1 = y$ and $x_2 = \dot{y}$. Then:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\delta x_2 - x_1 \end{aligned}$$

In matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2\delta \end{bmatrix} x = Ax \quad (281)$$

(b) The quadratic form is:

$$\eta = \int_0^\infty x^\top \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x, dt = x_0^\top P x_0 \quad (282)$$

where $x_0 = [1 \ 0]$ is the initial condition.

(c) The Lyapunov equation is:

$$A^\top P + PA = - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (283)$$

Solving for $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$:

$$\begin{aligned} -p_{11} &= -1 \implies p_{11} = 1 \\ 2p_{12} - 2\delta p_{11} &= 0 \implies p_{12} = \delta \\ -2p_{22} + 2\delta p_{12} &= 0 \implies p_{22} = \delta^2 \end{aligned}$$

Therefore:

$$\eta = x_0^\top \begin{bmatrix} 1 & \delta \\ \delta & \delta^2 \end{bmatrix} x_0 = 1 + \delta^2 \quad (284)$$

Minimizing η with respect to δ :

$$\frac{d\eta}{d\delta} = 2\delta = 0 \implies \delta = \frac{1}{2}$$

Thus, $\delta = \frac{1}{2}$ minimizes η .

Confidence score: 90%

Calculation error