

5.19 PI Controller

Consider the following first order system:

$$\dot{y} = -0.5y + 2u \quad y(0) = 0$$

with a proportional-integral (PI) control law:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau$$

with $e(t) = r(t) - y(t)$ is the tracking error. Combining the system model and PI controller leads to the closed-loop system in the following form:

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{r} + b_0 r$$

How do the damping ratio and natural frequency depend on K_p and K_i ? What is the steady state error if r is a unit step?

Solution :

The second order system in the above ODE form gives us:

$$a_0 = 2K_p + 1 = 2 \quad a_1 = 2K_i \quad b_0 = 2K_i \quad b_1 = 2K_p$$

The characteristic polynomial then is:

$$s^2 + s(2K_p + \frac{1}{2}) + 2K_i \implies \omega_n = \sqrt{2K_i} \quad \text{and} \quad \zeta = \frac{4K_p + 1}{4\sqrt{2K_i}}$$

For a unit-step reference signal this results in zero steady state error.

GPT Answer :

To address this question, let's start by analyzing the system and the control law provided.

Given the system:

$$\dot{y} = -0.5y + 2u \quad (1)$$

with initial condition $y(0) = 0$ and the PI control law:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau \quad (2)$$

where $e(t) = r(t) - y(t)$ is the tracking error. The system dynamics with the controller can be expressed as a second-order differential equation:

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{r} + b_0 r \quad (3)$$

Substituting the PI controller into the system equation, we can identify the coefficients a_1 , a_0 , b_1 , and b_0 in terms of K_p and K_i . The natural frequency ω_n and damping ratio ζ for a second-order system are generally defined by the standard form:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \text{forcing terms}$$

Without going into the intricate algebraic manipulations, we can state that ω_n and ζ would depend on the coefficients a_1 and a_0 , which are in turn functions of K_p and K_i . Specifically, $\omega_n = \sqrt{a_0}$ and $\zeta = \frac{a_1}{2\sqrt{a_0}}$. Hence, both the natural frequency and damping ratio depend on the values of K_p and K_i , but determining the exact dependence requires substitution and simplification that we haven't explicitly performed here.

For the steady-state error analysis with $r(t)$ being a unit step, the final value theorem in Laplace transform is typically used, which states:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

where $F(s)$ is the Laplace transform of $f(t)$. For a system with PI control, the presence of the integral component (K_i) ensures that the steady-state error for a step input is zero. This is because the integral term accumulates the error over time, adjusting the control action to eliminate steady-state error. Therefore, for a unit step input $r(t)$, the steady-state error is 0.

In summary, the damping ratio ζ and natural frequency ω_n of the closed-loop system depend on the proportional (K_p) and integral (K_i) gains of the PI controller. The exact relationship involves the coefficients of the second-order differential equation representing the closed-loop system dynamics. The steady-state error for a unit step input is zero due to the integral action of the PI controller.

Confidence Level: 85%

Correct

Claude Answer :

To determine how the damping ratio and natural frequency depend on K_p and K_i , we first need to find the values of a_1 , a_0 , b_1 , and b_0 in terms of K_p and K_i .

From the closed-loop system equation:

$$\ddot{y} + a_1\dot{y} + a_0y = b_1\dot{r} + b_0r$$

Substituting the system model and PI controller:

$$\ddot{y} + a_1(-0.5y + 2u) + a_0y = b_1\dot{r} + b_0r$$

$$\ddot{y} + a_1(-0.5y + 2(K_p e + K_i \int_0^t e(\tau) d\tau)) + a_0y = b_1\dot{r} + b_0r$$

Comparing the coefficients, we get:

$$a_1 = 0.5 + 2K_p \quad a_0 = 2K_i \quad b_1 = 2K_p \quad b_0 = 2K_i$$

Now, the characteristic equation of the closed-loop system is:

$$s^2 + a_1s + a_0 = 0$$

The natural frequency ω_n and damping ratio ζ are given by:

$$\omega_n = \sqrt{a_0} = \sqrt{2K_i} \quad \zeta = \frac{a_1}{2\sqrt{a_0}} = \frac{0.5 + 2K_p}{2\sqrt{2K_i}}$$

Therefore, the natural frequency depends only on K_i , while the damping ratio depends on both K_p and K_i .

To find the steady-state error for a unit step input, we can use the final value theorem:

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{R(s) - Y(s)}{1}$$

For a unit step input, $R(s) = \frac{1}{s}$. The transfer function of the closed-loop system is:

$$\frac{Y(s)}{R(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} = \frac{2K_p s + 2K_i}{s^2 + (0.5 + 2K_p)s + 2K_i}$$

Therefore,

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{\frac{1}{s} - \frac{2K_p s + 2K_i}{s^2 + (0.5 + 2K_p)s + 2K_i}}{1} = 0$$

The steady-state error for a unit step input is zero, regardless of the values of K_p and K_i .

Confidence Level: 95%

Correct