

9.8 Phase Margin

Show that for the transfer function $KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$, the phase margin is independent of ω_n and is given as $\tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right)$.

Solution :

Given,

$$KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

To calculate the phase margin, we first find the gain-crossover-frequency (ω_c) :

$$\begin{aligned} |KG(j\omega)|_{\omega=\omega_c} = 1 &\Rightarrow \frac{\omega_n^2}{|-\omega_c^2 + 2j\omega_n\omega_c\zeta|} = 1 \\ &\Rightarrow \frac{\omega_n^2}{\sqrt{\omega_c^4 + 4\omega_n^2\omega_c^2\zeta^2}} = 1 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega_n^4 &= \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 \\ \Leftrightarrow \omega_n^4 + (2\zeta^2\omega_n^2)^2 &= \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 + (2\zeta^2\omega_n^2)^2 \\ \Leftrightarrow \omega_n^4 (1 + 4\zeta^4) &= (\omega_c^2 + 2\zeta^2\omega_n^2)^2 \end{aligned}$$

Thus, excluding the negative root (why?),

$$\begin{aligned} \omega_c^2 &= -2\zeta^2\omega_n^2 + \omega_n^2\sqrt{4\zeta^4 + 1} \\ &= \left(\sqrt{4\zeta^4 + 1} - 2\zeta^2\right)\omega_n^2 \end{aligned}$$

So we get,

$$KG(j\omega) = \frac{\omega_n^2}{-\omega_c^2 + 2j\zeta\omega_n\omega_c} \quad \text{and} \quad \angle KG(j\omega) = -\tan^{-1} \left(\frac{2\zeta\omega_n\omega_c}{-\omega_c^2} \right)$$

which is

$$\tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}} \right)$$

Note that $\theta = \tan^{-1} x \iff \pi + \theta = \tan^{-1}(x)$.

If we ask for formulas, Claude-3 gives $PM = \pi - \arctan\left(\frac{2\zeta}{\sqrt{1-2\zeta^2} + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}\right) = 180^\circ - \tan^{-1}\left(\frac{2\zeta(\zeta + \sqrt{\zeta^2 + 1})}{(\zeta + \sqrt{\zeta^2 + 1})^2 - 2\zeta(\zeta + \sqrt{\zeta^2 + 1})}\right)$, which is close but still wrong.

GPT Answer :

To show that the phase margin of the given transfer function $KG(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$ is independent of ω_n and to derive the given expression for it, we follow these steps:

1. **Find the Phase of $G(s)$** : The phase of $G(s)$ is obtained by substituting $s = j\omega$, where j is the imaginary unit and ω is the frequency. This yields the phase angle of the transfer function.
2. **Determine the Phase Margin**: The phase margin (PM) is the amount by which the phase angle is less than -180° at the gain crossover frequency, where the magnitude of $G(j\omega)$ is 1 (or 0 dB).

Let's start by substituting $s = j\omega$ into $KG(s)$ to determine its phase:

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega}$$

The phase of $G(j\omega)$ is the arctan of the imaginary part divided by the real part:

$$\text{Phase}(G(j\omega)) = \arctan\left(\frac{2\zeta\omega_n\omega}{-\omega^2}\right) = \arctan(-2\zeta\omega_n\omega / \omega^2)$$

Since we're interested in the phase margin, we need the phase at the gain crossover frequency, where $|G(j\omega)| = 1$. However, for this transfer function, determining the exact gain crossover frequency is not necessary for understanding the phase margin's dependence on ζ and its independence from ω_n . Instead, we consider the phase margin conceptually as the phase at the point where the magnitude of the transfer function equals unity, plus 180° .

To make this determination, note that the phase of a transfer function as it approaches the gain crossover frequency involves evaluating the contribution of each term in the denominator to the overall phase. The phase margin is then:

$$PM = 180^\circ + \text{Phase}(G(j\omega))$$

However, to directly show the expression for the phase margin given in the question:

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4 + 1} - 2\zeta^2}}\right)$$

This expression is derived from a more detailed analysis that takes into account the specific relationship between the phase and magnitude of the system at the gain crossover frequency, and it effectively demonstrates the phase margin's dependency solely on ζ , not on ω_n . The derivation of this exact expression requires an understanding of the system's response in both the time and frequency domains, specifically how the phase and magnitude relate at the point of unity gain.

Confidence Level: 70%

Correct

Claude Answer :

To show that the phase margin for the given transfer function is independent of ω_n and is given by the expression $\tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}} \right)$, let's follow these steps:

Step 1: Write the transfer function in the standard form. $KG(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2-\omega_n^2} = \frac{\omega_n^2}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}$

Step 2: Find the frequency at which the magnitude of the transfer function is 1 (0 dB), i.e., the gain crossover frequency ω_{gc} . $|KG(j\omega_{gc})| = 1 \Rightarrow \frac{\omega_n^2}{(\omega_{gc})^2+2\zeta\omega_n\omega_{gc}} = 1 \Rightarrow \omega_{gc} = \omega_n \sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$

Step 3: Calculate the phase angle at the gain crossover frequency. $\angle KG(j\omega_{gc}) = -\tan^{-1} \left(\frac{2\zeta\omega_n\omega_{gc}}{(\omega_{gc})^2} \right) = -\tan^{-1} \left(\frac{2\zeta\omega_n^2\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}{\omega_n^2(\sqrt{4\zeta^4+1}-2\zeta^2)} \right) = -\tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}} \right)$

Step 4: The phase margin (PM) is the difference between the phase angle at the gain crossover frequency and -180° . $PM = 180^\circ + \angle KG(j\omega_{gc}) = 180^\circ - \tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}} \right) = \tan^{-1} \left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}} \right)$

As we can see, the phase margin expression does not contain ω_n , which means it is independent of ω_n and depends only on the damping ratio ζ .

Confidence Level: 95%

Correct