

1.7 Equilibrium

Consider the scalar ODE

$$\dot{x} = x^2 - 1 \quad (27)$$

- Find all the equilibria
- Compute the linearization of the ODE about each equilibrium and determine the stability.

Solution :

- The equilibria occur where $\dot{x} = 0$. Solving the equation $x^2 - 1 = 0$, we find the equilibrium points are $x = 1$ and $x = -1$.
- The linearization involves computing the derivative of $f(x) = x^2 - 1$ with respect to x , which is $f'(x) = 2x$. Evaluating this derivative at each equilibrium point:
 - For $x = 1$, $f'(1) = 2$. Since $f'(1) > 0$, the equilibrium at $x = 1$ is unstable.
 - For $x = -1$, $f'(-1) = -2$. Since $f'(-1) < 0$, the equilibrium at $x = -1$ is stable.

Thus, the equilibrium at $x = 1$ is unstable, while the equilibrium at $x = -1$ is stable.

GPT Answer :

Part (a): Finding All the Equilibria

The equilibria of an ODE are found by setting the derivative to zero, $\dot{x} = 0$. For the given ODE,

$$\dot{x} = x^2 - 1, \quad (28)$$

setting $\dot{x} = 0$ gives us

$$x^2 - 1 = 0. \quad (29)$$

Solving this equation for x yields

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1. \quad (30)$$

Thus, there are two equilibria: $x_1 = 1$ and $x_2 = -1$.

Part (b): Linearization and Stability

The linearization of an ODE around an equilibrium point x_0 is given by the first derivative of the right-hand side of the ODE evaluated at x_0 , denoted as $f'(x_0)$. For the given ODE,

$$\dot{x} = x^2 - 1, \quad (31)$$

the derivative with respect to x is

$$f'(x) = 2x. \quad (32)$$

At $x_1 = 1$:

$$f'(1) = 2 \times 1 = 2. \quad (33)$$

Since $f'(1) > 0$, the equilibrium at $x_1 = 1$ is unstable.

At $x_2 = -1$:

$$f'(-1) = 2 \times (-1) = -2. \quad (34)$$

Since $f'(-1) < 0$, the equilibrium at $x_2 = -1$ is stable.

Conclusion and Confidence Level

We found that the given scalar ODE has two equilibria at $x = \pm 1$, with the equilibrium at $x = 1$ being unstable and the equilibrium at $x = -1$ being stable. My confidence level in the accuracy of this answer is 95%.

Confidence Level: 95%

Correct

Claude Answer :

(a) The equilibria of the ODE $\dot{x} = x^2 - 1$ are the points where $\dot{x} = 0$. Solving $x^2 - 1 = 0$ gives the two equilibria: $x_1 = -1$ and $x_2 = 1$.

(b) To determine the stability, we linearize the ODE about each equilibrium:

At $x_1 = -1$: Let $x = -1 + \epsilon$, where ϵ is a small perturbation. Substituting into the ODE:

$$\dot{\epsilon} = (-1 + \epsilon)^2 - 1 = \epsilon^2 - 2\epsilon$$

Linearizing by neglecting the ϵ^2 term:

$$\dot{\epsilon} \approx -2\epsilon$$

The linearized system has a negative coefficient, indicating that $x_1 = -1$ is a stable equilibrium.

At $x_2 = 1$: Let $x = 1 + \epsilon$. Substituting into the ODE:

$$\dot{\epsilon} = (1 + \epsilon)^2 - 1 = \epsilon^2 + 2\epsilon$$

Linearizing:

$$\dot{\epsilon} \approx 2\epsilon$$

The linearized system has a positive coefficient, indicating that $x_2 = 1$ is an unstable equilibrium.

Confidence level: 95/100

Correct