

5.3 Minimal-Order Control Design

Consider a nominal model for a plant given by:

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{1}{(s+1)(s-2)} \quad (163)$$

The closed-loop characteristic polynomial is defined as:

$$A_{cl}(s) = (s+2)(s+3)(s+4)(s+5) \quad (164)$$

Design a controller $C(s)$ that satisfies the following conditions:

- Results in the given closed-loop characteristic polynomial.
- Has the minimum order numerator to ensure system stability.
- Express both the numerator and the denominator of $C(s)$ as expanded polynomials.

Provide the transfer function of the designed controller in the format: " $C(s) = \frac{P(s)}{L(s)} = \frac{\text{Numerator}}{\text{Denominator}}$ ", where Numerator and Denominator are fully expanded polynomials.

Solution :

The degree of $A_{cl}(s)$ is four and the degree of $A_o(s)$ is two. Hence $L(s)$ should have a degree equal to two. Then

$$C(s) = \frac{p_2 s^2 + p_1 s + p_0}{s^2 + \lambda_1 s + \lambda_0} \quad (165)$$

The corresponding pole assignment equation becomes

$$A_o(s)L(s) + B_o(s)P(s) = (s+2)(s+3)(s+4)(s+5) \quad (166)$$

$$(s+1)(s-2)(s^2 + \lambda_1 s + \lambda_0) + (p_2 s^2 + p_1 s + p_0) = (s+2)(s+3)(s+4)(s+5) \quad (167)$$

$$s^4 + (\lambda_1 - 1)s^3 + (p_2 - \lambda_1 + \lambda_0 - 2)s^2 + (p_1 - \lambda_0 - 2\lambda_1)s + (p_0 - 2\lambda_0) = s^4 + 14s^3 + 71s^2 + 154s + 120 \quad (168)$$

This polynomial identity leads to the equations

$$\lambda_1 - 1 = 14 \quad (169)$$

$$p_2 - \lambda_1 + \lambda_0 - 2 = 71 \quad (170)$$

$$p_1 - \lambda_0 - 2\lambda_1 = 154 \quad (171)$$

$$p_0 - 2\lambda_0 = 120 \quad (172)$$

The solution of these equations is

$$\lambda_1 = 15 \quad (173)$$

$$p_2 = 73 + \lambda_1 - \lambda_0 = 88 - \lambda_0 \quad (174)$$

$$p_1 = 154 + 2\lambda_1 + \lambda_0 = 184 + \lambda_0 \quad (175)$$

$$p_0 = 120 + 2\lambda_0 \quad (176)$$

We thus observe that there is an infinite number of solutions. Every choice of λ_0 leads to a different stabilizing controller, for example, to force integration in the controller we can choose $\lambda_0 = 0$

$$C(s) = \frac{88s^2 + 184s + 120}{s^2 + 15s} \quad (177)$$

In MATLAB we can solve this problem with the command line

```
>> [L,P]=paq([1 -1 -2],1,[1 14 71 154 120])
```

This leads to

$$C(s) = \frac{272s + 296}{s^2 + 15s + 88} \quad (178)$$

This is equivalent to choosing $\lambda_0 = 88$. The algorithm in `paq.m` has been designed to yield a minimum degree $P(s)$.

GPT Answer:

To design a controller $C(s)$ that satisfies the given conditions, we start with the relationship between the plant $G_o(s)$, the controller $C(s)$, and the closed-loop characteristic polynomial $A_{cl}(s)$. The closed-loop transfer function $T(s)$ for a unity feedback system is given by:

$$T(s) = \frac{G_o(s)C(s)}{1 + G_o(s)C(s)}$$

The characteristic equation of the closed-loop system is then:

$$1 + G_o(s)C(s) = 0$$

Substituting the given $G_o(s)$ into the characteristic equation:

$$1 + \frac{C(s)}{(s+1)(s-2)} = 0$$

The desired closed-loop characteristic polynomial $A_{cl}(s)$ is:

$$A_{cl}(s) = (s+2)(s+3)(s+4)(s+5)$$

To ensure that the closed-loop system has the characteristic polynomial $A_{cl}(s)$, the denominator of the open-loop transfer function (which includes the controller and the plant) must match $A_{cl}(s)$ when set equal to zero. Thus, we set:

$$G_o(s)C(s) = A_{cl}(s) - A_o(s)$$

Given that $A_o(s) = (s + 1)(s - 2)$ and knowing $G_o(s) = \frac{B_o(s)}{A_o(s)}$, we can rearrange the equation to solve for $C(s)$:

$$C(s) = \frac{A_{cl}(s) - A_o(s)}{G_o(s)} = \frac{A_{cl}(s)}{G_o(s)} - 1$$

Since $G_o(s) = \frac{1}{(s+1)(s-2)}$, we find:

$$C(s) = (s + 2)(s + 3)(s + 4)(s + 5)(s + 1)(s - 2) - 1$$

To proceed with the solution, let's calculate the product and subtract 1, resulting in the expanded form of $C(s)$.

Let's calculate this:

The transfer function of the designed controller $C(s)$ is:

$$C(s) = \frac{s^6 + 13s^5 + 55s^4 + 55s^3 - 176s^2 - 428s - 241}{1}$$

This controller satisfies the given conditions, with the numerator as the expanded polynomial representing the controller's action, and the denominator being 1, indicating a direct controller without additional dynamics.

Confidence level: 95%

Reasoning Error

Claude Answer:

$$\text{Given: } G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{1}{(s+1)(s-2)} \quad A_{cl}(s) = (s + 2)(s + 3)(s + 4)(s + 5)$$

Step 1: Expand the polynomials $A_o(s)$, $B_o(s)$, and $A_{cl}(s)$:

$$A_o(s) = s^2 - s - 2 \quad B_o(s) = 1 \quad A_{cl}(s) = s^4 + 14s^3 + 71s^2 + 150s + 120$$

Step 2: Solve the Diophantine equation by equating the coefficients of like powers of s :

$$A_{cl}(s) = A_o(s)L(s) + B_o(s)P(s)$$

Let $L(s) = l_2 s^2 + l_1 s + l_0$ and $P(s) = p_2 s^2 + p_1 s + p_0$ (minimum order numerator for stability).

Equating the coefficients:

$$s^4 : l_2 = 1 \quad s^3 : -l_2 + l_1 = 14 \quad s^2 : -2l_2 - l_1 + l_0 + p_2 = 71 \quad s^1 : 2l_2 - l_0 + p_1 = 150 \quad s^0 : 2l_0 + p_0 = 120$$

Solving the system of equations:

$$l_2 = 1, l_1 = 15, l_0 = 60, p_2 = 58, p_1 = 92, p_0 = 0$$

Therefore, the designed controller $C(s)$ is:

$$C(s) = \frac{P(s)}{L(s)} = \frac{58s^2 + 92s}{s^2 + 15s + 60}$$

Calculation Error

Confidence level: 95%