① ~~ ~~ 4^{th} order effective Hamiltonian

$$T_n = \sum_{\alpha_1,\ldots\alpha_{n-1}} \frac{\langle f|V|\alpha_1\rangle\langle\alpha_1|V|\alpha_2\rangle\ldots\langle\alpha_{n-1}|V|i\rangle}{(E_{f\alpha_1}+i\epsilon)\ldots(E_{f\alpha_{n-1}}+i\epsilon)}$$

$$V_{eff} = H_1 + H_2 + \ldots$$

From these : $\langle f|T_1|i\rangle = \langle f|V|i\rangle$

To ensure the effective ~~potential~~ theory has a ~~ ~~ Hamiltonian which matches the transition matrix :

$$\langle f|H_1|i\rangle_{eff} = \langle f|V|i\rangle$$

For second order we have

$$\langle f|T_2|i\rangle = \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_{f\alpha_1}}$$

Lower energy states ($\alpha<$) are already accounted for by H_1 so for H_2

$$\langle f|H_2|i\rangle_{eff} = \sum_{\alpha}^{>} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_{f\alpha}}$$

For third order we get a new term β

$$\langle f|T_3|i\rangle = \sum_{\alpha,\beta} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{E_{f\alpha}E_{f\beta}}$$

We do the same as we did previously eliminating terms that we've already accounted for but maintaing the high energy states we havent

(2) 4^{th} order effective Hamiltonian

First term $\rightarrow$ High energy states

$$\langle f|H_3|i\rangle_{eff} = \sum^{>}_{\alpha,\beta} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{E_{f\alpha}\, E_{f\beta}}$$

$$- \sum^{<}_{\alpha}\sum^{>}_{\beta} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{E_{\alpha\beta}\, E_{f\beta}}$$

Produced by the lower order effective operators so we remove them

~~we~~ the subtraction term isn't explicitly here for H_3, its here to remove terms introduced by H_1 and H_2 and prevent double counting

Table of combinations:

	α	β	
1	$>$	$>$	We want this
2	$<$	$>$	Appears through H_1 and H_2 so eliminate
3	$>$	$<$	$\beta <$ never appears in H_3 to begin wit
4	$<$	$<$	Low energy (H_1) and not double cou

3 and 4 dont appear in H_3

3

4^{th} order effective Hamiltonian

$$\langle f | T_4 | i \rangle = \sum_{\alpha,\beta,\gamma} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|\gamma\rangle\langle\gamma|V|i\rangle}{E_{f\alpha}\, E_{f\beta}\, E_{f\gamma}}$$

$$= \sum_{\alpha,\beta,\gamma} \frac{V_{f\alpha}\, V_{\alpha\beta}\, V_{\beta\gamma}\, V_{\gamma i}}{E_{f\alpha}\, E_{f\beta}\, E_{f\gamma}} = \sum_{\alpha,\beta,\gamma} \frac{V_4}{E_4}$$

α	β	γ		α	β	γ	
>	–	–	×	>	<	–	×
–	>	–	×	>	–	<	×
–	–	>	×	–	>	<	×
>	>	–	×	<	>	–	×
>	–	>	×	<	–	>	×
–	>	>	×	–	<	>	×
>	>	>	✓				
–	–	–	×				
<	–	–	×				
–	<	–	×	>	<	<	✓
–	–	<	×	<	>	<	✓
<	<	–	×	<	<	>	✓
<	–	<	×	>	>	<	✓
–	<	<	×	>	<	>	✓
<	<	<	× 1	<	>	<	✓

Anything with – eliminated because don't want full range for any. 1 eliminated because all < so low energy handled already

4^{th} order effective Hamiltonian

α	β	γ	
>	>	>	1— Main high energy stuff
>	<	<	2— Already generated by H_1, H_2, H_3 as previous
<	>	<	
<	<	>	
>	>	<	3— Parts removed by ↑ but we still need
>	<	>	
<	>	>	

$$\langle f | H_4 | i \rangle = \sum_{\alpha,\beta,\gamma}^{>} \frac{V_{f\alpha} V_{\alpha\beta} V_{\beta\gamma} V_{\gamma i}}{E_{f\alpha} E_{f\beta} E_{f\gamma}} \quad \Big\} \; 1$$

$$- \sum_{\alpha}^{>} \sum_{\beta\gamma}^{<} \frac{V_4}{E_4} - \sum_{\beta}^{>} \sum_{\alpha\gamma}^{<} \frac{V_4}{E_4} - \sum_{\gamma}^{>} \sum_{\alpha\beta}^{<} \frac{V_4}{E_4} \quad \Big\} \; 2$$

$$+ \sum_{\alpha\beta}^{>} \sum_{\gamma}^{<} \frac{V_4}{E_4} + \sum_{\alpha\gamma}^{>} \sum_{\beta}^{<} \frac{V_4}{E_4} + \sum_{\beta\gamma}^{>} \sum_{\alpha}^{<} \frac{V_4}{E_4} \quad \Big\} \; 3$$